# A preservation theorem for theories without the tree property of the first kind 

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#### Abstract

We prove that the $\mathrm{NTP}_{1}$ property of a geometric theory $T$ is inherited by theories of lovely pairs and $H$-structures associated to $T$. We also provide a class of examples of nonsimple geometric $\mathrm{NTP}_{1}$ theories.


## 0 Introduction

One theme of research in model theory is to inquire whether some well-known properties are preserved under a certain unary predicate expansions of a given structure. One of the motivations for this is that positive theorems of this kind often allows us to obtain interesting and complicatedlooking theories which still satisfy some strong tameness conditions.

The study of expansions by unary predicates reaches back to the paper of Poizat on beautiful pairs [10], and has been developed in various directions ever since. Remarkable papers on this subject include for example [5], where the stability condition is examined, and [1], which is dedicated to studying expansions in simple theories (which generalize the stable ones).

The well-known equivalence $\mathrm{TP} \Leftrightarrow \mathrm{TP}_{1} \vee \mathrm{TP}_{2}$ due to Shelah [11] (where TP denotes the tree property while $\mathrm{TP}_{1}$ and $\mathrm{TP}_{2}$ denote the tree properties of the first and second kind, respectively), suggests two natural generalizations of simple theories, namely $\mathrm{NTP}_{1}$ theories and $\mathrm{NTP}_{2}$ theories (i.e., theories without $\mathrm{TP}_{1}$ and $\mathrm{TP}_{2}$, respectively). So far, $\mathrm{NTP}_{1}$ and $\mathrm{NTP}_{2}$ theories have been studied much less extensively than the simple ones (i.e. theories without TP). However, recently some interesting results on these theories began to appear, notably [6] and [7]. In particular, natural examples of non-simple $\mathrm{NTP}_{1}$ thoeries were provided in [6, namely: $\omega$-free PAC fields, linear spaces with a generic bilinear form and a class of theories obtained by the "pfc" construction.

The study of expansions in the $\mathrm{NTP}_{2}$ context was undertaken in [2], where it was shown that the $\mathrm{NTP}_{2}$ property is preserved under "dense and codense" unary predicate expansions of geometric structures where the unary predicate is assumed to define either an algebraically independent subset or an elementary substructure. In the present paper, we prove that the $\mathrm{NTP}_{1}$ property is also preserved under such expansions. One of the main ingredients in our proof is the recently proved fact (due to Chernikov and Ramsey [6]) that the $\mathrm{TP}_{1}$ property can, in any $\mathrm{TP}_{1}$ theory, always be witnessed by some formula in a single free variable. We also prove (in Section (4) that an $\mathrm{NTP}_{1}$ nonsimple geometric theory can be obtained from any Fraïssé limit which has a simple

[^0]theory by applying some constructions from [6]. This yields a large class of nonsimple (so also not $\mathrm{NTP}_{2}$ ) theories satisfying the assumptions of our main theorem.

Our paper is organized as follows. In Section [1, we review some essential facts about dense codense predicate expansions from [2]. In Section 2, we state slightly modified versions of some results from [6] concerning $\mathrm{SOP}_{2}$ (equivalently $\mathrm{TP}_{1}$ ). In Section 3, we prove our main result, namely, that $\mathrm{NTP}_{1}$ is preserved under the unary predicate expansions defined in Section 1 . In Section 4 , we show that the "pfc" construction from [6] preserves a certain strengthening of geometricity, and conclude from this that a class of geometric nonsimple $\mathrm{NTP}_{1}$ structures can be obtained via imaginary cover and "pfc" operations.

## 1 Dense codense predicate expansion

In this section, we review some basic facts about the dense codense predicate expansions.
Recall that a theory is called geometric if (1) it eliminates the quantifier $\exists^{\infty}$, and (2) the algebraic closure satisfies the exchange property. Examples of geometric theories include all SUrank 1 theories (in particular, strongly minimal theories). For nonsimple NTP $_{1}$ examples see Section 4.

Throughout, unless stated otherwise, variables may have an arbitrary length. The symbol $\downarrow$ denotes the algebraic independence relation.

Definition 1.1 Let $T$ be a geometric complete theory in a language $\mathcal{L}$, and let $\mathcal{L}_{H}:=\mathcal{L} \cup\{H\}$ be the extended language obtained by adding a new unary predicate symbol $H$. For any model $M \vDash T$, let $(M, H(M))$ denote an expansion of $M$ to the extended language $\mathcal{L}_{H}$, where $H(M):=\{x \in M \mid$ $H(x)\}$.

1. We say $(M, H(M))$ is dense codense if every non-algebraic 1-type in $\mathcal{L}$ over any finite dimensional subset $A \subseteq M$ has realizations both in $H(M)$ and in $M \backslash \operatorname{acl}_{T}(A \cup H(M))$.
2. A dense codense expansion $(M, H(M))$ is called a lovely pair (resp. H-structure) if $H(M)$ happens to be an elementary substructure (resp. algebraically independent subset) of $M$.

