Choice principles in local mantles

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Abstract

Assume ZFC. Let κ be a cardinal. A < κ -ground is a transitive proper class W modelling ZFC such that V is a generic extension of W via a forcing $\mathbb{P} \in W$ of cardinality < κ . The κ -mantle \mathscr{M}_{κ} is the intersection of all < κ -grounds.

We prove that certain partial choice principles in \mathcal{M}_{κ} are the consequence of κ being inaccessible/weakly compact, and some other related facts.

1 Introduction

Let us recall some standard notions from set-theoretic geology. We generally assume ZFC, though at times (in particular in §2) we will also consider a weaker theory T_1 (which includes AC).

Given a transitive model W^1 of ZFC and a forcing $\mathbb{P} \in W$, a (W, \mathbb{P}) -generic is a filter $G \subseteq \mathbb{P}$ which is generic with respect to W. For a cardinal κ , a $< \kappa$ -ground of V is a transitive proper class $W \models$ ZFC such that there is $\mathbb{P} \in W$ with \mathbb{P} of cardinality $< \kappa$ (with cardinality as computed in W, or equivalently, in V) and a (W, \mathbb{P}) -generic filter G such that V = W[G]. A ground is a $< \kappa$ -ground for some cardinal κ .² The mantle \mathscr{M} is the intersection of all grounds. The κ -mantle \mathscr{M}_{κ} is the intersection of all $< \kappa$ -grounds.

By [4], as refined in [1], there is a formula $\varphi(x, y)$ in two free variables such that (i) for all $r, W_r = \{x \mid \varphi(r, x)\}$ is a ground (possibly $W_r = V$), and (ii) for every ground W there is r such that $W = W_r$. Therefore we can discuss grounds uniformly, and \mathscr{M} and \mathscr{M}_{κ} are transitive classes which are definable (\mathscr{M}_{κ} from parameter κ).

In §2 we will give the proof of ground definability, but from somewhat less than ZFC: we show that it holds under a certain theory T_1 (see 2.2), which is true in \mathcal{H}_{κ} whenever κ is a strong limit cardinal (assuming ZFC). The proof is essentially the usual ZFC proof, however.

From now on, we take W_r to be defined as in §2, by which $r = (\mathcal{H}_{\gamma^+})^W$ for some $\gamma \geq \omega$ for which there is $\mathbb{P} \in r$ and a (W, \mathbb{P}) -generic G with W[G] = V.

Let θ be a strong limit cardinal. By Usuba [10], the grounds are set-directed. By [10] and [8], this is moreover reasonably local, and in particular if $X \in \mathcal{H}_{\theta}$, then there

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¹Here we are not specific about exactly what formalization of classes we use. We could work in some class set theory, which allows quantification over such classes W, or we could with more care restrict to classes definable from parameters; for us a class must have the property that the structure (V, \in, W) satisfies ZFC in the language with symbols \in, W which interpret \in and W.

 $^{^2\}mathrm{Throughout},$ we consider only set-forcing, no class-forcing.

is $s \in \mathcal{H}_{\theta}$ with $W_s \subseteq \bigcap_{r \in X} W_r$. (For following Usuba's proof of [10, Proposition 5.1], note that we can take the regular cardinal κ of that proof with $\kappa < \theta$, and then the model W constructed there satisfies the κ^{++} -uniform covering property for V. Usuba then uses Bukovsky's theorem, [10, Fact 3.9], to deduce that W is a ground of V. But by [8, Theorem 3.11], the forcing for this can be taken of size $2^{\kappa^{++}}$ in W[g] = V.)

Also by [10], $\mathscr{M} \models \operatorname{ZFC}$, and by [11, §2], $\mathscr{M}_{\theta} \models \operatorname{ZF}$ (so note $\mathscr{M}_{\theta} \models ``\theta$ is a strong limit cardinal", in the ZF sense that \mathscr{M}_{θ} has no surjection $\pi : V_{\alpha} \to \theta$ with $\alpha < \theta$). If $V_{\theta} \preccurlyeq_n V$ with *n* large enough, then $V_{\theta}^{\mathscr{M}_{\theta}} = V_{\theta}^{\mathscr{M}}$, and hence $V_{\theta}^{\mathscr{M}_{\theta}} \models \operatorname{AC.}^3$ Usuba showed in [11] that if κ is an extendible cardinal then $\mathscr{M}_{\kappa} = \mathscr{M}$, so in this case, $\mathscr{M}_{\kappa} \models \operatorname{ZFC}$. Hence Usuba asked in [11] about whether $\mathscr{M}_{\kappa} \models \operatorname{ZFC}$ in general. We consider related questions in this paper. Let us first sketch some further history.

Suppose now κ is inaccessible. Then $V_{\kappa}^{\mathscr{M}_{\kappa}} \models \operatorname{ZFC}$. For note that by inaccessibility and the remarks above, for each $\alpha < \kappa$ there is some $r \in V_{\kappa}$ such that $V_{\alpha}^{W_r} = V_{\alpha}^{\mathscr{M}_{\kappa}}$. Since each $W_r \models \operatorname{ZFC}$, it follows that $V_{\kappa}^{\mathscr{M}_{\kappa}} \models \operatorname{ZFC}$. Clearly $\mathscr{M}_{\kappa} \models "\kappa$ is inaccessible", and if κ is Mahlo then $\mathscr{M}_{\kappa} \models "\kappa$ is Mahlo".

However, A. Lietz ([5]) answered Usuba's question above negatively (assuming large cardinals), showing that in fact it is consistent relative to a Mahlo cardinal that κ is Mahlo but $\mathscr{M}_{\kappa} \models ``\kappa\text{-AC}$ fails". In fact, Lietz constructs a forcing extension L[G] of L in which κ is Mahlo and $\mathscr{M}_{\kappa}^{L[G]}$ satisfies "there is a function $f : \kappa \to \mathcal{H}_{\kappa^+}$ for which there is no choice function". He also proved other related things.

In the last few years, the theory of Varsovian models has also been developed by Fuchs, Schindler, Sargsyan and more recently the author. Here, among other things, full mantles \mathscr{M} of certain fully iterable mice have been analyzed, and shown to be strategy mice, hence satisfying ZFC. Analysis of natural κ -mantles of those mice was, however, missing. But using Varsovian model techniques, the author then analyzed the κ_0 -mantle of the mouse M_{swsw} (Definition 3.1), showing that it is a strategy mouse, modelling ZFC + GCH. A very brief outline is given in §3 (but the other results in the note do not rely on this, and no inner model theory appears elsewhere in the paper). The argument has elements in common with Usuba's extendibility proof.

Schindler then showed that if κ is measurable then $\mathcal{M}_{\kappa} \models AC$, hence ZFC; see [8]. In this note we adapt this argument, deducing that fragments of choice hold in \mathcal{M}_{κ} from the weak compactness and inaccessibility of κ respectively.

1.1 Definition. Given an ordinal α and set X, let (α, X) -*Choice* be the assertion that for every function $f : \alpha \to X$, there is a choice function for f. And $(< \alpha, X)$ -*Choice* is the assertion that (β, X) -Choice holds for all $\beta < \alpha$.

Part 4 of the following theorem applies to the kind of functions involved in the failure of κ -AC in Lietz' example, but now with domain $< \kappa$. Note that we assume ZFC except where otherwise stated; κ -amenable-closure is defined in 2.18.

Theorem (3.15). If κ be inaccessible then:

- 1. $\mathscr{M}_{\kappa} \models ``\kappa \text{ is inaccessible}" and <math>\mathcal{H}_{\kappa}^{\mathscr{M}_{\kappa}} = V_{\kappa}^{\mathscr{M}_{\kappa}} \models \operatorname{ZFC}.$
- 2. \mathcal{M}_{κ} is κ -amenably-closed.
- 3. $\mathscr{M}_{\kappa} \models ``(\kappa, \mathcal{H}_{\kappa})$ -Choice" $\iff \mathscr{M}_{\kappa} \models ``\mathcal{H}_{\kappa}$ is wellordered".
- 4. $\mathscr{M} \models (< \kappa, \mathcal{H}_{\kappa^+})$ -Choice holds, so $(\mathcal{H}_{\kappa^+})^{<\kappa} \subseteq \mathcal{H}_{\kappa^+}$.

³An earlier draft asserted here that ([10] shows) $\mathcal{H}_{\theta}^{\mathcal{M}_{\theta}} = \mathcal{H}_{\theta}^{\mathcal{M}}$ for every strong limit cardinal θ . This was however not used anywhere, and it is false, at least if M_1 exists; see [9].

1.2 Remark. In part 3, the " κ^+ " and " \mathcal{H}_{κ^+} " are both in the sense of \mathscr{M}_{κ} . However, it can be that κ is Mahlo and $\mathscr{M}_{\kappa} \models (\kappa, \mathcal{H}_{\kappa^+})$ -Choice fails, and $(\mathcal{H}_{\kappa^+})^{\kappa} \not\subseteq \mathcal{H}_{\kappa^+}$; indeed, note that this occurs in Lietz' example L[G] mentioned above.

In the following theorem, the initial observation that $\mathscr{M}_{\kappa} \models \mathscr{H}_{\kappa}$ is wellordered" was due to Lietz:

Theorem (3.14). ⁴ Let κ be weakly compact. Then:

- 1. $\mathcal{M}_{\kappa} \models \kappa$ -DC + " κ is weakly compact".⁵
- 2. for each $A \in \mathscr{M}_{\kappa} \cap \mathcal{H}_{\kappa^+}$, $\mathscr{M}_{\kappa} \models ``A$ is wellordered''.
- 3. if $\mathcal{P}(\kappa)^{\mathscr{M}_{\kappa}}$ has cardinality κ then (i) κ is measurable in \mathscr{M}_{κ} , and (ii) $x^{\#}$ exists for every $x \in \mathcal{P}(\kappa)^{\mathscr{M}_{\kappa}}$, and $x^{\#} \in \mathscr{M}_{\kappa}$.
- 4. If $\mathcal{M}_{\kappa} \models \mu$ is a countably complete ultrafilter over $\gamma \leq \kappa$, then the ultrapower $\mathrm{Ult}(\mathscr{M}_{\kappa},\mu)$ is wellfounded and the ultrapower embedding

$$i_{\mu}^{\mathcal{M}_{\kappa}} : \mathcal{M}_{\kappa} \to \mathrm{Ult}(\mathcal{M}_{\kappa},\mu)$$

is fully elementary.

As a corollary to Schindler's proof, one easily gets:

Fact (3.3). Let κ be measurable and μ be a normal measure on κ . Then for μ measure one many $\gamma < \kappa$, $\mathscr{M}_{\gamma} \models "V_{\gamma+1}$ is wellorderable".

As mentioned above, Usuba showed that $\mathcal{M} = \mathcal{M}_{\kappa}$ assuming κ is extendible. The next result indicates that there are signs of this in the leadup to an extendible cardinal (for the definition of a Σ_2 -strong cardinal, see 3.5):

Theorem (3.9). Suppose κ is Σ_2 -strong. Then $V_{\kappa+1}^{\mathscr{M}_{\kappa}} = V_{\kappa+1}^{\mathscr{M}}$.

Analogously, down lower:

Theorem (3.4). Let A be a set such that $A^{\#}$ exists. Let κ be an A-indiscernible. Then $V_{\kappa+1}^{\mathscr{M}_{\kappa}^{L(A)}} = V_{\kappa+1}^{\mathscr{M}_{\kappa+1}^{L(A)}}$ and this set is wellordered in $\mathscr{M}_{\kappa}^{L(A)}$.

For further related results, which involve some inner model theory, see [9].

Before beginning our discussion of these results, we go through some background set-theoretic geology, including a proof of the the definability of grounds from a theory modelled by \mathcal{H}_{κ} whenever κ is a strong limit cardinal.

