# ULTRAFILTER EXTENSIONS DO NOT PRESERVE ELEMENTARY EQUIVALENCE 

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#### Abstract

We show that there exist models $\mathcal{M}_{1}$ and $\mathcal{M}_{2}$ such that $\mathcal{M}_{1}$ elementarily embeds into $\mathcal{M}_{2}$ but their ultrafilter extensions $\boldsymbol{\beta}\left(\mathcal{M}_{1}\right)$ and $\boldsymbol{\beta}\left(\mathcal{M}_{2}\right)$ are not elementarily equivalent.


## 1. Introduction

The ultrafilter extension of a first-order model is a model in the same vocabulary, the universe of which consists of all ultrafilters on the universe of the original model, and which extends the latter in a canonical way. This construction was introduced in [1]. The article [2] is an expanded version of [1]; it contains a list of problems, one of which is solved here.

The main precursor of the general construction was the ultrafilter extension of semigroups, called often the Čech-Stone compactification of semigroups. This particular case was discovered in 1970s and became since then an important tool for getting various Ramsey-theoretic results in combinatorics, algebra, and dynamics; the textbook [3] is a comprehensive treatise of this area. For theory of ultrafilters and for model theory we refer the reader to the standard textbooks [4] and 5], respectively.

Recall the construction of ultrafilter extensions and related basic facts.
Definition 1. For a set $M$, an ultrafilter $D$ on $M$, and a formula $\varphi(x, \ldots)$ with parameters $x, \ldots$, we let

$$
\left(\forall^{D} x\right) \varphi(x, \ldots) \text { if and only if }\{a \in M: \varphi(a, \ldots)\} \in D .
$$

It is easy to see that the ultrafilter quantifier is self-dual: it coincides with $\left(\exists^{D} x\right)$, defined as $\neg\left(\forall^{D} x\right) \neg$, since $D$ is ultra. Note also that if $D$ is the principal ultrafilter given by some $a \in M$, then $\left(\forall^{D} x\right) \varphi(x, \ldots)$ is reduced to $\varphi(a, \ldots)$, and that, e.g., $\left(\forall^{D_{1}} x_{1}\right)\left(\forall^{D_{2}} x_{2}\right) \varphi\left(x_{1}, x_{2}, \ldots\right)$ means $\left\{a_{1} \in M\right.$ : $\left.\left\{a_{2} \in M: \varphi\left(a_{1}, a_{2}, \ldots\right)\right\} \in D_{2}\right\} \in D_{1}$.

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Definition 2. Let $\mathcal{M}$ be a model in a vocabulary $\tau$ with the universe $M$. Define the model $\boldsymbol{\beta}(\mathcal{M})$ and the function $j_{M}$ as follows:
(a) the universe of $\boldsymbol{\beta}(\mathcal{M})$ is $\boldsymbol{\beta}(M)$, the set of ultrafilters on $M$,
(b) $j_{M}: M \rightarrow \boldsymbol{\beta}(M)$ is such that for all $a \in M, j_{M}(a)$ is the principal ultrafilter on $M$ given by $a$, i.e., $j_{M}(a)=\{A \subseteq M: a \in A\}$,
(c) if $P \in \tau$ is an $n$-ary predicate symbol (other than the equality symbol), let

$$
P^{\boldsymbol{\beta}(\mathcal{M})}=\left\{\left(D_{1}, \ldots, D_{n}\right):\left(\forall^{D_{1}} x_{1}\right) \ldots\left(\forall^{D_{n}} x_{n}\right) P^{\mathcal{M}}\left(x_{1}, \ldots, x_{n}\right)\right\},
$$

(d) if $F \in \tau$ is an $n$-ary function symbol, let

$$
\begin{gathered}
F^{\boldsymbol{\beta ( \mathcal { M } )}}\left(D_{1}, \ldots, D_{n}\right)=D \text { if and only if } \\
(\forall A \subseteq M)\left(A \in D \Leftrightarrow\left(\forall^{D_{1}} x_{1}\right) \ldots\left(\forall^{D_{n}} x_{n}\right) F^{\mathcal{M}}\left(x_{1}, \ldots, x_{n}\right) \in A\right) .
\end{gathered}
$$

The model $\boldsymbol{\beta}(\mathcal{M})$ is the ultrafilter extension of the model $\mathcal{M}$, and $j_{M}$ is the natural embedding of $\mathcal{M}$ into $\boldsymbol{\beta}(\mathcal{M})$.

The using of words "extension" and "embedding" is easily justified:
Proposition 1. If $\mathcal{M}$ is a model in a vocabulary $\tau$, then
(a) $\boldsymbol{\beta}(\mathcal{M})$ is also a model in $\tau$, and
(b) $j_{M}$ isomorphically embeds $\mathcal{M}$ into $\boldsymbol{\beta}(\mathcal{M})$.

