Refining the arithmetical hierarchy of classical principles

Fujiwara, Makoto

Kurahashi, Taishi

(Citation)
Mathematical Logic Quarterly,68(3):318-345

(Issue Date)
2022-08

(Resource Type)
journal article

(Version)
Accepted Manuscript

Refining the arithmetical hierarchy of classical principles

Makoto Fujiwara[*†] and Taishi Kurahashi[‡§]

Abstract

We refine the arithmetical hierarchy of various classical principles by finely investigating the derivability relations between these principles over Heyting arithmetic. We mainly investigate some restricted versions of the law of excluded middle, de Morgan's law, the double negation elimination, the collection principle and the constant domain axiom.

1 Introduction

The interrelations between weak logical principles over intuitionistic arithmetic have been studied extensively in these three decades (cf. [1, 6, 8, 10, 11, 14, 17]). In particular, Akama et al. [1] systematically studied the structure of the law of excluded middle **LEM** and the double negation elimination **DNE** restricted to prenex formulas and some related principles over intuitionistic first-order arithmetic HA. Interestingly, the derivability relation between them forms a beautiful hierarchy as presented in Figure 1 (cf. [1, Figure 2]).

By the prenex normal form theorem, which is first presented in [1] and corrected recently in [13], this arithmetical hierarchy covers **LEM** for arbitrary formulas. In this sense, the infinite hierarchy in Figure 1 represents a gradual transition of strength of semi-classical arithmetic from HA to the classical arithmetic $\mathsf{PA} = \mathsf{HA}+\mathbf{LEM}$. This hierarchy plays an important role in several aspects. First, it is employed for the relativization of the relation between classical and intuitionistic arithmetic into the context of semi-classical arithmetic. For example, PA is Π_{k+2}-conservative over $\mathsf{HA}+\Sigma_k$-**LEM** for all natural numbers k (see [13, Section 6] and [2, 12]). In addition, for any theory T in-between HA and PA, the prenex normal form theorem for the classes of formulas $\mathrm{U}_{k'}$ (introduced in [1]) and $\Pi_{k'}$ holds in T for all $k' \leq k$, if and only if, T proves $(\Pi_k \vee \Pi_k)$-**DNE** (see [13, Section 7]). Then the refinement of the hierarchy is also important for analyzing

[*]Email: makotofujiwara@rs.tus.ac.jp

[†]Department of Applied Mathematics, Faculty of Science Division I, Tokyo University of Science, 1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan.

[‡]Email: kurahashi@people.kobe-u.ac.jp

[§]Graduate School of System Informatics, Kobe University, 1-1 Rokkodai, Nada, Kobe 657-8501, Japan.

$$
\begin{tikzcd}
& \Sigma_k\text{-}\mathbf{LEM} \arrow[ddl] \arrow[dr] & \\
& & \Pi_k\text{-}\mathbf{LEM} \arrow[d] \\
\Sigma_k\text{-}\mathbf{DNE} \arrow[dr] & & (\Pi_k \vee \Pi_k)\text{-}\mathbf{DNE} \arrow[dl] \\
& \Delta_k\text{-}\mathbf{LEM} \arrow[d] & \\
& \Sigma_{k-1}\text{-}\mathbf{LEM} \arrow[dl, dash] \arrow[dr, dash] & \\
{} & & {}
\end{tikzcd}
$$

Figure 1: An arithmetical hierarchy of classical principles

the results on the relation between classical and intuitionistic arithmetic in more detail. Secondly, the hierarchy is employed as a framework for a sort of constructive reverse mathematics over HA (cf. [3, 4, 19]). For example, Ramsey's theorem for pairs and recursive assignments of 2 colors is located in the place of $(\Pi_3 \vee \Pi_3)$-**DNE** (see [3]). Despite the fact that mathematical statements are usually not in prenex normal form, many of them are shown to be equivalent to some restricted logical principle in the arithmetical hierarchy (seemingly because the prenex normal form theorem is partly available in semi-classical arithmetic containing such logical principles). Then the refinement of the hierarchy makes it possible to classify the logical strength of mathematical statements in finer classes. After [1], in connection with the development of constructive reverse mathematics [15] over intuitionistic second-order arithmetic, further fine-grained analysis has been done for the principles with $k = 1$ in the hierarchy ([8, 11, 17]). More recently, some connection between those principles and some other principles has been also found ([6, 10]). Then it should be expected to recast the hierarchy in [1] based on these recent developments. The history of the research of this line until [11] is summarized in [11, Section 2.1].

Motivated from them, we study the interrelations between various principles from the previous research and the related principles comprehensively in the context of HA. In particular, we investigate principles more finely and more systematically than ever before. Such a fine-grained analysis reveals a more detailed hierarchical structure which the logical principles have. In addition to the principles dealt with in [1], we deal with de Morgan's law **DML**, the (contrapositive) collection principle $\mathbf{COLL^{cp}}$ and the constant domain axiom **CD** systematically. Among many other things, we show that $(\Pi_k \vee \Pi_k)$-**DNE**, Σ_k-**DML** with respect to duals (which is Σ_k-**LLPO** in [1]), Σ_k-**DML** + Σ_{k-1}-**DNE**, Π_k-$\mathbf{COLL^{cp}}$ and (Π_k, Π_k)-**CD** are pairwise equivalent over HA for all natural numbers k greater than 0 (see Corollary 7.6).

The structure of the paper is as follows. In Section 3, we extract and investigate the principles concerning duals $\varphi^\perp$ (which are prenex formulas classically equivalent to $\neg\varphi$) of prenex formulas φ. In Section 4, we investigate variants of **LEM**. Section 5 is devoted to investigate several variations of **DML**. In particular, **LEM** for negated formulas is shown to be a variation of **DML**. In Section 6, we investigate variants of **DNE**. In particular, **DML** is shown to be a variation of **DNE**. Finally, we investigate **CD** in Section 7. The results established in this paper are summarized in Section 8, to which we refer the reader who merely wants to consult the results.

2 Preliminaries

In this paper, we work within the framework of first-order intuitionistic arithmetic with the logical connectives $\wedge, \vee, \to, \exists, \forall$ and $\perp$, where $\neg\varphi$ is the abbreviation of $\varphi \to \perp$. We may assume that the language of first-order arithmetic contains function symbols corresponding to all primitive recursive functions. Heyting arithmetic HA is an intuitionistic theory in the language of first-order arithmetic consisting of basic axioms for arithmetic, induction axiom scheme and axioms corresponding to defining equations of primitive recursive functions (see [16, Section 3.2]). Recall that $\varphi \to \neg\neg\varphi$, $(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$, $\neg\neg(\varphi \to \psi) \leftrightarrow (\neg\neg\varphi \to \neg\neg\psi)$, $\neg\neg\neg\varphi \to \neg\varphi$ and $\forall x \neg\varphi \leftrightarrow \neg\exists x\varphi$ etc. are intuitionistically derivable. For more information about the logical implications over intuitionistic logic, we refer the reader to [20, Section 6.2].

Throughout this paper, we assume that k always denotes a natural number $k \geq 0$. We define the family $\{\Sigma_k, \Pi_k : k \geq 0\}$ of sets of formulas inductively as follows:

- Let $\Sigma_0 = \Pi_0$ be the set of all quantifier-free formulas;
- $\Sigma_{k+1} := \{\exists x_1 \cdots \exists x_n \varphi \mid \varphi \in \Pi_k,\ n \geq 1 \text{ and } x_1, \ldots, x_n \text{ are variables}\}$;
- $\Pi_{k+1} := \{\forall x_1 \cdots \forall x_n \varphi \mid \varphi \in \Sigma_k,\ n \geq 1 \text{ and } x_1, \ldots, x_n \text{ are variables}\}$.

For convenience, we assume that Σ_m and Π_m denote the empty set for any negative integer m. We say that a formula is in *prenex normal form* if it is in Σ_k or Π_k for some k. Let $\mathrm{FV}(\varphi)$ denote the set of all free variables in φ. It is known that every formula φ in Σ_{k+1} (resp. Π_{k+1}) is HA-equivalent to a formula ψ in Σ_{k+1} (resp. Π_{k+1}) such that $\mathrm{FV}(\varphi) = \mathrm{FV}(\psi)$ and ψ is of the form $\exists x\psi'$ (resp. $\forall x\psi'$) where ψ' is Π_k (resp. Σ_k).

Let Γ and Θ be sets of formulas. We define $\Gamma \vee \Theta$, Γ^{n} and Γ^{dn} to be the sets $\{\varphi \vee \psi \mid \varphi \in \Gamma \text{ and } \psi \in \Theta\}$, $\{\neg\varphi \mid \varphi \in \Gamma\}$ and $\{\neg\neg\varphi \mid \varphi \in \Gamma\}$ of formulas, respectively. We adopt a convention that we write $\Gamma \subseteq \Theta$ if for any formula $\varphi \in \Gamma$, there exists a formula $\psi \in \Theta$ such that $\mathrm{FV}(\varphi) = \mathrm{FV}(\psi)$ and HA proves $\varphi \leftrightarrow \psi$. Then it is shown that $\Sigma_k \subseteq \Sigma_{k+1} \cap \Pi_{k+1}$ and $\Pi_k \subseteq \Sigma_{k+1} \cap \Pi_{k+1}$ (cf. [13]).

We introduce several principles which give semi-classical arithmetic as follows:

Definition 2.1. Let Γ be any set of formulas.

Γ-**LEM**	$\varphi \vee \neg\varphi$	$(\varphi \in \Gamma)$
Δ_k-**LEM**	$(\varphi \leftrightarrow \psi) \to \varphi \vee \neg\varphi$	$(\varphi \in \Sigma_k$ and $\psi \in \Pi_k)$
Γ-**DNE**	$\neg\neg\varphi \to \varphi$	$(\varphi \in \Gamma)$

For each theory T and principle P, let $T + P$ denote the theory obtained from T by adding universal closures of all instances of P as axioms. Since HA proves $\varphi \vee \neg\varphi \to (\neg\neg\varphi \to \varphi)$ for any formula φ, the following fact trivially holds.

Fact 2.2. *For any set Γ of formulas,* $\mathsf{HA} + \Gamma\text{-}\mathbf{LEM} \vdash \Gamma\text{-}\mathbf{DNE}$.

Nontrivial implications between the principles defined in Definition 2.1 are investigated by Akama et al. [1]. The following fact is visualized in Figure 1 in Section 1.

Fact 2.3 (Akama et al. [1])**.**

1. Σ_k-**LEM** *and* Π_k-**LEM** $+$ Σ_k-**DNE** *are equivalent over* HA*;*
2. $\mathsf{HA} + \Pi_k\text{-}\mathbf{LEM} \vdash (\Pi_k \vee \Pi_k)\text{-}\mathbf{DNE}$*;*
3. $\mathsf{HA} + (\Pi_k \vee \Pi_k)\text{-}\mathbf{DNE} \vdash \Delta_k\text{-}\mathbf{LEM}$*;*
4. $\mathsf{HA} + \Sigma_k\text{-}\mathbf{DNE} \vdash \Delta_k\text{-}\mathbf{LEM}$*;*
5. $\mathsf{HA} + \Delta_{k+1}\text{-}\mathbf{LEM} \vdash \Sigma_k\text{-}\mathbf{LEM}$*;*
6. Σ_k-**DNE** *and* Π_{k+1}-**DNE** *are equivalent over* HA.

In the present paper, we also deal with other important principles based on such as the double negation shift, de Morgan's law and the constant domain axiom.

Definition 2.4. Let Γ and Θ be any sets of formulas.

Γ-**DNS**	$\forall x\, \neg\neg\varphi(x) \to \neg\neg\, \forall x \varphi(x)$	$(\varphi(x) \in \Gamma)$
Γ-**DML**	$\neg(\varphi \wedge \psi) \to \neg\varphi \vee \neg\psi$	$(\varphi, \psi \in \Gamma)$
(Γ, Θ)-**CD**	$\forall x(\varphi \vee \psi(x)) \to \varphi \vee \forall x \psi(x)$	$(\varphi \in \Gamma,\ \psi(x) \in \Theta$ and $x \notin \mathrm{FV}(\varphi))$

The principle Σ_k-**DML** is introduced in [3]. The principles defined in Definition 2.4 have mainly been investigated for $k = 1$ in the literature. For example, Σ_1-**DML** and Π_1-**DML** correspond to the principle **LLPO** and disjunctive Markov's principle, respectively (see [14]). Also the principle Δ_1-**LEM** corresponds to the principle (IIIa) in [8] and to the principle Δ_a-**LEM** in [11]. Notice that [8, 10, 14] are studied in the context of second-order arithmetic. We have the following results from the proofs of the corresponding results in these papers.

Fact 2.5 (Ishihara [14, Proposition 1])**.**

1. $\mathsf{HA} + \Sigma_1\text{-}\mathbf{DNE} \vdash \Pi_1\text{-}\mathbf{DML}$*;*

2. $\mathsf{HA} + \Sigma_1$-**DML** $\vdash \Pi_1$-**DML**.

Fact 2.6 (Fujiwara, Ishihara and Nemoto [8, Proposition 2])**.** $\mathsf{HA}+\Pi_1$-**DML** $\vdash$ Δ_1-**LEM**.

Fact 2.7 (Fujiwara and Kawai [10, Proposition 4.2])**.** (Π_1, Π_1)-**CD** *and* Σ_1-**DML** *are equivalent over* HA.

In the following sections, we investigate those principles more finely than ever before. In the process of the investigation, we also generalize the facts stated above.

Concerning Γ-**DNS**, we easily obtain the following proposition.

Proposition 2.8.

1. $\mathsf{HA} + \Sigma_k$-**DNE** $\vdash \Sigma_k$-**DNS***;*
2. Σ_k-**DNS** *and* Π_{k+1}-**DNS** *are equivalent over* HA.

Proof. 1. Let φ be any Σ_k formula. Then $\mathsf{HA} + \Sigma_k$-**DNE** $\vdash \forall x \neg\neg\varphi \to \forall x\varphi$. We obtain $\mathsf{HA} + \Sigma_k$-**DNE** $\vdash \forall x \neg\neg\varphi \to \neg\neg \forall x\varphi$.

2. We prove $\mathsf{HA} + \Sigma_k$-**DNS** $\vdash \Pi_{k+1}$-**DNS**. Let $\forall y\varphi(x, y)$ be any Π_{k+1} formula where $\varphi(x, y) \in \Sigma_k$. Then $\mathsf{HA} \vdash \forall x \neg\neg \forall y\varphi(x, y) \to \forall x\forall y \neg\neg\varphi(x, y)$. Let $(z)_0$ and $(z)_1$ be primitive recursive inverse functions of a fixed pairing function which calculate the first and the second components of z as a pair, respectively. Then $\mathsf{HA} \vdash \forall x \neg\neg \forall y\varphi(x, y) \to \forall z \neg\neg\varphi((z)_0, (z)_1)$. By applying Σ_k-**DNS**, we obtain $\mathsf{HA} + \Sigma_k$-**DNS** $\vdash \forall x \neg\neg \forall y\varphi(x, y) \to \neg\neg \forall z\varphi((z)_0, (z)_1)$. We conclude $\mathsf{HA} + \Sigma_k$-**DNS** $\vdash \forall x \neg\neg \forall y\varphi(x, y) \to \neg\neg \forall x\forall y\varphi(x, y)$. $\square$

A detailed investigation of the principle Σ_1-**DNS** including Proposition 2.8.1 for $k = 1$ is in [11].

3 The dual principles

In [13], the following result is proved.

Fact 3.1 (Fujiwara and Kurahashi [13, Lemma 4.7])**.**

1. *For any Σ_k formula φ, there exists a Π_k formula φ' such that* $\mathsf{HA} + \Sigma_{k-1}$-**DNE** $\vdash \neg\varphi \leftrightarrow \varphi'$*;*
2. *For any Π_k formula φ, there exists a Σ_k formula φ' such that* $\mathsf{HA} + \Sigma_k$-**DNE** $\vdash \neg\varphi \leftrightarrow \varphi'$.

In this section, we investigate the dual principles and the weak dual principles (see Definitions 3.2 and 3.10) motivated from Fact 3.1.

3.1 The dual principles

First, we recall the notion of duals of formulas in prenex normal form, which is defined in [1] informally.

Definition 3.2 (cf. [1])**.** For any formula φ in prenex normal form, we define the dual $\varphi^\perp$ of φ inductively as follows:

1. $\varphi^\perp :\equiv \neg\varphi$ if φ is quantifier-free;
2. $(\forall x \varphi(x))^\perp :\equiv \exists x \varphi^\perp(x)$;
3. $(\exists x \varphi(x))^\perp :\equiv \forall x \varphi^\perp(x)$.

The following proposition is a basic property of duals.

Proposition 3.3. *Let φ be any formula in prenex normal form.*

1. *If φ is Σ_k (resp. Π_k), then $\varphi^\perp$ is Π_k (resp. Σ_k);*
2. $\mathsf{HA} \vdash \varphi^{\perp\perp} \leftrightarrow \varphi$*;*
3. $\mathsf{HA} \vdash \varphi^\perp \to \neg\varphi$*;*
4. $\mathsf{HA} \vdash \neg(\varphi \wedge \varphi^\perp)$.

Proof. 1. Trivial.

2. It is known that if φ is Σ_0, then $\mathsf{HA} \vdash \neg\neg\varphi \leftrightarrow \varphi$. Then clause 2 is proved by induction on the number of quantifiers contained in φ.

3. Notice that HA proves the formulas $\exists x \neg\varphi \to \neg \forall x \varphi$ and $\forall x \neg\varphi \to \neg \exists x \varphi$. Then clause 3 is also proved by induction on the number of quantifiers in φ.

4. This is because $\mathsf{HA} \vdash \varphi \wedge \varphi^\perp \to \varphi \wedge \neg\varphi$ by clause 3. $\square$

From Propositions 3.3.(1) and (2), we have that the mapping $(\cdot)^\perp$ is a bijection between Σ_k (resp. Π_k) and Π_k (resp. Σ_k) modulo HA-provable equivalence.

Remark 3.4. It is possible to extend the notion of duals in Definition 3.2 (from [1]) to arbitrary formulas by the operation $(\cdot)^d$ defined inductively as

1. $\varphi^d :\equiv \neg\varphi$ if φ is prime;
2. $(\varphi \wedge \psi)^d :\equiv \varphi^d \vee \psi^d$;
3. $(\varphi \vee \psi)^d :\equiv \varphi^d \wedge \psi^d$;
4. $(\varphi \to \psi)^d :\equiv \varphi \wedge \psi^d$;
5. $(\forall x \varphi(x))^d :\equiv \exists x \varphi^d(x)$;
6. $(\exists x \varphi(x))^d :\equiv \forall x \varphi^d(x)$.

In fact, φ^d is HA-equivalent to $\neg\varphi$ for quantifier-free φ, and hence, φ^d is HA-equivalent to $\varphi^\perp$ for prenex φ. On the one hand, clauses 3 and 4 in Proposition 3.3 hold for the operation $(\cdot)^d$. On the other hand, for clause 2, $\varphi \to \left(\varphi^d\right)^d$ is not provable in HA for some (non-prenex) φ whereas the converse is always provable in HA.

In contrast to Proposition 3.3.(3), the formula $\neg\varphi \to \varphi^\perp$ cannot be proved in HA even for some prenex φ. For example, $\neg\mathrm{Con}(\mathsf{HA}) \to \mathrm{Con}(\mathsf{HA})^\perp$ is not provable in HA, where $\mathrm{Con}(\mathsf{HA})$ is a conventional Π_1 consistency statement of HA (cf. [18, Section 4]). Thus, we introduce the following principle.

Definition 3.5 (The dual principles)**.** Let Γ be any set of formulas in prenex normal form.

Γ-**DUAL** $\quad \neg\varphi \to \varphi^\perp \quad (\varphi \in \Gamma)$

The principle Σ_1-**DUAL** is provable in HA.

Proposition 3.6. $\mathsf{HA} \vdash \Sigma_1$-**DUAL**.

Proof. Let $\varphi \equiv \exists x \psi$ be any Σ_1 formula where ψ is Σ_0. Then $\varphi^\perp$ is $\forall x \neg\psi$, and hence $\neg\varphi$ is equivalent to $\varphi^\perp$ over HA. □

Proposition 3.7. *The following are equivalent over* HA*:*

1. Σ_{k+1}-**DUAL**.
2. Π_k-**DUAL**.
3. Σ_k-**DNE**.

Proof. It is trivial that $\mathsf{HA} + \Sigma_{k+1}$-**DUAL** proves Π_k-**DUAL** because $\Pi_k \subseteq \Sigma_{k+1}$.

We prove $\mathsf{HA} + \Pi_k$-**DUAL** $\vdash \Sigma_k$-**DNE**. Let φ be any Σ_k formula. By Proposition 3.3.(3), we have $\mathsf{HA} \vdash \varphi^\perp \to \neg\varphi$. Then $\mathsf{HA} \vdash \neg\neg\varphi \to \neg\varphi^\perp$. Since $\varphi^\perp$ is Π_k by Proposition 3.3.(1), $\mathsf{HA} + \Pi_k$-**DUAL** proves $\neg\varphi^\perp \to \varphi^{\perp\perp}$. By Proposition 3.3.(2), we conclude $\mathsf{HA} + \Pi_k$-**DUAL** $\vdash \neg\neg\varphi \to \varphi$.

Finally, we prove $\mathsf{HA} + \Sigma_k$-**DNE** $\vdash \Sigma_{k+1}$-**DUAL** by induction on k. The case $k = 0$ follows from Proposition 3.6. Suppose that the statement holds for all $k' < k + 1$, and we prove $\mathsf{HA} + \Sigma_{k+1}$-**DNE** $\vdash \Sigma_{k+2}$-**DUAL**.