Fact/Definition [3, 4] Let $T$ be any geometric complete theory. Then all of its lovely pairs have the same theory, i.e., they are elementarily equivalent to one another. The same holds for $H$-structures. We let $T_{P}$ and $T^{\text {ind }}$ denote the common complete theories of the lovely pairs and $H$-structures, respectively, associated with $T$. By $T^{*}$, we shall mean either $T_{P}$ or $T^{\text {ind }}$.

For the remainder of this section, we shall work inside some fixed, sufficiently saturated model $(M, H(M)) \vDash T^{*}$ unless stated otherwise. When $x$ is a tuple of variables, $H(x)$ shall mean the conjunction $H\left(x_{1}\right) \wedge \cdots \wedge H\left(x_{n}\right)$ where $x_{i}$ 's are the variables occurring in $x$. When $A$ is a subset of $M, H(A)$ denotes the set $\{x \in A \mid H(x)\}$.

Definition 1.2 For any subset $B \subseteq M$, we define

$$
\operatorname{scl}(B):=\operatorname{acl}_{T}(B \cup H(M))
$$

which is called the small closure of $B$. If $A$ is any subset of $\operatorname{scl}(B)$, we shall say that $A$ is $B$-small.
Definition 1.3 $A$ subset $A \subseteq M$ is said to be $H$-independent if $A \downarrow_{H(A)} H(M)$.
The following two facts will be important tools in the proof of our main result.

Fact $1.4([2])$ For any $\mathcal{L}_{H}$-formula $\varphi(x, a)$ where $a$ is $H$-independent, there exists some $\mathcal{L}$-formula $\psi(x, a)$ such that

$$
\varphi(x, a) \wedge H(x) \leftrightarrow \psi(x, a) \wedge H(x) .
$$

Fact $1.5([2])$ For any $\mathcal{L}_{H}$-formula $\varphi(x, a)$ where $x$ is a single variable and $a$ is $H$-independent, there exists some $\mathcal{L}$-formula $\psi(x, a)$ such that the symmetric difference $\varphi(x, a) \triangle \psi(x, a)$ defines an a-small set.

We will also use the following observations:
Fact $1.6([2])$ 1. For any finite tuple $c$, there exists some finite tuple $h$ in $H(M)$ such that $c \downarrow_{h} H(M)$.
2. For any $H$-independent tuple $c$ and any tuple $h$ in $H(M)$, ch is $H$-independent.

We end this section by remarking that all the results and their proofs in this paper may be carried over to many-sorted contexts. However, for the sake of simplifying our arguments, we shall assume that our theory $T$ is one-sorted throughout the paper.

## 2 Overview of some results on $\mathrm{SOP}_{2}$ from [6]

In this section, we state some results about $\mathrm{SOP}_{2}$ from [6] in slightly modified ('localized') versions which we will need later. But first, let us quickly review some basic terminologies. We consider the language $L_{0}:=\left\{\triangleleft,<_{\text {lex }}, \wedge\right\}$ where $\triangleleft$ and $<_{\text {lex }}$ are binary relation symbols and $\wedge$ is a binary function symbol. Then any set $\alpha^{<\beta}$ (where $\alpha$ and $\beta$ are ordinals) admits a natural $\mathcal{L}_{0}$-structure whereby $\triangleleft$ is interpreted as the prefix partial order, $<_{l e x}$ as the lexicographic order and $\wedge$ as the infimum function (with respect to the prefix order). We will use the following 'localized' version of $\mathrm{SOP}_{2}$ :

Definition 2.1 A formula $\phi(x ; y)$ is said to have $S O P_{2}$ inside a type $q(x)$ if there are tuples $\left(a_{\eta}\right)_{\eta \in 2<\omega}$ satisfying the following two properties:

1. For every $\xi \in 2^{\omega}$, the set $q(x) \cup\left\{\phi\left(x, a_{\xi \mid n}\right): n<\omega\right\}$ is consistent;
2. For every pair of $\triangleleft$-incomparable elements $\eta, \nu \in 2^{<\omega}$, the formula $\phi\left(x ; a_{\eta}\right) \wedge \phi\left(x ; a_{\nu}\right)$ is inconsistent.

And a theory has $S O P_{2}$ inside of $q(x)$ if some formula has it inside $q(x)$.
(The original, non-localized definition of $\mathrm{SOP}_{2}$ is obtained by setting $q=\emptyset$.)
By compactness, we easily get:
Remark 2.2 If $\phi(x ; y)$ has $S O P_{2}$ inside a type $q(x)$ witnessed by $\left(a_{\eta}\right)_{\eta \in 2^{<\omega}}$, then for every $\xi \in 2^{\omega}$ the type $q(x) \cup\left\{\phi\left(x, a_{\xi \mid n}\right): n<\omega\right\}$ is nonalgebraic.

Now, let us recall the notion of modeling property on strongly indiscernible trees, which we will use repeatedly in the paper.