Proof. See [1], [2].
The following result, called the First Extension Theorem in [2], shows that the ultrafilter extension lifts certain relationships between models.

Theorem 1. Let $\mathcal{M}_{1}$ and $\mathcal{M}_{2}$ be two models in the same vocabulary with the universes $M_{1}$ and $M_{2}$, respectively, and let $h$ be a mapping of $M_{1}$ into $M_{2}$ and $\widetilde{h}$ its (unique) continuous extension of $\boldsymbol{\beta}\left(M_{1}\right)$ into $\boldsymbol{\beta}\left(M_{2}\right)$ :


If $h$ is a homomorphism (epimorphism, isomorphic embedding) of $\mathcal{M}_{1}$ into $\mathcal{M}_{2}$, then $\widetilde{h}$ is a homomorphism (epimorphism, isomorphic embedding) of $\boldsymbol{\beta}\left(\mathcal{M}_{1}\right)$ into $\boldsymbol{\beta}\left(\mathcal{M}_{2}\right)$.

Proof. See [1], [2].
Actually Theorem 1 is a special case of a stronger result, called the Second Extension Theorem in [2]. Here we omit its precise formulation, which involves topological concepts, and note only that it generalizes the standard topological fact stating that the Čech-Stone compactification is the largest one, to the case when the underlying discrete space $M$ carries
an arbitrary first-order structure. This confirms that the construction of ultrafilter extensions given in Definition 2 is canonical in a certain sense.

Theorem 1 holds also for certain other relationships between models (e.g., for so-called homotopies and isotopies, see [1], [2]). A natural task is a characterization of such relationships. In particular, one can ask whether elementary embeddings or elementary equivalence lift under ultrafilter extensions. This task was posed in [2] (see Problem 5.1 there and comments before it).

In this note, we answer this particular question in the negative. In fact, we establish a slightly stronger result:
Theorem 2 (the Main Theorem). There exist models $\mathcal{M}_{1}$ and $\mathcal{M}_{2}$ in the same vocabulary such that $\mathcal{M}_{1}$ elementarily embeds into $\mathcal{M}_{2}$ but their ultrafilter extensions $\boldsymbol{\beta}\left(\mathcal{M}_{1}\right)$ and $\boldsymbol{\beta}\left(\mathcal{M}_{2}\right)$ are not elementarily equivalent:


Of course, it follows that neither elementary embeddings nor elementary equivalence are preserved under ultrafilter extensions. The construction of such models $\mathcal{M}_{1}$ and $\mathcal{M}_{2}$ will be provided in the next section.

We conclude this section with the following natural questions on possible general results in this direction.

Problem 1. Characterize (or at least, provide interesting necessary or sufficient conditions on) theories $T$ such that the implication

$$
\mathcal{M}_{1} \equiv \mathcal{M}_{2} \Rightarrow \beta\left(\mathcal{M}_{1}\right) \equiv \beta\left(\mathcal{M}_{2}\right)
$$

holds for all $\mathcal{M}_{1}, \mathcal{M}_{2} \vDash T$.
Problem 2. The same question for elementary embeddings.

## 2. Proof of the Main Theorem

First we define a vocabulary $\tau$ and construct two specific models $\mathcal{M}_{1}$ and $\mathcal{M}_{2}$ in $\tau$. Then we shall show that these models are as required.

Definition 3. Let $\tau$ be the vocabulary consisting of two unary predicate symbols $P_{1}$ and $P_{2}$, two binary predicate symbols $R_{1}$ and $R_{2}$, and one binary function symbol $F$.