$\mathbf{2}$ Grounds and mantles

We discuss here some background, starting with the key fact of the definability of set-forcing grounds under ZFC, proved by some combination of Laver, Woodin and Hamkins:

2.1 Fact. Let M, N be proper class transitive inner models of ZFC and $\gamma \in OR$ with $\mathcal{P}(\gamma) \cap M = \mathcal{P}(\gamma) \cap N$. Let $\mathbb{P} \in M$ and $\mathbb{Q} \in N$, with $\mathbb{P}, \mathbb{Q} \subseteq \gamma$, and let G be (M, \mathbb{P}) -generic and H be (N, \mathbb{Q}) -generic and suppose M[G] = N[H] = V. Then M = N.

⁴Regarding part 2, the author initially observed that a variant of Schindler's argument gives that $\mathcal{M}_{\kappa} \models \kappa$ -DC, and then Lietz and the author independently noticed that the argument can be adjusted to show that every set in $\mathcal{H}_{\kappa^+} \cap \mathscr{M}_{\kappa}$ is wellordered in \mathscr{M}_{κ} . ⁵So also $\mathscr{M}_{\kappa} \models "\kappa^+$ is regular and $\mathcal{H}_{\kappa^+} \models \text{ZFC}^-$ ". ⁶Note that the " κ^+ " and " \mathcal{H}_{κ^+} " here are computed in V, not \mathscr{M}_{κ} .

We will discuss the proof of the result above, for two purposes. The fact is central to our concerns, and the proof contains elements which will come up in various places later, so it is natural to collect all these things together. Second, we wish to prove a version which assumes less background theory (than ZFC). The authors of [3] make use of an analysis of the complexity of the definability of grounds. As shown there, each ground W is, in particular, Σ_2 in a parameter r. However, the Σ_2 definition given there is not particularly local: to compute V_{α}^W , they work in V_{β} , for a significantly larger ordinal β . So for [3, Theorem 4], they adopt the background theory ZFC $_{\delta}$. We show here that the ground definability can be done much more locally (though still requiring Σ_2 complexity), hence requiring significantly less than ZFC $_{\delta}$.

2.2 Definition. Let T_1^- be the following theory in the language of set theory. The axioms are Extensionality, Foundation, Pairing, Union, Infinity, "Every set is bijectable with an ordinal", Σ_1 -Separation and Σ_1 -Collection. Now let

$$T_1 = T_1^- +$$
Powerset. \dashv

Note that $T_1^- \models$ AC. We will show that models of T_1 can uniformly define their grounds from parameters. First we give some lemmas.

2.3 Lemma. Assume ZFC. Then for every cardinal $\kappa \geq \omega$, (i) $\mathcal{H}_{\kappa} \models T_1^-$, and (ii) $\mathcal{H}_{\kappa} \models T_1$ iff κ is a strong limit cardinal.

The usual proofs from ZFC easily adapt to give:

2.4 Lemma. Assume T_1 . Then (i) for each ordinal ξ , \mathcal{H}_{ξ} exists, (ii) $V = \bigcup_{\xi \in OR} \mathcal{H}_{\xi}$, (iii) $\mathcal{H}_{\xi} \preccurlyeq_1 V$, (iv) $\mathcal{H}_{\xi} \models T_1^-$, (iv) the Lowenheim-Skolem theorem holds.

In the following lemma, the forcing relations \square_{Σ_i} for $i \in \{0, 1\}$, and \square_{Π_1} , are the relations defined in a first-order manner over M in the usual manner, and the strong- Σ_{i+1} -forcing relation $\square_{\Sigma_{i+1}}^*$ is the relation for which, given a Π_i formula $\psi(\vec{x}, \vec{y})$ with free variables \vec{x}, \vec{y} , and given $\vec{\tau} \in (M^{\mathbb{P}})^{<\omega}$, we say $p \square_{\Sigma_{i+1}}^* \exists \vec{y} \ \psi(\vec{\tau}, \vec{y})$ iff there is $\vec{\sigma} \in (M^{\mathbb{P}})^{<\omega}$ such that $p \square_{\Pi_i} \ \psi(\vec{\tau}, \vec{\sigma})$.

2.5 Lemma (Forcing over T_1^- and T_1). Let $M \models T_1^-$. Let $\mathbb{P} \in M$ be a poset with $\mathbb{P} \subseteq \gamma \in OR^M$ and G be (M, \mathbb{P}) -generic. Then:

- 1. We have:
 - (a) The Σ_0 -forcing relation \square_{Σ_0} for (M, \mathbb{P}) is $\Delta_1^M(\{\mathbb{P}\})$, uniformly.
 - (b) The Σ_1 -forcing relation $-_{\Sigma_1}$ for (M, \mathbb{P}) is $\Sigma_1^M(\{\mathbb{P}\})$, uniformly.⁷
 - (c) The Π_1 -forcing relation \square_{Π_1} for (M, \mathbb{P}) is $\Pi_1^M(\{\mathbb{P}\})$, uniformly.

Hence, $\|_{\Sigma_0}$, $\|_{\Sigma_1}$ and $\|_{\Pi_1}$ are absolute to \mathcal{H}^M_{κ} , for *M*-cardinals $\kappa > \gamma$.

- 2. The strong- Σ_2 -forcing relation $\models^*_{\Sigma_2}$ for (M, \mathbb{P}) is $\Sigma_2^M(\{\mathbb{P}\})$, uniformly.
- 3. The forcing theorem for Σ_0 , Σ_1 , Π_1 formulas holds for M[G], with respect to \llbracket_{Σ_0} , \llbracket_{Σ_1} , \llbracket_{Π_1} ; likewise for Σ_2 and $\llbracket_{\Sigma_2}^*$. That is, if φ is Σ_i , where $i \in \{0, 1\}$, and $\vec{\tau} \in (M^{\mathbb{P}})^{<\omega}$, then

$$M[G] \models \varphi(\vec{\tau}_G) \iff \exists p \in G \left[M \models "p \middle\|_{\Sigma_i} \varphi(\vec{\tau})" \right]$$

⁷In a previous draft of this document, it mistakenly said that the Σ_1 -forcing relation is Δ_1^M -definable, which is clearly false, since in the case of trivial forcing, it would imply that $\Sigma_1^M = \Delta_1^M$.

Likewise for Π_1 with $-_{\Pi_1}$, and for Σ_2 with $-_{\Sigma_2}^*$.

- 4. $M[G] \models T_1^-$, and if $M \models T_1$ then $M[G] \models T_1$.
- 5. M and M[G] have the same cardinals $\kappa > \gamma$,
- 6. for each *M*-cardinal $\kappa > \gamma$, we have $\mathcal{H}_{\kappa}^{M[G]} = \mathcal{H}_{\kappa}^{M}[G]$.

Such local forcing calculations are very common in the literature, in particular in fine structure theory, where much more local calculations are often used. But we include a proof in case the reader has not seen these before.

Proof. Parts 1, 3 for Σ_0 : The usual internal definition of the Σ_0 -forcing relation $-_0$ works locally; in fact, for each $\xi \in OR^M$ with $\xi \geq \gamma$, the Σ_0 -forcing relation for names in \mathcal{H}_{ξ} , is $\Delta_1^{\mathcal{H}_{\xi}}(\{\mathbb{P}\})$, uniformly in ξ . This gives the Forcing Theorem for Σ_0 formulas in the usual manner.

Parts 1, 3 for Σ_1 : We defined the strong- Σ_1 -forcing relation \models_1^* over M above. Using the Σ_0 -Forcing Theorem, note that $M[G] \models \exists y \ \varphi(y, \tau_G)$ iff there is $p \in G$ such that $M \models "p \models_1^* \exists y \ \varphi(y, \tau)$ ". Moreover, \models_1^* is uniformly $\Sigma_1^M(\{\mathbb{P}\})$ -definable.

Note that we take $-_1$ defined over M as follows: Working in M, for φ being Σ_1 and $\tau \in M^{\mathbb{P}}$, set

$$p \Vdash_{1} \varphi(\tau) \iff \forall q \le p \; \exists r \le q \; \Big[r \Vdash_{1}^{*} \varphi(\tau) \Big].$$

We claim that $p \models_1 \varphi(\tau)$ iff $p \models_1^* \varphi(\tau)$. For the non-trivial direction, suppose $p \models_1 \varphi(\tau)$. Then working in M, using Σ_1 -Collection and AC, we can put together a name $\sigma \in M^{\mathbb{P}}$ showing that $p \models_1^* \varphi(\tau)$. This completes the calculation for Σ_1 .

Parts 1, 3 for Π_1 : \square_{Π_1} is defined as usual: Working in M, for φ being Π_1 and $\tau \in M^{\mathbb{P}}$, say $p \amalg_{\Pi_1} \varphi(\tau)$ iff there is no $q \leq p$ such that $q \amalg_{\Sigma_1} \neg \varphi(\tau)$. So \amalg_{Π_1} is $\Pi_1^M(\{\mathbb{P}\})$. If $p \in G$ and $p \amalg_{\Pi_1} \varphi(\tau)$, then clearly $M[G] \models \varphi(\tau_G)$. So suppose $M[G] \models \varphi(\tau_G)$ where φ is Π_1 . Let

$$D = \{ p \in \mathbb{P} \mid p \mid \neg_{\Sigma_1} \neg \varphi(\tau) \}.$$

By Σ_1 -Separation, $D \in M$. Let $D' = D \cup \{p \in \mathbb{P} \mid \neg \exists q \in D \ [q \leq p]\}$, then $D' \in M$, and since D' is dense, this easily suffices.

Parts 1, 3 for Σ_2 : Here we only consider the strong- Σ_2 forcing relation $\models_{\Sigma_2}^*$, and the claims regarding this follow immediately just like for $\models_{\Sigma_1}^*$.

Part 4: Most of the axioms are routine consequences of the previous parts. Let us verify that $M[G] \models \Sigma_1$ -Collection. Fix a Σ_0 formula φ and $\sigma, \tau \in M^{\mathbb{P}}$. Let $t \in M$ be the transitive closure of $\{\sigma, \tau\}$. Then there is $w \in M$ such that for all $p \in \mathbb{P}$ and $\varrho \in t$, if

$$p \mid_{\Sigma_1} \quad "\varrho \in \sigma \text{ and } \exists y \varphi(\varrho, \tau, y) ",$$

then there is $y \in M^{\mathbb{P}} \cap w$ such that $p \models_{\Sigma_0} {}^{\circ} \varrho \in \sigma$ and $\varphi(\varrho, \tau, y)$. But then using w, we easily get a bound on witnesses in M[G], as desired. This and the Σ_0 -Forcing Theorem easily yields Σ_1 -Separation.

The remaining parts follow from routine calculations with nice names.

2.6 Definition. Let $(M, E) \models T_1^-$. A ground of M is a $W \subseteq M$ such that:

- 1. $(W, E \upharpoonright W)$ is *M*-transitive; that is, for all $x \in W$ and all $y \in M$, if yEx then $y \in W$,
- 2. $W \models T_1^-$,
- 3. there is $\mathbb{P} \in W$ and a (W, \mathbb{P}) -generic $G \in M$ such that M = W[G].
- 4. If $(M, E) \models T_1$ then $(W, E \upharpoonright W) \models T_1$.

We now prove that T_1 suffices for the definability of grounds (in the sense of the definition above). The proof is essentially that due to some combination of Laver, Woodin and Hamkins. In the proof we make implicit use of Lemma 2.5, to allow the forcing calculations:

 \neg

2.7 Theorem (Ground definability under T_1). Assume T_1 . Let $\gamma \in OR$, $H \subseteq \mathcal{H}_{\gamma^+}$ and $\kappa \geq \gamma^+$ a cardinal. Then there is at most one transitive $M \subseteq \mathcal{H}_{\kappa}$ such that $M \models T_1^-, (\mathcal{H}_{\gamma^+})^M = H$, and M is a ground for \mathcal{H}_{κ} via some $\mathbb{P} \in H$.