Definition 2.3 We say that a tree $\left(a_{\eta}\right)_{\eta \in S}$ of compatible tuples of elements of a model $M$ is strongly indiscernible over a set $C \subseteq M$, if

$$
q f p_{L_{0}}\left(\eta_{0}, \ldots, \eta_{n-1}\right)=\operatorname{qftp}_{L_{0}}\left(\nu_{0}, \ldots, \nu_{n-1}\right)
$$

implies $\operatorname{tp}\left(a_{\eta_{0}}, \ldots, a_{\eta_{n-1}} / C\right)=\operatorname{tp}\left(a_{\nu_{0}}, \ldots, a_{\nu_{n-1}} / C\right)$ for all $n<\omega$ and all tuples $\left(\eta_{0}, \ldots, \eta_{n-1}\right),\left(\nu_{0}, \ldots, \nu_{n-1}\right)$ of elements of $S$.

The following fact comes from [13].
Fact 2.4 Let $\mathfrak{C}$ be a monster model of a complete theory. Then for any tree of parameters $\left(a_{\eta}\right)_{\eta \in \omega<\omega}$ from $\mathfrak{C}$ there is a strongly indiscernible tree $\left(b_{\eta}\right)_{\eta \in \omega<\omega}$ based on the tree $\left(a_{\eta}\right)_{\eta \in \omega<\omega}$, which means that for every $\eta_{0}, \ldots, \eta_{n-1} \in \omega^{<\omega}$ there are $\mu_{0}, \ldots, \mu_{n-1} \in \omega^{<\omega}$ such that qftp $p_{L_{0}}\left(\eta_{0}, \ldots, \eta_{n-1}\right)=$ $q f t_{L_{0}}\left(\mu_{0}, \ldots, \mu_{n-1}\right)$ and $\operatorname{tp}\left(b_{\eta_{0}}, \ldots, b_{\eta_{n-1}}\right)=\operatorname{tp}\left(a_{\mu_{0}}, \ldots, a_{\mu_{n-1}}\right)$.

Notice that if $q$ is over $\emptyset$, then the consistency condition in Definition 2.1 is preserved under tree modeling. Hence, inside of such a $q, \mathrm{SOP}_{2}$ is always witnessed by a strongly indiscernible tree of parameters.

Remark 2.5 With the notation from the above definition, the set $q(x) \cup\left\{\phi\left(x, a_{\xi \mid n}\right): n<\omega\right\}$ has infinitely many realizations for any $\xi \in 2^{\omega}$.

By a very slight modification of the proof of Lemma 4.6 for [6], we get:
Fact 2.6 Suppose $\left(a_{\eta}\right)_{\eta \in 2^{<\omega}}$ is a tree strongly indiscernible over $C$ such that ( $a_{0^{\alpha}}: 0<\alpha<\omega$ ) is indiscernible over $c C$. Let

$$
p(y ; z)=\operatorname{tp}\left(c ;\left(a_{\left.0 \frown 0^{\gamma}\right)_{\gamma<\omega}}\right) / C\right)
$$

and let $q(y)$ be a type over $\emptyset$ contained in $\operatorname{tp}(c)$. Then, if

$$
q(y) \cup p\left(y ;\left(a_{0} \frown 0^{\gamma}\right)_{\gamma<\omega}\right) \cup p\left(y ;\left(a_{1 \frown 0^{\gamma}}\right)_{\gamma<\omega}\right)
$$

is inconsistent, then $T$ has $S O P_{2}$ inside of $q$.
Proof. By naming parameters we can assume that $C=\emptyset$. Suppose the type $q(y) \cup p\left(y ;\left(a_{0} \frown 0^{\gamma}\right)_{\gamma<\omega}\right) \cup$ $p\left(y ;\left(a_{0}{ }^{1 \gamma}\right)_{\gamma<\omega}\right)$ is not consistent. By compactness and indiscernibility there is a formula $\psi \in p$ such that

$$
q(y) \cup\left\{\psi\left(y, a_{0}, \ldots, a_{0 \frown 0^{n-1}}\right), \psi\left(y, a_{1}, \ldots, a_{1 \frown 0^{n-1}}\right)\right\}
$$

is inconsistent. Then as in [6], the $n$-fold elongation (see Definition 2.6 from [6]) of $\left(a_{\eta}\right)_{\eta \in 2<\omega}$ witnesses that $\psi$ has $\mathrm{SOP}_{2}$ inside of $q$.

Using the above fact and modifying the proof of Theorem 4.8 from [6] in the same manner as above (i.e., replacing any set of formulas related to a consistency condition by its union with an appropriate type over $\emptyset$ ), we obtain:

Fact 2.7 Suppose a theory $T$ has $S O P_{2}$ inside of some type $q\left(x_{0}, \ldots, x_{n-1}\right)=\bigcup_{i<n} q_{i}\left(x_{i}\right)$. Then, for some $i<n, T$ has $S O P_{2}$ inside of $q_{i}\left(x_{i}\right)$.