Definition 4. Let $\mathcal{M}_{1}$ be a model in $\tau$ having the universe $M_{1}$ and defined as follows:
(a) $M_{1}=\mathbb{N} \sqcup \mathcal{P}(\mathbb{N})$, the disjoint sum of $\mathbb{N}$ and $\mathcal{P}(\mathbb{N})$ (which we shall identify with their disjoint copies),
(b) $P_{1}^{\mathcal{M}_{1}}=\mathbb{N}$,
(c) $P_{2}^{\mathcal{M}_{1}}=\mathcal{P}(\mathbb{N})$,
(d) $R_{1}^{\mathcal{M}_{1}}=\{(n, a): n \in \mathbb{N} \wedge a \in \mathcal{P}(\mathbb{N}) \wedge n \in a\}$, i.e., the intersection of the membership relation with $\mathbb{N} \times \mathcal{P}(\mathbb{N})$,
(e) $R_{2}^{\mathcal{M}_{1}}$ is a relation such that
( $\alpha) R_{2}^{\mathcal{M}_{1}} \cap(\mathbb{N} \times \mathbb{N})$ is the usual order on $\mathbb{N}$,
( $\beta$ ) $R_{2}^{\mathcal{M}_{1}} \cap(\mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N}))$ is a linear order on $\mathcal{P}(\mathbb{N})$ with no endpoints,
$(\gamma)$ if $a \in \mathbb{N} \Leftrightarrow b \notin \mathbb{N}$ then $R_{2}^{\mathcal{M}_{1}}(a, b)$ is defined arbitrarily (really this case will not be used),
(f) $F^{\mathcal{M}_{1}}$ is an unordered pairing function mapping $\mathbb{N}$ into $\mathbb{N}$ and $\mathcal{P}(\mathbb{N})$ into $\mathcal{P}(\mathbb{N})$, i.e., satisfying the following conditions:
$(\alpha)$ if either $a_{1}, b_{1}, a_{2}, b_{2} \in \mathbb{N}$ or $a_{1}, b_{1}, a_{2}, b_{2} \in \mathcal{P}(\mathbb{N})$, then $F^{\mathcal{M}_{1}}\left(a_{1}, b_{1}\right)=F^{\mathcal{M}_{1}}\left(a_{2}, b_{2}\right) \Leftrightarrow\left\{a_{1}, b_{1}\right\}=\left\{a_{2}, b_{2}\right\}$,
$(\beta)$ if $a, b \in \mathbb{N}$ then $F^{\mathcal{M}_{1}}(a, b) \in \mathbb{N}$,
$(\gamma)$ if $a, b \in \mathcal{P}(\mathbb{N})$ then $F^{\mathcal{M}_{1}}(a, b) \in \mathcal{P}(\mathbb{N})$,
( $\delta$ ) if $a \in \mathbb{N} \Leftrightarrow b \notin \mathbb{N}$ then $F^{\mathcal{M}_{1}}(a, b)$ is defined arbitrarily (really this case will not be used).

Proposition 2. Assume $\lambda \geq 2^{\aleph_{0}}$. Then there exists a model $\mathcal{M}_{2}$ in $\tau$ such that $\mathcal{M}_{1} \prec \mathcal{M}_{2}$ and $\left|P_{1}^{\mathcal{M}_{2}}\right|=\left|P_{2}^{\mathcal{M}_{2}}\right|=\lambda$.
Proof. Let $\mathcal{M}_{3}$ be $\lambda$-saturated and $\mathcal{M}_{1} \prec \mathcal{M}_{3}$. By the $\lambda$-saturatedness, for each $i \in\{1,2\}$ we have $\left|P_{i}^{\mathcal{M}_{3}}\right| \geq \lambda$, so we can pick $A_{i} \subseteq P_{i}^{\mathcal{M}_{3}}$ with $\left|A_{i}\right|=\lambda$. By the downward Löwenheim-Skolem Theorem, there exists a model $\mathcal{M}_{2}$ with the universe $M_{2}$ such that:
(a) $\mathcal{M}_{2} \prec \mathcal{M}_{3}$,
(b) $M_{1} \cup A_{1} \cup A_{2} \subseteq M_{2}$,
(c) $\left|M_{2}\right|=\lambda$,
whence it follows that $\mathcal{M}_{2}$ is a required model.
Alternatively, we can use a version of the upward Löwenheim-Skolem Theorem by picking two sets of constants, $C_{1}$ and $C_{2}$, with $\left|C_{1}\right|=\left|C_{2}\right|=\lambda$ and adding to the elementary diagram of $\mathcal{M}_{1}$ the formulas $P_{i}\left(c_{i}\right)$ for all $c_{i} \in C_{i}, i \in\{1,2\}$. The obtained theory is consistent (by compactness), so extract its submodel of cardinality $\lambda$ (by the downward LöwenheimSkolem Theorem) and reduce it to the required model $\mathcal{M}_{2}$ in the original vocabulary $\tau$.

Clearly, this observation is of a general character; a similar argument allows to get, for any model, its elementary extension in which all predicate symbols are interpreted by relations of the same cardinality.

To simplify reading, we slightly shorthand the notation for the ultrafilter extensions of the models $\mathcal{M}_{1}$ and $\mathcal{M}_{2}$ as follows:
Definition 5. For $\ell \in\{1,2\}$, let
(a) $\mathcal{N}_{\ell}=\boldsymbol{\beta}\left(\mathcal{M}_{\ell}\right)$,
(b) $N_{\ell}=\beta\left(M_{\ell}\right)$,
(c) $j_{\ell}=j_{M_{\ell}}$.