Proof. We proceed by induction on κ . For $\kappa = \gamma^+$ it is trivial.

Suppose κ is a limit cardinal, and that for each cardinal $\theta \in [\gamma^+, \kappa)$, there is a (unique) model M_{θ} of ordinal height θ with the stated properties. Then clearly $M = \bigcup_{\theta < \kappa} M_{\theta}$ is the unique candidate at κ . To see that M works, we just need to verify that M is indeed a set-ground of \mathcal{H}_{κ} via some $\mathbb{P} \in H$; i.e. there is $\mathbb{P} \in H$ and an (M, \mathbb{P}) -generic $G \subseteq \mathbb{P}$ such that $M[G] = \mathcal{H}_{\kappa}$. But we can use any (\mathbb{P}, G) which worked at some earlier θ . For let $\theta_0 \leq \theta_1 < \kappa$, and let $(\mathbb{P}_0, G_0), (\mathbb{P}_1, G_1)$ work for $M_0 = M_{\theta_0}$ and $M_1 = M_{\theta_1}$. So G_0 is also (M_1, \mathbb{P}_0) -generic, and vice versa. And since $\mathcal{H}_{\gamma^+}^{M_0} = H = \mathcal{H}_{\gamma^+}^{M_1}$, and $H[G_0] = \mathcal{H}_{\gamma^+} = H[G_1]$, it follows that $\mathcal{H}_{\kappa} = M_0[G_0] = M_0[G_1]$ and $M_1[G_0] = M_1[G_1] = \mathcal{H}_{\kappa}$, so the specific choice of (\mathbb{P}, G) is irrelevant.

So consider $\kappa = \theta^+ > \gamma^+$. Let M, N be grounds of \mathcal{H}_{κ} with the stated properties. By induction, $M \cap \mathcal{H}_{\theta} = N \cap \mathcal{H}_{\theta}$. It just remains to verify that $\mathcal{P}(\theta) \cap M = \mathcal{P}(\theta) \cap N$. The proof is, however, not by contradiction; we will not assume that $M \neq N$. Fix (\mathbb{P}, G) such that $\mathbb{P} \in H$ and G is (M, \mathbb{P}) -generic and $M[G] = \mathcal{H}_{\kappa}$.

Suppose first that $cof(\theta) > \gamma$, as this case is easier; however, it is in the end subsumed into the general case. Let $A \subseteq \theta$. Then:

CLAIM 1. $A \in M$ iff $A \cap \alpha \in M$ for all $\alpha < \theta$.

Proof. For the non-trivial direction, suppose $A \cap \alpha \in M$ for every $\alpha < \theta$. Let $f: \theta \to M$ be $f(\alpha) = A \cap \alpha$. Then $f \in \mathcal{H}_{\kappa}$. So there is a \mathbb{P} -name $\dot{f} \in M$ with $\dot{f}_G = f$. Working in M, for $p \in \mathbb{P}$, compute

$$D_p = \{ \alpha < \theta \mid \exists x \ [p \mid -\dot{f}(\check{\alpha}) = \check{x}] \},$$

and let $f_p: D_p \to \theta$ be the function

$$f_p(\alpha) =$$
 unique x such that $p - \dot{f}(\check{\alpha}) = \check{x}$.

So $\langle D_p, f_p \rangle_{p \in \mathbb{P}} \in M$, and because $\operatorname{cof}(\theta) > \gamma$, there is $p \in G$ such that D_p is cofinal in θ . Then $f = \left(\bigcup_{\alpha \in D_p} f_p(\alpha)\right) \in M$.

We now argue in general.

CLAIM 2. Let $A \subseteq \theta$. Then $A \in M$ iff for every $X \in \mathcal{P}(\theta) \cap M$ such that $\operatorname{card}(X) < (\gamma^+) = (\gamma^+)^M$, we have $A \cap X \in M$.

Proof. The forward direction is trivial. So let $A \subseteq \theta$ with $A \notin M$. Let $\dot{A} \in M$ be a \mathbb{P} -name and $p_0 \in G$ such that $p_0 \models \dot{A} \subseteq \check{\theta}$. For each $q \leq p_0$, if there is $\alpha < \theta$ such that

$$q \not\vdash \check{\alpha} \in \dot{A} \text{ and } q \not\vdash \check{\alpha} \notin \dot{A},$$

then let α_q be the least such α ; otherwise α_q is undefined. Let D be the set of all $q \leq p_0$ such that α_q exists. Then $G \subseteq D$, because otherwise q decides all elements of \dot{A} , so $A \in M$.

In M, let $X = \{\alpha_q \mid q \in D\}$. Then $X \in M$, $\operatorname{card}^M(X) \leq \gamma$ and $X \cap A \notin M$, as desired. For given $Y \in \mathcal{P}(X) \cap M$, an easy density argument shows that $Y \neq X \cap A$.

CLAIM 3. Let $X \subseteq \theta$ with $\operatorname{card}(X) < \gamma^+$. Then $X \in M$ iff $X \in N$.

Proof. Suppose $X_0 = X \in N$. Let $\dot{X} \in M$ be a P-name for X. Using the forcing relation and \dot{X} , there is a set $X_1 \in \mathcal{P}(\theta) \cap M$ with $X_0 \subseteq X_1$ and $\operatorname{card}(X_1) < (\gamma^+)^V$. Proceeding back-and-forth, construct (in V) a continuous sequence of sets $\langle X_\alpha \rangle_{\alpha < \gamma^+}$ such that (i) $X_0 = X$, (ii) $X_{\omega\alpha+2n+1} \in M$ and $X_{\omega\alpha+2n+2} \in N$, and (iii) $\operatorname{card}(X_\alpha) < (\gamma^+)^V$.

Now $\gamma^+ < \kappa$, so $\langle X_{\alpha} \rangle_{\alpha < \gamma^+} \in \mathcal{H}_{\kappa}$, so M, N have names for this sequence. So as in the $\operatorname{cof}(\theta) > \gamma$ case, we get a cofinal set $D_M \subseteq \gamma^+$ such that $D_M \in M$ and $\langle X_{\alpha} \rangle_{\alpha \in D_M} \in M$. Likewise with a cofinal set $D_N \in N$. Let D'_M be the set of limit points of D_M , and D'_N likewise. So these are club in γ^+ . Let $\alpha \in D'_M \cap D'_N$. Then note that

$$X_{\alpha} = \left(\bigcup_{\beta \in D_{M} \cap \alpha} X_{\beta}\right) = \left(\bigcup_{\beta \in D_{N} \cap \alpha} X_{\beta}\right) \in M \cap N.$$

Let $\pi: \xi \to X_{\alpha}$ be the increasing enumeration of X_{α} . Then $\xi < \gamma^+$ and $\pi \in M \cap N$. We have $X \subseteq \operatorname{rg}(\pi)$. Let $\overline{X} = \pi^{-1}(X)$. Then $\overline{X} \in N$. But $\mathcal{H}^M_{\gamma^+} = H = \mathcal{H}^N_{\gamma^+}$, so $\overline{X} \in M$. So $\pi^* \overline{X} = X \in M$, as desired.

This completes the proof of ground definability under T_1 .

2.8 Remark. If $M \models T_1^- +$ "there is a largest cardinal κ , and κ is regular", then grounds of M via forcings \mathbb{P} of M-cardinality $< \kappa$ are also definable from parameters over M, by arguing much as above.

2.9 Definition. Assume T_1 . Let $\varphi_{\text{grd}}(r, x)$ be the formula "there are γ , \mathbb{P} , G, M, κ such that $\gamma < \kappa$ are cardinals, $M \subseteq \mathcal{H}_{\kappa}$ is transitive, $M \models T_1^-$, $\mathbb{P} \in r = (\mathcal{H}_{\gamma^+})^M$, G is (M, \mathbb{P}) -generic, $\mathcal{H}_{\kappa} = M[G]$ and $x \in M$ ".

We write $W'_r = \{x \mid \varphi_{\text{grd}}(r, x)\}$. We say r is a *true index* iff W'_r is proper class. We write $W_r = W'_r$ for true indices r, and $W_r = V$ otherwise.

2.10 Corollary. Assume ZFC+GCH and let λ be a limit cardinal. Then the grounds of \mathcal{H}_{λ} are definable from parameters over \mathcal{H}_{λ} .

2.11 Remark. Assume ZFC+GCH. Then for each limit ordinal ξ , $V_{\omega+\xi}$ is equivalent in the codes to $\mathcal{H}_{\aleph_{\xi}}$. So one can correctly formulate "grounds" of $V_{\omega+\xi}$, and they are definable over $V_{\omega+\xi}$ from parameters.

So we have the standard uniform definability of grounds, just assuming T_1 : **2.12 Lemma.** Let $M \models T_1$. Then $\{W_r^M \mid r \in M\}$ enumerates exactly the grounds of M (with repetitions, including M itself). **2.13 Remark.** Assume T_1 . Note that φ_{grd} is Σ_2 , and the assertion "r is a true index" is Π_2 . (In fact, there are fixed Σ_2 and Π_2 formulas, such that T_1 proves that these fixed formulas always work.) Moreover, letting $\xi = \operatorname{card}(trcl(\{r, x\}))$, note that $\varphi_{\text{grd}}(r, x)$ is absolute between V and $\mathcal{H}_{(2^{\xi})^+}$. (It is witnessed by some (\mathcal{H}_{ξ^+}, M) , a structure of size 2^{ξ} .) Therefore:

2.14 Fact (Local definability of grounds). Assume T_1 + "There is a proper class of strong limit cardinals". Let λ be a strong limit cardinal. Let $r \in \mathcal{H}_{\lambda}$ be a true index. Then $\mathcal{H}_{\lambda} \models "r$ is a true index" and $W_r^{\mathcal{H}_{\lambda}} = W_r \cap \mathcal{H}_{\lambda} = \mathcal{H}_{\lambda}^{W_r}$.

It seems it might be possible, however, that $\mathcal{H}_{\lambda} \models "r$ is a true index" while r fails to be a true index in V.

The remaining facts in this section, and the rest of the paper, have a background theory of ZFC. We have not investigated to what extent things go through under T_1 . By [10, Proposition 5.1] and an examination of its proof, we have:

2.15 Fact (Local set-directedness of grounds (Usuba)). (Assume ZFC.) Let θ be a strong limit cardinal and $R \in \mathcal{H}_{\theta}$. Then there is $t \in \mathcal{H}_{\theta}$ with $t \in \bigcap_{r \in R} W_r$ and $W_t \subseteq W_r$ and $W_t = W_t^{W_r}$ for each $r \in R$. In particular, $W_t \subseteq \bigcap_{r \in R} W_r$.

Proof. We refer here to the λ -uniform covering property for V; see [10, Definition (4.2) or [8, Definition 2.1]. Let us set up some of the notation from the proof of [10, 10]Proposition 5.1]. Let X = R (following the notation from [10]).⁸We may assume that X is a set of true indices r. For $r \in X$ let $\mathbb{P}_r \in W_r$ be a forcing witnessing that r is a true index. Let κ be a regular cardinal with $\kappa > \operatorname{card}(X)$ and $\kappa > \operatorname{card}(\mathbb{P}_r)$ for each r (so it suffices if $\kappa > \operatorname{card}(trcl(X)))$). Then the proof of [10, Proposition 5.1] constructs a ground $W \subseteq \bigcap_{r \in X} W_r$ with the $\lambda = \kappa^{++}$ -uniform covering property for V. Therefore by [7, Theorem 3.3], there is $\mathbb{P} \in W$ such that $W \models \text{``card}(\mathbb{P}) = 2^{2^{<\lambda}}$, and W is a ground of V via \mathbb{P} . Let $\gamma_0 = \text{card}^W(\mathbb{P})$ and $t_0 = (\mathcal{H}_{\gamma_0^+})^W$. So $\gamma_0 < \theta$, t_0 is a true index and $W = W_{t_0}$. Let $\mathbb{B} \in W$ be such that $W \models \mathbb{B}$ is the complete Boolean algebra determined by \mathbb{P} " (so \mathbb{P} is a dense sub-order of \mathbb{B}). So card^W(\mathbb{B}) \leq $(2^{\gamma_0})^W < \theta$. Then by [2, Lemma 15.43] (or [10, Fact 3.1]) for each $r \in X$ there is some $\mathbb{B}_r \in W$ with $\mathbb{B}_r \subseteq \mathbb{B}$ and there is a (W, \mathbb{B}_r) -generic G_r such that $W[G_r] = W_r$. So letting $\gamma = (2^{\gamma_0})^W$, then $t = (\mathcal{H}_{\gamma^+})^W$ is as desired.