Let $\exists x \forall y \psi$ be any Σ_{k+2} formula where ψ is Σ_k. Since $\mathsf{HA} + \Sigma_k$-**DNE** proves $\neg\neg\psi \to \psi$, we have $\mathsf{HA} + \Sigma_k$-**DNE** $\vdash \neg\, \exists x \forall y \psi \to \neg\, \exists x \forall y\, \neg\neg\psi$. Then,

$$\mathsf{HA} + \Sigma_k\text{-}\mathbf{DNE} \vdash \neg\, \exists x \forall y \psi \to \forall x\, \neg\neg\, \exists y\, \neg\psi.$$

By induction hypothesis, $\mathsf{HA} + \Sigma_{k-1}$-**DNE** $\vdash \neg\psi \to \psi^\perp$. Then,

$$\mathsf{HA} + \Sigma_k\text{-}\mathbf{DNE} \vdash \neg\, \exists x \forall y \psi \to \forall x\, \neg\neg\, \exists y \psi^\perp.$$

Since $\exists y \psi^\perp \equiv (\forall y \psi)^\perp$ is Σ_{k+1},

$$\mathsf{HA} + \Sigma_{k+1}\text{-}\mathbf{DNE} \vdash \neg\, \exists x \forall y \psi \to \forall x (\forall y \psi)^\perp.$$

We conclude $\mathsf{HA} + \Sigma_{k+1}$-**DNE** $\vdash \neg\, \exists x \forall y \psi \to (\exists x \forall y \psi)^\perp$. □

From Propositions 3.3.(3) and 3.7, we obtain Fact 3.1.

We may introduce the following Δ_k-variations of the dual principle.

Definition 3.8 (Δ_k dual principles)**.**

Δ_k-$\mathbf{DUAL}^\Sigma$	$(\varphi \leftrightarrow \psi) \to (\neg\varphi \to \varphi^\perp)$	$(\varphi \in \Sigma_k$ and $\psi \in \Pi_k)$
Δ_k-$\mathbf{DUAL}^\Pi$	$(\varphi \leftrightarrow \psi) \to (\neg\psi \to \psi^\perp)$	$(\varphi \in \Sigma_k$ and $\psi \in \Pi_k)$

However, each of them is trivially equivalent to the corresponding original dual principle.

Proposition 3.9.

1. *Δ_k-$\mathbf{DUAL}^\Sigma$ is equivalent to Σ_k-$\mathbf{DUAL}$ over* HA*;*

2. *Δ_k-$\mathbf{DUAL}^\Pi$ is equivalent to Π_k-$\mathbf{DUAL}$ over* HA*.*

Proof. 1. $\mathsf{HA} + \Sigma_k$-$\mathbf{DUAL}$ obviously proves Δ_k-$\mathbf{DUAL}^\Sigma$. On the other hand, let φ be any Σ_k formula. Then $\mathsf{HA} \vdash \neg\varphi \to (\varphi \leftrightarrow \bot)$. Hence $\mathsf{HA} + \Delta_k$-$\mathbf{DUAL}^\Sigma$ proves $\neg\varphi \to (\neg\varphi \to \varphi^\perp)$. We conclude $\mathsf{HA} + \Delta_k$-$\mathbf{DUAL}^\Sigma \vdash \neg\varphi \to \varphi^\perp$.

2 is proved in a similar way. $\square$

Thus it follows from Proposition 3.7 that Δ_k-$\mathbf{DUAL}^\Sigma$ and Δ_k-$\mathbf{DUAL}^\Pi$ are HA-equivalent to Σ_{k-1}-$\mathbf{DNE}$ and Σ_k-$\mathbf{DNE}$, respectively. In fact, Δ_1-$\mathbf{DUAL}^\Pi$ corresponds to the principle (VIb) in [8], and it is proved to be HA-equivalent to Σ_1-$\mathbf{DNE}$ (see [8, Proposition 1]).

3.2 The weak dual principles

In this subsection, we investigate weak variations of the dual principle, which we call the weak dual principles.

Definition 3.10 (The weak dual principles)**.** Let Γ be any set of formulas in prenes normal form.

Γ-$\mathbf{WDUAL}$	$\neg\varphi^\perp \to \neg\neg\varphi$	$(\varphi \in \Gamma)$

Of course Γ-$\mathbf{DUAL}$ implies Γ-$\mathbf{WDUAL}$ over HA. It is known that Σ_1-$\mathbf{DNE}$ is not provable in HA (cf. [1]), and so is Π_1-$\mathbf{DUAL}$ by Proposition 3.7. On the other hand, the following proposition shows that Π_1-$\mathbf{WDUAL}$ is HA-provable.

Proposition 3.11.

1. $\mathsf{HA} \vdash \Sigma_1$-$\mathbf{WDUAL}$*;*

2. $\mathsf{HA} \vdash \Pi_1$-$\mathbf{WDUAL}$.

Proof. 1. This follows from Proposition 3.6.

2. Let $\forall x\varphi$ be any Π_1 formula where φ is Σ_0. Since $\neg(\forall x\varphi)^\perp \equiv \neg\,\exists x\,\neg\varphi$, we have

$$\begin{aligned} \mathsf{HA} \vdash \neg(\forall x\varphi)^\perp &\to \forall x\,\neg\neg\varphi, \\ &\to \forall x\varphi, && \text{(because } \varphi \in \Sigma_0) \\ &\to \neg\neg\forall x\varphi. && \square \end{aligned}$$

Unlike the situation of the dual principles, we show that Σ_{k+1}-**WDUAL** and Π_{k+1}-**WDUAL** are equivalent over HA.

Proposition 3.12. *The following are equivalent over* HA*:*

1. Σ_{k+1}-**WDUAL**.
2. Π_{k+1}-**WDUAL**.
3. Σ_k-**DNS**.

Proof. First, we prove $\mathsf{HA} + \Sigma_{k+1}\text{-}\mathbf{WDUAL} \vdash \Sigma_k\text{-}\mathbf{DNS}$. Let φ be any Σ_k formula. Since $\exists x \varphi^\perp$ is Σ_{k+1},

$$\mathsf{HA} + \Sigma_{k+1}\text{-}\mathbf{WDUAL} \vdash \neg(\exists x \varphi^\perp)^\perp \to \neg\neg\,\exists x \varphi^\perp.$$

By Propositions 3.3.(2) and 3.3.(3), $\mathsf{HA}+\Sigma_{k+1}\text{-}\mathbf{WDUAL} \vdash \neg\,\forall x \varphi \to \neg\neg\,\exists x\,\neg\varphi$, and thus $\mathsf{HA} + \Sigma_{k+1}\text{-}\mathbf{WDUAL}$ proves $\neg\,\exists x\,\neg\varphi \to \neg\neg\,\forall x \varphi$. Then, we obtain

$$\mathsf{HA} + \Sigma_{k+1}\text{-}\mathbf{WDUAL} \vdash \forall x\,\neg\neg\varphi \to \neg\neg\,\forall x \varphi.$$

Secondly, we prove $\mathsf{HA} + \Pi_{k+1}\text{-}\mathbf{WDUAL} \vdash \Sigma_k\text{-}\mathbf{DNS}$. Let φ be any Σ_k formula. By Proposition 3.3.(3), $(\forall x \varphi)^\perp \equiv \exists x \varphi^\perp$ implies $\exists x\,\neg\varphi$ in HA. Thus $\mathsf{HA} \vdash \neg\,\exists x\,\neg\varphi \to \neg(\forall x\varphi)^\perp$. Since $\forall x\varphi$ is Π_{k+1}, we obtain

$$\mathsf{HA} + \Pi_{k+1}\text{-}\mathbf{WDUAL} \vdash \forall x\,\neg\neg\varphi \to \neg\neg\,\forall x \varphi.$$

Finally, we show that $\mathsf{HA} + \Sigma_k\text{-}\mathbf{DNS}$ proves both Σ_{k+1}-**WDUAL** and Π_{k+1}-**WDUAL** by induction on k. The case $k = 0$ follows from Proposition 3.11. Suppose that the statement holds for k, and we prove

(i) $\mathsf{HA} + \Sigma_{k+1}\text{-}\mathbf{DNS} \vdash \Sigma_{k+2}\text{-}\mathbf{WDUAL}$; and

(ii) $\mathsf{HA} + \Sigma_{k+1}\text{-}\mathbf{DNS} \vdash \Pi_{k+2}\text{-}\mathbf{WDUAL}$.

(i): Let $\exists x\varphi$ be any Σ_{k+2} formula where φ is Π_{k+1}. By induction hypothesis,

$$\mathsf{HA} + \Sigma_k\text{-}\mathbf{DNS} \vdash \neg\varphi^\perp \to \neg\neg\varphi.$$

Then, $\mathsf{HA} + \Sigma_k\text{-}\mathbf{DNS}$ proves the formula $\neg\varphi \to \neg\neg\varphi^\perp$, and hence it proves $\forall x\,\neg\varphi \to \forall x\,\neg\neg\varphi^\perp$. Since $\varphi^\perp$ is Σ_{k+1}, by applying Σ_{k+1}-**DNS**, we obtain

$$\mathsf{HA} + \Sigma_{k+1}\text{-}\mathbf{DNS} \vdash \forall x\,\neg\varphi \to \neg\neg\,\forall x\varphi^\perp.$$

Then $\mathsf{HA} + \Sigma_{k+1}\text{-}\mathbf{DNS} \vdash \neg\,\forall x\varphi^\perp \to \neg\,\forall x\,\neg\varphi$. Therefore we conclude

$$\mathsf{HA} + \Sigma_{k+1}\text{-}\mathbf{DNS} \vdash \neg(\exists x\varphi)^\perp \to \neg\neg\,\exists x\varphi.$$

(ii): Let $\forall x\varphi$ be any Π_{k+2} formula where φ is Σ_{k+1}. Since $\neg(\forall x\varphi)^\perp \equiv \neg\,\exists x\varphi^\perp$ implies $\forall x\,\neg\varphi^\perp$ in HA, by induction hypothesis, we obtain

$$\mathsf{HA} + \Sigma_k\text{-}\mathbf{DNS} \vdash \neg(\forall x\varphi)^\perp \to \forall x\,\neg\neg\varphi.$$

Since φ is Σ_{k+1}, we conclude

$$\mathsf{HA} + \Sigma_{k+1}\text{-}\mathbf{DNS} \vdash \neg(\forall x\varphi)^\perp \to \neg\neg\,\forall x\varphi. \qquad \square$$

As in the case of the dual principles, we can introduce the Δ_k-variations of the weak dual principle, namely, Δ_k-$\mathbf{WDUAL}^{\Sigma}$ and Δ_k-$\mathbf{WDUAL}^{\Pi}$. Notice that any instance of Γ-$\mathbf{WDUAL}$ is HA-equivalent to a formula of the form $\neg\varphi \to \neg\neg\varphi^{\perp}$. Then, as in the proof of Proposition 3.9, it is shown that Δ_k-$\mathbf{WDUAL}^{\Sigma}$ and Δ_k-$\mathbf{WDUAL}^{\Pi}$ are equivalent to Σ_k-$\mathbf{WDUAL}$ and Π_k-$\mathbf{WDUAL}$ over HA, respectively. So they are also equivalent to Σ_{k-1}-$\mathbf{DNS}$ by Proposition 3.12.

4 The law of excluded middle

In this section, we investigate variations of the law of excluded middle. This section consists of two subsections. First, we investigate the law of excluded middle with respect to duals. Secondly, we investigate the law of excluded middle for negated formulas.

4.1 The law of excluded middle with respect to duals

From the observations in Section 3, $\varphi^{\perp}$ is stronger than $\neg\varphi$. Hence by replacing $\neg\varphi$ in Γ-$\mathbf{LEM}$ with $\varphi^{\perp}$, we can expect to get a stronger principle. As an example of an application of the investigations in Section 3, in this subsection, we study this kind of variation of the law of excluded middle.

Definition 4.1 (The law of excluded middle with respect to duals)**.** Let Γ be any set of formulas in prenex normal form.

Γ-$\mathbf{LEM}^{\perp}$	$\varphi \vee \varphi^{\perp}$	$(\varphi \in \Gamma)$
Δ_k-$\mathbf{LEM}^{\perp,\Sigma}$	$(\varphi \leftrightarrow \psi) \to \varphi \vee \varphi^{\perp}$	$(\varphi \in \Sigma_k$ and $\psi \in \Pi_k)$
Δ_k-$\mathbf{LEM}^{\perp,\Pi}$	$(\varphi \leftrightarrow \psi) \to \psi \vee \psi^{\perp}$	$(\varphi \in \Sigma_k$ and $\psi \in \Pi_k)$

The principle Δ_1-$\mathbf{LEM}^{\perp,\Pi}$ corresponds to the principle (IIIb) in [8] and to the principle Δ_b-$\mathbf{LEM}$ in [11]. The following fact is already known.

Fact 4.2 (Fujiwara, Ishihara and Nemoto [8, Proposition 1])**.** *The following are equivalent over* HA*:*

1. Δ_1-$\mathbf{LEM}^{\perp,\Pi}$.
2. Σ_1-$\mathbf{DNE}$.

The following proposition shows interrelations between the laws of excluded middle and their counterparts with respect to duals.

Proposition 4.3. *Let Γ be any set of formulas in prenex normal form.*

1. Γ-$\mathbf{LEM}^{\perp}$ *is equivalent to* Γ-$\mathbf{LEM} + \Gamma$-$\mathbf{DUAL}$ *over* HA*;*
2. $\mathsf{HA} + \Delta_k$-$\mathbf{LEM}^{\perp,\Sigma} \vdash \Delta_k$-$\mathbf{LEM}$*;*
3. $\mathsf{HA} + \Delta_k$-$\mathbf{LEM}^{\perp,\Pi} \vdash \Delta_k$-$\mathbf{LEM}$*;*
4. $\mathsf{HA} + \Delta_k$-$\mathbf{LEM} + \Sigma_k$-$\mathbf{DUAL} \vdash \Delta_k$-$\mathbf{LEM}^{\perp,\Sigma}$*;*

5. $\mathsf{HA} + \Delta_k\text{-}\mathbf{LEM} + \Pi_k\text{-}\mathbf{DUAL} \vdash \Delta_k\text{-}\mathbf{LEM}^{\perp,\Pi}$.

Proof. 1. By Proposition 3.3.(3), $\mathsf{HA} + \Gamma\text{-}\mathbf{LEM}^{\perp} \vdash \Gamma\text{-}\mathbf{LEM}$. Also $\mathsf{HA} + \Gamma\text{-}\mathbf{LEM}^{\perp} \vdash \Gamma\text{-}\mathbf{DUAL}$ is evident because HA proves $\varphi \vee \varphi^{\perp} \to (\neg\varphi \to \varphi^{\perp})$. On the other hand, $\mathsf{HA} + \Gamma\text{-}\mathbf{LEM} + \Gamma\text{-}\mathbf{DUAL} \vdash \Gamma\text{-}\mathbf{LEM}^{\perp}$ is easily obtained.

Clauses 2, 3, 4 and 5 are proved similarly. $\square$

From Proposition 4.3, we obtain the exact strengths of the principles defined in Definition 4.1.

Proposition 4.4.

1. $\Sigma_k\text{-}\mathbf{LEM}^{\perp}$ *is equivalent to* $\Sigma_k\text{-}\mathbf{LEM}$ *over* HA*;*

2. $\Pi_k\text{-}\mathbf{LEM}^{\perp}$ *is equivalent to* $\Sigma_k\text{-}\mathbf{LEM}$ *over* HA*;*

3. $\Delta_k\text{-}\mathbf{LEM}^{\perp,\Sigma}$ *is equivalent to* $\Delta_k\text{-}\mathbf{LEM}$ *over* HA*;*

4. $\Delta_k\text{-}\mathbf{LEM}^{\perp,\Pi}$ *is equivalent to* $\Sigma_k\text{-}\mathbf{DNE}$ *over* HA.

Proof. 1. By Proposition 4.3.(1), $\Sigma_k\text{-}\mathbf{LEM}^{\perp}$ is equivalent to $\Sigma_k\text{-}\mathbf{LEM}+\Sigma_k\text{-}\mathbf{DUAL}$. Since $\mathsf{HA}+\Sigma_k\text{-}\mathbf{LEM}$ proves $\Sigma_k\text{-}\mathbf{DUAL}$ by Fact 2.3 and Proposition 3.7, $\Sigma_k\text{-}\mathbf{LEM}^{\perp}$ is equivalent to $\Sigma_k\text{-}\mathbf{LEM}$.

2. Since $\mathsf{HA} + \Pi_k\text{-}\mathbf{LEM}^{\perp}$ proves $\varphi^{\perp} \vee \varphi^{\perp\perp}$ for each Σ_k sentence φ, $\mathsf{HA} + \Pi_k\text{-}\mathbf{LEM}^{\perp} \vdash \Sigma_k\text{-}\mathbf{LEM}^{\perp}$ follows from Proposition 3.3.(2). In a similar way, we have $\mathsf{HA} + \Sigma_k\text{-}\mathbf{LEM}^{\perp} \vdash \Pi_k\text{-}\mathbf{LEM}^{\perp}$. Hence by clause 1, $\Pi_k\text{-}\mathbf{LEM}^{\perp}$ equivalent to $\Sigma_k\text{-}\mathbf{LEM}$ over HA.

3. Since $\mathsf{HA} + \Delta_k\text{-}\mathbf{LEM} \vdash \Sigma_{k-1}\text{-}\mathbf{DNE}$, this is immediately obtained from Propositions 3.7, 4.3.(2) and 4.3.(4).

4. Since $\mathsf{HA} + \Sigma_k\text{-}\mathbf{DNE}$ proves $\Delta_k\text{-}\mathbf{LEM}$ and $\Pi_k\text{-}\mathbf{DUAL}$ by Fact 2.3 and Proposition 3.7, we obtain $\mathsf{HA}+\Sigma_k\text{-}\mathbf{DNE} \vdash \Delta_k\text{-}\mathbf{LEM}^{\perp,\Pi}$ by Proposition 4.3.(5).

On the other hand, we prove $\mathsf{HA} + \Delta_k\text{-}\mathbf{LEM}^{\perp,\Pi} \vdash \Sigma_k\text{-}\mathbf{DNE}$. Let φ be any Σ_k formula. Since $\neg\neg\varphi \to \neg\varphi^{\perp}$ is HA-provable by Proposition 3.3.(3), we obtain $\mathsf{HA} \vdash \neg\neg\varphi \to (\varphi^{\perp} \leftrightarrow \perp)$. Since $\varphi^{\perp} \in \Pi_k$ and $\perp \in \Sigma_k$,

$$\mathsf{HA} + \Delta_k\text{-}\mathbf{LEM}^{\perp,\Pi} \vdash \neg\neg\varphi \to \varphi^{\perp} \vee \varphi^{\perp\perp}.$$

Since $\mathsf{HA} + \Delta_k\text{-}\mathbf{LEM}^{\perp,\Pi} \vdash \neg\neg\varphi \to \neg\varphi \vee \varphi$ by Proposition 3.3, we conclude that $\mathsf{HA} + \Delta_k\text{-}\mathbf{LEM}^{\perp,\Pi}$ proves $\neg\neg\varphi \to \varphi$. $\square$

Proposition 4.4.(4) is a generalization of Fact 4.2.

4.2 The law of excluded middle for negated formulas

In this subsection, we investigate the law of excluded middle for negated formulas, which are investigated in [6, 8] for $k = 1$.

Definition 4.5 (The law of excluded middle for negated formulas)**.** Let Γ be any set of formulas.

Γ^{n}-**LEM**	$\neg\varphi \vee \neg\neg\varphi$	($\varphi \in \Gamma$, in other words, $\neg\varphi \in \Gamma^{\mathrm{n}}$)
Δ^{n}_k-**LEM**	$(\varphi \leftrightarrow \psi) \to \neg\varphi \vee \neg\neg\varphi$	($\varphi \in \Sigma_k$ and $\psi \in \Pi_k$)

Although the definition of Γ^{n}-**LEM** is included in Definition 2.1, we defined it individually to pay attention to its properties. The principle Δ^{n}_1-**LEM** corresponds to the principle (IVa) in [8] and Δ_a-**WLEM** in [6]. The following fact is already obtained.

Fact 4.6 (Fujiwara, Ishihara and Nemoto [8, Proposition 3])**.** *The following are equivalent over* HA*:*

1. Δ^{n}_1-**LEM**.
2. Δ_1-**LEM**.

Obviously, Γ^{n}-**LEM** is weaker than Γ-**LEM**, and we obtain the following proposition. Proposition 4.7.(2) is a generalization of Fact 4.6.

Proposition 4.7. *Let* Γ *be any set of formulas.*

1. Γ^{n}-**LEM** $+$ Γ-**DNE** *is equivalent to* Γ-**LEM** *over* HA*;*
2. Δ^{n}_k-**LEM** $+$ Σ_{k-1}-**DNE** *is equivalent to* Δ_k-**LEM** *over* HA*;*
3. $\mathsf{HA} + \Sigma^{\mathrm{n}}_k$-**LEM** $\vdash \Delta^{\mathrm{n}}_k$-**LEM***;*
4. $\mathsf{HA} + \Pi^{\mathrm{n}}_k$-**LEM** $\vdash \Delta^{\mathrm{n}}_k$-**LEM**.

Proof. 1. This follows from Fact 2.2.

2. This is a consequence of Facts 2.2 and 2.3.

3 and 4 are obvious. □

From Fact 2.3, Σ_k-**LEM** and Π_k-**LEM** are equivalent modulo Σ_k-**DNE**. We prove an analogous result concerning Σ^{n}_k-**LEM** and Π^{n}_k-**LEM**.

Proposition 4.8. *The following are equivalent over* $\mathsf{HA} + \Sigma_{k-1}$-**DNS***:*

1. Σ^{n}_k-**LEM**.
2. Π^{n}_k-**LEM**.