It is easy to observe the following:
(a) $P_{1}^{\mathcal{N}_{\ell}}$ consists of all ultrafilters $D$ on $M_{\ell}$ such that $P_{1}^{\mathcal{M}_{\ell}} \in D$ (so for $\ell=1$ this means $\mathbb{N} \in D$ ), and $P_{1}^{\mathcal{N}_{\ell}} \backslash\left\{j_{\ell}(n): n \in P_{1}^{\mathcal{M}_{\ell}}\right\}$ consists of all such non-principal ultrafilters,
(b) $P_{2}^{\mathcal{N}_{\ell}}$ consists of all ultrafilters $D$ on $M_{\ell}$ such that $P_{2}^{\mathcal{M}_{\ell}} \in D$ (so for $\ell=1$ this means $\mathcal{P}(\mathbb{N}) \in D$ ), and $P_{2}^{\mathcal{V}_{\ell}} \backslash\left\{j_{\ell}(A): A \in P_{2}^{\mathcal{M}_{\ell}}\right\}$ consists of all such non-principal ultrafilters.
Now we are going to construct a specific sentence $\psi$ which will be satisfied in $\mathcal{N}_{1}$ but not in $\mathcal{N}_{2}$. First we define two auxiliary formulas $\varphi_{1}$ and $\varphi_{2}$.
Definition 6. For $i \in\{1,2\}$, let $\varphi_{i}(x)$ be the following formula in $\tau$ :

$$
P_{i}(x) \wedge \forall y\left(P_{i}(y) \rightarrow F(x, y)=F(y, x)\right) .
$$

Thus $\varphi_{i}(x)$ means that $x$ is in the center in a sense. Actually, only $\varphi_{2}$ will be used to construct $\psi$.
Proposition 3. Assume $i, \ell \in\{1,2\}$. For every $D \in N_{\ell}$,

$$
\mathcal{N}_{\ell} \vDash \varphi_{i}(D) \text { if and only if } D \in\left\{j_{\ell}(a): a \in P_{i}^{\mathcal{M}_{\ell}}\right\} .
$$

Proof. This follows from the four lemmas below.
Lemma 1. If $D \notin P_{i}^{\mathcal{N}_{\ell}}$ then $\mathcal{N}_{\ell} \vDash \neg \varphi_{i}(D)$.
Proof. By the first conjunct in $\varphi_{i}$.
Lemma 2. If $D_{1} \in P_{i}^{\mathcal{N}_{\ell}}$ and $D_{2}=j_{\ell}(a)$ for some $a \in P_{i}^{\mathcal{M}_{\ell}}$, then

$$
\mathcal{N}_{\ell} \vDash F\left(D_{1}, D_{2}\right)=F\left(D_{2}, D_{1}\right) .
$$

Proof. We must check that $F^{\mathcal{N}_{\ell}}\left(D_{1}, D_{2}\right)=F^{\mathcal{N}_{\ell}}\left(D_{2}, D_{1}\right)$. It suffices to show that, for any $A \subseteq P_{i}^{\mathcal{M}_{\ell}}$, the following equivalence holds:

$$
A \in F^{\mathcal{N}_{\ell}}\left(D_{1}, D_{2}\right) \Leftrightarrow A \in F^{\mathcal{N}_{\ell}}\left(D_{2}, D_{1}\right)
$$

By Definition 2, we have

$$
A \in F^{\mathcal{N}_{\ell}}\left(D_{1}, D_{2}\right) \Leftrightarrow\left(\forall^{D_{1}} x_{1}\right)\left(\forall^{D_{2}} x_{2}\right) F^{\mathcal{M}_{\ell}}\left(x_{1}, x_{2}\right) \in A .
$$

But $D_{2}=j_{\ell}(a)$ for an $a \in P_{i}^{\mathcal{M}_{\ell}}$, i.e., $D_{2}$ is a principal ultrafilter given by $a$. Hence $\forall^{D_{2}} x_{2}$ is reduced by replacing the bounded occurrence of the variable $x_{2}$ with $a$ (as we have noted after Definition (1), whence we get

$$
A \in F^{\mathcal{N}_{\ell}}\left(D_{1}, D_{2}\right) \Leftrightarrow\left(\forall^{D_{1}} x_{1}\right) F^{\mathcal{M}_{\ell}}\left(x_{1}, a\right) \in A
$$

Similarly we get

$$
A \in F^{\mathcal{N}_{\ell}}\left(D_{2}, D_{1}\right) \Leftrightarrow\left(\forall^{D_{1}} x_{1}\right) F^{\mathcal{M}_{\ell}}\left(a, x_{1}\right) \in A
$$

Since $a \in P_{i}^{\mathcal{M}_{\ell}}$, we have $F^{\mathcal{M}_{\ell}}(a, b)=F^{\mathcal{M}_{\ell}}(b, a)$ for all $b \in P_{i}^{\mathcal{M}_{\ell}}$ by Definition $4(\mathrm{f})(\alpha)$. And since $P_{i}^{\mathcal{M}_{\ell}} \in D_{1}$, the required equivalence follows.