## 3 The main result

The aim of this section is to prove our main result, i.e. Theorem 3.6. First, we give a characterization of $\mathrm{SOP}_{2}$ thoeries that we will need later.

Proposition 3.1 $A$ theory $T$ has $S O P_{2}$ if there is a formula $\phi(x, y)$ and a strongly indiscernible tree $\left(a_{\eta}\right)_{\eta \in 2^{2}<\omega}$, such that the set $\left\{\phi\left(x, a_{0^{n}}\right): n<\omega\right\}$ has infinitely many realizations, and the formula $\phi\left(x, a_{0}\right) \wedge \phi\left(x, a_{1}\right)$ has finitely many realizations.

Proof. Let $\phi(x, y)$ and $\left(a_{\eta}\right)_{\eta \in 2<\omega}$ be as above. For a set $B \subseteq 2^{<\omega}$, we put

$$
A_{B}:=\bigcap_{\eta \in B} \phi\left(\mathfrak{C}, a_{\eta}\right),
$$

and for a tuple $b=\left(\eta_{0}, \ldots, \eta_{n-1}\right) \in\left(2^{<\omega}\right)^{n}$, we put

$$
A_{b}:=\bigcap_{i<n} \phi\left(\mathfrak{C}, a_{\eta_{i}}\right)
$$

Claim 1 We can assume that

$$
\begin{equation*}
(\forall n, m \in \omega \backslash\{0\})\left(A_{0,1}=A_{0^{n}, 1^{m}}\right) . \tag{*}
\end{equation*}
$$

Proof of Claim 1. Since the set $A_{\left\{0^{k}, k^{k}: k=1,2,3, \ldots\right\}}$ is finite, by compactness, it is equal to $D:=$ $A_{\left\{0^{k}, 1^{k}: k=1,2,3, \ldots K\right\}}$ for some $K<\omega$. So for any positive, pairwise distinct $k_{1}, \ldots, k_{2 K}<\omega$, the set $A_{0^{k_{1}}, \ldots, 0^{k_{K}}, 1^{k_{K+1}}, \ldots, 1^{k_{2 K}}}$ is contained in $D$, but by indiscernibility it has the same (finite) cardinality as $D$, so it is equal to $D$. Replacing $\phi(x, y)$ by $\psi\left(x, y_{0}, \ldots, y_{K-1}\right)=\bigwedge_{i<K} \phi\left(x, y_{i}\right)$ and each $a_{\epsilon_{0}, \ldots, \epsilon_{m-1}}$ by $a_{\epsilon_{0}^{K}, \ldots, \epsilon_{m-1}^{K}}$ we obtain a tree with the desired properties.
So we will assume that (*) holds.
Claim 2 We can assume (in addition to (*)) that

$$
\begin{equation*}
A_{\left\{0^{n} 1: n<\omega\right\}}=\emptyset . \tag{**}
\end{equation*}
$$

Proof of Claim 2. Since the set $A_{\left\{0^{n} 1: n<\omega\right\}}$ is finite, it is equal to $D:=A_{\left\{0^{n} 1: n=0,1,2, \ldots K-1\right\}}$ for some $K<\omega$. Then $D=A_{0^{k_{0} 1, \ldots, 0^{k} K-11}}$ for any pairwise distinct $k_{i}$ 's. Since the tree $\left.\left(a_{0^{K}}{ }^{\prime}\right)_{\eta}\right)_{\eta \in 2^{2}<\omega}$ is strongly indiscernible over $\left\{a_{0^{k} 1}: k=0,1,2, \ldots K-1\right\}$, we can replace $\phi(x, y)$ by $\phi(x, y) \wedge$ $\neg \psi\left(x, y_{0}, \ldots, y_{K-1}\right)$, where $\psi\left(x, a_{1}, \ldots, a_{0^{k-1} 1}\right)$ defines $D$, and $a_{\eta}$ by $\left(a_{0^{K} \neg \eta}, a_{1}, \ldots, a_{0^{K-1} 1}\right)$, guaranteeing $(* *)$ while preserving $(*)$ and strong indiscernibility of the tree.
Let us assume that $(*)$ and $(* *)$ hold. Then there is a maximal $n$ such that $A_{\left\{1,01, \ldots, 0^{n-1} 1,0^{n}, 0^{n+1}, \ldots\right\}}$ is nonempty. Put $c=\left(a_{1}, \ldots, a_{0^{n-1} 1}\right), \psi\left(x, y_{0}, \ldots, y_{n-1}\right)=\bigwedge_{i<K} \phi\left(x, y_{i}\right)$ and $\phi^{\prime}\left(x, y, y_{0}, \ldots, y_{n-1}\right)=$ $\phi(x, y) \wedge \psi\left(x, y_{0}, \ldots, y_{n-1}\right)$. We claim that the strongly indiscernible tree $\left(b_{\eta}\right)_{\eta \in 2<\omega}$, where $b_{\eta}:=$ $a_{0^{n} \frown \eta} c$, witnesses $\mathrm{SOP}_{2}$ of $\phi^{\prime}(x, y)$. Indeed, by the choice of $n$ (and by strong indiscernibility) all paths are consistent. Moreover, by maximality of $n$, the set $\left\{\phi^{\prime}\left(x, b_{1} c\right)\right\} \cup\left\{\phi^{\prime}\left(x, b_{0^{k}} c\right): k=1,2, \ldots\right\}$ is inconsistent, but by $(*)$ (and by the strong indiscernibility of the tree $\left(a_{\eta}\right)$ ) this set is equivalent to the formula $\phi^{\prime}\left(x, b_{0} c\right) \wedge \phi^{\prime}\left(x, b_{1} c\right)$, so we get that the latter formula is inconsistent, and we are done. (Note that, a posteriori, by Remark [2.2, the $n$ chosen above must be equal to zero, i.e. the formula $\phi\left(x, a_{0}\right) \wedge \phi\left(x, a_{1}\right)$ is already inconsistent if we assume $(*)$ and $(* *)$.)