An easy corollary of local set-directedness is:

2.16 Fact (Invariance of \mathcal{M}_{κ}). Let κ be a strong limit cardinal and $r \in \mathcal{H}_{\kappa}$. Then $\mathscr{M}^{W_r}_{\kappa} = \mathscr{M}^{\cdot}_{\kappa}.$

2.17 Lemma (Absoluteness of \mathcal{M}_{κ}). Let $\kappa < \lambda$ be strong limit cardinals and suppose $\mathcal{H}_{\lambda} = V_{\lambda} \preccurlyeq_2 V$. Then for each $r \in \mathcal{H}_{\kappa}$, we have:

(i) < κ -grounds and \mathcal{M}_{κ} are absolute to V_{λ} :

$$W_r^{V_\lambda} = W_r \cap V_\lambda = V_\lambda^{W_r}$$
 and $\mathscr{M}_\kappa^{V_\lambda} = \mathscr{M}_\kappa \cap V_\lambda = V_\lambda^{\mathscr{M}_\kappa}$,

- (ii) $V_{\lambda}^{W_r} \preccurlyeq_2 W_r$,
- (iii) $\mathcal{M}_{\kappa}^{V_{\lambda}^{W_{r}}} = \mathcal{M}_{\kappa}^{W_{r}} \cap V_{\lambda}^{W_{r}} = \mathcal{M}_{\kappa} \cap V_{\lambda} = \mathcal{M}_{\kappa}^{V_{\lambda}}.$ ⁸We wrote *R* in the statement of the fact for consistency with later notation.

Proof. Part (i): The absoluteness of W_r is because the class true indices r is Π_2 , and each W_r is $\Sigma_2(\{r\})$. But then clearly

$$\mathscr{M}_{\kappa}^{V_{\lambda}} = \bigcap_{r \in V_{\kappa}} W_{r}^{V_{\lambda}} = \bigcap_{r \in V_{\kappa}} V_{\lambda}^{W_{r}} = V_{\lambda}^{\mathscr{M}_{\kappa}}.$$

Part (ii): If $W_r = V$ then this is trivial. Suppose $W_r \subsetneq V$ and let φ be Σ_2 and $x \in W_r \cap V_\lambda$ and suppose that $W_r \models \varphi(x)$. Then by Fact 2.14, $V \models \psi(x)$ where ψ asserts "There is a strong limit cardinal ξ such that $W_r^{\mathcal{H}_{\xi}} \models \varphi(x)$ ", but this is also Σ_2 , so $V_\lambda \models \psi(x)$, so letting $\xi < \lambda$ witness this, again by Fact 2.14, we get $W_r \cap \mathcal{H}_{\xi} \models \varphi(x)$, so $W_r \cap V_\lambda \models \varphi(x)$.

Part (iii): This follows from the previous parts and Fact 2.16.

2.18 Definition. Let N be an inner model. Let $f : \kappa \to N$. Say that f is amenable to N iff $f \upharpoonright \alpha \in N$ for every $\alpha < \kappa$. Say that N is κ -amenably-closed iff for every $f : \kappa \to N$, if f is amenable to N then $f \in N$. Say that N is κ -stationarily-computing (κ -unboundedly-computing) iff for every $f : \kappa \to N$, there is a stationary (unbounded) $A \subseteq \kappa$ such that $f \upharpoonright A \in N$.

2.19 Lemma. Let N be an inner model of ZF and $\kappa > \omega$ be regular. If N is κ -stationarily-computing then N is κ -unboundedly-computing. If N is κ -unboundedly-computing then N is κ -amenably-closed.

2.20 Lemma. Let W be a < κ -ground of V, where $\kappa > \omega$ is regular. Then W is κ -stationarily-computing.

2.21 Lemma. The intersection of any family of κ -amenably-closed structures is κ -amenably-closed.

2.22 Lemma. If κ is inaccessible then \mathcal{M}_{κ} is κ -amenably-closed.

3 Choice principles in the κ -mantle

As mentioned above, from now on we have ZFC as background theory.

The first positive results along the lines of what we will prove here (regarding about κ -mantles when $\kappa < \infty$), consists in Usuba's work, including his extendibility result. This was followed by Lietz' negative results [5]. Some time after this, using the general theory of [6], the author showed that the κ_0 -mantle $\mathscr{M}^M_{\kappa_0}$ of $M = M_{\text{swsw}}$ (see below) is a strategy mouse. We give an outline of this argument, but it is primarily intended for the reader familiar with inner model theory, and can be safely skipped over, as the remainder of the paper does not depend on it. We omit all specifics to do with Varsovian models, just mentioning enough to indicate what is relevant here. The full proof will appear in [6].

3.1 Definition. M_{swsw} denotes the least iterable proper class mouse with ordinals $\delta_0 < \kappa_0 < \delta_1 < \kappa_1$ satisfying "each δ_i is Woodin and each κ_i is strong". \dashv

The Varsovian model analysis produces a mouse M_{∞} , which is the direct limit of (pseudo-)iterates P of M via correct iteration trees \mathcal{T} on M, with $\mathcal{T} \in M|\kappa_0$, and which are based on $M|\delta_0$. It also defines a certain fragment Σ of the iteration strategy for M_{∞} , yielding a strategy mouse $M_{\infty}[\Sigma]$. It turns out that $M_{\infty}[\Sigma]$ has universe $\mathcal{M}_{\kappa_0}^M$. What is relevant here is the proof that $\mathscr{M}^{M}_{\kappa_{0}} \subseteq M_{\infty}[\Sigma]$. Let $X \in \mathscr{M}^{M}_{\kappa_{0}}$ be a set of ordinals. We must see that $X \in M_{\infty}[\Sigma]$. ⁹ Now κ_{0} is measurable in M. Let E be a normal measure on κ_0 , in the extender sequence of M, and let

$$j: M \to U = \text{Ult}(M, E)$$

be the ultrapower map. By elementarity, $j(X) \in \mathscr{M}_{j(\kappa_0)}^U$. With methods from the Varsovian model analysis, one can then construct a specific $\langle j(\kappa_0)$ -ground W of U, with $W \subseteq M_{\infty}[\Sigma]$. So

$$j(X) \in \mathscr{M}_{j(\kappa_0)}^U \subseteq W \subseteq M_\infty[\Sigma].$$

Other facts from Varsovian model analysis give $j \mid \alpha \in M_{\infty}[\Sigma]$ for each $\alpha \in OR$. But then $X \in M_{\infty}[\Sigma]$, as desired, since

$$\beta \in X \iff j(\beta) \in j(X).$$

The preceding argument has structural similarities to Usuba's extendibility proof (see [11]). Schindler then found the following result (see [8]). We will use an adaptation of the proof for Theorem 3.14 later, so we present this one first as a warmup, and in order to note a simple corollary. We give essentially Schindler's proof, although the precise implementation might differ slightly.

3.2 Fact (Schindler). Let κ be measurable. Then $\mathscr{M}_{\kappa} \models AC$, so $\mathscr{M}_{\kappa} \models ZFC$.

Proof. Let $A \in \mathscr{M}_{\kappa}$. We will find a wellorder $<_A$ of A with $<_A \in \mathscr{M}_{\kappa}$. Let μ be a normal measure on κ , $M = Ult(V, \mu)$ and $j = i_{\mu}^{V} : V \to M$ the ultrapower map. So $\kappa = \operatorname{cr}(j)$ and $j(A) \in \mathscr{M}^{M}_{i(\kappa)}$.

CLAIM 1. We have:

1. $\mathcal{M}_{i(\kappa)}^{M} \subseteq \mathcal{M}_{\kappa}^{M} \subseteq \mathcal{M}_{\kappa}$, and

2. $j \upharpoonright \mathscr{M}_{\kappa}$ is amenable to \mathscr{M}_{κ} .

Proof. Part 1: The first \subseteq is immediate. For the second, we have

$$\mathscr{M}_{\kappa} = \bigcap_{r \in V_{\kappa}} W_r$$
 and $\mathscr{M}^M_{\kappa} = \bigcap_{r \in V_{\kappa}} W^M_r$.

Let $\mu_r = \mu \cap W_r$. Then by standard forcing calculations and elementarity, we get $\mu_r \in W_r$ and

$$W_r^M = j(W_r) = \text{Ult}(W_r, \mu)^V = \text{Ult}(W_r, \mu_r)^{W_r},$$

so $W_r^M \subseteq W_r$, so $\mathscr{M}_{\kappa}^M \subseteq \mathscr{M}_{\kappa}$ as desired. Part 2: Let $r \in V_{\kappa}$. Then calculations as above give $i_{\mu_r}^{W_r} \upharpoonright W_r \subseteq j$. But $\mathscr{M}_{\kappa} \subseteq W_r$, and so $j \upharpoonright \mathscr{M}_{\kappa}$ is amenable to W_r . Therefore $j \upharpoonright \mathscr{M}_{\kappa}$ is amenable to \mathscr{M}_{κ} , as desired. \square

Since κ is a strong limit, Fact 2.15 gives $s \in V_{i(\kappa)}^{M}$ such that

$$\mathscr{M}^M_{j(\kappa)} \subseteq W = W^M_s \subseteq \mathscr{M}^M_\kappa$$

So $j(A) \in W \models$ ZFC, so there is a wellorder $<^*$ of j(A) with $<^* \in W$. But $W \subseteq \mathscr{M}^M_{\kappa}$, so $<^* \in \mathscr{M}^M_{\kappa} \subseteq \mathscr{M}_{\kappa}$. Now working in \mathscr{M}_{κ} , where we have $k = j \upharpoonright A$ and j(A) and $<^*$, we can define a

wellorder $<_A$ of A by setting, for $x, y \in A$:

$$x <_A y \iff k(x) <^* k(y).$$

This completes the proof.

 $^{^{9}}$ What blocks the more obvious attempt to prove this is that it is not clear that the iteration maps i_{PQ} between the iterates P, Q of the direct limit system eventually fix X.

As a corollary to the proof above, we observe:

3.3 Corollary. Let κ be measurable and μ be a normal measure on κ . Then for μ -measure one many $\gamma < \kappa$, $\mathscr{M}_{\gamma} \models ``V_{\gamma+1}$ is wellorderable".

Proof. Continue with the notation from the proof of Fact 3.2. We show $\mathscr{M}^{M}_{\kappa} \models "V_{\kappa+1}$ is wellorderable".

CLAIM. $V_{\kappa+1} \cap \mathscr{M}_{j(\kappa)}^M = V_{\kappa+1} \cap \mathscr{M}_{\kappa}^M = V_{\kappa+1} \cap \mathscr{M}_{\kappa}.$

Proof. We have $V_{\kappa+1} \cap \mathscr{M}_{\kappa} \subseteq V_{\kappa+1} \cap \mathscr{M}_{j(\kappa)}^{M}$ since $j \upharpoonright \mathscr{M}_{\kappa} : \mathscr{M}_{\kappa} \to \mathscr{M}_{j(\kappa)}^{M}$ is elementary and $\kappa = \operatorname{cr}(j)$. By Claim 1 of the proof of Fact 3.2, this suffices.