Lemma 3. If $D_{1} \in P_{1}^{\mathcal{N}_{1}} \backslash\left\{j_{1}(n): n \in P_{1}^{\mathcal{M}_{1}}\right\}$, then there exists $D_{2} \in P_{1}^{\mathcal{N}_{1}}$ such that

$$
F^{\mathcal{N}_{1}}\left(D_{1}, D_{2}\right) \neq F^{\mathcal{N}_{1}}\left(D_{2}, D_{1}\right) .
$$

Proof. Actually we shall prove a bit stronger assertion: if $D_{1}, D_{2} \in P_{1}^{\mathcal{N}_{1}} \backslash$ $\left\{j_{1}(n): n \in P_{1}^{\mathcal{M}_{1}}\right\}$ are such that $D_{1} \neq D_{2}$, then

$$
F^{\mathcal{N}_{1}}\left(D_{1}, D_{2}\right) \neq F^{\mathcal{N}_{1}}\left(D_{2}, D_{1}\right) .
$$

So assume that $D_{1}, D_{2}$ are distinct non-principal ultrafilters on $M_{1}$ such that $\mathbb{N} \in D_{1} \cap D_{2}$. By $D_{1} \neq D_{2}$, there is $A_{1} \in \mathcal{P}(\mathbb{N})$ such that $A_{1} \in D_{1}$ and $A_{2}=\mathbb{N} \backslash A_{1} \in D_{2}$. Let

$$
\begin{aligned}
& B_{1}=\left\{F^{\mathcal{M}_{1}}\left(n_{1}, n_{2}\right): n_{1} \in A_{1} \wedge n_{2} \in A_{2} \wedge\left(n_{1}, n_{2}\right) \in R_{2}^{\mathcal{M}_{1}}\right\}, \\
& B_{2}=\left\{F^{\mathcal{M}_{1}}\left(n_{1}, n_{2}\right): n_{1} \in A_{1} \wedge n_{2} \in A_{2} \wedge\left(n_{2}, n_{1}\right) \in R_{2}^{\mathcal{M}_{1}}\right\} .
\end{aligned}
$$

Recall that $R_{2}^{\mathcal{M}_{1}} \cap(\mathbb{N} \times \mathbb{N})$ is the usual order $<$ on $\mathbb{N}$, so the last conjuncts in the definition of $B_{1}$ and $B_{2}$ mean just $n_{1}<n_{2}$ and $n_{2}<n_{1}$, respectively.

Now our stronger assertion clearly follows from claims (a)-(c) below:
(a) $B_{1} \cap B_{2}=\emptyset$,
(b) $B_{1} \in F^{\mathcal{N}_{1}}\left(D_{1}, D_{2}\right)$,
(c) $B_{2} \in F^{\mathcal{N}_{1}}\left(D_{2}, D_{1}\right)$.

It remains to verify these claims.
For (a), note that if there is some $c \in B_{1} \cap B_{2}$, then:
$(\alpha)$ since $c \in B_{1}$, we can find $n_{1}<n_{2}$ such that $F^{\mathcal{M}_{1}}\left(n_{1}, n_{2}\right)=c$, $n_{1} \in A_{1}, n_{2} \in A_{2}$,
$(\beta)$ since $c \in B_{2}$, we can find $m_{2}<m_{1}$ such that $F^{\mathcal{M}_{1}}\left(m_{1}, m_{2}\right)=c$, $m_{1} \in A_{1}, m_{2} \in A_{2}$.
So, since by Definition $\boxed{4}(\mathrm{f})(\alpha), F^{\mathcal{M}_{1}}$ is an unordered pairing function, we conclude $\left\{n_{1}, n_{2}\right\}=\left\{m_{1}, m_{2}\right\}$. However, then $n_{1}<n_{2}$ and $m_{2}<m_{1}$ imply $n_{1}=m_{2}$ and $n_{2}=m_{1}$, which contradicts to $n_{1} \in A_{1}, m_{2} \in A_{2}$.