Proof. First, we show $\mathsf{HA} + \Sigma_{k-1}$-**DNS** $+ \Sigma^{\mathrm{n}}_k$-**LEM** $\vdash \Pi^{\mathrm{n}}_k$-**LEM**. Let φ be any Π_k formula. Since $\varphi^\perp$ is Σ_k, we have

$$\mathsf{HA} + \Sigma^{\mathrm{n}}_k\text{-}\mathbf{LEM} \vdash \neg\varphi^\perp \vee \neg\neg\varphi^\perp.$$

Then, $\mathsf{HA} + \Sigma^{\mathrm{n}}_k$-**LEM** $\vdash \neg\varphi^\perp \vee \neg\varphi$ by Proposition 3.3.(3). Since Π_k-**WDUAL** is equivalent to Σ_{k-1}-**DNS** over HA by Proposition 3.12, we obtain

$$\mathsf{HA} + \Sigma_{k-1}\text{-}\mathbf{DNS} + \Sigma^{\mathrm{n}}_k\text{-}\mathbf{LEM} \vdash \neg\neg\varphi \vee \neg\varphi.$$

In a similar way, it is proved that $\mathsf{HA} + \Sigma_{k-1}$-**DNS** $+ \Pi^{\mathrm{n}}_k$-**LEM** proves Σ^{n}_k-**LEM** because Σ_k-**WDUAL** is also equivalent to Σ_{k-1}-**DNS** over HA by Proposition 3.12. □

From Fact 2.3.(6), Propositions 2.8.(1), 4.7 and 4.8, we obtain the following corollaries.

Corollary 4.9. *The following are equivalent over* HA*:*

1. Π_k-**LEM**.
2. Σ_k^{n}-**LEM** $+ \Sigma_{k-1}$-**DNE**.
3. Π_k^{n}-**LEM** $+ \Sigma_{k-1}$-**DNE**.

Corollary 4.10. *The following are equivalent over* HA*:*

1. Σ_k-**LEM**.
2. Σ_k^{n}-**LEM** $+ \Sigma_k$-**DNE**.
3. Π_k^{n}-**LEM** $+ \Sigma_k$-**DNE**.

5 De Morgan's law

In this section, we extensively investigate principles based on de Morgan's law.

Definition 5.1 (De Morgan's law)**.** Let Γ and Θ be any sets of formulas.

(Γ, Θ)-**DML**	$\neg(\varphi \wedge \psi) \to \neg\varphi \vee \neg\psi$	$(\varphi \in \Gamma$ and $\psi \in \Theta)$
Δ_k-**DML**	$(\varphi \leftrightarrow \varphi') \wedge (\psi \leftrightarrow \psi')$ $\to (\neg(\varphi \wedge \psi) \to \neg\varphi \vee \neg\psi)$	$(\varphi, \psi \in \Sigma_k$ and $\varphi', \psi' \in \Pi_k)$
(Δ_k, Θ)-**DML**	$(\varphi \leftrightarrow \varphi') \to (\neg(\varphi \wedge \psi) \to \neg\varphi \vee \neg\psi)$	$(\varphi \in \Sigma_k$, $\varphi' \in \Pi_k$ and $\psi \in \Theta)$

Several variations of Δ_1-**DML** are extensively investigated in [6]. As in the case of the law of excluded middle, we also deal with the principles of the forms $(\Gamma^{\mathrm{n}}, \Theta)$-**DML**, $(\Delta_k^{\mathrm{n}}, \Theta)$-**DML**, and so on. Of course, (Γ, Θ)-**DML** and (Θ, Γ)-**DML** are equivalent.

This section consists of four subsections. First, we investigate several basic implications between the principles. Secondly, we study the interrelationship between de Morgan's law and the contrapositive version of the collection principle. Thirdly, Δ_k and Δ_k^{n} variants of de Morgan's law are explored. Finally, we investigate de Morgan's law with respect to duals.

5.1 Basic implications

In this subsection, we organize several versions of de Morgan's law. Some arguments in this subsection for $k = 1$ can be found in [6]. The following proposition is trivially obtained.

Proposition 5.2. *Let* $\Gamma \in \{\Sigma_k, \Pi_k\}$ *and* Θ *be any set of formulas.*

1. $\mathsf{HA} + (\Gamma, \Theta)$-**DML** $\vdash (\Delta_k, \Theta)$-**DML***;*
2. $\mathsf{HA} + (\Gamma^{\mathrm{n}}, \Theta)$-**DML** $\vdash (\Delta_k^{\mathrm{n}}, \Theta)$-**DML**.

We show that Γ^{n}**-LEM** and Δ_k^{n}**-LEM** are stronger than several versions of de Morgan's law.

Proposition 5.3. *Let Γ and Θ be any sets of formulas.*

1. $\mathsf{HA} + \Gamma^{\mathrm{n}}$-**LEM** $\vdash (\Gamma, \Theta)$-**DML**;
2. $\mathsf{HA} + \Gamma^{\mathrm{n}}$-**LEM** $\vdash (\Gamma^{\mathrm{n}}, \Theta)$-**DML**;
3. $\mathsf{HA} + \Delta_k^{\mathrm{n}}$-**LEM** $\vdash (\Delta_k, \Theta)$-**DML**;
4. $\mathsf{HA} + \Delta_k^{\mathrm{n}}$-**LEM** $\vdash (\Delta_k^{\mathrm{n}}, \Theta)$-**DML**.

Proof. 1. Let $\varphi \in \Gamma$ and $\psi \in \Theta$. Since $\mathsf{HA} \vdash \neg(\varphi \wedge \psi) \to \neg(\neg\neg\varphi \wedge \psi)$, we get

$$\mathsf{HA} \vdash (\neg\varphi \vee \neg\neg\varphi) \to (\neg(\varphi \wedge \psi) \to \neg\varphi \vee \neg\psi).$$

It follows that $\mathsf{HA} + \Gamma^{\mathrm{n}}$-**LEM** proves (Γ, Θ)-**DML**.

2, 3 and 4 are proved as for clause 1. □

Corollary 5.4.

1. *For any set Γ of formulas,* $\mathsf{HA} + \Gamma^{\mathrm{n}}$-**LEM** *proves* Γ-**DML** *and* Γ^{n}-**DML**;
2. $\mathsf{HA} + \Delta_k^{\mathrm{n}}$-**LEM** *proves* Δ_k-**DML** *and* Δ_k^{n}-**DML**.

Conversely, we show that the principles Γ^{n}**-LEM** and Δ_k^{n}**-LEM** are equivalent to some variations of de Morgan's law.

Proposition 5.5. *For any set Γ of formulas, the following are equivalent over* HA*:*

1. Γ^{n}-**LEM**.
2. $(\Gamma, \Gamma^{\mathrm{n}})$-**DML**.

Proof. By Proposition 5.3, $\mathsf{HA} + \Gamma^{\mathrm{n}}$-**LEM** $\vdash (\Gamma, \Gamma^{\mathrm{n}})$-**DML**. On the other hand, let φ be any Γ formula. Since $\mathsf{HA} \vdash \neg(\varphi \wedge \neg\varphi)$, we obtain $\mathsf{HA} + (\Gamma, \Gamma^{\mathrm{n}})$-**DML** $\vdash$ $\neg\varphi \vee \neg\neg\varphi$. □

Proposition 5.6. *For $\Gamma \in \{\Sigma_k, \Pi_k\}$, the following are equivalent over* HA*:*

1. Δ_k^{n}-**LEM**.
2. $(\Delta_k, \Gamma^{\mathrm{n}})$-**DML**.
3. $(\Delta_k^{\mathrm{n}}, \Gamma)$-**DML**.
4. $(\Delta_k, \Delta_k^{\mathrm{n}})$-**DML**.

Proof. By Proposition 5.3, Δ_k^{n}**-LEM** entails $(\Delta_k, \Gamma^{\mathrm{n}})$-**DML** and $(\Delta_k^{\mathrm{n}}, \Gamma)$-**DML**. By Proposition 5.2, each of $(\Delta_k, \Gamma^{\mathrm{n}})$-**DML** and $(\Delta_k^{\mathrm{n}}, \Gamma)$-**DML** implies $(\Delta_k, \Delta_k^{\mathrm{n}})$-**DML**. On the other hand, we can show that $\mathsf{HA} + (\Delta_k, \Delta_k^{\mathrm{n}})$-**DML** proves Δ_k^{n}**-LEM** as in the proof of Proposition 5.5. □

Here we investigate several equivalences of some variations of de Morgan's law over the theory $\mathsf{HA} + \Sigma_{k-1}$-**DNS**.

Proposition 5.7. *Let Θ be any set of formulas.*

1. $(\Sigma_k^{\mathrm{n}}, \Theta)$-**DML** *is equivalent to* (Π_k, Θ)-**DML** *over* $\mathsf{HA} + \Sigma_{k-1}$-**DNS**;
2. $(\Pi_k^{\mathrm{n}}, \Theta)$-**DML** *is equivalent to* (Σ_k, Θ)-**DML** *over* $\mathsf{HA} + \Sigma_{k-1}$-**DNS**.

Proof. Recall that each of Σ_k-**WDUAL** and Π_k-**WDUAL** is HA-equivalent to Σ_{k-1}-**DNS** (Proposition 3.12). Then for any $\varphi \in \Sigma_k$ and $\psi \in \Pi_k$, $\mathsf{HA} + \Sigma_{k-1}$-**DNS** proves $\neg\varphi^\perp \leftrightarrow \neg\neg\varphi$ and $\neg\psi^\perp \leftrightarrow \neg\neg\psi$. Then clauses 1 and 2 follow from this observation and the fact that HA proves $\neg(\xi \wedge \delta) \leftrightarrow \neg(\neg\neg\xi \wedge \delta)$. $\square$

From Proposition 5.7, we obtain several equivalences over $\mathsf{HA} + \Sigma_{k-1}$-**DNS**.

Corollary 5.8.

1. Σ_k^{n}-**LEM**, Π_k^{n}-**LEM**, $(\Sigma_k, \Sigma_k^{\mathrm{n}})$-**DML**, $(\Pi_k, \Pi_k^{\mathrm{n}})$-**DML**, (Σ_k, Π_k)-**DML** *and* $(\Sigma_k^{\mathrm{n}}, \Pi_k^{\mathrm{n}})$-**DML** *are equivalent over* $\mathsf{HA} + \Sigma_{k-1}$-**DNS**;
2. Σ_k-**DML**, $(\Sigma_k, \Pi_k^{\mathrm{n}})$-**DML** *and* Π_k^{n}-**DML** *are equivalent over* $\mathsf{HA}+\Sigma_{k-1}$-**DNS**;
3. Π_k-**DML**, $(\Pi_k, \Sigma_k^{\mathrm{n}})$-**DML** *and* Σ_k^{n}-**DML** *are equivalent over* $\mathsf{HA}+\Sigma_{k-1}$-**DNS**;
4. *For* $\Gamma \in \{\Sigma_k, \Pi_k, \Sigma_k^{\mathrm{n}}, \Pi_k^{\mathrm{n}}\}$, *each of* (Δ_k, Γ)-**DML** *and* $(\Delta_k^{\mathrm{n}}, \Gamma)$-**DML** *is equivalent to* Δ_k^{n}-**LEM** *over* $\mathsf{HA} + \Sigma_{k-1}$-**DNS**.

Proof. 1. This is a consequence of Propositions 4.8, 5.5 and 5.7.

2 and 3 are immediate from Proposition 5.7.

4. The principles (Δ_k, Σ_k)-**DML**, (Δ_k, Π_k)-**DML**, $(\Delta_k^{\mathrm{n}}, \Sigma_k^{\mathrm{n}})$-**DML** and $(\Delta_k^{\mathrm{n}}, \Pi_k^{\mathrm{n}})$-**DML** are equivalent to $(\Delta_k, \Pi_k^{\mathrm{n}})$-**DML**, $(\Delta_k, \Sigma_k^{\mathrm{n}})$-**DML**, $(\Delta_k^{\mathrm{n}}, \Pi_k)$-**DML** and $(\Delta_k^{\mathrm{n}}, \Sigma_k)$-**DML** over $\mathsf{HA} + \Sigma_{k-1}$-**DNS**, respectively. Then, by Proposition 5.6, each of them is equivalent to Δ_k^{n}-**LEM**. $\square$

From Corollaries 4.9, 4.10, 5.8 and Proposition 5.5, we also obtain the following.

Corollary 5.9. *Let P be one of* $(\Sigma_k, \Sigma_k^{\mathrm{n}})$-**DML**, $(\Pi_k, \Pi_k^{\mathrm{n}})$-**DML**, (Σ_k, Π_k)-**DML** *and* $(\Sigma_k^{\mathrm{n}}, \Pi_k^{\mathrm{n}})$-**DML**.

1. $P + \Sigma_{k-1}$-**DNE** *is equivalent to* Π_k-**LEM** *over* HA;
2. $P + \Sigma_k$-**DNE** *is equivalent to* Σ_k-**LEM** *over* HA.

The following corollary follows from Propositions 5.6, 4.7.(2) and Corollary 5.8.(4).

Corollary 5.10. *Let* $\Gamma \in \{\Sigma_k, \Pi_k, \Sigma_k^{\mathrm{n}}, \Pi_k^{\mathrm{n}}\}$. *Let P be one of the principles* (Δ_k, Γ)-**DML**, $(\Delta_k^{\mathrm{n}}, \Gamma)$-**DML** *and* $(\Delta_k, \Delta_k^{\mathrm{n}})$-**DML**. *Then* $P + \Sigma_{k-1}$-**DNE** *is equivalent to* Δ_k-**LEM** *over* HA.

We get the following corollary.

Corollary 5.11. *Let* $\Gamma \in \{\Sigma_k, \Pi_k, \Sigma_k^{\mathrm{n}}, \Pi_k^{\mathrm{n}}\}$.

1. $\mathsf{HA} + \Gamma$-$\mathbf{DML} + \Sigma_{k-1}$-$\mathbf{DNS} \vdash \Delta_k^{\mathrm{n}}$-$\mathbf{LEM}$;
2. $\mathsf{HA} + \Gamma$-$\mathbf{DML} + \Sigma_{k-1}$-$\mathbf{DNE} \vdash \Delta_k$-$\mathbf{LEM}$.

Proof. 1. Since Γ-$\mathbf{DML}$ implies (Δ_k, Γ)-$\mathbf{DML}$ by Proposition 5.2, the statement immediately follows from Corollary 5.8.(4).

2. This follows from Corollary 5.10. □

Corollary 5.11.(2) generalizes Fact 2.6. Also we generalize Fact 2.5.(1).

Proposition 5.12. $\mathsf{HA} + \Sigma_k$-$\mathbf{DNE} \vdash \Pi_k$-$\mathbf{DML}$.

Proof. Since $\mathsf{HA} + \Sigma_k$-$\mathbf{DNE}$ proves Σ_{k-1}-$\mathbf{DNS}$, it is sufficient to show that $\mathsf{HA} + \Sigma_k$-$\mathbf{DNE} \vdash \Sigma_k^{\mathrm{n}}$-$\mathbf{DML}$ by Corollary 5.8.(3). Let φ and ψ be any Σ_k formulas. Since $\mathsf{HA} \vdash \neg(\neg\varphi \land \neg\psi) \to \neg\neg(\varphi \lor \psi)$ and $\varphi \lor \psi$ is HA-equivalent to some Σ_k formula, we obtain

$$\mathsf{HA} + \Sigma_k\text{-}\mathbf{DNE} \vdash \neg(\neg\varphi \land \neg\psi) \to \varphi \lor \psi.$$

Therefore

$$\mathsf{HA} + \Sigma_k\text{-}\mathbf{DNE} \vdash \neg(\neg\varphi \land \neg\psi) \to \neg\neg\varphi \lor \neg\neg\psi.$$

□

By combining Corollary 5.11.(2) and Proposition 5.12, we obtain a proof of Fact 2.3.(4).

5.2 The collection principles and de Morgan's law

In this subsection, we investigate the so-called collection principles. The following proposition is stated in [5].

Proposition 5.13. *For any formula* $\varphi(y, z)$,

$$\mathsf{HA} \vdash \forall y < x\, \exists z\, \varphi(y, z) \to \exists w\, \forall y < x\, \exists z < w\, \varphi(y, z).$$

Proof. Let $\psi(x)$ be the formula

$$\forall y < x\, \exists z\, \varphi(y, z) \to \exists w\, \forall y < x\, \exists z < w\, \varphi(y, z),$$

and this proposition is proved by applying the induction axiom for $\psi(x)$. □

We introduce the following contrapositive version of the collection principle.

Definition 5.14 (The contrapositive collection principles)**.** Let Γ be any set of formulas.

Γ-$\mathbf{COLL^{cp}}$ $\quad \forall w\, \exists y < x\, \forall z < w\, \varphi(y, z) \to \exists y < x\, \forall z\, \varphi(y, z) \qquad (\varphi(y, z) \in \Gamma)$

Proposition 5.15. *The following are equivalent over* HA*:*

1. Π_{k+1}-$\mathbf{COLL^{cp}}$.
2. Σ_k-$\mathbf{COLL^{cp}}$.

Proof. By using a primitive recursive pairing function, it is easy to show that for any Σ_k formula $\varphi(y, z_0, z_1)$, $\mathsf{HA} + \Sigma_k$-$\mathbf{COLL^{cp}}$ proves

$$\forall w\, \exists y < x\, \forall z_0 < w\, \forall z_1 < w\, \varphi(y, z_0, z_1) \to \exists y < x\, \forall z_0\, \forall z_1\, \varphi(y, z_0, z_1). \quad (1)$$

From this observation, the equivalence of Σ_k-$\mathbf{COLL^{cp}}$ and Π_{k+1}-$\mathbf{COLL^{cp}}$ immediately follows. $\square$

The following proposition extends [10, Corollary 4.5].

Proposition 5.16. $\mathsf{HA} + \Sigma_{k+1}$-$\mathbf{DML} + \Sigma_k$-$\mathbf{DNE} \vdash \Pi_{k+1}$-$\mathbf{COLL^{cp}}$.

Proof. We simultaneously prove the following two statements by induction on k:

(i) $\mathsf{HA} + \Sigma_{k+1}$-$\mathbf{DML} + \Sigma_k$-$\mathbf{DNE} \vdash \Pi_{k+1}$-$\mathbf{COLL^{cp}}$;

(ii) For any Π_{k+1} formula $\varphi(y)$, there exists a Π_{k+1} formula $\psi(x)$ such that

$$\mathsf{HA} + \Sigma_{k+1}\text{-}\mathbf{DML} + \Sigma_k\text{-}\mathbf{DNE} \vdash \exists y < x\, \varphi(y) \leftrightarrow \psi(x).$$

We suppose that our statements hold for all $k' < k$, and we prove (i) and (ii).

(i): Prior to proving our statement, we show that for any Π_k formula $\varphi(y, z)$,

$$\mathsf{HA} + \Sigma_{k+1}\text{-}\mathbf{DML} + \Sigma_{k-1}\text{-}\mathbf{DNE} \vdash \neg \forall y < x\, \exists z\, \varphi(y, z) \to \exists y < x\, \forall z\, \neg \varphi(y, z), \quad (2)$$

which is a generalization of [10, Lemma 4.4].

Let $\psi(x)$ be the formula

$$\neg \forall y < x\, \exists z\, \varphi(y, z) \to \exists y < x\, \forall z\, \neg \varphi(y, z),$$

and we show that $\forall x \psi(x)$ is derivable by applying the induction axiom for $\psi(x)$. Since $\mathsf{HA} \vdash \neg y < 0$, we have $\mathsf{HA} \vdash \forall y < 0\, \exists z\, \varphi(y, z)$. Thus we obviously obtain $\mathsf{HA} \vdash \psi(0)$.

We prove induction step. We have

$$\mathsf{HA} \vdash \neg \forall y \leq x\, \exists z\, \varphi(y, z) \to \neg(\forall y < x\, \exists z\, \varphi(y, z) \land \exists z\, \varphi(x, z)).$$

By Proposition 5.13, the formula $\forall y < x\, \exists z\, \varphi(y, z)$ is HA-equivalent to the formula $\exists w\, \forall y < x\, \exists z < w\, \varphi(y, z)$. If $k = 0$, the formula $\exists z < w\, \varphi(y, z)$ is HA-provably equivalent to some Π_0 formula $\rho(y, w)$. If $k > 0$, by induction hypothesis (ii) for $k - 1$, the formula $\exists z < w\, \varphi(y, z)$ is equivalent to some Π_k formula $\rho(y, w)$ in $\mathsf{HA} + \Sigma_k$-$\mathbf{DML} + \Sigma_{k-1}$-$\mathbf{DNE}$. Also $\exists w\, \forall y < x\, \rho(y, w)$ is HA-equivalent to a Σ_{k+1} formula. Thus $\forall y < x\, \exists z\, \varphi(y, z)$ can be regarded as a Σ_{k+1}

formula in $\mathsf{HA}+\Sigma_k$-**DML**$+\Sigma_{k-1}$-**DNE**. Then $\mathsf{HA}+\Sigma_{k+1}$-**DML**$+\Sigma_{k-1}$-**DNE** proves

$$\neg\,\forall y \le x\,\exists z\,\varphi(y,z) \to \neg\,\forall y < x\,\exists z\,\varphi(y,z) \vee \neg\,\exists z\,\varphi(x,z).$$

Hence it also proves

$$\psi(x) \wedge \neg\,\forall y \le x\,\exists z\,\varphi(y,z) \to \exists y < x\,\forall z\,\neg\varphi(y,z) \vee \forall z\,\neg\varphi(x,z).$$

It follows that the theory proves

$$\psi(x) \wedge \neg\,\forall y \le x\,\exists z\,\varphi(y,z) \to \exists y \le x\,\forall z\,\neg\varphi(y,z).$$

This means $\mathsf{HA}+\Sigma_{k+1}$-**DML**$+\Sigma_{k-1}$-**DNE** $\vdash \psi(x) \to \psi(x+1)$. We have proved (2).