By Fact 3.2, $\mathscr{M}_{\kappa} \models AC$, so $\mathscr{M}_{j(\kappa)}^{M} \models AC$ also. Let $\langle \ast \in \mathscr{M}_{j(\kappa)}^{M}$ be a wellorder of $V_{\kappa+1} \cap \mathscr{M}_{j(\kappa)}^{M}$. Then $\langle \ast \in \mathscr{M}_{\kappa}^{M}$ and $\langle \ast \rangle$ is a wellorder of $V_{\kappa+1} \cap \mathscr{M}_{\kappa}^{M}$.

We next use the simple idea above to prove that certain cardinals are "stable" with respect to the mantle. The first observation is:

3.4 Theorem. Let A be a set such that $A^{\#}$ exists. Let κ be an A-indiscernible. Then $V_{\kappa+1}^{\mathscr{M}_{\kappa}^{L(A)}} = V_{\kappa+1}^{\mathscr{M}^{L(A)}}$ and this set is wellordered in $\mathscr{M}_{\kappa}^{L(A)}$.

Proof. Let $j: L(A) \to L(A)$ be elementary with $\operatorname{cr}(j) = \kappa$. We write \mathscr{M}_{κ} for $\mathscr{M}_{\kappa}^{L(A)}$; likewise $\mathscr{M}_{j(\kappa)}$. Now $j \upharpoonright \mathscr{M}_{\kappa} : \mathscr{M}_{\kappa} \to \mathscr{M}_{j(\kappa)}$ is elementary. Clearly $\mathscr{M}_{j(\kappa)} \subseteq \mathscr{M}_{\kappa}$. But also, $B = V_{\kappa+1}^{\mathscr{M}_{\kappa}} \subseteq V_{\kappa+1}^{\mathscr{M}_{j(\kappa)}}$ as in the previous proof. So $V_{\kappa+1}^{\mathscr{M}_{j(\kappa)}} = B$. But $V_{j(\kappa)}^{\mathscr{M}_{j(\kappa)}} \models \operatorname{ZFC}$, so there is a wellorder of B in $\mathscr{M}_{j(\kappa)} \subseteq \mathscr{M}_{\kappa}$.

It now follows that $V_{\kappa+1}^{\mathscr{M}_{\kappa}} = V_{\kappa+1}^{\mathscr{M}}$, because we can take $j(\kappa)$ as large as we like, hence past any true index.

3.5 Definition. A cardinal κ is Σ_2 -strong iff for every $\alpha \in OR$ there is an elementary embedding $j: V \to M$ with $\alpha < j(\kappa)$ and $V_\alpha \subseteq M$ and $\operatorname{Th}_{\Sigma_2}^M(V_\alpha) = \operatorname{Th}_{\Sigma_2}^V(V_\alpha)$.¹⁰ An embedding $j: V \to M$ is superstrong iff $V_{j(\kappa)} \subseteq M$. A cardinal κ is ∞ -

An embedding $j: V \to M$ is superstrong iff $V_{j(\kappa)} \subseteq M$. A cardinal κ is ∞ -superstrong iff for every $\alpha \in OR$ there is a superstrong embedding j with $cr(j) = \kappa$ and $j(\kappa) > \alpha$.

A superstrong extender is the V_{β} -extender derived from a superstrong embedding $j: V \to M$ where $\beta = j(\kappa)$ and $\kappa = \operatorname{cr}(j)$.

Note that:

3.6 Lemma. If E is a superstrong extender and $W \models \text{ZFC}$ is a transitive proper class with $E \in W$, then $W \models "E$ is a superstrong extender".

3.7 Remark. Say that a cardinal κ is ∞ -1-extendible iff for every $\alpha \in OR$ there is $\beta \in OR$ with $\beta \geq \alpha$ and and an elementary $j: V_{\kappa+1} \to V_{\beta+1}$ (hence $j(\kappa) = \beta$) with $\operatorname{cr}(j) = \kappa$.

3.8 Theorem. We have:

- 1. Every extendible cardinal is ∞ -1-extendible and carries a normal measure concentrating on ∞ -1-extendible cardinals.
- 2. Every ∞-1-extendible cardinal is ∞-superstrong and carries a normal measure concentrating on ∞-superstrong cardinals.

¹⁰That is, for each Σ_2 formula φ and all $\vec{x} \in (V_\alpha)^{<\omega}$, we have $M \models \varphi(\vec{x})$ iff $V \models \varphi(\vec{x})$.

 Every ∞-superstrong cardinal is Σ₂-strong and carries a normal measure concentrating on Σ₂-strong cardinals.

Proof. Part 1: This is routine and left to the reader.

Part 2: Let κ be ∞ -1-extendible. Let $j : V_{\kappa+1} \to V_{\beta+1}$ be elementary with $\operatorname{cr}(j) = \kappa$. Let E be the extender derived from j with support V_{β} . Let $M = \operatorname{Ult}(V, E)$ and $k : V \to M$ be the ultrapower map. Then one can show that k is a superstrong embedding with $k(\kappa) = \beta$ and that $M \models "\kappa$ is ∞ -superstrong". (For the last clause, consider ultrapowers $\operatorname{Ult}(M, E \upharpoonright \alpha)$ where $\alpha < \beta$, and show that unboundedly many of these produce superstrong embeddings in M, and also use that β is 1- ∞ -extendible in M.)

Part 3: Let κ be ∞ -superstrong. We show first that κ is Σ_2 -strong. So let $\alpha \in \text{OR}$. We may assume that $V_{\alpha} \preccurlyeq_2 V$. Let $j : V \to M$ be any superstrong embedding with $\operatorname{cr}(j) = \kappa$ and $\alpha < j(\kappa)$. It suffices to verify: CLAIM 1. $\operatorname{Th}_{\Sigma_2}^M(V_{\alpha}) = \operatorname{Th}_{\Sigma_2}^V(V_{\alpha})$.

Proof. Let φ be Σ_2 and $\vec{x} \in (V_\alpha)^{<\omega}$. If $V \models \varphi(\vec{x})$ then $V_\alpha \models \varphi(\vec{x})$, which implies $M \models \varphi(\vec{x})$. Conversely, suppose $M \models \varphi(\vec{x})$. Because κ is ∞ -superstrong, it is clearly strong, which implies that $V_{\kappa} \preccurlyeq_2 V$. Therefore $V_{j(\kappa)}^M \preccurlyeq_2 M$. Therefore $V_{j(\kappa)}^M \models \varphi(\vec{x})$. But $V_{j(\kappa)}^M = V_{j(\kappa)}$, so $V_{j(\kappa)} \models \varphi(\vec{x})$, so $V \models \varphi(\vec{x})$, as desired.

Now let $j: V \to M$ be a superstrong embedding with $\operatorname{cr}(j) = \kappa$. We will show that $M \models "\kappa$ is Σ_2 -strong", which completes the proof.

CLAIM 2. $M \models "\kappa \text{ is } < \beta - \Sigma_2 \text{-strong"}$, where $\beta = j(\kappa)$. That is, for each $\alpha < \beta$, M has an elementary $k : M \to N$ with $\operatorname{cr}(k) = \kappa$ and $V_{\alpha} \subseteq N$ and $\operatorname{Th}_{\Sigma_2}^N(V_{\alpha}) = \operatorname{Th}_{\Sigma_2}^M(V_{\alpha})$.

Proof. Since $M \models ``\beta$ is strong", $V_{\beta}^{M} \preccurlyeq_{2} M$ and there are club many $\alpha < \beta$ such that $V_{\alpha}^{M} = V_{\alpha} \preccurlyeq_{2} M$. Fix some such α . Let E_{α} be the extender derived from j with support V_{α} . Then $E_{\alpha} \in V_{\beta} \subseteq M$, and $M \models ``E_{\alpha}$ is an extender". Moreover, letting $N_{\alpha} = \text{Ult}(M, E_{\alpha})$, we have $V_{\alpha} \subseteq N_{\alpha}$ and

$$\operatorname{Th}_{\Sigma_2}^{N_{\alpha}}(V_{\alpha}) = \operatorname{Th}_{\Sigma_2}^M(V_{\alpha}).$$

For let $t = \operatorname{Th}_{\Sigma_2}^V(V_{\kappa}) = \operatorname{Th}_{\Sigma_2}^M(V_{\kappa})$. Then letting $k_{\alpha} : M \to N_{\alpha}$ be the ultrapower map,

$$j(t) = \operatorname{Th}_{\Sigma_2}^M(V_\beta)$$
 and $k_\alpha(t) = \operatorname{Th}_{\Sigma_2}^N(V_{k_\alpha(\kappa)}^N)$.

So $\operatorname{Th}_{\Sigma_2}^M(V_\alpha) = j(t) \cap V_\alpha = k_\alpha(t) \cap V_\alpha = \operatorname{Th}_{\Sigma_2}^N(V_\alpha).$

Now since κ is Σ_2 -strong, $M \models ``\beta = j(\kappa)$ is Σ_2 -strong". So let $\alpha \in OR$ be a strong limit cardinal. Then M has an embedding $\ell : M \to N$ with $\operatorname{cr}(\ell) = \beta$ and $V^M_{\alpha} = V^N_{\alpha}$ and $\operatorname{Th}^M_{\Sigma_2}(V^M_{\alpha}) = \operatorname{Th}^N_{\Sigma_2}(V^M_{\alpha})$. By the claim and elementarity, $N \models ``\kappa$ is $< \ell(\beta)$ - Σ_2 -strong". But then extenders in N which witness $< \alpha$ - Σ_2 -strength in N also witness this in M. Since α was arbitrary, we are done.

We now prove an analogue of Usuba's extendibility result down lower: **3.9 Theorem.** Suppose κ is Σ_2 -strong. Then $V_{\kappa+1}^{\mathcal{M}_{\kappa}} = V_{\kappa+1}^{\mathcal{M}}$.

Proof. Suppose not and let r be such that $V_{\kappa+1}^{W_r} \subsetneq V_{\kappa+1}^{\mathscr{M}_{\kappa}}$. Let $\lambda \in OR$ be such that $\beth_{\lambda} = \lambda$ and $r \in V_{\lambda}$. Let $j: V \to M$ witness Σ_2 -strength with respect to λ .

Since the class of true indices is Π_2 , $M \models "r$ is a true index". Also, by the local definability of grounds,

$$W_r^M \cap V_\lambda = W_r^{V_\lambda^M} = W_r^{V_\lambda} = W_r \cap V_\lambda.$$

In particular, $V_{\kappa+1}^{W_r^M} = V_{\kappa+1}^{W_r} \subsetneq V_{\kappa+1}^{\mathscr{M}_\kappa}$. Since $r \in V_\lambda \subseteq V_{j(\kappa)}^M$, therefore $\mathscr{M}_{j(\kappa)}^M \cap V_{\kappa+1} \subsetneq \mathscr{M}_\kappa \cap V_{\kappa+1}$. But since $\operatorname{cr}(j) = \kappa$, as in the proof of Theorem 3.2, we have

$$\mathscr{M}_{\kappa} \cap V_{\kappa+1} \subseteq \mathscr{M}_{j(\kappa)}^{M} \cap V_{\kappa+1},$$

a contradiction.

3.10 Question. Suppose κ is strong. Is $V_{\kappa+1}^{\mathcal{M}} = V_{\kappa+1}^{\mathcal{M}_{\kappa}}$?

We now move toward the positive results in the cases that κ is inaccessible and/or weakly compact. Toward these we first prove a couple of lemmas.