For (b), note that $\left\{n_{2} \in A_{2}: n_{2}>n_{1}\right\} \in D_{2}$ because of $A_{2} \in D_{2}$ and $D_{2}$ is non-principal. It follows $\left(\forall D^{2} n_{2}\right) F\left(n_{1}, n_{2}\right) \in B_{1}$. But $A_{1} \in D_{1}$, so we get

$$
\left(\forall^{D_{1}} n_{1}\right)\left(\forall^{D_{2}} n_{2}\right) F\left(n_{1}, n_{2}\right) \in B_{1} .
$$

By Definition 2(d), this gives claim (b).
For (c), argue similarly.
The fourth lemma (and its proof) generalizes the previous one.
Lemma 4. If $i, \ell \in\{1,2\}$ and $D_{1} \in P_{i}^{\mathcal{N}_{\ell}} \backslash\left\{j_{\ell}(a): a \in P_{i}^{\mathcal{M}_{\ell}}\right\}$, then there exists $D_{2} \in P_{i}^{\mathcal{N}_{\ell}}$ such that

$$
F^{\mathcal{N}_{\ell}}\left(D_{1}, D_{2}\right) \neq F^{\mathcal{N}_{\ell}}\left(D_{2}, D_{1}\right) .
$$

Proof. Let $D_{1}$ be a non-principal ultrafilter on $P_{i}^{\mathcal{M}_{\ell}}$. It follows from Definition $4(\mathrm{e})$ and $\mathcal{M}_{1} \prec \mathcal{M}_{2}$ that $R_{2}^{\mathcal{M}_{\ell}}$ is a linear order on $P_{i}^{\mathcal{M}_{\ell}}$. One of the two following possibilities occurs:
(a) there is an initial segment $I$ of the linearly ordered set $\left(P_{i}^{\mathcal{M}_{\ell}}, R_{2}^{\mathcal{M}_{\ell}}\right)$ such that $I \in D_{1}$ but if $I_{1} \subset I$ is another initial segment of the set then $I_{1} \notin D_{1}$ (this $I$ necessarily has no last element);
(b) there is a final segment $J$ of the linearly ordered set $\left(P_{i}^{\mathcal{M}_{\ell}}, R_{2}^{\mathcal{M}_{\ell}}\right)$ such that $J \in D_{1}$ but if $J_{1} \subset J$ is another final segment of the set then $J_{1} \notin D_{1}$ (this $J$ necessarily has no first element).
To see, notice the following general facts. If $(X,<)$ is a linearly ordered set, for any ultrafilter $D$ on $X$ define the initial segment $I_{D}$ and the final segment $J_{D}$ of $(X,<)$ as follows:

$$
\begin{aligned}
I_{D} & =\bigcap\{I \in D: I \text { is an initial segment of }(X,<)\}, \\
J_{D} & =\bigcap\{J \in D: J \text { is a final segment of }(X,<)\} .
\end{aligned}
$$

As easy to see, if $D$ is principal then $I_{D} \cap J_{D}=\{x\}$ for $\{x\} \in D$; and if $D$ is non-principal then $\left(I_{D}, J_{D}\right)$ is a cut and either $I_{D}$ or $J_{D}$, but not both, is in $D$. Furthermore, if $I_{D}$ is in $D$, then so are all final segments of $I_{D}$, $S \cap I_{D}$ is cofinal in $I_{D}$ for all $S \in D$, and $I_{D}$ does not have a greatest element whenever $D$ is non-principal; and symmetrically for $J_{D}$ in $D$. (More details related to ultrafilter extensions of linearly ordered sets can be found in 6].)

In our situation, $D_{1}$ is non-principal, so we have either $I_{D_{1}} \in D_{1}$, in which case we get possibility (a) with $I=I_{D_{1}}$, or $J_{D_{1}} \in D_{1}$, in which case we get possibility (b) with $J=J_{D_{1}}$.

For (a), choose an ultrafilter $D_{2}$ on $P_{i}^{\mathcal{M}_{\ell}}$ such that
( $\alpha$ ) $I \in D_{2}$,
( $\beta$ ) if $I_{1} \subset I$ is an initial segment of $\left(P_{i}^{\mathcal{M}_{\ell}}, R_{2}^{\mathcal{M}_{\ell}}\right)$ then $I_{1} \notin D_{2}$,
( $\gamma$ ) $D_{2} \neq D_{1}$.
Now we can repeat the proof of Lemma 3 mutatis mutandis, i.e., we can find $A_{1} \in D_{1} \backslash D_{2}$ such that $A_{1} \subseteq I$ and $A_{2}=I \backslash A_{1} \in D_{2}$, then define

$$
\begin{aligned}
& B_{1}=\left\{F^{\mathcal{M}_{\ell}}\left(a_{1}, a_{2}\right): a_{1} \in A_{1} \wedge a_{2} \in A_{2} \wedge\left(a_{1}, a_{2}\right) \in R_{2}^{\mathcal{M}_{\ell}}\right\}, \\
& B_{2}=\left\{F^{\mathcal{M}_{\ell}}\left(a_{1}, a_{2}\right): a_{1} \in A_{1} \wedge a_{2} \in A_{2} \wedge\left(a_{2}, a_{1}\right) \in R_{2}^{\mathcal{M}_{\ell}}\right\},
\end{aligned}
$$

etc.
For (b), the proof is symmetric: we only replace $I$ with $J$, initial segments with final ones, and $x R_{2}^{\mathcal{M}_{\ell}} y$ with $y R_{2}^{\mathcal{M}_{\ell}} x$.