We prove $\mathsf{HA} + \Sigma_{k+1}$-**DML** $+ \Sigma_k$-**DNE** $\vdash \Pi_{k+1}$-$\mathbf{COLL^{cp}}$. It suffices to prove Σ_k-$\mathbf{COLL^{cp}}$ by Proposition 5.15. Let $\varphi(y,z)$ be any Σ_k formula. By Proposition 5.13 for the formula $\varphi^\perp(y,z)$, we have

$$\mathsf{HA} \vdash \neg\,\exists w\,\forall y < x\,\exists z < w\,\varphi^\perp(y,z) \to \neg\,\forall y < x\,\exists z\,\varphi^\perp(y,z).$$

In the light of Proposition 3.3.(3), we obtain

$$\mathsf{HA} \vdash \forall w\,\exists y < x\,\forall z < w\,\varphi(y,z) \to \neg\,\exists w\,\forall y < x\,\exists z < w\,\varphi^\perp(y,z).$$

Therefore

$$\mathsf{HA} \vdash \forall w\,\exists y < x\,\forall z < w\,\varphi(y,z) \to \neg\,\forall y < x\,\exists z\,\varphi^\perp(y,z).$$

Since $(\varphi(y,z))^\perp$ is Π_k, from (2), we obtain that $\mathsf{HA}+\Sigma_{k+1}$-**DML**$+\Sigma_{k-1}$-**DNE** proves

$$\forall w\,\exists y < x\,\forall z < w\,\varphi(y,z) \to \exists y < x\,\forall z\,\neg\varphi^\perp(y,z).$$

Since Σ_k-**DNE** proves Π_k-**DUAL**, we conclude that $\mathsf{HA}+\Sigma_{k+1}$-**DML**$+\Sigma_k$-**DNE** proves

$$\forall w\,\exists y < x\,\forall z < w\,\varphi(y,z) \to \exists y < x\,\forall z\,\varphi(y,z)$$

by Proposition 3.3.(2). This completes the proof of (i).

(ii): Let $\forall z\varphi(y,z)$ be any Π_{k+1} formula where $\varphi(y,z)$ is Σ_k. Since $\varphi^\perp(y,z)$ is Π_k, by induction hypothesis (ii) for $k-1$, there exists a Π_k formula $\psi(y,w)$ such that

$$\mathsf{HA} + \Sigma_k\text{-}\mathbf{DML} + \Sigma_{k-1}\text{-}\mathbf{DNE} \vdash \exists z < w\,\varphi^\perp(y,z) \leftrightarrow \psi(y,w).$$

This is also the case for $k=0$. Then

$$\mathsf{HA} + \Sigma_k\text{-}\mathbf{DML} + \Sigma_{k-1}\text{-}\mathbf{DNE} \vdash \forall z < w\,\neg\varphi^\perp(y,z) \leftrightarrow \neg\psi(y,w).$$

Since Σ_k-**DNE** implies Π_k-**DUAL**, we obtain

$$\mathsf{HA} + \Sigma_k\text{-}\mathbf{DML} + \Sigma_k\text{-}\mathbf{DNE} \vdash \forall z < w\,\varphi(y,z) \leftrightarrow \psi^\perp(y,w).$$

By (i), we have that $\mathsf{HA} + \Sigma_{k+1}\text{-}\mathbf{DML} + \Sigma_k\text{-}\mathbf{DNE}$ proves

$$\exists y < x\, \forall z\, \varphi(y, z) \leftrightarrow \forall w\, \exists y < x\, \forall z < w\, \varphi(y, z).$$

Therefore we obtain that $\mathsf{HA} + \Sigma_{k+1}\text{-}\mathbf{DML} + \Sigma_k\text{-}\mathbf{DNE}$ also proves

$$\exists y < x\, \forall z\, \varphi(y, z) \leftrightarrow \forall w\, \exists y < x\, \psi^{\perp}(y, w).$$

This completes the proof of (ii). $\square$

Remark 5.17. By Proposition 5.15, $\Pi_0\text{-}\mathbf{COLL}^{\mathbf{cp}}$ is equivalent to $\Pi_1\text{-}\mathbf{COLL}^{\mathbf{cp}}$ over HA. We will show in Proposition 5.22 that $\mathsf{HA} + \Pi_1\text{-}\mathbf{COLL}^{\mathbf{cp}} \vdash \Sigma_1\text{-}\mathbf{DML}$. Therefore $\mathsf{HA} \nvdash \Pi_0\text{-}\mathbf{COLL}^{\mathbf{cp}}$ because it is known that $\mathsf{HA} \nvdash \Sigma_1\text{-}\mathbf{DML}$ (cf. [1]). Thus the statement of Proposition 5.16 for $k = -1$ does not holds.

Corollary 5.18.

1. *For any Π_k formula $\varphi(y)$, there exists a Π_k formula $\psi(x)$ such that*

$$\mathsf{HA} + \Sigma_k\text{-}\mathbf{DML} + \Sigma_{k-1}\text{-}\mathbf{DNE} \vdash \exists y < x\, \varphi(y) \leftrightarrow \psi(x);$$

2. *For any Σ_k formula $\varphi(y)$, there exists a Σ_k formula $\psi(x)$ such that*

$$\mathsf{HA} + \Sigma_{k-1}\text{-}\mathbf{DML} + \Sigma_{k-2}\text{-}\mathbf{DNE} \vdash \forall y < x\, \varphi(y) \leftrightarrow \psi(x).$$

Proof. 1. For $k = 0$, this is trivial. For $k > 0$, the statement is already proved in the proof of Proposition 5.16.

2. Since the statement obviously holds for $k = 0$, we may assume $k > 0$. Let $\exists z \varphi(y, z)$ be any Σ_k formula where $\varphi(y, z)$ is Π_{k-1}. By Proposition 5.13, we have

$$\mathsf{HA} \vdash \forall y < x\, \exists z\, \varphi(y, z) \leftrightarrow \exists w\, \forall y < x\, \exists z < w\, \varphi(y, z).$$

By clause 1, there exists a Π_{k-1} formula $\psi(y, w)$ such that

$$\mathsf{HA} + \Sigma_{k-1}\text{-}\mathbf{DML} + \Sigma_{k-2}\text{-}\mathbf{DNE} \vdash \exists z < w\, \varphi(y, z) \leftrightarrow \psi(y, w).$$

Hence

$$\mathsf{HA} + \Sigma_{k-1}\text{-}\mathbf{DML} + \Sigma_{k-2}\text{-}\mathbf{DNE} \vdash \forall y < x\, \exists z\, \varphi(y, z) \leftrightarrow \exists w\, \forall y < x\, \psi(y, w).$$

Since $\exists w\, \forall y < x\, \psi(y, w)$ is obviously equivalent to a Σ_k formula, this completes our proof of clause 2. $\square$

Corollary 5.18 is very useful for exploring principles containing bounded quantifiers. For instance, it can be applied to the study of the least number principle.

Definition 5.19 (The least number principle)**.** Let Γ be a set of formulas.

Γ-**LN** $\quad \exists x \varphi(x) \rightarrow \exists x(\varphi(x) \wedge \forall y < x\, \neg\varphi(y)) \qquad (\varphi \in \Gamma)$

Theorem 5.20. *Let Γ be either Σ_k or Π_k. Then Γ-**LN** and Γ-**LEM** are equivalent over* HA.

Proof. First, we prove $\mathsf{HA} + \Gamma\text{-}\mathbf{LN} \vdash \Gamma\text{-}\mathbf{LEM}$. Let φ be any Γ formula and let $\psi(x)$ be a Γ formula HA-equivalent to $\varphi \vee 0 < x$, where x does not occur freely in φ. Notice that $0 < x \wedge \forall y < x \neg\psi(y)$ implies $\neg\psi(0)$ which implies $\neg\varphi$. Hence we have

$$\mathsf{HA} \vdash (\varphi \vee 0 < x) \wedge \forall y < x \neg\psi(y) \to \varphi \vee \neg\varphi,$$

and thus

$$\mathsf{HA} \vdash \exists x(\psi(x) \wedge \forall y < x \neg\psi(y)) \to \varphi \vee \neg\varphi.$$

Since $\mathsf{HA} \vdash \exists x\psi(x)$, we have $\mathsf{HA} + \Gamma\text{-}\mathbf{LN} \vdash \exists x(\psi(x) \wedge \forall y < x \neg\psi(y))$. Therefore we obtain $\mathsf{HA} + \Gamma\text{-}\mathbf{LN} \vdash \varphi \vee \neg\varphi$.

Secondly, we prove $\mathsf{HA} + \Pi_k\text{-}\mathbf{LEM} \vdash \Pi_k\text{-}\mathbf{LN}$. A proof for $\mathsf{HA} + \Sigma_k\text{-}\mathbf{LEM} \vdash \Sigma_k\text{-}\mathbf{LN}$ is similar. Let $\varphi(x)$ be any Π_k formula, and let $\psi(z)$ be the formula

$$\exists x < z\, \varphi(x) \to \exists x < z\, (\varphi(x) \wedge \forall y < x \neg\varphi(y)).$$

We prove $\mathsf{HA} + \Pi_k\text{-}\mathbf{LEM} \vdash \forall z\psi(z)$ by applying the induction axiom for $\psi(z)$. Since $\mathsf{HA} \vdash \neg\, \exists x < 0\, \varphi(x)$, we obtain $\mathsf{HA} \vdash \psi(0)$.

We prove induction step. Notice $\mathsf{HA}+\Pi_k\text{-}\mathbf{LEM}$ proves $\Sigma_k\text{-}\mathbf{DML}+\Sigma_{k-1}\text{-}\mathbf{DNE}$ by Corollary 4.9 and Proposition 5.3.(1). Thus by Corollary 5.18.(1), the formula $\exists x < z\, \varphi(x)$ is equivalent to some Π_k formula in $\mathsf{HA}+\Pi_k\text{-}\mathbf{LEM}$. Therefore

$$\mathsf{HA} + \Pi_k\text{-}\mathbf{LEM} \vdash \exists x < z\, \varphi(x) \vee \neg\, \exists x < z\, \varphi(x). \tag{3}$$

Since $\mathsf{HA} \vdash \exists x \leq z\, \varphi(x) \leftrightarrow (\exists x < z\, \varphi(x) \vee \varphi(z))$, we obtain

$$\mathsf{HA} \vdash \exists x \leq z\, \varphi(x) \wedge \neg\, \exists x < z\, \varphi(x) \to \varphi(z) \wedge \forall x < z \neg\varphi(x),$$

and hence

$$\mathsf{HA} \vdash \exists x \leq z\, \varphi(x) \wedge \neg\, \exists x < z\, \varphi(x) \to \exists x \leq z\, (\varphi(x) \wedge \forall y < x \neg\varphi(y)).$$

On the other hand, we obviously obtain

$$\mathsf{HA} \vdash \psi(z) \wedge \exists x < z\, \varphi(x) \to \exists x \leq z\, (\varphi(x) \wedge \forall y < x \neg\varphi(y)).$$

Then by (3), we have

$$\mathsf{HA} + \Pi_k\text{-}\mathbf{LEM} \vdash \psi(z) \wedge \exists x \leq z\, \varphi(x) \to \exists x \leq z\, (\varphi(x) \wedge \forall y < x \neg\varphi(y)).$$

It follows $\mathsf{HA}+\Pi_k\text{-}\mathbf{LEM} \vdash \psi(z) \to \psi(z+1)$. We have completed our proof. $\square$

By using Corollary 5.18 and Theorem 5.20, we are able to generalize Fact 2.5.(2). The proof is similar to that of the implication $2 \Rightarrow 1$ of [8, Proposition 2].

Proposition 5.21. $\mathsf{HA} + \Sigma_k\text{-}\mathbf{DML} + \Sigma_{k-1}\text{-}\mathbf{DNE} \vdash \Pi_k\text{-}\mathbf{DML}$.

Proof. We may assume $k > 0$. Let $\forall x \varphi(x)$ and $\forall y \psi(y)$ be any Π_k formulas where $\varphi(x)$ and $\psi(y)$ are Σ_{k-1}. We define the formulas $\xi(x)$ and $\eta(y)$ as follows:

- $\xi(x) :\equiv \forall z < x(\varphi(z) \wedge \psi(z)) \wedge \varphi^\perp(x)$;
- $\eta(y) :\equiv \forall z < y(\varphi(z) \wedge \psi(z)) \wedge \psi^\perp(y) \wedge \varphi(y)$.

Since $\varphi(z) \wedge \psi(z)$ is HA-equivalent to a Σ_{k-1} formula, by Corollary 5.18.(2), $\forall z < x(\varphi(z) \wedge \psi(z))$ is equivalent to some Σ_{k-1} formula in $\mathsf{HA} + \Sigma_{k-2}$-$\mathbf{DML}$ + Σ_{k-3}-$\mathbf{DNE}$. Thus the formula $\exists x \xi(x)$ is equivalent to a Σ_k formula in the theory. Similarly, $\exists y \eta(y)$ is also equivalent to some Σ_k formula in the theory.

By the definitions of $\xi(x)$ and $\eta(y)$, we obtain

- $\mathsf{HA} \vdash \xi(x) \wedge \eta(y) \wedge x \leq y \to \varphi^\perp(x) \wedge \varphi(x)$, and
- $\mathsf{HA} \vdash \xi(x) \wedge \eta(y) \wedge y < x \to \psi(y) \wedge \psi^\perp(y)$.

Thus by Proposition 3.3.(4) and $\mathsf{HA} \vdash x \leq y \vee y < x$, we have that HA proves $\neg(\exists x \xi(x) \wedge \exists y \eta(y))$. Then from the above observations, we obtain

$$\mathsf{HA} + \Sigma_k\text{-}\mathbf{DML} + \Sigma_{k-3}\text{-}\mathbf{DNE} \vdash \neg \exists x \xi(x) \vee \neg \exists y \eta(y). \tag{4}$$

Note that $\mathsf{HA} + \Sigma_k$-$\mathbf{DML} + \Sigma_{k-1}$-$\mathbf{DNE}$ proves Σ_{k-1}-$\mathbf{DUAL}$, Π_{k-1}-$\mathbf{DUAL}$ and Π_{k-1}-$\mathbf{LEM}$. Then $\mathsf{HA} + \Sigma_k$-$\mathbf{DML} + \Sigma_{k-1}$-$\mathbf{DNE}$ proves

$$\begin{aligned} \exists x \neg\varphi(x) &\to \exists x \varphi^\perp(x), && (\text{by } \Sigma_{k-1}\text{-}\mathbf{DUAL}) \\ &\to \exists x[\varphi^\perp(x) \wedge \forall z < x \neg\varphi^\perp(z)], && (\text{by } \Pi_{k-1}\text{-}\mathbf{LEM} \text{ and Theorem 5.20}) \\ &\to \exists x[\varphi^\perp(x) \wedge \forall z < x\, \varphi(z)]. && (\text{by } \Pi_{k-1}\text{-}\mathbf{DUAL} \text{ and Proposition 3.3.(2)}) \end{aligned}$$

Hence, by the definition of the formula $\xi(x)$, we have

$$\mathsf{HA} + \Sigma_k\text{-}\mathbf{DML} + \Sigma_{k-1}\text{-}\mathbf{DNE} \vdash \exists x \neg\varphi(x) \wedge \forall y \psi(y) \to \exists x \xi(x).$$

Since $\mathsf{HA} \vdash \neg \exists x \neg\varphi(x) \to \forall x \neg\neg\varphi(x)$ and $\mathsf{HA} + \Sigma_{k-1}$-$\mathbf{DNE}$ implies Σ_{k-1}-$\mathbf{DNS}$ by Proposition 2.8.(1), we obtain

$$\mathsf{HA} + \Sigma_k\text{-}\mathbf{DML} + \Sigma_{k-1}\text{-}\mathbf{DNE} \vdash \forall y \psi(y) \wedge \neg \exists x \xi(x) \to \neg\neg \forall x \varphi(x).$$

On the other hand,

$$\mathsf{HA} \vdash \neg(\forall x \varphi(x) \wedge \forall y \psi(y)) \wedge \forall y \psi(y) \to \neg \forall x \varphi(x).$$

Therefore we obtain

$$\mathsf{HA} + \Sigma_k\text{-}\mathbf{DML} + \Sigma_{k-1}\text{-}\mathbf{DNE} \vdash \neg(\forall x \varphi(x) \wedge \forall y \psi(y)) \wedge \neg \exists x \xi(x) \to \neg \forall y \psi(y). \tag{5}$$

In a similar way, we obtain

$$\mathsf{HA} + \Sigma_k\text{-}\mathbf{DML} + \Sigma_{k-1}\text{-}\mathbf{DNE} \vdash \neg(\forall x \varphi(x) \wedge \forall y \psi(y)) \wedge \neg \exists y \eta(y) \to \neg \forall x \varphi(x). \tag{6}$$

By combining (4), (5) and (6), we conclude

$$\mathsf{HA} + \Sigma_k\text{-}\mathbf{DML} + \Sigma_{k-1}\text{-}\mathbf{DNE} \vdash \neg(\forall x \varphi(x) \wedge \forall y \psi(y)) \to \neg \forall x \varphi(x) \vee \neg \forall y \psi(y). \quad \square$$

Finally, we prove that the converse of Proposition 5.16 also holds. This is closely related to [5, Theorem 4.5].

Proposition 5.22. $\mathsf{HA} + \Pi_k\text{-}\mathbf{COLL^{cp}} \vdash \Sigma_k\text{-}\mathbf{DML} + \Sigma_{k-1}\text{-}\mathbf{LEM}$.

Proof. We prove by induction on k. For $k = 0$, our statement obviously holds. Suppose that the statement holds for k, and we prove the case of $k + 1$. We prove the following two statements:

(i) $\mathsf{HA} + \Pi_{k+1}\text{-}\mathbf{COLL^{cp}} \vdash \Sigma_k\text{-}\mathbf{LEM}$;

(ii) $\mathsf{HA} + \Pi_{k+1}\text{-}\mathbf{COLL^{cp}} \vdash \Sigma_{k+1}\text{-}\mathbf{DML}$.

(i): Let $\exists x\varphi$ be any Σ_k formula where φ is Π_{k-1}. By induction hypothesis, $\mathsf{HA} + \Pi_k\text{-}\mathbf{COLL^{cp}} \vdash \Sigma_k\text{-}\mathbf{DML} + \Sigma_{k-1}\text{-}\mathbf{LEM}$. By Fact 2.3, $\mathsf{HA} + \Pi_k\text{-}\mathbf{COLL^{cp}}$ also proves $\Pi_{k-1}\text{-}\mathbf{LEM}$ and $\Sigma_{k-1}\text{-}\mathbf{DNE}$. It follows from Corollary 5.18.(1), we have that $\exists x < z\, \varphi$ is equivalent to some Π_{k-1} formula in $\mathsf{HA} + \Pi_k\text{-}\mathbf{COLL^{cp}}$. Then by applying $\Pi_{k-1}\text{-}\mathbf{LEM}$, we obtain

$$\mathsf{HA} + \Pi_k\text{-}\mathbf{COLL^{cp}} \vdash \exists x < z\, \varphi \lor \neg \exists x < z\, \varphi.$$

Then

$$\mathsf{HA} + \Pi_k\text{-}\mathbf{COLL^{cp}} \vdash \exists w < 2\, [(w = 0 \to \exists x < z\, \varphi) \land (w = 1 \to \neg \exists x < z\, \varphi)].$$

Since $\mathsf{HA} + \Pi_k\text{-}\mathbf{COLL^{cp}}$ proves $\Pi_{k-1}\text{-}\mathbf{DUAL}$, we obtain

$$\mathsf{HA} + \Pi_k\text{-}\mathbf{COLL^{cp}} \vdash \exists w < 2\, [(w = 0 \to \exists x < z\, \varphi) \land (w = 1 \to \forall x < z\, \varphi^{\perp})].$$

Hence

$$\mathsf{HA} + \Pi_k\text{-}\mathbf{COLL^{cp}} \vdash \forall z\, \exists w < 2\, \forall x < z\, [(w = 0 \to \exists x\varphi) \land (w = 1 \to \varphi^{\perp})].$$

Since $(w = 0 \to \exists x\varphi) \land (w = 1 \to \varphi^{\perp})$ is equivalent to some Σ_k formula, by Proposition 5.15,

$$\mathsf{HA} + \Pi_{k+1}\text{-}\mathbf{COLL^{cp}} \vdash \exists w < 2\, \forall x\, [(w = 0 \to \exists x\varphi) \land (w = 1 \to \varphi^{\perp})].$$

Then

$$\mathsf{HA} + \Pi_{k+1}\text{-}\mathbf{COLL^{cp}} \vdash \exists w < 2\, [(w = 0 \to \exists x\varphi) \land (w = 1 \to \forall x\varphi^{\perp})].$$

Thus we obtain $\mathsf{HA} + \Pi_{k+1}\text{-}\mathbf{COLL^{cp}} \vdash \exists x\varphi \lor \neg \exists x\varphi$ by Proposition 3.3.(3). This means $\mathsf{HA} + \Pi_{k+1}\text{-}\mathbf{COLL^{cp}} \vdash \Sigma_k\text{-}\mathbf{LEM}$.