3.11 Lemma (κ -uniform hulls). Let κ be inaccessible. For true indices $r \in V_{\kappa}$, let (\mathbb{P}_r, G_r) witness this, and otherwise let $\mathbb{P}_r = G_r = \emptyset$. Let $\lambda = \beth_{\lambda}$ with $\operatorname{cof}(\lambda) > \kappa$ and $V_{\lambda} \preccurlyeq_2 V$. Let $S \in V_{\lambda}$. Then there is \widetilde{X} such that, letting $\widetilde{X}_r = \widetilde{X} \cap V_{\lambda}^{W_r}$ for $r \in V_{\kappa}$, we have:

- 1. $V_{\kappa} \cup \{S, \kappa\} \subset \widetilde{X} \preccurlyeq V_{\lambda} \text{ and } \widetilde{X}^{<\kappa} \subset \widetilde{X} \text{ and } |\widetilde{X}| = \kappa$,
- 2. $\widetilde{X}_r \in W_r$ and $\widetilde{X}_r \preccurlyeq V_{\lambda}^{W_r} \preccurlyeq_2 W_r$,

and letting X be the transitive collapse of \widetilde{X} and $\sigma: X \to \widetilde{X}$ the uncollapse and X_r, σ_r likewise, then:

- 3. $X_r \subset X$ and in fact, $X_r = W_r^X$,
- 4. $\sigma: X \to V_{\lambda}$ is fully elementary with $\operatorname{cr}(\sigma) > \kappa$,
- 5. $\sigma_r: X_r \to V_{\lambda}^{W_r}$ is fully elementary and $\sigma_r \subseteq \sigma$,
- 6. G_r is (X_r, \mathbb{P}_r) -generic and $X = X_r[G_r]$,
- 7. $\mathcal{M}_{\kappa}^{X} = \mathcal{M}_{\kappa}^{X_{r}} = \bigcap_{s \in V_{\kappa}} X_{s}$; hence $\mathcal{M}_{\kappa}^{X} \in \mathcal{M}_{\kappa}$,
- 8. $X^{<\kappa} \subseteq X$ and $X_r^{<\kappa} \cap W_r \subseteq X_r$ and $(\mathcal{M}_{\kappa}^X)^{<\kappa} \cap \mathcal{M}_{\kappa} \subseteq \mathcal{M}_{\kappa}^X$,
- 9. $\sigma \upharpoonright \mathscr{M}_{\kappa}^{X} = \sigma_{r} \upharpoonright \mathscr{M}_{\kappa}^{\widetilde{X}_{r}}$; hence $\sigma \upharpoonright \mathscr{M}_{\kappa}^{X} \in \mathscr{M}_{\kappa}$,
- 10. $\sigma \upharpoonright \mathscr{M}^X_{\kappa} : \mathscr{M}^X_{\kappa} \to \mathscr{M}^{V_{\lambda}}_{\kappa}$ is fully elementary.
- 11. $V_{\lambda}, \tilde{X}, X, \tilde{X}_r, X_r$ each satisfy T_1 and the following statements:
 - (a) "There are unboundedly many η such that $\eta = \beth_{\eta}$ ",
 - (b) "Fact 2.15".
 - (c) "There is $\xi = \beth_{\xi}$ such that for each $r \in V_{\kappa}$ and $s \in V_{\kappa}^{W_r}$, we have $W_r \models$ "s is a index" iff $V_{\xi}^{W_r} \models$ "s is a true index".

Proof. The fact that $V_{\lambda}^{W_r} \preccurlyeq_2 W_r$ is by Lemma 2.17.

Construct an increasing sequence $\left\langle \widetilde{X}_{\alpha} \right\rangle_{\alpha < \kappa}$ such that $\widetilde{X}_{\alpha} \preccurlyeq V_{\lambda}$ and $V_{\kappa} \cup \{x\} \subseteq \widetilde{X}_{\alpha}$ and $\widetilde{X}_{\alpha}^{<\kappa} \subseteq \widetilde{X}_{\alpha}$ and $|\widetilde{X}_{\alpha}| = \kappa$, and such that for each $r \in V_{\kappa}$ there are cofinally many $\alpha < \kappa$ such that $\widetilde{X}_{\alpha} \cap W_r \in W_r$.

To construct this sequence, suppose we have constructed \widetilde{X}_{α} , and let $r \in V_{\kappa}$. Let $X = \widetilde{X}_{\alpha} \cap W_r$. By elementarity, $X \preccurlyeq V_{\lambda}^{W_r}$ and

$$\widetilde{X}_{\alpha} = X[G_r] = \{ \tau_{G_r} \mid \tau \in \widetilde{X} \}.$$

Since $|X| = \kappa$, there is some $\widetilde{X'} \in W_r$ with $|\widetilde{X'}| = \kappa$ (hence $W_r \models ``|\widetilde{X'}| = \kappa$ "), and $X \subseteq \widetilde{X'}$, so there is also $\widetilde{X''} \in W_r$ with $\widetilde{X''} \preccurlyeq V_{\lambda}^{W_r}$ and $\widetilde{X'} \subseteq \widetilde{X''}$ and $|\widetilde{X''}| = \kappa$ (in V and W_r) and such that $W_r \models ``(\widetilde{X''})^{<\kappa} \subseteq (\widetilde{X''})$ ". It easily follows that

$$\widetilde{X}_{\alpha} \subseteq \widetilde{X}''[G_r] = \{\tau_{G_r} \mid \tau \in \widetilde{X}''\} \preccurlyeq V_{\lambda}$$

and $\widetilde{X}''[G_r] \cap W_r = \widetilde{X}''$. We set $\widetilde{X}_{\alpha+1} = \widetilde{X}''[G_r]$. Then everything is clear except for the requirement that $\widetilde{X}_{\alpha+1}^{<\kappa} \subseteq \widetilde{X}_{\alpha+1}$. So let $f: \gamma \to \widetilde{X}_{\alpha+1}$ where $\gamma < \kappa$ (with $f \in V$); we claim that $f \in \widetilde{X}_{\alpha+1}$. Let $g: \gamma \to \widetilde{X}''$ be such that $g(\alpha)_{G_r} = f(\alpha)$ for each $\alpha < \gamma$. So $g \in V$, but we don't know that $g \in W_r$. But there is a \mathbb{P}_r -name $\dot{g} \in V_{\lambda}^{W_r}$ such that $\dot{g}_{G_r} = g$. And $\widetilde{X}'' \in W_r$, so there is $p_0 \in G_r$ forcing that $\operatorname{rg}(\dot{g}) \subseteq \widetilde{X}''$. Working in W_r then, we may fix for each $\alpha < \gamma$ an antichain $A_\alpha \subseteq \mathbb{P}_r$ maximal below p_0 and for each $p \in A_\alpha$ some $\tau_{\alpha p} \in \widetilde{X}''$ such that p forces that $\dot{g}(\alpha) = \tau_{\alpha p}$. Then the sequence $\langle \tau_{\alpha p} \rangle_{(\alpha, p) \in I}$, where

$$I = \{ (\alpha, p) \mid \alpha < \gamma \text{ and } p \in A_{\alpha} \}$$

is $\subseteq \widetilde{X}''$, and hence in \widetilde{X}'' . Since $W_r \models "(\widetilde{X}'')^{<\kappa} \subseteq (\widetilde{X}'')$ ", this gives a name $\dot{g}'' \in \widetilde{X}''$ such that p_0 forces $\dot{g}'' = \dot{g}$, and therefore

$$g = \dot{g}_{G_r} = \dot{g}_{G_r}'' \in \widetilde{X}''[G_r] = \widetilde{X}_{\alpha+1}.$$

But since $G_r \in \widetilde{X}_{\alpha+1}$, therefore $f \in \widetilde{X}_{\alpha+1}$, so $\widetilde{X}_{\alpha+1}^{<\kappa} \subseteq \widetilde{X}_{\alpha+1}$ as desired. With some simple bookkeeping then, we get an appropriate sequence.

Let now $X = \bigcup_{\alpha < \kappa} X_{\alpha}$. We claim that X is as desired. The only thing we need to verify is that for each $r \in V_{\kappa}$, we have

$$\widetilde{X}_r = \widetilde{X} \cap W_r \in W_r.$$

Fix r. There is a \mathbb{P}_r -name $\tau \in W_r$ such that $\tau_{G_r} = \left\langle \widetilde{X}_{\alpha} \right\rangle_{\alpha < \kappa}$, and for cofinally many $\alpha < \kappa$ there is $p_{\alpha} \in G_r$ and $\widetilde{X}_{\alpha}^r \in W_r$ such that

$$p_{\alpha} - \tau_{\alpha} \cap W_r = \tilde{X}_{\alpha}^r$$

(hence $\widetilde{X}_{\alpha}^{r} = \widetilde{X}_{\alpha} \cap W_{r}$). Since $\mathbb{P}_{r} \in V_{\kappa}$, there is therefore a fixed $p \in \mathbb{P}_{r}$ such that $p_{\alpha} = p$ for cofinally many α . So $\widetilde{X}_{r} = \bigcup_{\alpha \in I} \widetilde{X}_{\alpha}^{r}$ where

$$I = \{ \alpha < \kappa \mid \exists x \ [p \vdash \tau_{\alpha} = \check{x}] \},\$$

so $\widetilde{X}_r \in W_r$.

This completes the construction. The verification of the remaining properties is now straightforward. We omit discussing them, other than two remarks. In part 8, the third statement follows directly from the first two together with part 7; the first two follow readily from the construction. And in part 11, note that ξ exists because $\operatorname{cof}(\lambda) > \kappa = |V_{\kappa}|$.

3.12 Fact. Let κ be weakly compact. Then X be transitive with $\kappa \in X$ and $X^{<\kappa} \subseteq X$ and $|X| = \kappa$.¹¹ Then there is a non-principal X- κ -complete X-normal¹² ultrafilter μ over κ such that letting $Y = \text{Ult}(X, \mu)$ and i_{μ}^{X} the ultrapower embedding, then Y is wellfounded. Moreover, i_{μ}^{X} is Σ_1 -elementary and cofinal and $\text{cr}(i_{\mu}^{X}) = \kappa$.

Proof. Let $\pi : X \to Z$ be any elementary embedding with Z transitive and $cr(\pi) = \kappa$. Let μ be the normal measure derived from π . Note that μ works.

We now extend the situation above, adding the assumption that κ is weakly compact.

3.13 Lemma (κ -uniform weak compactness embedding). Adopt the assumptions and notation from the statement and proof of Lemma 3.11. Assume further that κ is weakly compact. Let $\pi : X \to Y$ witness the weak compactness of κ in V, with $Y = \text{Ult}(X, \mu)$ for an X- κ -complete X-normal ultrafilter μ over κ , and $\pi = i_{\mu}^{X}$. For $r \in V_{\kappa}$, let $\mu_r = \mu \cap X_r$. Then:

1. $\mu_r \in W_r$ and μ_r is an X_r - κ -complete ultrafilter over κ ; let

$$Y_r = \text{Ult}(X_r, \mu_r) \text{ and } \pi_r : X_r \to Y_r$$

the ultrapower map; so $Y_r, \pi_r \in W_r$,

- 2. μ is the X-ultrafilter generated by μ_r (μ_r is dense in μ).
- 3. For each $f: \kappa \to X_r$ with $f \in X$, there is $f_r \in X_r$ with $f_r: \kappa \to X_r$ and $f_r(\alpha) = f(\alpha)$ for μ -measure one many $\alpha < \kappa$.¹³
- 4. The ultrapowers satisfy Los' theorem for Σ_1 formulas, and π_r , π are Σ_2 -elementary.
- 5. $Y, Y_r \models T_1$ and Y_r is transitive, $Y_r = W_r^Y$, and $Y = Y_r[G_r]$.
- 6. $\pi_r \subseteq \pi$.
- 7. $\mathscr{M}_{\pi(\kappa)}^{Y} = \mathscr{M}_{\pi_{\kappa}(\kappa)}^{Y_{r}} \in W_{r}$; hence this belongs to \mathscr{M}_{κ} .
- 8. $\pi \upharpoonright \mathscr{M}_{\kappa}^{X} : \mathscr{M}_{\kappa}^{X} \to \mathscr{M}_{\pi(\kappa)}^{Y}$ is cofinal Σ_{1} -elementary; this map belongs to \mathscr{M}_{κ} .