These four lemmas complete the proof of Proposition 3.
Now everything is ready in order to provide a sentence $\psi$ having the required property.

Definition 7. Let $\psi$ be the following sentence in $\tau$ :

$$
\begin{aligned}
\left(\forall x_{1}\right)\left(\forall x_{2}\right)\left(P_{1}\left(x_{1}\right) \wedge P_{1}\left(x_{2}\right)\right. & \wedge x_{1} \neq x_{2} \\
& \left.\rightarrow(\exists y) \varphi_{2}(y) \wedge R_{1}\left(x_{1}, y\right) \wedge \neg R_{1}\left(x_{2}, y\right)\right) .
\end{aligned}
$$

Proposition 4. Let $\ell \in\{1,2\}$. Then

$$
\mathcal{N}_{\ell} \vDash \psi \text { if and only if } \ell=1 .
$$

Proof. 1. First we show that $\mathcal{N}_{1} \vDash \psi$.
Let $D_{1}, D_{2}$ satisfy the antecedent of $\psi$, i.e., $D_{1}, D_{2} \in P_{1}^{\mathcal{N}_{1}}$ and $D_{1} \neq D_{2}$. We should find $b \in N_{1}$ such that

$$
\mathcal{N}_{1} \vDash \varphi_{2}(b) \wedge R_{1}\left(D_{1}, b\right) \wedge \neg R_{1}\left(D_{2}, b\right) .
$$

Since $D_{1}, D_{2}$ are distinct ultrafilters on $M_{1}$ such that $P_{1}^{\mathcal{M}_{1}} \in D_{1} \cap D_{2}$, we can choose $A_{1} \subseteq P_{1}^{\mathcal{M}_{1}}$ such that $A_{1} \in D_{1}$ and $A_{1} \notin D_{2}$. Then $A_{1} \in P_{2}^{\mathcal{M}_{1}}$ clearly follows from Definition 4 (b), (c). So $b=j_{1}\left(A_{1}\right) \in P_{2}^{\mathcal{N}_{1}}$, and hence, by the "if" part of Proposition 3, $\mathcal{N}_{1} \vDash \varphi_{2}(b)$.

It remains to show the conjunction

$$
\left(D_{1}, b\right) \in R_{1}^{\mathcal{N}_{1}} \text { and }\left(D_{2}, b\right) \notin R_{1}^{\mathcal{N}_{1}} .
$$

To this end, note that for any ultrafilter $D$ concentrated on $P_{1}^{\mathcal{M}_{1}}$ and any $A \in P_{2}^{\mathcal{M}_{1}}$, by Definition 2(c), the formula $\left(D, j_{1}(A)\right) \in R_{1}^{\mathcal{N}_{1}}$ means

$$
\left(\forall^{D} n\right)\left(\forall^{j(A)} B\right)(n, B) \in R_{1}^{\mathcal{M}_{1}}
$$

Recalling that $R_{1}^{\mathcal{M}_{1}}$ is the membership relation (Definition $4(\mathrm{~d})$ ) and reducing $\left(\forall^{j(A)} B\right)$, we see that the latter formula is equivalent to $\left(\forall^{D} n\right) n \in A$, and so, to $A \in D$. Since we have $A_{1} \in D_{1}$ and $A_{1} \notin D_{2}$, this gives the required conjunction.
2. Now we show that $\mathcal{N}_{2} \vDash \neg \psi$.