(ii): Let $\exists x\varphi$ and $\exists y\psi$ be any Σ_{k+1} formulas where φ and ψ are Π_k. We have $\mathsf{HA} \vdash \neg(\exists x\varphi \land \exists y\psi) \to \neg(\exists x < z\, \varphi \land \exists y < z\, \psi)$. From (i), we have that $\mathsf{HA} + \Pi_{k+1}\text{-}\mathbf{COLL^{cp}}$ proves $\Sigma_k\text{-}\mathbf{LEM}$. By Fact 2.3, Propositions 5.21 and 3.7, the theory also proves $\Sigma_k\text{-}\mathbf{DML}$, $\Sigma_k\text{-}\mathbf{DNE}$, $\Pi_k\text{-}\mathbf{DML}$ and $\Pi_k\text{-}\mathbf{DUAL}$. Then by Corollary 5.18.(1), both $\exists x < z\, \varphi$ and $\exists y < z\, \psi$ are equivalent to some Π_k

formulas in $\mathsf{HA}+\Pi_{k+1}$-$\mathbf{COLL^{cp}}$. By applying Π_k-$\mathbf{DML}$, $\mathsf{HA}+\Pi_{k+1}$-$\mathbf{COLL^{cp}}$ proves

$$\begin{aligned}\neg(\exists x\varphi \wedge \exists y\psi) &\to \neg\,\exists x<z\,\varphi \vee \neg\,\exists y<z\,\psi,\\ &\to \exists w<2\,[(w=0\to\neg\,\exists x<z\,\varphi)\wedge(w=1\to\neg\,\exists y<z\,\psi)],\\ &\to \exists w<2\,[(w=0\to\forall x<z\,\varphi^{\perp})\wedge(w=1\to\forall y<z\,\psi^{\perp})],\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(by }\Pi_k\text{-}\mathbf{DUAL}\text{)}\\ &\to \exists w<2\,\forall x<z\,\forall y<z\,[(w=0\to\varphi^{\perp})\wedge(w=1\to\psi^{\perp})].\end{aligned}$$

Thus we have that $\mathsf{HA}+\Pi_{k+1}$-$\mathbf{COLL^{cp}}$ proves

$$\neg(\exists x\varphi \wedge \exists y\psi) \to \forall z\,\exists w<2\,\forall x<z\,\forall y<z\,[(w=0\to\varphi^{\perp})\wedge(w=1\to\psi^{\perp})].$$

Then, in the light of (1), $\mathsf{HA}+\Pi_{k+1}$-$\mathbf{COLL^{cp}}$ proves

$$\begin{aligned}\neg(\exists x\varphi \wedge \exists y\psi) &\to \exists w<2\,\forall x\,\forall y\,[(w=0\to\varphi^{\perp})\wedge(w=1\to\psi^{\perp})],\\ &\to \exists w<2\,[(w=0\to\forall x\varphi^{\perp})\wedge(w=1\to\forall y\psi^{\perp})],\\ &\to \exists w<2\,[(w=0\to\neg\,\exists x\varphi)\wedge(w=1\to\neg\,\exists y\psi)],\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(by Proposition 3.3.(3))}\\ &\to \neg\,\exists x\varphi \vee \neg\,\exists y\psi.\end{aligned}$$

Therefore $\mathsf{HA}+\Pi_{k+1}$-$\mathbf{COLL^{cp}} \vdash \Sigma_{k+1}$-$\mathbf{DML}$. □

From Propositions 5.16, 5.22 and Fact 2.3, we get the following corollary.

Corollary 5.23. *The following are equivalent over* HA*:*

1. Π_{k+1}-$\mathbf{COLL^{cp}}$.
2. Σ_{k+1}-$\mathbf{DML}+\Sigma_k$-$\mathbf{LEM}$.
3. Σ_{k+1}-$\mathbf{DML}+\Sigma_k$-$\mathbf{DNE}$.

5.3 The principles Δ_k-DML and Δ_k^{n}-DML

In this subsection, we mainly investigate the principles Δ_k-$\mathbf{DML}$ and Δ_k^{n}-$\mathbf{DML}$.

Proposition 5.24.

1. $\mathsf{HA}+\Delta_{k+1}$-$\mathbf{DML}+\Sigma_{k-1}$-$\mathbf{DNS}\vdash\Sigma_k^{\mathrm{n}}$-$\mathbf{LEM}$*;*
2. $\mathsf{HA}+\Delta_{k+1}^{\mathrm{n}}$-$\mathbf{DML}+\Sigma_{k-1}$-$\mathbf{DNS}\vdash\Sigma_k^{\mathrm{n}}$-$\mathbf{LEM}$.

Proof. Let φ be any Σ_k formula.

1. By Proposition 3.3.(4), $\mathsf{HA}\vdash\neg(\varphi\wedge\varphi^{\perp})$. Since both φ and $\varphi^{\perp}$ are Δ_{k+1}, $\mathsf{HA}+\Delta_{k+1}$-$\mathbf{DML}\vdash\neg\varphi\vee\neg\varphi^{\perp}$. Then $\mathsf{HA}+\Delta_{k+1}$-$\mathbf{DML}+\Sigma_{k-1}$-$\mathbf{DNS}$ proves $\neg\varphi\vee\neg\neg\varphi$ by Proposition 3.12.

2. Since $\mathsf{HA}\vdash\neg(\neg\varphi\wedge\neg\neg\varphi)$, $\mathsf{HA}+\Sigma_{k-1}$-$\mathbf{DNS}\vdash\neg(\neg\varphi\wedge\neg\varphi^{\perp})$. Then $\mathsf{HA}+\Delta_{k+1}^{\mathrm{n}}$-$\mathbf{DML}+\Sigma_{k-1}$-$\mathbf{DNS}\vdash\neg\neg\varphi\vee\neg\neg\varphi^{\perp}$. We conclude that the theory proves $\neg\varphi\vee\neg\neg\varphi$. □

From Corollaries 4.9, 4.10 and Proposition 5.24, we obtain the following.

Corollary 5.25. *Let* $\Gamma \in \{\Delta_{k+1}, \Delta^{\mathrm{n}}_{k+1}\}$.

1. $\mathsf{HA} + \Gamma\text{-}\mathbf{DML} + \Sigma_{k-1}\text{-}\mathbf{DNE} \vdash \Pi_k\text{-}\mathbf{LEM}$;
2. $\mathsf{HA} + \Gamma\text{-}\mathbf{DML} + \Sigma_{k}\text{-}\mathbf{DNE} \vdash \Sigma_k\text{-}\mathbf{LEM}$.

Furthermore, we prove the following proposition by adapting the proofs of Proposition 5.21 and [6, Lemma 2.14].

Proposition 5.26. $\mathsf{HA} + \Delta_k\text{-}\mathbf{DML} + \Sigma_{k-1}\text{-}\mathbf{DNE} \vdash \Delta^{\mathrm{n}}_k\text{-}\mathbf{DML}$.

Proof. We may assume $k > 0$. Let $\exists x\varphi(x)$ and $\exists y\psi(y)$ be any Σ_k formulas where $\varphi(x)$ and $\psi(y)$ are Π_{k-1}, and let φ' and ψ' be any Π_k formulas. Let χ denote the formula $(\exists x\varphi(x) \leftrightarrow \varphi') \wedge (\exists y\psi(y) \leftrightarrow \psi')$. We define the formulas $\xi(x)$ and $\eta(y)$ as follows:

- $\xi(x) :\equiv \forall z < x(\varphi^{\perp}(z) \wedge \psi^{\perp}(z)) \wedge \varphi(x)$;
- $\eta(y) :\equiv \forall z < y(\varphi^{\perp}(z) \wedge \psi^{\perp}(z)) \wedge \psi(y) \wedge \varphi^{\perp}(y)$.

As in the proof of Proposition 5.21, the formulas $\exists x\xi(x)$ and $\exists y\eta(y)$ are equivalent to some Σ_k formulas in the theory $\mathsf{HA} + \Sigma_{k-2}\text{-}\mathbf{DML} + \Sigma_{k-3}\text{-}\mathbf{DNE}$ which is included in $\mathsf{HA} + \Sigma_{k-1}\text{-}\mathbf{DNE}$ by Fact 2.3, Corollary 4.10 and Proposition 5.3. Also

$$\mathsf{HA} \vdash \neg(\exists x\xi(x) \wedge \exists y\eta(y)). \tag{7}$$

By Corollary 5.25.(1), $\mathsf{HA} + \Delta_k\text{-}\mathbf{DML} + \Sigma_{k-2}\text{-}\mathbf{DNE}$ proves $\Pi_{k-1}\text{-}\mathbf{LEM}$. Since $\Sigma_{k-1}\text{-}\mathbf{DNE}$ implies $\Pi_{k-1}\text{-}\mathbf{DUAL}$, by Theorem 5.20, we obtain

$$\mathsf{HA} + \Delta_k\text{-}\mathbf{DML} + \Sigma_{k-1}\text{-}\mathbf{DNE} \vdash \exists x\varphi(x) \to \exists x[\varphi(x) \wedge \forall z < x\, \varphi^{\perp}(z)]. \tag{8}$$

In a similar way, we have

$$\mathsf{HA} + \Delta_k\text{-}\mathbf{DML} + \Sigma_{k-1}\text{-}\mathbf{DNE} \vdash \exists y < x\, \psi(y) \to \exists y < x\, [\psi(y) \wedge \forall z < y\, \psi^{\perp}(z)].$$

Then by the definition of $\eta(y)$,

$$\mathsf{HA} + \Delta_k\text{-}\mathbf{DML} + \Sigma_{k-1}\text{-}\mathbf{DNE} \vdash \forall z < x\, \varphi^{\perp}(z) \wedge \exists z < x\, \psi(z) \to \exists y\eta(y).$$

From this with (8), $\mathsf{HA} + \Delta_k\text{-}\mathbf{DML} + \Sigma_{k-1}\text{-}\mathbf{DNE}$ proves

$$\exists x\varphi(x) \wedge \neg\exists y\eta(y) \to \exists x[\varphi(x) \wedge \forall z < x\, \varphi^{\perp}(z) \wedge \forall z < x\, \psi^{\perp}(z)].$$

It follows that the theory proves $\exists x\varphi(x) \wedge \neg\exists y\eta(y) \to \exists x\xi(x)$. On the other hand, HA proves $\exists x\xi(x) \to \exists x\varphi(x) \wedge \neg\exists y\eta(y)$ from (7). Therefore $\mathsf{HA} + \Delta_k\text{-}\mathbf{DML} + \Sigma_{k-1}\text{-}\mathbf{DNE}$ proves

$$\chi \to [\exists x\xi(x) \leftrightarrow (\varphi' \wedge \forall y\eta^{\perp}(y))].$$

Also $\varphi' \wedge \forall y\eta^{\perp}(y)$ is HA-provably equivalent to some Π_k formula.

In a similar way, we obtain that $\mathsf{HA} + \Delta_k$-**DML** $+ \Sigma_{k-1}$-**DNE** proves

$$\chi \to [\exists y \eta(y) \leftrightarrow (\psi' \wedge \forall x \xi^{\perp}(x))]$$

and $\psi' \wedge \forall x \xi^{\perp}(x)$ is HA-provably equivalent to some Π_k formula.

Then by applying Δ_k-**DML** to (7),

$$\mathsf{HA} + \Delta_k\text{-}\mathbf{DML} + \Sigma_{k-1}\text{-}\mathbf{DNE} \vdash \chi \to \neg \exists x \xi(x) \vee \neg \exists y \eta(y). \tag{9}$$

From (8) and the definition of $\xi(x)$,

$$\mathsf{HA} + \Delta_k\text{-}\mathbf{DML} + \Sigma_{k-1}\text{-}\mathbf{DNE} \vdash \exists x \varphi(x) \wedge \forall y \psi^{\perp}(y) \to \exists x \xi(x).$$

Then

$$\mathsf{HA} + \Delta_k\text{-}\mathbf{DML} + \Sigma_{k-1}\text{-}\mathbf{DNE} \vdash \neg \exists x \xi(x) \wedge \neg \exists y \psi(y) \to \neg \exists x \varphi(x).$$

Therefore we obtain

$$\mathsf{HA}+\Delta_k\text{-}\mathbf{DML}+\Sigma_{k-1}\text{-}\mathbf{DNE} \vdash \neg(\neg \exists x \varphi(x) \wedge \neg \exists y \psi(y)) \wedge \neg \exists x \xi(x) \to \neg\neg \exists y \psi(y). \tag{10}$$

In a similar way, we obtain

$$\mathsf{HA}+\Delta_k\text{-}\mathbf{DML}+\Sigma_{k-1}\text{-}\mathbf{DNE} \vdash \neg(\neg \exists x \varphi(x) \wedge \neg \exists y \psi(y)) \wedge \neg \exists y \eta(y) \to \neg\neg \exists x \varphi(x). \tag{11}$$

By combining (9), (10) and (11), we conclude that $\mathsf{HA}+\Delta_k$-**DML**$+\Sigma_{k-1}$-**DNE** proves

$$\chi \to [\neg(\neg \exists x \varphi(x) \wedge \neg \exists y \psi(y)) \to \neg\neg \exists x \varphi(x) \vee \neg\neg \exists y \psi(y)]. \qquad \square$$

5.4 De Morgan's law with respect to duals

In [1], principles based on de Morgan's law with respect to duals are introduced.

Definition 5.27 (De Morgan's law with respect to duals)**.** Let Γ and Θ be any sets of formulas in prenex normal form.

Γ-$\mathbf{DML}^{\perp}$	$\neg(\varphi \wedge \psi) \to \varphi^{\perp} \vee \psi^{\perp}$	$(\varphi, \psi \in \Gamma)$
(Γ, Θ)-$\mathbf{DML}^{\perp}$	$\neg(\varphi \wedge \psi) \to \varphi^{\perp} \vee \psi^{\perp}$	$(\varphi \in \Gamma$ and $\psi \in \Theta)$
Δ_k-$\mathbf{DML}^{\perp}$	$(\varphi \leftrightarrow \varphi') \wedge (\psi \leftrightarrow \psi')$ $\to (\neg(\varphi \wedge \psi) \to \varphi^{\perp} \vee \psi^{\perp})$	$(\varphi, \psi \in \Sigma_k$ and $\varphi', \psi' \in \Pi_k)$
(Δ_k, Γ)-$\mathbf{DML}^{\perp,\Sigma}$	$(\varphi \leftrightarrow \varphi') \to (\neg(\varphi \wedge \psi) \to \varphi^{\perp} \vee \psi^{\perp})$	$(\varphi \in \Sigma_k$, $\varphi' \in \Pi_k$ and $\psi \in \Gamma)$
(Δ_k, Γ)-$\mathbf{DML}^{\perp,\Pi}$	$(\varphi \leftrightarrow \varphi') \to (\neg(\varphi \wedge \psi) \to (\varphi')^{\perp} \vee \psi^{\perp})$	$(\varphi \in \Sigma_k$, $\varphi' \in \Pi_k$ and $\psi \in \Gamma)$

Our Σ_k-$\mathbf{DML}^{\perp}$ is called Σ_k-**LLPO** in [1]. As in the case of Γ-$\mathbf{LEM}^{\perp}$ (Proposition 4.3), we show that the principles defined in Definition 5.27 are exactly de Morgan's laws equipped with the dual principles.

Proposition 5.28. *Let Γ and Θ be any sets of formulas in prenex normal form.*

1. (Γ,Θ)-$\mathbf{DML}^{\perp}$ *is equivalent to* (Γ,Θ)-$\mathbf{DML}+\Gamma$-$\mathbf{DUAL}+\Theta$-$\mathbf{DUAL}$ *over* HA*;*

2. (Δ_k,Θ)-$\mathbf{DML}^{\perp,\Sigma}$ *is equivalent to* (Δ_k,Θ)-$\mathbf{DML}+\Sigma_k$-$\mathbf{DUAL}+\Theta$-$\mathbf{DUAL}$ *over* HA*;*

3. (Δ_k,Θ)-$\mathbf{DML}^{\perp,\Pi}$ *is equivalent to* (Δ_k,Θ)-$\mathbf{DML}+\Pi_k$-$\mathbf{DUAL}+\Theta$-$\mathbf{DUAL}$ *over* HA*.*

Proof. 1. By Proposition 3.3.(3), $\mathsf{HA}+(\Gamma,\Theta)$-$\mathbf{DML}^{\perp}\vdash(\Gamma,\Theta)$-$\mathbf{DML}$. Let $\varphi\in\Gamma$. Since $\mathsf{HA}\vdash\neg\varphi\to\neg(\varphi\wedge\neg\perp)$, we have that $\mathsf{HA}+(\Gamma,\Theta)$-$\mathbf{DML}^{\perp}$ proves the formula $\neg\varphi\to\varphi^{\perp}\vee(\neg\perp)^{\perp}$. Thus $\mathsf{HA}+(\Gamma,\Theta)$-$\mathbf{DML}^{\perp}\vdash\neg\varphi\to\varphi^{\perp}$, and this means that Γ-$\mathbf{DUAL}$ is provable. Similarly, Θ-$\mathbf{DUAL}$ is also provable. On the other hand, (Γ,Θ)-$\mathbf{DML}^{\perp}$ is easily proved in $\mathsf{HA}+(\Gamma,\Theta)$-$\mathbf{DML}+\Gamma$-$\mathbf{DUAL}+\Theta$-$\mathbf{DUAL}$.

2 and 3 are proved in a similar way. □

Summarizing the results so far, we obtain the following corollary.

Corollary 5.29.

1. Σ_k-$\mathbf{DML}^{\perp}$ *is equivalent to* Σ_k-$\mathbf{DML}+\Sigma_{k-1}$-$\mathbf{DNE}$ *over* HA*;*

2. (Σ_k,Π_k)-$\mathbf{DML}^{\perp}$ *is equivalent to* Σ_k-$\mathbf{LEM}$ *over* HA*;*

3. (Δ_k,Σ_k)-$\mathbf{DML}^{\perp,\Sigma}$ *is equivalent to* Δ_k-$\mathbf{LEM}$ *over* HA*;*

4. Δ_k-$\mathbf{DML}^{\perp}$ *is equivalent to* Δ_k-$\mathbf{DML}+\Sigma_{k-1}$-$\mathbf{DNE}$ *over* HA*;*

5. *Each of the principles* Π_k-$\mathbf{DML}^{\perp}$*,* (Δ_k,Σ_k)-$\mathbf{DML}^{\perp,\Pi}$*,* (Δ_k,Π_k)-$\mathbf{DML}^{\perp,\Sigma}$ *and* (Δ_k,Π_k)-$\mathbf{DML}^{\perp,\Pi}$ *is equivalent to* Σ_k-$\mathbf{DNE}$ *over* HA*.*

Proof. 1. This is a consequence of Propositions 3.7 and 5.28.(1).

2. By Propositions 3.7 and 5.28.(1), (Σ_k,Π_k)-$\mathbf{DML}^{\perp}$ is HA-equivalent to (Σ_k,Π_k)-$\mathbf{DML}+\Sigma_k$-$\mathbf{DNE}$. Then it is also HA-equivalent to Σ_k-$\mathbf{LEM}$ by Corollary 5.9.(2).

3. From Propositions 3.7 and 5.28.(2), (Δ_k,Σ_k)-$\mathbf{DML}^{\perp,\Sigma}$ is HA-equivalent to (Δ_k,Σ_k)-$\mathbf{DML}+\Sigma_{k-1}$-$\mathbf{DNE}$. Then it is HA-equivalent to Δ_k-$\mathbf{LEM}$ by Corollary 5.10.

4 is proved as in the proof of Proposition 5.28.

5. By Propositions 3.7 and 5.28.(1), Π_k-$\mathbf{DML}^{\perp}$ is HA-equivalent to Π_k-$\mathbf{DML}+\Sigma_k$-$\mathbf{DNE}$. Since $\mathsf{HA}+\Sigma_k$-$\mathbf{DNE}\vdash\Pi_k$-$\mathbf{DML}$ by Proposition 5.12, Π_k-$\mathbf{DML}^{\perp}$ is HA-equivalent to Σ_k-$\mathbf{DNE}$. Similarly, each of (Δ_k,Σ_k)-$\mathbf{DML}^{\perp,\Pi}$, (Δ_k,Π_k)-$\mathbf{DML}^{\perp,\Sigma}$ and (Δ_k,Π_k)-$\mathbf{DML}^{\perp,\Pi}$ is HA-equivalent to Σ_k-$\mathbf{DNE}$ because each of them implies Σ_k-$\mathbf{DNE}$ over HA by Proposition 5.28, and $\mathsf{HA}+\Sigma_k$-$\mathbf{DNE}$ proves (Δ_k,Σ_k)-$\mathbf{DML}$ and (Δ_k,Π_k)-$\mathbf{DML}$ by Fact 2.3.(4) and Proposition 5.3.(3). □

In [3, Theorem 14], it is proved that Σ_k-$\mathbf{DML}^{\perp}$ is equivalent to Σ_k-$\mathbf{DML}+\Sigma_{k-1}$-$\mathbf{LEM}$ over HA. This result follows from Corollaries 5.23 and 5.29.(1).

6 The double negation elimination

In this section, we explore variations of the double negation elimination. As in the previous sections, we deal with the principles of forms $(\Gamma^{\mathrm{n}} \vee \Theta)$-**DNE**, $(\Delta_k \vee \Theta)$-**DNE**, and so on. As in the case of de Morgan's law, $(\Gamma \vee \Theta)$-**DNE** is obviously equivalent to $(\Theta \vee \Gamma)$-**DNE**. Interestingly, de Morgan's law can be seen as a variation of the double negation elimination.

Proposition 6.1. *For any sets Γ and Θ of formulas, the following are equivalent over* HA*:*

1. (Γ, Θ)-**DML**.
2. $(\Gamma^{\mathrm{n}} \vee \Theta^{\mathrm{n}})$-**DNE**.

The analogous equivalences also hold for the versions of Δ_k and Δ_k^{n}.