9. $\mathscr{M}_{\kappa}^{Y} = \bigcap_{s \in V_{\kappa}} W_{s}^{Y} = \mathscr{M}_{\kappa}^{Y_{r}} \in W_{r}$; hence this belongs to \mathscr{M}_{κ} .

- 10. Y, Y_r each satisfy T_1 and the following statements:
 - (a) "There are unboundedly many η such that $\eta = \beth_{\eta}$ ",
 - (b) "Fact 2.15 holds at $\theta = \pi(\kappa) = \beth_{\pi(\kappa)}$ ",

¹¹In an earlier draft, the hypothesis " $|X| = \kappa$ " was accidentally omitted, which obviously makes the statement equivalent to measurability.

 $^{^{12}\}text{That}$ is, $\kappa\text{-completeness}$ and normality with respect to sequences in X.

¹³A draft assumed only $f: \kappa \to X$, not $f: \kappa \to X_r$, which obviously makes the statement false when $X \neq X_r$.

(c) "There is $\xi = \beth_{\xi}$ such that for each $r \in V_{\pi(\kappa)}$ and $s \in V_{\pi(\kappa)}^{W_r}$, we have $W_r \models$ "s is true" iff $V_{\varepsilon}^{W_r} \models$ "s is true".

Therefore there is $t \in V_{\pi(\kappa)}^Y$ with $W_t^Y \subseteq \mathscr{M}_{\kappa}^Y$.

Proof. Parts 1-3: These are simple variants of the version for measurable cardinals κ of V[G] via small forcing $\mathbb{P} \in V_{\kappa}$; one uses especially, however, the fact that X_r is $< \kappa$ -closed in W_r . We leave the details to the reader.

Part 4: Note that V_{λ} satisfies Σ_1 -Collection and "For all $\alpha \in OR$, V_{α} exists and $\beth_{\alpha} \in OR$ exists, and $OR = \beth_{OR}$, so X_r, X do also. Therefore if φ is Σ_0 and $x \in X$ and

$$X \models \forall \alpha < \kappa \; \exists y \; \varphi(x, y, \alpha)$$

then some $V_{\xi}^X \in X$ satisfies the same statement, and hence there is $f \in X$ picking witnesses y. This gives Los' theorem for Σ_1 formulas. The Σ_2 -elementarity of π : $X \to Y$ follows. Likewise for X_r, π_r .

Parts 5, 6: The fact that $Y, Y_r \models T_1$ follows from Σ_2 -elementarity and cofinality of π, π_r , and (for Σ_1 -Collection) that for each $\xi \in OR^X$, we have $\mathcal{H}^X_{\xi} \preccurlyeq_1 X$ and $\mathcal{H}_{\ell}^{X_r} \preccurlyeq_1 X_r$. The rest follows as usual from the fact that functions in X with codomain X_r are represented in X_r (part 3), and again the Σ_2 -elementarity of π, π_r .

Parts 7, 8: Basically by invariance of \mathcal{M}_{κ} (Fact 2.16), we have $\mathcal{M}_{\kappa}^{X} = \mathcal{M}_{\kappa}^{X_{r}}$, and by part 11 of 3.11, there is $\xi < \operatorname{OR}^{X}$ such that for each $r \in V_{\kappa}$ and $s \in V_{\kappa}^{W_{r}}$, we have $X_{r} \models "s$ is true" iff $V_{\xi}^{X_{r}} \models "s$ is true". Let

$$T_r = \{ s \in V_{\kappa}^{W_r} \mid W_r \models "s \text{ is true"} \}.$$

So $T_r \in X_r$ and has the same definition there; likewise for $T_r \in Y_r$, since π_r is Σ_2 -elementary. And because of the existence of ξ ,

$$\pi(T_r) = \{ s \in V_{\pi_r(\kappa)}^{Y_r} \mid Y_r \models "s \text{ is true"} \},\$$

and it follows (in the case of $r = \emptyset$, but similarly in general),

$$\mathscr{M}^{Y}_{\pi(\kappa)} = \left(\bigcap_{s \in V^{Y}_{\pi(\kappa)}} W^{Y}_{s}\right) = \left(\bigcup_{\zeta \in I} \pi(\mathscr{M}^{V_{\zeta}}_{\kappa})\right)$$

where I is the set of all $\zeta \in [\xi, \mathrm{OR}^X)$ such that $\beth_{\zeta}^X = \zeta$. But $\mathscr{M}_{\kappa}^X = \mathscr{M}_{\kappa}^{X_r}$ and $\pi_r \subseteq \pi$, so $\mathscr{M}^Y_{\pi(\kappa)} = \mathscr{M}^{Y_r}_{\pi_r(\kappa)}$. The calculations above also show that

$$\pi \upharpoonright \mathscr{M}^X_{\kappa} : \mathscr{M}^X_{\kappa} \to \mathscr{M}^Y_{\pi(\kappa)}$$

is cofinal Σ_1 -elementary, and likewise for $\pi_r \subseteq \pi$.

Part 9: By part 5, $W_s^Y = Y_s$, so $\mathcal{M}_{\kappa}^Y = \bigcap_{s \in V_{\kappa}} W_s^Y$. And note that the density of the grounds of X_r in the grounds of X is lifted to that for those of Y_r in those of Y. (That is, for example, if r, s are such that $X_r \subseteq X_s$, then $Y_r \subseteq Y_s$, as this is preserved by π .) So $\mathscr{M}_{\kappa}^{Y_r} = \mathscr{M}_{\kappa}^Y$, as desired. Part 10a: For each $\zeta \in X$ with $\zeta = \beth_{\zeta}^X$, we have $\pi(\zeta) = \beth_{\pi(\zeta)}^Y$. Part 10c: If ξ witnesses the corresponding statement in X, note that $\pi(\xi)$ works

in Y.

Part 10b: We consider literally Y, but the same proof works for Y_r . Note that there is a function $f: V_{\kappa} \to V_{\kappa}$ with $f \in X$, such that for each $R \in V_{\kappa}, X \models "t = f(R)$

is a true index and t witnesses Fact 2.15 for R" (f exists by the elementarity of σ). We claim that $\pi(f)$ has the same property for Y. For by Π_2 -elementarity, $Y \models$ "Every $t \in \operatorname{rg}(\pi(f))$ is a true index". Moreover, let ξ be as before. Then for each ζ such that $\xi < \zeta < \operatorname{OR}^X$ and $\zeta = \beth_{\zeta}^X$, V_{ζ}^X satisfies " $W_{f(R)} \subseteq W_r$ for each $R \in V_{\kappa}$ and $r \in R$ ". This lifts to Y under π , and since π is cofinal, this suffices.

We are now ready to prove the main theorem for weakly compact κ . The first proof that, under this assumption, $\mathscr{M}_{\kappa} \models "V_{\kappa}$ is wellordered" is due to Lietz:

3.14 Theorem. Let κ be weakly compact. Then:

- 1. $\mathcal{M}_{\kappa} \models \kappa$ -DC + " κ is weakly compact".¹⁴
- 2. for each $A \in \mathscr{M}_{\kappa} \cap \mathcal{H}_{\kappa^+}$, $\mathscr{M}_{\kappa} \models "A$ is wellordered". ¹⁵
- 3. if $\mathcal{P}(\kappa)^{\mathscr{M}_{\kappa}}$ has cardinality κ then (i) κ is measurable in \mathscr{M}_{κ} , and (ii) $x^{\#}$ exists for every $x \in \mathcal{P}(\kappa)^{\mathscr{M}_{\kappa}}$.
- 4. If $\mathcal{M}_{\kappa} \models ``\mu$ is a countably complete ultrafilter over $\gamma \leq \kappa$ '', then the ultrapower $\mathrm{Ult}(\mathscr{M}_{\kappa},\mu)$ is wellfounded and the ultrapower embedding

$$i_{\mu}^{\mathscr{M}_{\kappa}}:\mathscr{M}_{\kappa}\to \mathrm{Ult}(\mathscr{M}_{\kappa},\mu)$$

is fully elementary.

Proof. Part 4 follows directly from part 1, as the wellfoundedness of $Ult(\mathcal{M}_{\kappa},\mu)$ requires only ω -DC, and the proof of Los' theorem here only uses κ -choice. The conclusion that $x^{\#}$ exists in part 3 follows easily from the rest, using the elementarity of i_{μ} and that $\text{Ult}(\mathscr{M}_{\kappa},\mu)$ is wellfounded. To see that $\mathscr{M}_{\kappa}\models$ " κ is weakly compact", let $T \subseteq {}^{<\kappa}2$ be a tree in \mathscr{M}_{κ} . Then T has a cofinal branch b in V, by weak compactness in V. But $b \cap V_{\alpha} \in \mathscr{M}_{\kappa}$ for each $\alpha < \kappa$. Therefore by 2.20, $b \in \mathscr{M}_{\kappa}$.

Here is Lietz' argument that $\mathscr{M}_{\kappa} \models "V_{\kappa}$ is wellordered":¹⁶ Working in \mathscr{M}_{κ} , let T be the tree of all attempts to build a well order of V_{κ} . (For example, let $T \subseteq {}^{<\kappa}V_{\kappa}$ be the set of all functions $f : \alpha \to V_{\kappa}$ where $\alpha < \kappa$, such that for each $\beta < \alpha$, $f(\beta)$ is a wellorder of V_{β} , and for all $\beta_1 < \beta_2 < \alpha$, $f(\beta_2)$ is an end extension of $f(\beta_1)$.) Since $V_{\kappa}^{\mathscr{M}_{\kappa}} \models \operatorname{ZFC}$, T is unbounded in V_{κ} , and clearly $T \upharpoonright \alpha \in V_{\kappa}$ for each $\alpha < \kappa$. Therefore by weakly compactness in \mathcal{M}_{κ} , \mathcal{M}_{κ} has a T-cofinal branch, and clearly this gives a wellorder of $V_{\kappa} \cap \mathscr{M}_{\kappa}$.

We proceed now to the proof that $\mathscr{M}_{\kappa} \models \kappa$ -DC, and that every set $A \in \mathscr{M}_{\kappa} \cap \mathcal{H}_{\kappa^+}$ is wellordered in \mathscr{M}_{κ} . Let $\mathscr{T} \in \mathscr{M}$ be a κ -DC-tree,¹⁷ and let $A \in \mathscr{M}_{\kappa} \cap \mathcal{H}_{\kappa^+}$. Let $S = (\mathcal{T}, A) \in V_{\lambda}$ and X be a κ -uniform hull, etc, with $S \in X$ and everything as in Lemma 3.11, and let $\pi: X \to Y$, etc, be as in Lemma 3.13. So $\sigma: X \to V_{\lambda}$ is fully

elementary with $\kappa < \operatorname{cr}(\sigma)$. Let $\sigma(\bar{\mathscr{T}}) = \mathscr{T}$ and $\sigma(A) = A$. By 3.13, $\pi' = \pi \upharpoonright \mathscr{M}_{\kappa}^X : \mathscr{M}_{\kappa}^X \to \mathscr{M}_{\pi(\kappa)}^Y$ is cofinal Σ_1 -elementary, and these models and map belong to \mathscr{M}_{κ} . We have $A, \overline{\mathscr{T}} \in \mathscr{M}_{\kappa}^X$.

¹⁴So also $\mathscr{M}_{\kappa} \models ``\kappa^+$ is regular and $\mathcal{H}_{\kappa^+} \models \operatorname{ZFC}^-$ ". ¹⁵Note that the " κ^+ " and " \mathcal{H}_{κ^+} " here are computed in V, not \mathscr{M}_{κ} .

¹⁶The author first mistakenly thought that a similar argument worked with κ only inaccessible, but Lietz noted that one seems to need weak compactness for this.