For the remainder of this section, we will work inside a sufficiently saturated model $(M, H(M)) \vDash$ $T^{*}$.

Lemma 3.2 Let $\phi(x, y)$ be any $\mathcal{L}_{H}$-formula witnessing $S O P_{2}$. Then for some dummy variables $z$, the formula $\phi(x, y z)$ witnesses $S O P_{2}$ with some strongly indiscernible tree consisting of $H$ independent tuples.

Proof. Let $\left(a_{\eta}\right)_{\eta \in 2<\omega}$ be a strongly indiscernible tree witnessing that $\phi(x, y)$ is $\mathrm{SOP}_{2}$. By Fact 1.6. choose a finite tuple $h_{\emptyset}$ of elements of $H(M)$ such that $a_{\emptyset} h_{\emptyset}$ is $H$-independent. For any $\eta \in 2^{<\omega}$ let $h_{\eta}$ be a conjugate of $h_{\emptyset}$ under an automorphism (in the sense of $T_{H}$ ) sending $a_{\emptyset}$ to $h_{\emptyset}$. Then any indiscernible tree based on $\left(a_{\eta} h_{\eta}\right)_{\eta \in 2<\omega}$ will satisfy the conclusion.

We will need one more preparatory lemma.

Lemma 3.3 If there is some $\mathcal{L}_{H}$-formula $\phi(x, y)$ such that $\phi(x, y) \wedge H(x)$ witnesses $S O P_{2}$, then $T$ has $\mathrm{SOP}_{2}$.

Proof. By 2.7 (applied to $\phi(x, y)$ and $q(x):=H(x)$ ), we can assume that $x$ is a single variable. Let $\left(a_{\eta}\right)_{\eta \in 2<\omega}$ be a strongly indiscernible tree witnessing $\mathrm{SOP}_{2}$ of $\phi(x, y) \wedge H(x)$ such that $\left(a_{\eta}\right)$ is $H$-independent. By Fact 1.4, there is an $\mathcal{L}$-formula $\psi(x, y)$ agreeing with $\phi$ on $H$. For any $\eta \in 2^{<\omega}$, the formula $\psi\left(x, a_{\eta \frown 0}\right) \wedge \psi\left(x, a_{\eta \frown 1}\right)$ is algebraic, since otherwise, by the density of $H$, it would be realized inside of $H$, a contradiction. So, by Proposition [3.1, $\psi(x, y)$ has $\mathrm{SOP}_{2}$.

In the final proof we will use one more characterization of $\mathrm{TP}_{1}$ property, which was proved in [9]. First, we remind the definition of $k-\mathrm{TP}_{1}$ from there:

Definition 3.4 $A$ formula $\psi(x, y)$ has $k-T P_{1}$ if there are tuples $c_{\beta}, \beta \in \omega^{<\omega}$ such that for each $\beta \in \omega^{\omega}$ the set $\left\{\psi\left(x, c_{\beta_{\mid m}}\right): m<\omega\right\}$ is consistent, and for any pairwise incomparable elements $\beta_{0}, \ldots, \beta_{k-1} \in \omega^{<\omega}$ the set $\left\{\psi\left(x, c_{\beta_{i}}: i<k\right\}\right.$ is inconsistent.

Fact 3.5 Suppose that an $\mathcal{L}$ formula $\phi(x, y)$ and a tree $\left(a_{\eta}\right)_{\eta \in \omega<\omega}$ witness $k$ - $T P_{1}$ for some $k \geq 2$. Then for some $d<\omega$, the $\mathcal{L}$-formula $\psi\left(x, y_{0}, \ldots, y_{d}\right)=\phi\left(x, y_{0}\right) \wedge \cdots \wedge \phi\left(x, y_{d}\right)$ witnesses $2-T P_{1}$.