Proof. Let $\varphi \in \Gamma$ and $\psi \in \Theta$. Since $\mathsf{HA} \vdash \neg(\varphi \wedge \psi) \leftrightarrow \neg\neg(\neg\varphi \vee \neg\psi)$, HA proves

$$[\neg(\varphi \wedge \psi) \to \neg\varphi \vee \neg\psi] \leftrightarrow [\neg\neg(\neg\varphi \vee \neg\psi) \to \neg\varphi \vee \neg\psi].$$

The last statement is also proved in a similar way. □

We prove the following basic proposition concerning principles based on the double negation elimination.

Proposition 6.2. *Let $\Gamma \in \{\Sigma_k, \Pi_k, \Delta_k\}$ and let Θ be any set of formulas.*

1. $\mathsf{HA} + (\Gamma \vee \Theta)$-**DNE** $\vdash \Gamma$-**DNE***;*
2. *Suppose that for any $\varphi \in \Theta$, there exists $\psi \in \Sigma_k$ such that* $\mathsf{HA} + \Sigma_k$-**DNE** $\vdash \varphi \leftrightarrow \psi$*. Then $(\Sigma_k \vee \Theta)$-**DNE** is equivalent to Σ_k-**DNE** over* HA*;*
3. $(\Sigma_k^{\mathrm{n}} \vee \Theta)$-**DNE** $+ \Sigma_{k-1}$-**DNE** *is equivalent to* $(\Pi_k \vee \Theta)$-**DNE** *over* HA*;*
4. $(\Sigma_k^{\mathrm{n}} \vee \Gamma)$-**DNE** *is equivalent to* $(\Pi_k \vee \Gamma)$-**DNE** *over* HA*;*
5. $(\Pi_k^{\mathrm{n}} \vee \Theta)$-**DNE** $+ \Sigma_k$-**DNE** *is equivalent to* $(\Sigma_k \vee \Theta)$-**DNE** *over* HA*;*
6. $(\Sigma_k^{\mathrm{dn}} \vee \Theta)$-**DNE** *is equivalent to* $(\Pi_k^{\mathrm{n}} \vee \Theta)$-**DNE** *over* $\mathsf{HA} + \Sigma_{k-1}$-**DNS***;*
7. $(\Sigma_k^{\mathrm{dn}} \vee \Gamma)$-**DNE** *is equivalent to* $(\Pi_k^{\mathrm{n}} \vee \Gamma)$-**DNE** *over* HA*;*
8. $(\Pi_k^{\mathrm{dn}} \vee \Theta)$-**DNE** *is equivalent to* $(\Sigma_k^{\mathrm{n}} \vee \Theta)$-**DNE** *over* $\mathsf{HA} + \Sigma_{k-1}$-**DNS***;*
9. $(\Pi_k^{\mathrm{dn}} \vee \Gamma)$-**DNE** *is equivalent to* $(\Pi_k \vee \Gamma)$-**DNE** *over* HA*;*
10. $(\Delta_k^{\mathrm{dn}} \vee \Theta)$-**DNE** $+ \Sigma_{k-1}$-**DNE** *is equivalent to* $(\Delta_k \vee \Theta)$-**DNE** *over* HA*;*
11. $(\Delta_k^{\mathrm{dn}} \vee \Gamma)$-**DNE** *is equivalent to* $(\Delta_k \vee \Gamma)$-**DNE** *over* HA.

Also these statements hold even if $\Theta \in \{\Delta_k^{\mathrm{n}}, \Delta_k^{\mathrm{dn}}\}$.

Proof. 1. This is because $0 = 0 \in \Theta$ and for any $\varphi \in \Gamma$, $\varphi \vee 0 = 0$ is HA-provably equivalent to φ.

2. From clause 1, $\mathsf{HA} + (\Sigma_k \vee \Theta)$-**DNE** proves Σ_k-**DNE**. On the other hand, let φ and ψ be any Σ_k formulas. Notice that $\varphi \vee \psi$ is HA-equivalent to $\exists x((x = 0 \to \varphi) \wedge (x = 1 \to \psi))$. Then it is shown that $\varphi \vee \psi$ is provably equivalent to some Σ_k formula in HA (cf. [13, Lemma 4.4]). Therefore $\mathsf{HA} + \Sigma_k$-**DNE** proves $(\Sigma_k \vee \Theta)$-**DNE**.

3. By Propositions 3.3.(3) and 3.7, for any $\varphi \in \Sigma_k$ and $\psi \in \Pi_k$, $\mathsf{HA} + \Sigma_{k-1}$-**DNE** proves $\neg\varphi \leftrightarrow \varphi^{\perp}$ and $\neg\psi^{\perp} \leftrightarrow \psi$. Thus the principles $(\Sigma_k^{\mathrm{n}} \vee \Theta)$-**DNE** and $(\Pi_k \vee \Theta)$-**DNE** are equivalent over $\mathsf{HA} + \Sigma_{k-1}$-**DNE**. Also by clause 1, $\mathsf{HA} + (\Pi_k \vee \Theta)$-**DNE** proves Σ_{k-1}-**DNE**.

Clause 4 follows from clauses 1 and 3 because Γ-**DNE** entails Σ_{k-1}-**DNE**. Clause 5 is proved in a similar way as in the proof of clause 3. Clause 6 is a refinement of Proposition 5.7.(1) in the light of Proposition 6.1, and is proved in a similar way. Clause 7 follows from clause 6 and the fact that $\mathsf{HA} + \Gamma$-**DNE** proves Σ_{k-1}-**DNS**. Clause 8 is a refinement of Proposition 5.7.(2). Clause 9 follows from clause 8 because $\mathsf{HA} + \Gamma$-**DNE** proves Π_k-**DNE**. Clause 10 is proved in a similar way as in the proof of clause 3. Clause 11 follows from clause 10. $\square$

We have the following corollary which shows that Σ_k-**LEM** and Π_k-**LEM** are also variations of the double negation elimination. A part of Corollary 6.3.(4) is stated in [1].

Corollary 6.3.

1. *For* $\Gamma' \in \{\Sigma_k, \Delta_k, \Pi_k^{\mathrm{n}}, \Delta_k^{\mathrm{n}}, \Sigma_k^{\mathrm{dn}}, \Delta_k^{\mathrm{dn}}\}$, $(\Sigma_k \vee \Gamma')$-**DNE** *is equivalent to* Σ_k-**DNE** *over* HA*;*
2. Σ_k-**LEM**, $(\Sigma_k \vee \Pi_k)$-**DNE**, $(\Sigma_k \vee \Sigma_k^{\mathrm{n}})$-**DNE** *and* $(\Sigma_k \vee \Pi_k^{\mathrm{dn}})$-**DNE** *are equivalent over* HA*;*
3. Π_k-**LEM**, $(\Pi_k^{\mathrm{n}} \vee \Pi_k)$-**DNE** *and* $(\Sigma_k^{\mathrm{dn}} \vee \Pi_k)$-**DNE** *are equivalent over* HA*;*
4. Σ_k-**DML**$^{\perp}$, $(\Pi_k \vee \Pi_k)$-**DNE**, $(\Pi_k \vee \Sigma_k^{\mathrm{n}})$-**DNE** *and* $(\Pi_k \vee \Pi_k^{\mathrm{dn}})$-**DNE** *are equivalent over* HA*;*
5. *Let* $\Gamma' \in \{\Sigma_k, \Pi_k, \Delta_k, \Sigma_k^{\mathrm{n}}, \Pi_k^{\mathrm{n}}\}$ *and* $\Gamma'' \in \{\Delta_k, \Delta_k^{\mathrm{n}}, \Delta_k^{\mathrm{dn}}\}$*. Then* Δ_k-**LEM**, $(\Delta_k \vee (\Gamma')^{\mathrm{n}})$-**DNE** *and* $(\Gamma'' \vee \Pi_k)$-**DNE** *are equivalent over* HA*;*
6. $(\Delta_k \vee \Delta_k)$-**DNE**, $(\Delta_k \vee \Delta_k^{\mathrm{dn}})$-**DNE** *and* Δ_k^{n}-**DML** $+ \Sigma_{k-1}$-**DNE** *are equivalent over* HA.

Proof. 1. This follows from Proposition 6.2.(2).

2. By Corollary 5.9.(2), Σ_k-**LEM** is equivalent to $(\Sigma_k^{\mathrm{n}}, \Pi_k^{\mathrm{n}})$-**DML**$+\Sigma_k$-**DNE**. By Proposition 6.1, it is equivalent to $(\Sigma_k^{\mathrm{dn}} \vee \Pi_k^{\mathrm{dn}})$-**DNE**$+\Sigma_k$-**DNE**. By Propositions 6.2.(5), 6.2.(7) and 6.2.(9), it is equivalent to $(\Sigma_k \vee \Pi_k)$-**DNE**. Also by Propositions 6.2.(4) and 6.2.(9), each of $(\Sigma_k \vee \Sigma_k^{\mathrm{n}})$-**DNE** and $(\Sigma_k \vee \Pi_k^{\mathrm{dn}})$-**DNE** is equivalent to $(\Sigma_k \vee \Pi_k)$-**DNE**.

3. By Corollary 5.9.(1), Π_k-**LEM** is equivalent to $(\Sigma_k^{\mathrm{n}}, \Pi_k^{\mathrm{n}})$-**DML**+$\Sigma_{k-1}$-**DNE**, and it is equivalent to $(\Sigma_k^{\mathrm{dn}} \vee \Pi_k^{\mathrm{dn}})$-**DNE** + Σ_{k-1}-**DNE** by Proposition 6.1. By Propositions 6.2.(6) and 6.2.(9), it is equivalent to $(\Pi_k^{\mathrm{n}} \vee \Pi_k)$-**DNE**. By Proposition 6.2.(7), $(\Pi_k^{\mathrm{n}} \vee \Pi_k)$-**DNE** is equivalent to $(\Sigma_k^{\mathrm{dn}} \vee \Pi_k)$-**DNE**.

4. By Corollary 5.29.(1), Σ_k-**DML**$^\perp$ is equivalent to Σ_k-**DML**+Σ_{k-1}-**DNE**, and this is equivalent to $(\Sigma_k^{\mathrm{n}} \vee \Sigma_k^{\mathrm{n}})$-**DNE** + Σ_{k-1}-**DNE**. Then by Propositions 6.2.(3), it is equivalent to $(\Sigma_k^{\mathrm{n}} \vee \Pi_k)$-**DNE**. It is equivalent to $(\Pi_k \vee \Pi_k)$-**DNE** by Proposition 6.2.(4), and hence, also to $(\Pi_k \vee \Pi_k^{\mathrm{dn}})$-**DNE** by Proposition 6.2.(9).

5. By Corollary 5.10, Δ_k-**LEM** is equivalent to $(\Delta_k^{\mathrm{n}}, \Gamma')$-**DML**+$\Sigma_{k-1}$-**DNE**. And it is equivalent to $(\Delta_k^{\mathrm{dn}} \vee (\Gamma')^{\mathrm{n}})$-**DNE** + Σ_{k-1}-**DNE**. This is equivalent to $(\Delta_k \vee (\Gamma')^{\mathrm{n}})$-**DNE** by Proposition 6.2.(10). Also each of $(\Delta_k^{\mathrm{dn}} \vee \Pi_k)$-**DNE** and $(\Delta_k \vee \Pi_k)$-**DNE** is equivalent to $(\Delta_k \vee \Sigma_k^{\mathrm{n}})$-**DNE** by Propositions 6.2.(4) and 6.2.(10).

By Corollary 5.10, Δ_k-**LEM** is equivalent to (Δ_k, Σ_k)-**DML** + Σ_{k-1}-**DNE**, and it is equivalent to $(\Delta_k^{\mathrm{n}} \vee \Sigma_k^{\mathrm{n}})$-**DNE** + Σ_{k-1}-**DNE**. By Proposition 6.2.(3), it is equivalent to $(\Delta_k^{\mathrm{n}} \vee \Pi_k)$-**DNE**.

6. This is immediate from Propositions 6.1, 6.2.(10) and 6.2.(11). □

Corollary 6.4. $\mathsf{HA} + \Delta_k$-**LEM** $\vdash (\Delta_k \vee \Delta_k)$-**DNE**.

Proof. This is because $\mathsf{HA} + \Delta_k$-**LEM** $\vdash (\Delta_k \vee \Pi_k)$-**DNE** by Corollary 6.3.(5). □

In Akama et al. [1], it is shown that $\mathsf{HA} + \Delta_{k+1}$-**LEM** proves Σ_k-**LEM**. The following proposition is a refinement of their result from Corollary 6.4.

Proposition 6.5. $\mathsf{HA} + (\Delta_{k+1} \vee \Delta_{k+1})$-**DNE** $\vdash \Sigma_k$-**LEM**.

Proof. Let φ be any Σ_k formula. Since $\mathsf{HA} \vdash \neg(\neg\varphi \wedge \neg\neg\varphi)$, $\mathsf{HA} + \Sigma_{k-1}$-**DNS** $\vdash \neg(\neg\varphi \wedge \neg\varphi^\perp)$ by Proposition 3.12. Then $\mathsf{HA} + \Sigma_{k-1}$-**DNS** $\vdash \neg\neg(\varphi \vee \varphi^\perp)$. Since both φ and $\varphi^\perp$ are Δ_{k+1} and $\mathsf{HA} + (\Delta_{k+1} \vee \Delta_{k+1})$-**DNE** derives Σ_{k-1}-**DNS**, $\mathsf{HA} + (\Delta_{k+1} \vee \Delta_{k+1})$-**DNE** $\vdash \varphi \vee \varphi^\perp$. Hence the theory proves $\varphi \vee \neg\varphi$ by Proposition 3.3.(3). □

Finally, we introduce the following principle based on Peirce's law. We show that Peirce's law exactly corresponds to the double negation elimination.

Definition 6.6 (Peirce's law)**.** Let Γ be any set of formulas.

Γ-**PEIRCE** $\quad ((\varphi \to \psi) \to \varphi) \to \varphi \quad$ ($\varphi \in \Gamma$ and ψ is any formula)

Proposition 6.7. *For any set Γ of formulas,* Γ-**PEIRCE** *is equivalent to* Γ-**DNE** *over* HA.

Proof. First, we prove $\mathsf{HA} + \Gamma$-**PEIRCE** $\vdash \Gamma$-**DNE**. Let $\varphi \in \Gamma$. Since $\neg\neg\varphi$ is $(\varphi \to \bot) \to \bot$, $\mathsf{HA} \vdash \neg\neg\varphi \to ((\varphi \to \bot) \to \varphi)$. Thus $\mathsf{HA} + \Gamma$-**PEIRCE** $\vdash \neg\neg\varphi \to \varphi$.

Secondly, we prove $\mathsf{HA} + \Gamma$-**DNE** $\vdash \Gamma$-**PEIRCE**. Let φ be any Γ formula and ψ be arbitrary formula. Since HA proves $\neg\varphi \to (\varphi \to \psi)$, HA also proves $((\varphi \to \psi) \to \varphi) \wedge \neg\varphi \to \varphi$. Hence $\mathsf{HA} \vdash ((\varphi \to \psi) \to \varphi) \to \neg\neg\varphi$. We obtain $\mathsf{HA} + \Gamma$-**DNE** $\vdash ((\varphi \to \psi) \to \varphi) \to \varphi$. □

We get the table which summarizes principles equivalent to $(\Gamma \vee \Theta)$-**DNE** over the theory $\mathsf{HA} + \Sigma_{k-1}$-**DNS**. Notice that from Propositions 6.2.(6) and 6.2.(8), $(\Sigma_k^{\mathrm{dn}} \vee \Theta)$-**DNE** and $(\Pi_k^{\mathrm{dn}} \vee \Theta)$-**DNE** are equivalent to $(\Pi_k^{\mathrm{n}} \vee \Theta)$-**DNE** and $(\Sigma_k^{\mathrm{n}} \vee \Theta)$-**DNE** over $\mathsf{HA} + \Sigma_{k-1}$-**DNS**, respectively. So Σ_k^{dn} and Π_k^{dn} are excluded from the table.

Γ \ Θ	Σ_k	Π_k^{n}	Π_k	Σ_k^{n}
Σ_k	Σ_k-**DNE**	Σ_k-**DNE**	Σ_k-**LEM**	Σ_k-**LEM**
Π_k^{n}		Π_k-**DML**	Π_k-**LEM**	Σ_k^{n}-**LEM**
Π_k			Σ_k-$\mathbf{DML}^{\perp}$	Σ_k-$\mathbf{DML}^{\perp}$
Σ_k^{n}				Σ_k-**DML**

Γ \ Θ	Δ_k	Δ_k^{n}	Δ_k^{dn}
Σ_k	Σ_k-**DNE**	Σ_k-**DNE**	Σ_k-**DNE**
Π_k^{n}	Δ_k-**LEM**	Δ_k^{n}-**LEM**	Δ_k^{n}-**LEM**
Π_k	Δ_k-**LEM**	Δ_k-**LEM**	Δ_k-**LEM**
Σ_k^{n}	Δ_k-**LEM**	Δ_k^{n}-**LEM**	Δ_k^{n}-**LEM**
Δ_k	$(\Delta_k \vee \Delta_k)$-**DNE**	Δ_k-**LEM**	$(\Delta_k \vee \Delta_k)$-**DNE**
Δ_k^{n}		Δ_k-**DML**	Δ_k^{n}-**LEM**
Δ_k^{dn}			Δ_k^{n}-**DML**

Table 1: Principles equivalent to $(\Gamma \vee \Theta)$-**DNE** over $\mathsf{HA} + \Sigma_{k-1}$-**DNS**

7 The constant domain axiom

In this section, we investigate the principles of the form (Γ, Θ)-**CD** in Definition 2.4, and classify them in the arithmetical hierarchy of classical principles. Note that (Γ, Θ)-**CD** is not equivalent to (Θ, Γ)-**CD** in general.

In first-order intuitionistic Kripke semantics, the constant domain axiom corresponds to Kripke frames with constant domains (cf. [18, p. 328]). First of all, we show that in our framework of first-order intuitionistic arithmetic, the constant domain axiom is equivalent to the law of excluded middle despite its semantic origin. Let **LEM** and **CD** denote the principles Fml-**LEM** and (Fml, Fml)-**CD** respectively, where Fml is the set of all formulas.

Proposition 7.1. **CD** *is equivalent to* **LEM** *over* HA.

Proof. First, we prove $\mathsf{HA} + \mathbf{CD} \vdash \varphi \vee \neg\varphi$ for any formula φ by induction on the construction of φ. If φ is an atomic formula, then the statement is obvious.

Assume that $\mathsf{HA} + \mathbf{CD}$ proves $\psi \vee \neg\psi$ and $\rho \vee \neg\rho$, and suppose φ is one of the forms $\psi \wedge \rho$, $\psi \vee \rho$ and $\psi \to \rho$. Notice that $\neg\psi \vee \neg\rho \to \neg(\psi \wedge \rho)$, $\neg\psi \wedge \neg\rho \to \neg(\psi \vee \rho)$, $\neg\psi \vee \rho \to (\psi \to \rho)$ and $\psi \wedge \neg\rho \to \neg(\psi \to \rho)$ are provable in HA. Therefore $\varphi \vee \neg\varphi$ is also provable in $\mathsf{HA} + \mathbf{CD}$.

Assume that $\mathsf{HA} + \mathbf{CD}$ proves $\psi(x) \vee \neg\psi(x)$. Then $\forall x(\exists x\psi(x) \vee \neg\psi(x))$ and $\forall x(\psi(x) \vee \exists x \neg\psi(x))$ are also provable. By applying $\mathbf{CD}$, we obtain that $\mathsf{HA} + \mathbf{CD}$ proves $\exists x\psi(x) \vee \neg \exists x\psi(x)$ and $\forall x\psi(x) \vee \neg \forall x\psi(x)$. Therefore, if φ is of one of the forms $\exists x\psi(x)$ and $\forall x\psi(x)$, then $\varphi \vee \neg\varphi$ is provable in $\mathsf{HA} + \mathbf{CD}$.

Secondly, we prove $\mathsf{HA} + \mathbf{LEM} \vdash \mathbf{CD}$. Let φ and $\psi(x)$ be any formulas with $x \notin \mathrm{FV}(\varphi)$. We have $\mathsf{HA} \vdash \forall x(\varphi \vee \psi(x)) \wedge \neg\varphi \to \forall x\psi(x)$. Since $\mathsf{HA} + \mathbf{LEM}$ proves $\varphi \vee \neg\varphi$, we conclude that $\mathsf{HA} + \mathbf{LEM}$ also proves $\forall x(\varphi \vee \psi(x)) \to \varphi \vee \forall x\psi(x)$. □

Proposition 7.2.

1. *(Γ, Π_{k+1})-$\mathbf{CD}$ is equivalent to (Γ, Σ_k)-$\mathbf{CD}$ over* HA*;*
2. *$(\Gamma, \Sigma^{\mathrm{n}}_{k+1})$-$\mathbf{CD}$ is equivalent to $(\Gamma, \Pi^{\mathrm{n}}_k)$-$\mathbf{CD}$ over* HA*.*

Proof. These statements are proved by using a primitive recursive pairing function. □

As in the proof of Proposition 7.1, we can show that Γ-$\mathbf{LEM}$ and Δ_k-$\mathbf{LEM}$ are sufficiently strong for the constant domain axiom.

Proposition 7.3. *Let Γ and Θ be any sets of formulas.*

1. $\mathsf{HA} + \Gamma$-$\mathbf{LEM} \vdash (\Gamma, \Theta)$-$\mathbf{CD}$*;*
2. $\mathsf{HA} + \Delta_k$-$\mathbf{LEM} \vdash (\Delta_k, \Theta)$-$\mathbf{CD}$.

From the prenex normal form theorem proved in [1, Theorem 2.7] and [13, Theorem 5.7], $\mathbf{LEM}$ is equivalent to $\bigcup\{\Sigma_k\text{-}\mathbf{LEM} \mid k \geq 0\}$ over HA. Therefore, the following proposition can be regarded as a stratification of Proposition 7.1.