Define a function $G$ from $P_{1}^{\mathcal{N}_{2}}$ into $\mathcal{P}\left(P_{2}^{\mathcal{M}_{2}}\right)$ as follows:

$$
G(D)=\left\{b \in P_{2}^{\mathcal{M}_{2}}:\left\{a \in P_{1}^{\mathcal{M}_{2}}:(a, b) \in R_{1}^{\mathcal{M}_{2}}\right\} \in D\right\} .
$$

Recall that $\left|P_{1}^{\mathcal{M}_{2}}\right|=\left|P_{1}^{\mathcal{M}_{2}}\right|=\lambda$ (Proposition (2). Therefore,

$$
|\operatorname{dom}(G)|=\left|\boldsymbol{\beta}\left(\left|P_{1}^{\mathcal{M}_{2}}\right|\right)=|\boldsymbol{\beta}(\lambda)|=2^{2^{\lambda}}>2^{\lambda},\right.
$$

while

$$
|\operatorname{ran}(G)| \leq\left|\mathcal{P}\left(P_{2}^{\mathcal{M}_{2}}\right)\right|=|\mathcal{P}(\lambda)|=2^{\lambda},
$$

whence we conclude that $G$ is not one-to-one.
Take $S \in \mathcal{P}\left(P_{2}^{\mathcal{M}_{2}}\right)$ such that $\left|G^{-1}(S)\right|>1$, pick $D_{1}, D_{2} \in G^{-1}(S)$ such that $D_{1} \neq D_{2}$, and show that $D_{1}, D_{2}$ witness the failure of the sentence $\psi$.

Note that $\mathcal{N}_{2}$ satisfies the antecedent of $\psi$, i.e.,

$$
\mathcal{N}_{2} \vDash P_{1}\left(D_{1}\right) \wedge P_{1}\left(D_{2}\right) \wedge D_{1} \neq D_{2},
$$

by the condition $D_{1}, D_{2} \in G^{-1}(S) \subseteq P_{1}^{\mathcal{N}_{2}}$. So to finish, it suffices to show

$$
\mathcal{N}_{2} \vDash \neg(\exists y) \varphi_{2}(y) \wedge R_{1}\left(D_{1}, y\right) \wedge \neg R_{1}\left(D_{2}, y\right) .
$$

Toward a contradiction, assume that there is $b \in N_{2}$ such that

$$
\mathcal{N}_{2} \vDash \varphi_{2}(b) \wedge R_{1}\left(D_{1}, b\right) \wedge \neg R_{1}\left(D_{2}, b\right) .
$$

But since $\mathcal{N}_{2} \vDash \varphi_{2}(b)$, by the "only if" part of Proposition 3, we see that $b=j_{2}(A)$ for some $A \in P_{2}^{\mathcal{M}_{2}}$. So we obtain

$$
R_{1}^{\mathcal{N}_{2}}\left(D_{1}, j_{2}(A)\right) \text { and } \neg R_{1}^{\mathcal{N}_{2}}\left(D_{2}, j_{2}(A)\right)
$$

By Definition [2(c), $R_{1}^{\mathcal{N}_{2}}\left(D_{1}, j_{2}(A)\right)$ means $\left(\forall^{D_{1}} a\right)\left(\forall^{j_{2}}(A) b\right)(a, b) \in R_{1}^{\mathcal{M}_{2}}$, whence reducing $\left(\forall^{j_{2}}(A) b\right)$ we get $\left(\forall^{D_{1}} a\right)(a, A) \in R_{1}^{\mathcal{M}_{2}}$, i.e.,

$$
\left\{a \in P_{1}^{\mathcal{M}_{2}}:(a, A) \in R_{1}^{\mathcal{M}_{2}}\right\} \in D_{1} .
$$

Similarly, $R_{1}^{\mathcal{N}_{2}}\left(D_{2}, j_{2}(A)\right)$ is equivalent to $\left\{a \in P_{1}^{\mathcal{M}_{2}}:(a, A) \in R_{1}^{\mathcal{M}_{2}}\right\} \in D_{2}$, and hence, $\neg R_{1}^{\mathcal{N}_{2}}\left(D_{2}, j_{2}(A)\right)$ is equivalent to

$$
\left\{a \in P_{1}^{\mathcal{M}_{2}}:(a, A) \in R_{1}^{\mathcal{M}_{2}}\right\} \notin D_{2} .
$$

Therefore, $A \in G\left(D_{1}\right)$ and $A \notin G\left(D_{2}\right)$, which, however, contradicts to the choice of $D_{1}, D_{2}$.

This completes the proof.
So we have constructed two models $\mathcal{M}_{1}, \mathcal{M}_{2}$ in $\tau$ with

$$
\mathcal{M}_{1} \prec \mathcal{M}_{2}
$$

and a $\tau$-sentence $\psi$ such that $\mathcal{N}_{1}=\beta\left(\mathcal{M}_{1}\right) \vDash \psi$ and $\mathcal{N}_{2}=\beta\left(\mathcal{M}_{2}\right) \vDash \neg \psi$, thus witnessing

$$
\beta\left(\mathcal{M}_{1}\right) \not \equiv \beta\left(\mathcal{M}_{2}\right) .
$$

This proves the Main Theorem (Theorem (2).

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