Now we are in a position to prove the main result.
Theorem 3.6 If $T^{*}$ has $S O P_{2}$, then so does $T$.
Proof. Assume $T^{*}$ has $\mathrm{SOP}_{2}$ witnessed by an $\mathcal{L}_{H}$-formula $\phi(x, y)$, where $x$ is a single variable (we can assume that by 2.7) and a strongly indiscernible tree $A=\left\{\left(a_{\eta}\right)_{\eta \in 2<\omega}\right\}$, where each $a_{\eta}$ is $H$-independent.
Case 1: No realization of $\wedge_{i} \phi\left(x, a_{0^{i}}\right)$ is in $\operatorname{scl}(\mathrm{A})$. By Fact 1.5, let $\psi(x, y)$ be a formula such that for each $\eta, \phi\left(x, a_{\eta}\right) \Delta \psi\left(x, a_{\eta}\right)$ defines an $a_{\eta}$-small set. Then for any $\eta, \psi\left(x, a_{\eta-0}\right) \wedge \psi\left(x, a_{\eta-1}\right)$ has finitely many realizations, since otherwise, by the co-density condition, it would have a realization outside of $\operatorname{scl}(A)$, so realizing the formula $\phi\left(x, a_{\eta-0}\right) \wedge \phi\left(x, a_{\eta-1}\right)$. Also, every realization of $\wedge_{i} \phi\left(x, a_{0^{i}}\right)$ is a realization of $\wedge_{i} \psi\left(x, a_{0^{i}}\right)$, so we are done by Proposition 3.1.
Case 2: There is some $b \in \operatorname{scl}(A)$ satisfying $\wedge_{i} \phi\left(x, a_{0^{i}}\right)$. So $b$ realizes some algebraic formula $\theta(x, c, h)$, where $c$ and $h$ are tuples of elements of $A$ and $H$, respectively. We can assume that for any $c^{\prime}$ and $h^{\prime}$ the formula $\theta\left(x, c^{\prime}, h^{\prime}\right)$ has at most $k$ realizations, where $k<\omega$ is fixed. Choose $N<\omega$ such that $c$ is contained in $\left\{a_{\eta}: \eta \in 2^{<N}\right\}$. Put $d_{\eta}:=a_{0^{N}} \frown \eta$. Then $\phi(x, y)$ together with the tree $\left(d_{\eta}\right)_{\eta \in 2<\omega}$, which is strongly indiscernible over $c$, still witnesses $\mathrm{SOP}_{2}$. Put

$$
\mu(z, c, y):=H(z) \wedge \exists x(\theta(x, c, z) \wedge \phi(x, y))
$$

Then, since $\wedge_{n} \mu\left(z, c, d_{0^{n}}\right)$ is realized by $h$ (this is is witnessed by substituting $b$ for $x$ ), we get by the indiscernibility of $\left(d_{\eta}\right)_{\eta}$ over $c$ that $\wedge_{n} \mu\left(z, c, d_{\xi_{\mid n}}\right)$ is consistent for any $\xi \in 2^{\omega}$. Also, for any pairwise incomparable $\eta_{1}, \ldots, \eta_{n} \in 2^{<\omega}$, the set $\left\{\mu\left(z, c, d_{\eta_{i}}\right): i \leq n\right\}$ is $k+1$-inconsistent. Hence, by compactness, $\mu(z, x, y)$ has $(k+1)-T P_{1}$. It follows from Fact 3.5 that some $\mathcal{L}_{H}$ formula of the form $H(z) \wedge \nu(z)$ has $T P_{1}$, so also $\mathrm{SOP}_{2}$. We conclude by Lemma 3.3,

## 4 Examples of geometric nonsimple NTP $_{1}$ theories.

We start by outlining the "pfc" construction from Subsection 6.3 of [6]. For the reader's convenience, we repeat the definitions used there.

Definition 4.1 Suppose $K$ is a class of finite structures. We say that $K$ has the Strong Amalgamation Property (SAP) if given $A, B, C \in K$ and embeddings $e: A \rightarrow B$ and $f: A \rightarrow C$ there is $D \in K$ and embeddings $g: B \rightarrow D$ and $h: C \rightarrow D$ such that

1) $g e=h f$ and
2) $i m(g) \cap i m(h)=i m(g e)$ (and hence $=i m(h f)$ as well).

We will say that a theory is SAP if it has a countable ultrahomogeneous model whose age is SAP. The following criterion comes from [8].

Fact 4.2 Suppose $K$ is the age of a countable structure $M$. Then the following are equivalent:

1) $K$ has $S A P$
2) $M$ has no algebraicity

Let $K$ denote an SAP Fraïssé class in a finite relational language $\mathcal{L}=\left(R_{i}: i<k\right)$, where each $R_{i}$ has arity $n_{i}$. Denote by $T$ the theory of the Fraïssé limit of the class $K$. Then $\mathcal{L}_{p f c}$ is defined to be a two-sorted language, with the sorts denoted by $O$ and $P$, and relation symbols $R_{x}^{i}\left(x, y_{1}, y_{2}, \ldots, y_{n_{i}}\right)$, where $x$ is a variable of the sort $P$ and $y_{i}$ 's are variables of the sort $O$. Given an $\mathcal{L}_{p f c}$-structure $M=(A, B)$ and $b \in B$, the $\mathcal{L}$-structure associated to $b$ in $M$, denoted $A_{b}$, is defined to be the $\mathcal{L}$-structure interpreted in $M$ with domain $A$ and each $R_{i}$ interpreted as $R_{b}^{i}(A)$. Put

$$
K_{p f c}=\left\{M=(A, B) \in \operatorname{Mod}\left(L_{p f c}\right):|M|<\omega,(\forall b \in B)(\exists D \in K)\left(A_{b} \simeq D\right)\right\} .
$$