Proposition 7.4. *Let Θ be a set of formulas such that $\Sigma_{k-1} \subseteq \Theta$. Then the following are equivalent over* HA*:*

1. *(Σ_k, Θ)-$\mathbf{CD}$.*
2. *Σ_k-$\mathbf{LEM}$.*

Proof. First, we prove $\mathsf{HA} + (\Sigma_k, \Sigma_{k-1})$-$\mathbf{CD} \vdash \Sigma_k$-$\mathbf{LEM}$ by induction on k. For $k = 0$, the statement is trivial. Suppose that the statement holds for k, and we prove $\mathsf{HA} + (\Sigma_{k+1}, \Sigma_k)$-$\mathbf{CD} \vdash \Sigma_{k+1}$-$\mathbf{LEM}$. Let $\exists x\varphi(x)$ be any Σ_{k+1} formula with $\varphi(x) \in \Pi_k$. By induction hypothesis and Fact 2.3, $\mathsf{HA} + (\Sigma_k, \Sigma_{k-1})$-$\mathbf{CD}$ proves Π_k-$\mathbf{LEM} + \Sigma_k$-$\mathbf{DNE}$. Thus $\mathsf{HA} + (\Sigma_k, \Sigma_{k-1})$-$\mathbf{CD} \vdash \varphi(x) \vee \neg\varphi(x)$. We get $\mathsf{HA} + (\Sigma_k, \Sigma_{k-1})$-$\mathbf{CD} \vdash \forall x(\exists x\varphi(x) \vee \varphi^{\perp}(x))$ by using Π_k-$\mathbf{DUAL}$. Then

$$\mathsf{HA} + (\Sigma_{k+1}, \Sigma_k)\text{-}\mathbf{CD} \vdash \exists x\varphi(x) \vee \forall x\varphi^{\perp}(x).$$

This implies $\mathsf{HA} + (\Sigma_{k+1}, \Sigma_k)$-$\mathbf{CD} \vdash \exists x\varphi(x) \vee \neg \exists x\varphi(x)$.

On the other hand, $\mathsf{HA} + \Sigma_k$-$\mathbf{LEM} \vdash (\Sigma_k, \Theta)$-$\mathbf{CD}$ follows from Proposition 7.3.(1). □

Fact 2.7 states that (Π_1, Π_1)-**CD** is HA-equivalent to Σ_1-**DML**. By Corollary 5.29.(1), Σ_1-**DML** is HA-equivalent to Σ_1-$\mathbf{DML}^\perp$. So the following proposition is a generalization of Fact 2.7.

Proposition 7.5. *The following are equivalent over* HA*:*

1. (Π_k, Π_k)-**CD**.
2. Σ_k-$\mathbf{DML}^\perp$.

Proof. First, we prove $\mathsf{HA} + \Sigma_k\text{-}\mathbf{DML}^\perp \vdash (\Pi_k, \Pi_k)\text{-}\mathbf{CD}$. Let $\varphi, \psi(x) \in \Pi_k$ with $x \notin \mathrm{FV}(\varphi)$. Since $\mathsf{HA} \vdash \forall x(\varphi \vee \psi(x)) \wedge \neg\varphi \to \forall x\psi(x)$, HA proves $\forall x(\varphi \vee \psi(x)) \to \neg(\neg\varphi \wedge \neg\forall x\psi(x))$. By Proposition 3.3.(3), $\mathsf{HA} \vdash \forall x(\varphi \vee \psi(x)) \to \neg(\varphi^\perp \wedge (\forall x\psi(x))^\perp)$. Then we obtain

$$\mathsf{HA} + \Sigma_k\text{-}\mathbf{DML}^\perp \vdash \forall x(\varphi \vee \psi(x)) \to \varphi^{\perp\perp} \vee (\forall x\psi(x))^{\perp\perp}.$$

By Proposition 3.3.(2), we conclude

$$\mathsf{HA} + \Sigma_k\text{-}\mathbf{DML}^\perp \vdash \forall x(\varphi \vee \psi(x)) \to \varphi \vee \forall x\psi(x).$$

Secondly, we prove $\mathsf{HA} + (\Pi_k, \Pi_k)\text{-}\mathbf{CD} \vdash \Sigma_k\text{-}\mathbf{DML}^\perp$. We may assume $k > 0$. Let $\exists x\varphi(x)$ and $\exists y\psi(y)$ be any Σ_k formulas where $\varphi(x)$ and $\psi(y)$ are Π_{k-1}. Since $\psi(y)$ implies $\exists y\psi(y)$, we obtain

$$\mathsf{HA} \vdash \neg(\exists x\varphi(x) \wedge \exists y\psi(y)) \wedge \psi(y) \to \neg\exists x\varphi(x). \tag{12}$$

Since $\mathsf{HA}+(\Pi_k, \Pi_k)$-**CD** entails (Σ_{k-1}, Π_k)-**CD**, we obtain that $\mathsf{HA}+(\Pi_k, \Pi_k)$-**CD** proves Π_{k-1}-**LEM** $+ \Sigma_{k-1}$-**DNE** by Proposition 7.4 and Fact 2.3. Hence $\mathsf{HA} + (\Pi_k, \Pi_k)\text{-}\mathbf{CD} \vdash \psi(y) \vee \neg\psi(y)$. From this with (12), we have

$$\mathsf{HA} + (\Pi_k, \Pi_k)\text{-}\mathbf{CD} \vdash \neg(\exists x\varphi(x) \wedge \exists y\psi(y)) \to \forall y(\neg\exists x\varphi(x) \vee \neg\psi(y)).$$

By using Σ_k-**DUAL**, we get

$$\mathsf{HA} + (\Pi_k, \Pi_k)\text{-}\mathbf{CD} \vdash \neg(\exists x\varphi(x) \wedge \exists y\psi(y)) \to \forall y((\exists x\varphi(x))^\perp \vee \psi^\perp(y)).$$

Since $(\exists x\varphi(x))^\perp \in \Pi_k$ and $\psi^\perp(y) \in \Sigma_{k-1}$, we obtain

$$\mathsf{HA} + (\Pi_k, \Pi_k)\text{-}\mathbf{CD} \vdash \neg(\exists x\varphi(x) \wedge \exists y\psi(y)) \to (\exists x\varphi(x))^\perp \vee \forall y\psi^\perp(y).$$

Therefore

$$\mathsf{HA} + (\Pi_k, \Pi_k)\text{-}\mathbf{CD} \vdash \neg(\exists x\varphi(x) \wedge \exists y\psi(y)) \to (\exists x\varphi(x))^\perp \vee (\exists y\psi(y))^\perp. \qquad \square$$

From Corollaries 5.29.(1) and 6.3.(4) and Propositions 5.16, 5.22 and 7.5, we have the following result.

Corollary 7.6. *For $k \geq 1$, the following are equivalent over* HA*:*

1. Σ_k-**DML** $+ \Sigma_{k-1}$-**DNE**.

2. Σ_k-$\mathbf{DML}^{\perp}$.

3. (Π_k, Π_k)-$\mathbf{CD}$.

4. Π_k-$\mathbf{COLL}^{\mathbf{cp}}$.

5. $(\Pi_k \vee \Pi_k)$-$\mathbf{DNE}$.

Corollary 7.7. *For $k \geq 1$, each of Σ_k-$\mathbf{DML}^{\perp}$, (Π_k, Π_k)-$\mathbf{CD}$, Π_k-$\mathbf{COLL}^{\mathbf{cp}}$ and $(\Pi_k \vee \Pi_k)$-$\mathbf{DNE}$ implies Π_k-$\mathbf{DML}$ over HA.*

Proof. This is immediate from Proposition 5.21 and Corollary 7.6. □

Proposition 7.8. *Let Θ be a set of formulas such that $\Sigma_{k-1} \subseteq \Theta$. Then the following are equivalent over HA:*

1. (Δ_k, Θ)-$\mathbf{CD}$.

2. Δ_k-$\mathbf{LEM}$.

Proof. Notice that (Δ_k, Σ_{k-1})-$\mathbf{CD}$ implies $(\Sigma_{k-1}, \Sigma_{k-1})$-$\mathbf{CD}$. Then by Proposition 7.4 and Fact 2.3, $\mathsf{HA} + (\Delta_k, \Sigma_{k-1})$-$\mathbf{CD}$ proves Π_{k-1}-$\mathbf{LEM} + \Sigma_{k-1}$-$\mathbf{DNE}$. Therefore the statement $\mathsf{HA} + (\Delta_k, \Sigma_{k-1})$-$\mathbf{CD} \vdash \Delta_k$-$\mathbf{LEM}$ is proved as in the proof of Proposition 7.4. On the other hand, $\mathsf{HA} + \Delta_k$-$\mathbf{LEM} \vdash (\Delta_k, \Theta)$-$\mathbf{CD}$ follows from Proposition 7.3.(2). □

Next, we investigate the principles $(\Gamma^{\mathrm{n}}, \Theta)$-$\mathbf{CD}$ and $(\Delta_k^{\mathrm{n}}, \Theta)$-$\mathbf{CD}$. In the light of Proposition 7.3, they are derived from Γ^{n}-$\mathbf{LEM}$ and Δ_k^{n}-$\mathbf{LEM}$, respectively. In addition, for $\Theta = \Sigma_k^{\mathrm{n}}$, we obtain the following proposition.

Proposition 7.9. *Let Γ be any set of formulas.*

1. $\mathsf{HA} + (\Gamma, \Sigma_k)$-$\mathbf{DML} \vdash (\Gamma^{\mathrm{n}}, \Sigma_k^{\mathrm{n}})$-$\mathbf{CD}$;

2. $\mathsf{HA} + (\Delta_k, \Sigma_k)$-$\mathbf{DML} \vdash (\Delta_k^{\mathrm{n}}, \Sigma_k^{\mathrm{n}})$-$\mathbf{CD}$.

Proof. 1. By Proposition 7.2.(2), it suffices to show that $\mathsf{HA} + (\Gamma, \Sigma_k)$-$\mathbf{DML}$ proves $(\Gamma^{\mathrm{n}}, \Pi_{k-1}^{\mathrm{n}})$-$\mathbf{CD}$. Let $\varphi \in \Gamma$ and $\psi(x) \in \Pi_{k-1}$ with $x \notin \mathrm{FV}(\varphi)$. Then we have

$$\begin{aligned} \mathsf{HA} \vdash \forall x(\neg\varphi \vee \neg\psi(x)) &\to \forall x\, \neg(\varphi \wedge \psi(x)), \\ &\to \neg\, \exists x(\varphi \wedge \psi(x)), \\ &\to \neg(\varphi \wedge \exists x \psi(x)). \end{aligned}$$

Thus
$$\mathsf{HA} + (\Gamma, \Sigma_k)\text{-}\mathbf{DML} \vdash \forall x(\neg\varphi \vee \neg\psi(x)) \to \neg\varphi \vee \neg\, \exists x \psi(x).$$

We conclude

$$\mathsf{HA} + (\Gamma, \Sigma_k)\text{-}\mathbf{DML} \vdash \forall x(\neg\varphi \vee \neg\psi(x)) \to \neg\varphi \vee \forall x\, \neg\psi(x).$$

2 is proved similarly. □

With the help of Σ_{k-2}-**DNS**, the converse implications also hold.

Proposition 7.10.

1. $\mathsf{HA} + (\Pi_k^{\mathrm{n}}, \Sigma_k^{\mathrm{n}})$-**CD** $+ \Sigma_{k-2}$-**DNS** $\vdash (\Sigma_k, \Pi_k)$-**DML**;
2. $\mathsf{HA} + (\Sigma_k^{\mathrm{n}}, \Sigma_k^{\mathrm{n}})$-**CD** $+ \Sigma_{k-2}$-**DNS** $\vdash \Sigma_k$-**DML**;
3. $\mathsf{HA} + (\Delta_k^{\mathrm{n}}, \Sigma_k^{\mathrm{n}})$-**CD** $+ \Sigma_{k-2}$-**DNS** $\vdash (\Delta_k, \Sigma_k)$-**DML**.

Proof. 1. We prove by induction on $k \geq 0$. The statement for $k = 0$ is trivial. We assume that our statement holds for k, and we prove $\mathsf{HA}+(\Pi_{k+1}^{\mathrm{n}}, \Sigma_{k+1}^{\mathrm{n}})$-**CD**$+$ Σ_{k-1}-**DNS** $\vdash (\Sigma_{k+1}, \Pi_{k+1})$-**DML**. Let $\exists x\varphi(x) \in \Sigma_{k+1}$ and $\psi \in \Pi_{k+1}$ where $\varphi(x) \in \Pi_k$. We have

$$\begin{aligned}\mathsf{HA} \vdash \neg(\exists x\varphi(x) \wedge \psi) &\to \neg\, \exists x(\varphi(x) \wedge \psi),\\ &\to \forall x\, \neg(\varphi(x) \wedge \psi),\\ &\to \forall x\, \neg(\neg\neg\varphi(x) \wedge \neg\neg\psi).\end{aligned}$$

Then

$$\mathsf{HA} \vdash \neg(\exists x\varphi(x) \wedge \psi) \wedge \neg\neg\varphi(x) \to \neg\psi. \tag{13}$$

By induction hypothesis, $\mathsf{HA}+(\Pi_k^{\mathrm{n}}, \Sigma_k^{\mathrm{n}})$-**CD**$+\Sigma_{k-2}$-**DNS** $\vdash (\Sigma_k, \Pi_k)$-**DML**. By Corollary 5.8.(1), $\mathsf{HA} + (\Pi_k^{\mathrm{n}}, \Sigma_k^{\mathrm{n}})$-**CD** $+ \Sigma_{k-1}$-**DNS** proves Π_k^{n}-**LEM**. Thus we have that $\mathsf{HA} + (\Pi_k^{\mathrm{n}}, \Sigma_k^{\mathrm{n}})$-**CD** $+ \Sigma_{k-1}$-**DNS** proves $\neg\neg\varphi(x) \vee \neg\varphi(x)$. From this with (13), we obtain

$$\mathsf{HA} + (\Pi_k^{\mathrm{n}}, \Sigma_k^{\mathrm{n}})\text{-}\mathbf{CD} + \Sigma_{k-1}\text{-}\mathbf{DNS} \vdash \neg(\exists x\varphi(x) \wedge \psi) \to \forall x(\neg\psi \vee \neg\varphi(x)).$$

By applying $(\Pi_{k+1}^{\mathrm{n}}, \Sigma_{k+1}^{\mathrm{n}})$-**CD**, we have

$$\mathsf{HA} + (\Pi_{k+1}^{\mathrm{n}}, \Sigma_{k+1}^{\mathrm{n}})\text{-}\mathbf{CD} + \Sigma_{k-1}\text{-}\mathbf{DNS} \vdash \neg(\exists x\varphi(x) \wedge \psi) \to \neg\psi \vee \forall x\, \neg\varphi(x).$$

We conclude

$$\mathsf{HA} + (\Pi_{k+1}^{\mathrm{n}}, \Sigma_{k+1}^{\mathrm{n}})\text{-}\mathbf{CD} + \Sigma_{k-1}\text{-}\mathbf{DNS} \vdash \neg(\exists x\varphi(x) \wedge \psi) \to \neg\, \exists x\varphi(x) \vee \neg\psi.$$

2. We may assume $k > 0$. Let $\exists x\varphi(x)$ and $\exists y\psi(y)$ be any Σ_k formulas with $\varphi(x), \psi(y) \in \Pi_{k-1}$.

$$\begin{aligned}\mathsf{HA} \vdash \neg(\exists x\varphi(x) \wedge \exists y\psi(y)) &\to \neg\, \exists x\exists y(\varphi(x) \wedge \psi(y)),\\ &\to \forall x\forall y\, \neg(\varphi(x) \wedge \psi(y)).\end{aligned}$$

Since $(\Sigma_k^{\mathrm{n}}, \Sigma_k^{\mathrm{n}})$-**CD** entails $(\Pi_{k-1}^{\mathrm{n}}, \Sigma_{k-1}^{\mathrm{n}})$-**CD**, by clause 1, we have that $\mathsf{HA} +$ $(\Sigma_k^{\mathrm{n}}, \Sigma_k^{\mathrm{n}})$-**CD** $+ \Sigma_{k-3}$-**DNS** proves $(\Sigma_{k-1}, \Pi_{k-1})$-**DML**. Then by Corollary 5.8.(1), $\mathsf{HA} + (\Sigma_k^{\mathrm{n}}, \Sigma_k^{\mathrm{n}})$-**CD** $+ \Sigma_{k-2}$-**DNS** proves Π_{k-1}^{n}-**LEM**. By Proposition 5.3.(1), it also proves Π_{k-1}-**DML**. Thus

$$\mathsf{HA}+(\Sigma_k^{\mathrm{n}}, \Sigma_k^{\mathrm{n}})\text{-}\mathbf{CD}+\Sigma_{k-2}\text{-}\mathbf{DNS} \vdash \neg(\exists x\varphi(x)\wedge\exists y\psi(y)) \to \forall x\forall y(\neg\varphi(x)\vee\neg\psi(y)).$$

By applying $(\Sigma_k^{\mathrm{n}}, \Sigma_k^{\mathrm{n}})$-**CD** twice, we obtain

$$\mathsf{HA} + (\Sigma_k^{\mathrm{n}}, \Sigma_k^{\mathrm{n}})\text{-}\mathbf{CD} + \Sigma_{k-2}\text{-}\mathbf{DNS} \vdash \neg(\exists x \varphi(x) \wedge \exists y \psi(y)) \to \forall x \neg\varphi(x) \vee \forall y \neg\psi(y).$$

We conclude

$$\mathsf{HA} + (\Sigma_k^{\mathrm{n}}, \Sigma_k^{\mathrm{n}})\text{-}\mathbf{CD} + \Sigma_{k-2}\text{-}\mathbf{DNS} \vdash \neg(\exists x \varphi(x) \wedge \exists y \psi(y)) \to \neg\exists x \varphi(x) \vee \neg\exists y \psi(y).$$

3 is proved as in the proof of clause 2. □

We obtain the following corollary.

Corollary 7.11. *Let Θ be any set of formulas such that $\Pi_{k-1}^{\mathrm{n}} \subseteq \Theta$.*

1. $(\Pi_k^{\mathrm{n}}, \Sigma_k^{\mathrm{n}})$-**CD** *is equivalent to* (Σ_k, Π_k)-**DML** *over* $\mathsf{HA} + \Sigma_{k-2}$-**DNS**;
2. $(\Pi_k^{\mathrm{n}}, \Theta)$-**CD** *is equivalent to* Π_k^{n}-**LEM** *over* $\mathsf{HA} + \Sigma_{k-1}$-**DNS**;
3. $(\Sigma_k^{\mathrm{n}}, \Sigma_k^{\mathrm{n}})$-**CD** *is equivalent to* Σ_k-**DML** *over* $\mathsf{HA} + \Sigma_{k-2}$-**DNS**;
4. $(\Delta_k^{\mathrm{n}}, \Sigma_k^{\mathrm{n}})$-**CD** *is equivalent to* (Δ_k, Σ_k)-**DML** *over* $\mathsf{HA} + \Sigma_{k-2}$-**DNS**;
5. $(\Delta_k^{\mathrm{n}}, \Theta)$-**CD** *is equivalent to* Δ_k^{n}-**LEM** *over* $\mathsf{HA} + \Sigma_{k-1}$-**DNS**.

Proof. 1. This is immediate from Propositions 7.9.(1) and 7.10.(1).

2. From clause 1, Proposition 7.2 and Corollary 5.8.(1), we have that $\mathsf{HA} + (\Pi_k^{\mathrm{n}}, \Pi_{k-1}^{\mathrm{n}})$-**CD** $+ \Sigma_{k-1}$-**DNS** proves Π_k^{n}-**LEM**. On the other hand, $\mathsf{HA} + \Pi_k^{\mathrm{n}}$-**LEM** proves $(\Pi_k^{\mathrm{n}}, \Theta)$-**CD** by Proposition 7.3.(1).

3. This is a consequence of Propositions 7.9.(1) and 7.10.(2).

4. Immediate from Propositions 7.9.(2) and 7.10.(3).

5. As in the proof of clause 2, we obtain the statement from clause 4, Propositions 7.2, 7.3.(2) and Corollary 5.8.(4), □

Problem 7.12.

- *Is there a set Θ of formulas such that* $\mathsf{HA} + (\Pi_k, \Theta)$-**CD** *proves* Π_k-**LEM**?
- *Is there a set Θ of formulas such that* $\mathsf{HA} + (\Sigma_k^{\mathrm{n}}, \Theta)$-**CD** $+ \Sigma_{k-1}$-**DNS** *proves* Σ_k^{n}-**LEM**?