¹⁷That is, a set \mathscr{F} of functions f such that dom $(f) < \kappa$, with \mathscr{F} closed under initial segment, and no maximal elements; that is, for every $f \in \mathscr{F}$ there is $g \in \mathscr{F}$ with dom(f) < dom(g) and $f = g \restriction \text{dom}(f)$. Note that κ -DC is just the assertion that for every κ -DC tree \mathscr{T} , there is a \mathscr{T} -maximal branch; that is, a function $f \notin \mathscr{T}$ such that $f \upharpoonright \alpha \in \mathscr{T}$ for all $\alpha < \operatorname{dom}(f)$.

We first find a wellorder of A in \mathscr{M}_{κ} , by arguing as in Schindler's proof of Fact 3.2, but using the weak compactness embedding. We have $\pi'(A) \in \mathscr{M}_{\pi(\kappa)}^{Y}$. By 3.13, there is a ground W of $\mathscr{M}^{Y}_{\pi(\kappa)}$ such that

$$\mathscr{M}^{Y}_{\pi(\kappa)} \subseteq W \subseteq \mathscr{M}^{Y}_{\kappa} \in \mathscr{M}_{\kappa}.$$

So $W \models AC$ and $\pi'(A) \in W$. Let $<^* \in W$ be a wellorder of $\pi'(A)$. So $<^* \in \mathscr{M}_{\kappa}$. Working in \mathcal{M}_{κ} , we can therefore wellorder A by setting, for $x, y \in A$:

$$x <_A y \iff \pi'(x) <^* \pi'(y).$$

We now find a branch through $\bar{\mathscr{T}}$ in \mathscr{M}_{κ} , with length κ . Let $B \in \mathscr{M}_{\kappa}^{X}$ be the field of $\bar{\mathscr{T}}$. As above, there is a wellorder $<^{*}$ of B in \mathscr{M}_{κ} . Working in \mathscr{M}_{κ} , we recursively construct a sequence $\langle x_{\alpha} \rangle_{\alpha < \kappa}$ constituting a branch through $\bar{\mathscr{T}}$, using $<^{*}$ to pick next elements, and noting that at limit stages $\eta < \kappa$, we get $\langle x_{\alpha} \rangle_{\alpha < \eta} \in \mathscr{M}_{\kappa}^{X}$, because by 3.13 part 8 we have $(\mathscr{M}_{\kappa}^{X})^{<\kappa} \cap \mathscr{M}_{\kappa} \subseteq \mathscr{M}_{\kappa}^{X}$. By 3.11, $\sigma' = \sigma \upharpoonright \mathscr{M}_{\kappa}^{X} \in \mathscr{M}_{\kappa}$, and note that $\langle \sigma'(x_{\alpha}) \rangle_{\alpha < \kappa}$ is a cofinal branch through \mathscr{T} , as desired.

Part 3: Now suppose $\mathcal{P}(\kappa) \cap \mathscr{M}_{\kappa} \in \mathcal{H}_{\kappa^+}$. Then we may assume that $A = \mathcal{P}(\kappa) \cap \mathscr{M}_{\kappa}$ above. Therefore $\pi' : \mathscr{M}_{\kappa}^X \to \mathscr{M}_{\pi(\kappa)}^Y$ is \mathscr{M}_{κ} -total. Therefore κ is measurable in \mathcal{M}_{κ} . Since $\mathcal{M}_{\kappa} \models \kappa$ -DC, the rest now follows, as discussed in the first paragraph of the proof.

Recall (α, X) -Choice from Definition 1.1:

3.15 Theorem. Let κ be inaccessible (so $\mathcal{M}_{\kappa} \models "\kappa$ is inaccessible"). Then:

- 1. \mathcal{M}_{κ} is κ -amenably-closed.
- 2. $\mathcal{M}_{\kappa} \models "(\kappa, \mathcal{H}_{\kappa})$ -Choice" iff $\mathcal{M}_{\kappa} \models "V_{\kappa}$ is wellordered".
- 3. $\mathscr{M} \models (< \kappa, \mathcal{H}_{\kappa^+})$ -Choice holds, and hence, $(\mathcal{H}_{\kappa^+})^{<\kappa} \subseteq \mathcal{H}_{\kappa^+}$ ".

3.16 Remark. Note that in part 3, the " κ^+ " and " \mathcal{H}_{κ^+} " are both in the sense of \mathscr{M}_{κ} . Note that also, as κ is inaccessible, $V_{\kappa}^{\mathscr{M}_{\kappa}} \models \text{ZFC}$, $\mathscr{M}_{\kappa} \models "\kappa$ is inaccessible", and \mathcal{M}_{κ} is κ -amenable closed, by Lemma 2.22.

Proof. Part 1 was Lemma 2.22, and since $V_{\kappa}^{\mathscr{M}_{\kappa}} = \mathcal{H}_{\kappa}^{\mathscr{M}_{\kappa}} \models$ ZFC, part 2 is easy. Part 3: Let $\gamma < \kappa$ and $f \in \mathscr{M}_{\kappa}$ with $f : \gamma \to (\mathcal{H}_{\kappa^+})^{\mathscr{M}_{\kappa}}$. We find a choice function for f in \mathcal{M}_{κ} . Write $f_{\alpha} = f(\alpha)$. Fix a function $g: \gamma \to \mathcal{M}_{\kappa}$ with

$$g_{\alpha} = g(\alpha) : \kappa \to \operatorname{trancl}(f_{\alpha})$$

surjective for each $\alpha < \gamma$. Let $c : \gamma \to \mathscr{M}_{\kappa}$ be $c_{\alpha} = c(\alpha) \subseteq \kappa$ the induced code for g_{α} (so $c_{\alpha}, g_{\alpha} \in \mathscr{M}_{\kappa}$, but note we don't know that $c, g \in \mathscr{M}_{\kappa}$). Fix λ and a κ -uniform hull $X \preccurlyeq V_{\lambda}$ with $f, c, g \in X$ and everything else as in 3.11. So $\sigma(f, c, g) = (f, c, g)$. Fix a club C of $\bar{\kappa} < \kappa$ such that $\gamma < \bar{\kappa}$ and $V_{\bar{\kappa}} \preccurlyeq V_{\kappa}$ and such that we get a corresponding system of structures $X_r^{\bar{\kappa}}$ and elementary embeddings $\pi_r^{\bar{\kappa}} : X_r^{\bar{\kappa}} \to X_r$, for $r \in V_{\bar{\kappa}}$, with $X_r^{\bar{\kappa}}, \pi_r^{\bar{\kappa}} \in W_r, X_r^{\bar{\kappa}}$ of cardinality $\bar{\kappa}$ in W_r , $\operatorname{cr}(\pi_r^{\bar{\kappa}}) = \bar{\kappa}$ and $\pi_r^{\bar{\kappa}}(\bar{\kappa}) = \kappa$, and each $X_r^{\bar{\kappa}}[G_r] = X_{\theta\bar{\kappa}}$ and $\pi_r^{\bar{\kappa}} \subseteq \pi_{\theta\bar{\kappa}}$, and with $f, c_\alpha, g_\alpha \in \operatorname{rg}(\pi_r^{\bar{\kappa}})$ for each $\alpha < \gamma$. Write $\pi_r^{\bar{\kappa}}(f^{\bar{\kappa}}, c_\alpha^{\bar{\kappa}}, g_\alpha^{\bar{\kappa}}) = (f, c_\alpha, g_\alpha)$. So $c_\alpha^{\bar{\kappa}} = c_\alpha \cap \bar{\kappa}$, so $c_\alpha^{\bar{\kappa}}, g_\alpha^{\bar{\kappa}} \in (\mathcal{H}_{\bar{\kappa}^+})^{\mathscr{M}_{\kappa}}$, and $f^{\bar{\kappa}} : \gamma \to (\mathcal{H}_{\bar{\kappa}^+})^{\mathscr{M}_{\kappa}}$ with $f_\alpha^{\bar{\kappa}} \subseteq \operatorname{rg}(g_\alpha^{\bar{\kappa}})$. Let $c^{\bar{\kappa}} : \gamma \to \mathscr{M}_{\kappa}$ be $c^{\bar{\kappa}}(\alpha) = c_\alpha^{\bar{\kappa}}$ and likewise for $g^{\bar{\kappa}}$.

In V, pick a sequence $\langle <_{\bar{\kappa}} \rangle_{\bar{\kappa} \in C}$ of wellorders $<_{\bar{\kappa}}$ of $(\mathcal{H}_{\bar{\kappa}^+})^{\mathscr{M}_{\kappa}}$ with $<_{\bar{\kappa}} \in \mathscr{M}_{\kappa}$. Let $z_{\alpha}^{\bar{\kappa}}$ be the $<_{\bar{\kappa}}$ -least element of $f_{\alpha}^{\bar{\kappa}}$, and let $\xi_{\alpha}^{\bar{\kappa}} < \bar{\kappa}$ be the least ξ with $g_{\alpha}^{\bar{\kappa}}(\xi) = z_{\alpha}^{\bar{\kappa}}$.

Let S be the stationary set of all strong limit cardinals $\bar{\kappa} \in C$ of cofinality γ^+ . Enumerate $\gamma \kappa$ as $\{s_\beta\}_{\beta < \kappa}$, with $\gamma \bar{\kappa} = \{s_\beta\}_{\beta < \bar{\kappa}}$ for each $\bar{\kappa} \in S$. For $\bar{\kappa} \in S$, let $\beta_{\bar{\kappa}}$ be the $\beta < \bar{\kappa}$ such that $s_\beta = \langle \xi_{\alpha}^{\bar{\kappa}} \rangle_{\alpha < \gamma}$. Let $S' \subseteq S$ be stationary and such that $\beta_{\bar{\kappa}}$ is constant for $\bar{\kappa} \in S'$.

Let $d: \gamma \to \mathscr{M}_{\kappa}$ be the choice function for f given by $d(\alpha) = \pi_{\emptyset \bar{\kappa}}(z_{\alpha}^{\bar{\kappa}})$, whenever $\bar{\kappa} \in S'$. This is independent of $\bar{\kappa} \in S'$. For if $\bar{\kappa}_0, \bar{\kappa}_1 \in S'$ with $\bar{\kappa}_0 < \bar{\kappa}_1$, then for each $\alpha < \gamma$, we have $\xi = \xi_{\alpha}^{\bar{\kappa}_0} = \xi_{\alpha}^{\bar{\kappa}_1}$, so

$$\pi_{\emptyset\bar{\kappa}_0}(z_{\alpha}^{\bar{\kappa}_0}) = \pi_{\emptyset\bar{\kappa}_0}(g_{\alpha}^{\bar{\kappa}_0}(\xi)) = g_{\alpha}(\xi) = \pi_{\emptyset\bar{\kappa}_1}(g_{\alpha}^{\bar{\kappa}_1}(\xi)) = \pi_{\emptyset\bar{\kappa}_1}(z_{\alpha}^{\bar{\kappa}_1}).$$

But $d \in \mathscr{M}_{\kappa}$. For given $r \in V_{\kappa}$, let $\bar{\kappa} \in S'$ with $r \in V_{\bar{\kappa}}$. Then $f^{\bar{\kappa}} \in X_r^{\bar{\kappa}}$ and $\pi_r^{\bar{\kappa}}(f^{\bar{\kappa}}) = f$, since $\pi_r^{\bar{\kappa}} \subseteq \pi_{\emptyset\bar{\kappa}}$. And $X_r^{\bar{\kappa}} \in W_r$, so $f^{\bar{\kappa}} \in W_r$. But $\langle_{\bar{\kappa}}$ is in W_r , and so $d^{\bar{\kappa}} = \langle z_{\alpha}^{\bar{\kappa}} \rangle_{\alpha < \gamma} \in W_r$. And since $\pi_r^{\bar{\kappa}} \subseteq \pi_{\emptyset\bar{\kappa}}, \pi_r^{\bar{\kappa}}(d^{\bar{\kappa}}) = d$. Since $\pi_r^{\bar{\kappa}} \in W_r$, therefore $d \in W_r$. So $d \in \mathscr{M}_{\kappa} \models "d$ is a choice function for f", so we are done.

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