Fact 4.3 ([6]) $K_{p f c}$ is a Fraïssé class satisfying SAP.
Thanks to the above fact, there is a unique countable ultrahomogeneous $\mathcal{L}_{p f c}$-structure with age $K_{p f c}$. Let $T_{p f c}$ denote its theory. Then $T_{p f c}$ has quantifier elimination. Let us recall two facts from [6] that will be crucial for us.

Fact 4.4 Suppose $(A, B) \models T_{p f c}$. Then, for all $b \in B, A_{b} \models T$.
Fact 4.5 Suppose $T$ is a simple theory which is the theory of a Fraïssé limit of a SAP Fraïssé class $K$. Then $T_{p f c}$ is $N S O P_{1}$. Moreover, if the D-rank of $T$ is at least 2, then $T_{p f c}$ is not simple.

Now, we aim to prove that the "pfc" construction applied to a geometric theory satisfying condition $\operatorname{acl}(A)=A$ for any $A$, gives a geometric theory.

Let $N=(A, B)$ be a monster model of $T_{p f c}$.
Lemma 4.6 For any $A_{0} \subseteq A$ and $B_{0} \subseteq B$ we have that $\operatorname{acl}\left(A_{0} B_{0}\right) \cap B=B_{0}$.
Proof. Clearly we can assume that both $A_{0}$ and $B_{0}$ are finite. Put $C=A_{0} B_{0}$, take any $b \in B \backslash B_{0}$ and fix any natural number $n$. We will show that the orbit of $b$ over $C$ has at least $n$ elements. To see this, consider a finite $\mathcal{L}_{p f c}$-superstructure $E=\left(A_{0}, D\right)$ of $\left(A_{0}, B_{0}\right) \subseteq N$, where $D=B_{0} \cup\left\{b, d_{1}, \ldots, d_{n}\right\}$ with $d_{j}$ 's being pairwise distinct elements not belonging to $\{b\} \cup B_{0}$, and for each $i, j, R_{d_{j}}^{i}\left(A_{0}\right)$ is equal to $R_{b}^{i}\left(A_{0}\right)$ in the sense of $N$. Then clearly $E \in K_{p f c}$, so we can assume $E=\left(A_{0}, D\right)$ is a substructure of $N$ (by changing $d_{i}$ 's appropriately). Then, for every $j$, we have that $q f t p\left(d_{j} / C\right)=q f t p(b / C)$, so we are done due to the quantifier elemination in $T_{p f c}$.

Lemma 4.7 Suppose $\phi_{j}\left(x, a_{j}, b_{j}\right)$ for $j=1, \ldots, n$ are non-algebraic $\mathcal{L}_{p f c}$-formulas in a single variable of the first sort, where each $a_{j}$ is a tuple of elements of $A$ and $b_{j}$ are pairwise distinct elements of $B$. Then the conjunction $\phi:=\wedge_{j} \phi_{j}$ is non-algebraic.

Proof. Suppose for a contradiction that $\phi$ is algebraic, and denote by $A_{0}$ the finite set $\phi(A)$. We consider a finite $\mathcal{L}_{p c f}$-structure $E$ with universe $(C, D)$, where $C=A_{0} \cup\{c\}, c \notin A_{0}, D=\left\{b_{1}, \ldots, b_{n}\right\}$, and interpretation of symbols of $\mathcal{L}_{p c f}$ given as follows. For any $j \leq n$, by the non-algebraicity of $\phi_{i}$ we choose its realization $c_{j} \in A \backslash A_{0}$. Now, let $f_{j}: C \rightarrow A_{0} \cup\left\{c_{j}\right\}$ be the bijection whose restriction to $A_{0}$ is the identity. We interpret every $R_{b_{j}}^{i}$ in $E$ as $f_{j}^{-1}\left[R^{i}\left(A_{0} \cup\left\{c_{j}\right\}\right)\right]$. Then $E$ belongs to $K_{p f c}$ so it embeds in $N$ via some function $g$, and $g(c) \in N \backslash A_{0}$ is an realization of $\phi$. This is a contradiction to the choice of $A_{0}$, so the lemma is proved.

Corollary 4.8 If $T$ is geometric and satisfies condition $\operatorname{acl}(A)=A$ for any $A$, then $T_{p f c}$ is geometric satisfying the same condition.