The following figure (Figure 2) summarizes the situation for implications around the constant domain axioms for negated formulas. In [9, Example 10], it is shown that $\mathsf{HA} + \Sigma_k$-**DML** $+ \Sigma_k$-**DNE** does not prove Σ_k^{n}-**LEM** for $k \geq 1$. Therefore, in Figure 2, Σ_k-**DML** does not imply Σ_k^{n}-**LEM** even in the theory $\mathsf{HA} + \Sigma_k$-**DNE** for $k \geq 1$.

$$
\begin{tikzcd}
& \Sigma^{\mathrm{n}}_k\text{-}\mathbf{LEM} \arrow[dddl] \arrow[dd] \arrow[dr] \arrow[ddrr, bend right] & & \Pi^{\mathrm{n}}_k\text{-}\mathbf{LEM} \arrow[dl] \arrow[dd] \arrow[dddr] & \\
& & \Delta^{\mathrm{n}}_k\text{-}\mathbf{LEM} \arrow[dd] & & \\
& \Sigma_k\text{-}\mathbf{DML} \arrow[dr] \arrow[dd] & & (\Sigma_k, \Pi_k)\text{-}\mathbf{DML} \arrow[dl] \arrow[dd] \arrow[uull, dotted, bend right] \arrow[uu, dotted, bend left] & \\
(\Sigma^{\mathrm{n}}_k, \Theta)\text{-}\mathbf{CD} \arrow[dr] & & (\Delta_k, \Sigma_k)\text{-}\mathbf{DML} \arrow[dd] \arrow[uu, dotted, bend left] & & (\Pi^{\mathrm{n}}_k, \Theta)\text{-}\mathbf{CD} \arrow[dl] \\
& (\Sigma^{\mathrm{n}}_k, \Sigma^{\mathrm{n}}_k)\text{-}\mathbf{CD} \arrow[dr] \arrow[uu, dashed, bend left] & & (\Pi^{\mathrm{n}}_k, \Sigma^{\mathrm{n}}_k)\text{-}\mathbf{CD} \arrow[dl] \arrow[uu, dashed, bend left] & \\
& & (\Delta^{\mathrm{n}}_k, \Sigma^{\mathrm{n}}_k)\text{-}\mathbf{CD} \arrow[uu, dashed, bend left] & &
\end{tikzcd}
$$

Θ: A sufficiently large set of formulas

$\dashrightarrow$: Implication in $\mathsf{HA} + \Sigma_{k-2}\text{-}\mathbf{DNS}$

$\cdots\rightarrow$: Implication in $\mathsf{HA} + \Sigma_{k-1}\text{-}\mathbf{DNS}$

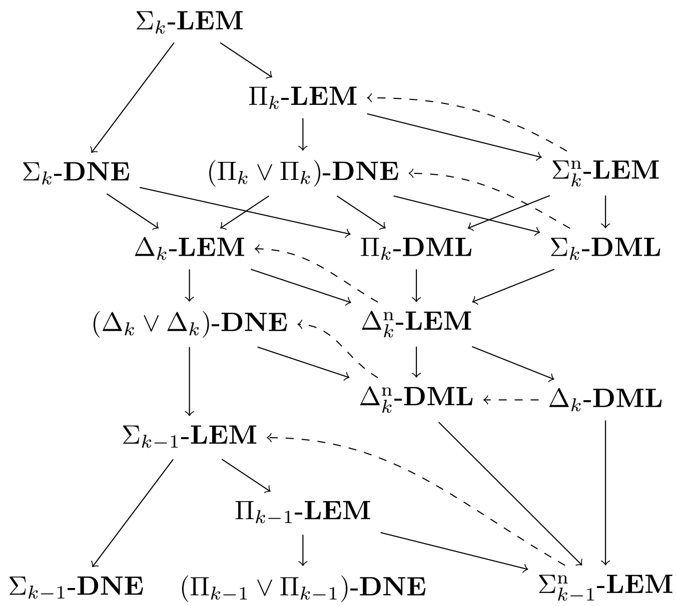
Figure 2: Implications around the constant domain axioms for negated formulas

8 Summary

As a summary, we illustrate the relationships between the principles we have dealt with so far. However, the structure of such relationships is somewhat complicated. As we have shown, some minor differences in some of the principles are smoothed out in the theory $\mathsf{HA} + \Sigma_{k-1}\text{-}\mathbf{DNS}$. Therefore, by illustrating the relationships between the principles in the theory $\mathsf{HA} + \Sigma_{k-1}\text{-}\mathbf{DNS}$, one can grasp the structure in perspective. In fact, in the presence of $\Sigma_1\text{-}\mathbf{DNS}$ (in second-order arithmetic), a lot of equivalences in classical reverse mathematics can be established even intuitionistically (cf. [11, Proposition 1.1] and [7, Theorem 2.10]).

Figure 3 summarizes the derivability relation between several principles over $\mathsf{HA} + \Sigma_{k-1}\text{-}\mathbf{DNS}$ with supplementary information about the situation over $\Sigma_{k-1}\text{-}\mathbf{DNE}$. In fact, except $\Sigma^{\mathrm{n}}_k\text{-}\mathbf{LEM} \to \Pi_k\text{-}\mathbf{DML}$, $\Sigma_k\text{-}\mathbf{DML} \to \Delta^{\mathrm{n}}_k\text{-}\mathbf{LEM}$, $\Pi_k\text{-}\mathbf{DML} \to \Delta^{\mathrm{n}}_k\text{-}\mathbf{LEM}$, $\Delta_k\text{-}\mathbf{DML} \to \Sigma^{\mathrm{n}}_{k-1}\text{-}\mathbf{LEM}$ and $\Delta^{\mathrm{n}}_k\text{-}\mathbf{DML} \to \Sigma^{\mathrm{n}}_{k-1}\text{-}\mathbf{LEM}$, all the (non-dashed) implications presented in Figure 3 are provable even in HA. However, one should note that the principle located at each vertex is one adequately selected from the equivalence class of principles modulo $\mathsf{HA} + \Sigma_{k-1}\text{-}\mathbf{DNS}$, and hence, the HA-provability depends on the choice of the representatives for the vertices. For instance, we can replace $\Sigma^{\mathrm{n}}_k\text{-}\mathbf{LEM}$ with $\Pi^{\mathrm{n}}_k\text{-}\mathbf{LEM}$ by Proposition 4.8. Then $\Pi^{\mathrm{n}}_k\text{-}\mathbf{LEM} \to \Pi_k\text{-}\mathbf{DML}$ is provable in HA while $\Pi^{\mathrm{n}}_k\text{-}\mathbf{LEM} \to \Sigma_k\text{-}\mathbf{DML}$ is so in $\mathsf{HA} + \Sigma_{k-1}\text{-}\mathbf{DNS}$.

As already mentioned so far, several underivability results are proved in the literature (cf. [1, 6, 8, 9, 16, 17]). In particular, Fujiwara et al. [9] recently intro-



$\longrightarrow$: Implication in $\mathsf{HA} + \Sigma_{k-1}$-**DNS**

$\dashrightarrow$: Implication in $\mathsf{HA} + \Sigma_{k-1}$-**DNE**

Figure 3: A refined arithmetical hierarchy of classical principles

duced a fairy useful method to separate Σ_k variants of the logical principles. All the underivability results in [1] obtained by using several kinds of functional interpretations can be proven uniformly in the methodology (see [9, Example 10]). Furthermore, as in [6, Section 4], one can also prove Σ_{k-1}-**LEM** $\not\to \Delta_k^{\mathrm{n}}$-**DML**, Σ_{k-1}-**LEM**+Δ_k^{n}-**DML** $\not\to \Delta_k$-**DML** and Σ_{k-1}-**LEM**+Δ_k-**DML** $\not\to \Delta_k$-**LEM** by this method. However, the separations of the principles which are equivalent only in the presence of Σ_{k-1}-**DNE** (or even Σ_{k-1}-**DNS**) are extremely delicate. One needs further effort for such separations.

In Section 5, we investigated the principles which are closely related to the induction principle such as the contrapositive collection principle and the least number principle over HA, which contains the full induction scheme, in order to examine the logical strength of them. Then we found that Π_k-$\mathbf{COLL^{cp}}$, Π_k-**LN** and Σ_k-**LN** are equivalent to Σ_k-**DML** + Σ_{k-1}-**DNE**, Π_k-**LEM** and Σ_k-**LEM** over HA, respectively (see Theorem 5.20 and Corollary 5.23). On the other hand, it is interesting to analyze the relationship between these principles and the induction principle over intuitionistic arithmetic only with restricted induction scheme.

Implications	Verifying theories	cf.
Σ_k-**LEM** $\to \Pi_k$-**LEM**	HA	Fact 2.3.(1)
Σ_k-**LEM** $\to \Sigma_k$-**DNE**	HA	Fact 2.3.(1)
Π_k-**LEM** $\to (\Pi_k \vee \Pi_k)$-**DNE**	HA	Fact 2.3.(2)
Π_k-**LEM** $\to \Sigma_k^{\mathrm{n}}$-**LEM**	$\mathsf{HA} + \Sigma_{k-1}$-**DNS**	Propositions 4.7.(1) and 4.8
Σ_k^{n}-**LEM** $\to \Pi_k$-**LEM**	$\mathsf{HA} + \Sigma_{k-1}$-**DNE**	Corollary 4.9
Σ_k-**DNE** $\to \Delta_k$-**LEM**	HA	Fact 2.3.(4)
Σ_k-**DNE** $\to \Pi_k$-**DML**	HA	Proposition 5.12
$(\Pi_k \vee \Pi_k)$-**DNE** $\to \Delta_k$-**LEM**	HA	Fact 2.3.(3)
$(\Pi_k \vee \Pi_k)$-**DNE** $\to \Pi_k$-**DML**	HA	Corollary 7.6 and Proposition 5.21
$(\Pi_k \vee \Pi_k)$-**DNE** $\to \Sigma_k$-**DML**	HA	Corollary 7.6
Σ_k-**DML** $\to (\Pi_k \vee \Pi_k)$-**DNE**	$\mathsf{HA} + \Sigma_{k-1}$-**DNE**	Corollary 7.6
Σ_k^{n}-**LEM** $\to \Pi_k$-**DML**	$\mathsf{HA} + \Sigma_{k-1}$-**DNS**	Proposition 4.8 and Corollary 5.4.(1)
Σ_k^{n}-**LEM** $\to \Sigma_k$-**DML**	HA	Corollary 5.4.(1)
Δ_k-**LEM** $\to (\Delta_k \vee \Delta_k)$-**DNE**	HA	Corollary 6.4
Δ_k-**LEM** $\to \Delta_k^{\mathrm{n}}$-**LEM**	HA	Proposition 4.7.(2)
Δ_k^{n}-**LEM** $\to \Delta_k$-**LEM**	$\mathsf{HA} + \Sigma_{k-1}$-**DNE**	Proposition 4.7.(2)
Π_k-**DML** $\to \Delta_k^{\mathrm{n}}$-**LEM**	$\mathsf{HA} + \Sigma_{k-1}$-**DNS**	Corollary 5.11.(1)
Σ_k-**DML** $\to \Delta_k^{\mathrm{n}}$-**LEM**	$\mathsf{HA} + \Sigma_{k-1}$-**DNS**	Corollary 5.11.(1)
$(\Delta_k \vee \Delta_k)$-**DNE** $\to \Sigma_{k-1}$-**LEM**	HA	Proposition 6.5
$(\Delta_k \vee \Delta_k)$-**DNE** $\to \Delta_k^{\mathrm{n}}$-**DML**	HA	Corollary 6.3.(6)
Δ_k^{n}-**DML** $\to (\Delta_k \vee \Delta_k)$-**DNE**	$\mathsf{HA} + \Sigma_{k-1}$-**DNE**	Corollary 6.3.(6)
Δ_k^{n}-**LEM** $\to \Delta_k^{\mathrm{n}}$-**DML**	HA	Corollary 5.4.(2)
Δ_k^{n}-**LEM** $\to \Delta_k$-**DML**	HA	Corollary 5.4.(2)
Δ_k^{n}-**DML** $\to \Sigma_{k-1}^{\mathrm{n}}$-**LEM**	$\mathsf{HA} + \Sigma_{k-2}$-**DNS**	Proposition 5.24.(2)
Δ_k-**DML** $\to \Delta_k^{\mathrm{n}}$-**DML**	$\mathsf{HA} + \Sigma_{k-1}$-**DNE**	Proposition 5.26
Δ_k-**DML** $\to \Sigma_{k-1}^{\mathrm{n}}$-**LEM**	$\mathsf{HA} + \Sigma_{k-2}$-**DNS**	Proposition 5.24.(1)
$\Sigma_{k-1}^{\mathrm{n}}$-**LEM** $\to \Sigma_{k-1}$-**LEM**	$\mathsf{HA} + \Sigma_{k-1}$-**DNE**	Corollary 4.10

Table 2: Implications in Figure 3

We close this paper with a list of principles which we have investigated.

Γ-**LEM**	$\varphi \vee \neg\varphi$	$(\varphi \in \Gamma)$
Γ-**LEM**$^{\perp}$	$\varphi \vee \varphi^{\perp}$	$(\varphi \in \Gamma)$
Δ_k-**LEM**	$(\varphi \leftrightarrow \psi) \to \varphi \vee \neg\varphi$	$(\varphi \in \Sigma_k$ and $\psi \in \Pi_k)$
Δ_k-**LEM**$^{\perp,\Sigma}$	$(\varphi \leftrightarrow \psi) \to \varphi \vee \varphi^{\perp}$	$(\varphi \in \Sigma_k$ and $\psi \in \Pi_k)$
Δ_k-**LEM**$^{\perp,\Pi}$	$(\varphi \leftrightarrow \psi) \to \psi \vee \psi^{\perp}$	$(\varphi \in \Sigma_k$ and $\psi \in \Pi_k)$
Δ_k^{n}-**LEM**	$(\varphi \leftrightarrow \psi) \to \neg\varphi \vee \neg\neg\varphi$	$(\varphi \in \Sigma_k$ and $\psi \in \Pi_k)$
Γ-**DNE**	$\neg\neg\varphi \to \varphi$	$(\varphi \in \Gamma)$
Γ-**PEIRCE**	$((\varphi \to \psi) \to \varphi) \to \varphi$	$(\varphi \in \Gamma$ and ψ is any formula$)$
Γ-**DNS**	$\forall x\, \neg\neg\varphi(x) \to \neg\neg\, \forall x \varphi(x)$	$(\varphi(x) \in \Gamma)$
Γ-**DML**	$\neg(\varphi \wedge \psi) \to \neg\varphi \vee \neg\psi$	$(\varphi, \psi \in \Gamma)$
(Γ, Θ)-**DML**	$\neg(\varphi \wedge \psi) \to \neg\varphi \vee \neg\psi$	$(\varphi \in \Gamma$ and $\psi \in \Theta)$
Δ_k-**DML**	$(\varphi \leftrightarrow \varphi') \wedge (\psi \leftrightarrow \psi')$ $\to (\neg(\varphi \wedge \psi) \to \neg\varphi \vee \neg\psi)$	$(\varphi, \psi \in \Sigma_k$ and $\varphi', \psi' \in \Pi_k)$
Δ_k^{n}-**DML**	$(\varphi \leftrightarrow \varphi') \wedge (\psi \leftrightarrow \psi')$ $\to (\neg(\neg\varphi \wedge \neg\psi) \to \neg\neg\varphi \vee \neg\neg\psi)$	$(\varphi, \psi \in \Sigma_k$ and $\varphi', \psi' \in \Pi_k)$
(Δ_k, Θ)-**DML**	$(\varphi \leftrightarrow \varphi') \to (\neg(\varphi \wedge \psi) \to \neg\varphi \vee \neg\psi)$	$(\varphi \in \Sigma_k$, $\varphi' \in \Pi_k$ and $\psi \in \Theta)$
Γ-**DML**$^{\perp}$	$\neg(\varphi \wedge \psi) \to \varphi^{\perp} \vee \psi^{\perp}$	$(\varphi, \psi \in \Gamma)$
(Γ, Θ)-**DML**$^{\perp}$	$\neg(\varphi \wedge \psi) \to \varphi^{\perp} \vee \psi^{\perp}$	$(\varphi \in \Gamma$ and $\psi \in \Theta)$
Δ_k-**DML**$^{\perp}$	$(\varphi \leftrightarrow \varphi') \wedge (\psi \leftrightarrow \psi')$ $\to (\neg(\varphi \wedge \psi) \to \varphi^{\perp} \vee \psi^{\perp})$	$(\varphi, \psi \in \Sigma_k$ and $\varphi', \psi' \in \Pi_k)$
(Δ_k, Γ)-**DML**$^{\perp,\Sigma}$	$(\varphi \leftrightarrow \varphi') \to (\neg(\varphi \wedge \psi) \to \varphi^{\perp} \vee \psi^{\perp})$	$(\varphi \in \Sigma_k$, $\varphi' \in \Pi_k$ and $\psi \in \Gamma)$
(Δ_k, Γ)-**DML**$^{\perp,\Pi}$	$(\varphi \leftrightarrow \varphi') \to (\neg(\varphi \wedge \psi) \to (\varphi')^{\perp} \vee \psi^{\perp})$	$(\varphi \in \Sigma_k$, $\varphi' \in \Pi_k$ and $\psi \in \Gamma)$
(Γ, Θ)-**CD**	$\forall x(\varphi \vee \psi(x)) \to \varphi \vee \forall x \psi(x)$	$(\varphi \in \Gamma$, $\psi(x) \in \Theta$ and $x \notin \mathrm{FV}(\varphi))$
Γ-**DUAL**	$\neg\varphi \to \varphi^{\perp}$	$(\varphi \in \Gamma)$
Δ_k-**DUAL**$^{\Sigma}$	$(\varphi \leftrightarrow \psi) \to (\neg\varphi \to \varphi^{\perp})$	$(\varphi \in \Sigma_k$ and $\psi \in \Pi_k)$
Δ_k-**DUAL**$^{\Pi}$	$(\varphi \leftrightarrow \psi) \to (\neg\psi \to \psi^{\perp})$	$(\varphi \in \Sigma_k$ and $\psi \in \Pi_k)$
Γ-**WDUAL**	$\neg\varphi^{\perp} \to \neg\neg\varphi$	$(\varphi \in \Gamma)$
Γ-**COLL**$^{\mathbf{cp}}$	$\forall w\, \exists y < x\, \forall z < w\, \varphi(y, z) \to \exists y < x\, \forall z\, \varphi(y, z)$	$(\varphi(y, z) \in \Gamma)$
Γ-**LN**	$\exists x \varphi(x) \to \exists x(\varphi(x) \wedge \forall y < x\, \neg\varphi(y))$	$(\varphi \in \Gamma)$

Acknowledgement

The authors thank to the anonymous referee for the valuable and insightful comments. The first author was supported by JSPS KAKENHI Grant Numbers JP19J01239 and JP20K14354, and the second author by JP19K14586.

References

[1] Yohji Akama, Stefano Berardi, Susumu Hayashi, and Ulrich Kohlenbach. An arithmetical hierarchy of the law of excluded middle and related principles. In *Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science, 2004*, pages 192–201, 2004.

[2] Stefano Berardi. A generalization of a conservativity theorem for classical versus intuitionistic arithmetic. *Mathematical Logic Quarterly (MLQ)*, 50(1):41–46, 2004.

[3] Stefano Berardi and Silvia Steila. Ramsey theorem for pairs as a classical principle in intuitionistic arithmetic. In Ralph Matthes and Aleksy Schubert, editors, *19th International Conference on Types for Proofs and Programs (TYPES 2013)*, volume 26 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 64–83, Dagstuhl, Germany, 2014. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

[4] Stefano Berardi and Silvia Steila. Ramsey's theorem for pairs and k colors as a sub-classical principle of arithmetic. *The Journal of Symbolic Logic*, 82(2):737–753, 2017.

[5] Wolfgang Burr. Fragments of Heyting arithmetic. *The Journal of Symbolic Logic*, 65(3):1223–1240, 2000.

[6] Makoto Fujiwara. Δ^0_1 variants of the law of excluded middle and related principles. *Archive for Mathematical Logic, to appear.*

[7] Makoto Fujiwara. Weihrauch and constructive reducibility between existence statements. *Computability*, 10(1):17–30, 2021.

[8] Makoto Fujiwara, Hajime Ishihara, and Takako Nemoto. Some principles weaker than Markov's principle. *Archive for Mathematical Logic*, 54(7-8):861–870, 2015.

[9] Makoto Fujiwara, Hajime Ishihara, Takako Nemoto, Nobu-Yuki Suzuki, and Keita Yokoyama. Extended frames and separations of logical principles. submitted. `https://researchmap.jp/makotofujiwara/misc/30348506`.

[10] Makoto Fujiwara and Tasuji Kawai. A logical characterization of the continuous bar induction. In Fenrong Liu, Hiroakira Ono, and Junhua Yu, editors, *Knowledge, Proof and Dynamics, The Fourth Asian Workshop on Philosophical Logic*, pages 25–33, 2020.

[11] Makoto Fujiwara and Ulrich Kohlenbach. Interrelation between weak fragments of double negation shift and related principles. *The Journal of Symbolic Logic*, 83(3):991–1012, 2018.

[12] Makoto Fujiwara and Taishi Kurahashi. Conservation results on semi-classical arithmetic. *The Journal of Symbolic Logic, to appear.*

[13] Makoto Fujiwara and Taishi Kurahashi. Prenex normal form theorems in semi-classical arithmetic. *The Journal of Symbolic Logic*, 86(3):1124–1153, 2021.

[14] Hajime Ishihara. Markov's principle, Church's thesis and Lindelöf's theorem. *Indagationes Mathematicae. New Series*, 4(3):321–325, 1993.

[15] Hajime Ishihara. Constructive reverse mathematics: compactness properties. In *From sets and types to topology and analysis*, volume 48 of *Oxford Logic Guides*, pages 245–267. Oxford Univ. Press, Oxford, 2005.

[16] Ulrich Kohlenbach. *Applied proof theory. Proof interpretations and their use in mathematics.* Berlin: Springer, 2008.

[17] Ulrich Kohlenbach. On the disjunctive Markov principle. *Studia Logica*, 103(6):1313–1317, 2015.

[18] Craig Smoryński. Applications of Kripke models. In *Metamathematical Investigation of Intuitionistic Arithmetic and Analysis*, volume 344 of *Lecture Notes in Mathematics*, pages 324–391. Springer, Berlin, Heidelberg, 1973.

[19] Michael Toftdal. A calibration of ineffective theorems of analysis in a hierarchy of semi-classical logical principles. In Josep Díaz, Juhani Karhumäki, Arto Lepistö, and Donald Sannella, editors, *Automata, Languages and Programming*, pages 1188–1200, Berlin, Heidelberg, 2004. Springer Berlin Heidelberg.

[20] Dirk van Dalen. *Logic and structure.* London: Springer, fifth edition, 2013.