Proof. First, we show the definability of infinity. If $\phi(x, y)$ is a formula with $x$ being a single variable of the sort $P$ (and $y$ of any lenght), then by Lemma 4.6, for any $c$, if $\phi(x, c)$ is algebraic then each of its realizations belongs to $c$, so $\phi(x, c)$ is algebraic iff it has at most $|y|$ realizations. Now, if $x$ is a variable of the sort $O$, then any formula in variable $x$ can be presented in the form $\phi\left(x, y_{j, l}, z_{j, l}\right)_{j, l}:=\vee_{l} \wedge_{j} \phi_{j, l}\left(x, y_{j, l}, z_{j, l}\right)$, where each $z_{j, l}$ is a single variable of the sort $P$, and $y_{j, l}$ are tuples of variables of the sort $O$ (we can obtain such a presentation since atomic formulas in $\mathcal{L}_{p f c}$ can involve only a single variable from the sort $P)$. Then $\phi\left(x, a_{j, l}, b_{j, l}\right)_{j, l}$ is algebraic if and only if for each $l_{0}$ the formula $\wedge_{j} \phi_{j, l_{0}}\left(x, a_{j, l_{0}}, b_{j, l_{0}}\right)$ is algebraic. But by Lemma 4.7 this holds iff there are $j_{1}, \ldots, j_{s}$ such that $b_{j_{1}, l_{0}}=b_{j_{2}, l_{0}}=\cdots=b_{j_{s}, l_{0}}$ and $\wedge_{t \leq s} \phi_{j, l_{0}}\left(x, a_{j_{t}, l_{0}}, b_{j_{t}, l_{0}}\right)$ is algebraic. By Fact 4.4 the latter is a definable condition on $a_{j, l_{0}}, b_{j, l_{0}}$, so we obtain the definability of infinity.

As to the condition $\operatorname{acl}(A)=A$, by Lemma 4.6 it is enough to check that for any finite $A_{0} \subset A$ and $B_{0} \subset B$ we have that $\operatorname{acl}\left(A_{0} B_{0}\right) \cap A=A_{0}$. Consider any $a \in \operatorname{acl}\left(A_{0} B_{0}\right) \cap A$. Then there are formulas $\phi_{j}\left(x, a_{j}, b_{j}\right)$ as in the statement of Lemma 4.7, such that the conjunction $\phi(x):=\wedge_{j} \phi_{j}\left(x, a_{j}, b_{j}\right)$ is algebraic and satisfied by $a$. By Lemma 4.7, for some $j$ the formula $\phi_{j}\left(x, a_{0}, b_{j}\right)$ is algebraic. By Fact 4.4 and the assumptions on $T$ this implies that $a$ in $\operatorname{acl}\left(a_{j}\right)=a_{j}$ in the sense of the structure $A_{b_{j}}$, so $a \in A_{0}$.

Remark 4.9 By a similar argument we can show, assuming only that $T$ is geometric, that in $T_{p f c}$ we have definability of infinity, and the following weaker form of exchange principle:

$$
a \in \operatorname{acl}(A b) \backslash \operatorname{acl}(A) \Longrightarrow b \in \operatorname{acl}(A a)
$$

for any parameter set $A$ and a belonging to the same sort as $b$. However, this condition seems not to be sufficient to prove a generalization of Theorem 3.6 by our methods.

Let us recall one more operation from [6] which we need to obtain examples of nonsimple NTP $_{1}$ theories. Given an $\mathcal{L}$-structure $M$, the imaginary cover $\tilde{M}$ of $M$ is defined to be the structure in language $\mathcal{L}^{\prime}$ obtained from $\mathcal{L}$ by adding a binary relation symbol $E$, constructed by replacing each element of $M$ with an infinite $E$-class and interpreting the symbols of $\mathcal{L}$ in the natural way. By Remark 6.19 from [6] we have:

Fact 4.10 If $T=T h(M)$ is simple and $S A P$, then $\tilde{T}:=T h(\tilde{M})$ is simple of $D$ rank at least 2 and SAP.

Let us notice the following.
Remark 4.11 The theory $\tilde{T}$ has definability of infinity and for any $A$ in a model of $\tilde{T}$ we have that $\operatorname{acl}(A)=A$.

Proof. The second clause is obvious, and for the first one notice that any atomic formula $\phi(x, a)$ in $\tilde{T}$ has either infinitely many or at most one realization. Hence, by the quantifier elimination we get definability of infinity for any formula.

Now we obtain the final corollary which yields a class of examples of nonsimple NTP $_{1}$ geometric theories.

Corollary 4.12 If $T$ is any SAP simple theory then $(\tilde{T})_{p f c}$ is a geometric nonsimple $N_{1} P_{1}$ theory.
Proof. By Remark 4.11, $\tilde{T}$ is geometric and satisfies the condition $\operatorname{acl}(A)=A$ for any $A$, so by Corollary 4.8 the same is true about $(\tilde{T})_{p f c}$. Moreover, $(\tilde{T})_{p f c}$ is $\mathrm{NTP}_{1}$ and nonsimple by Facts 4.5 and 4.10 .

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