# Choice principles in local mantles 

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#### Abstract

Assume ZFC. Let $\kappa$ be a cardinal. A $<\kappa$-ground is a transitive proper class $W$ modelling ZFC such that $V$ is a generic extension of $W$ via a forcing $\mathbb{P} \in W$ of cardinality $<\kappa$. The $\kappa$-mantle $\mathscr{M}_{\kappa}$ is the intersection of all $<\kappa$-grounds.

We prove that certain partial choice principles in $\mathscr{M}_{\kappa}$ are the consequence of $\kappa$ being inaccessible/weakly compact, and some other related facts.


## 1 Introduction

Let us recall some standard notions from set-theoretic geology. We generally assume ZFC, though at times (in particular in 82) we will also consider a weaker theory $T_{1}$ (which includes AC).

Given a transitive model $W^{11}$ of ZFC and a forcing $\mathbb{P} \in W$, a $(W, \mathbb{P})$-generic is a filter $G \subseteq \mathbb{P}$ which is generic with respect to $W$. For a cardinal $\kappa$, a $<\kappa$-ground of $V$ is a transitive proper class $W \models$ ZFC such that there is $\mathbb{P} \in W$ with $\mathbb{P}$ of cardinality $<\kappa$ (with cardinality as computed in $W$, or equivalently, in $V$ ) and a $(W, \mathbb{P})$-generic filter $G$ such that $V=W[G]$. A ground is a $<\kappa$-ground for some cardinal $\kappa]^{2}$ The mantle $\mathscr{M}$ is the intersection of all grounds. The $\kappa$-mantle $\mathscr{M}_{\kappa}$ is the intersection of all $<\kappa$-grounds.

By [4, as refined in [1], there is a formula $\varphi(x, y)$ in two free variables such that (i) for all $r, W_{r}=\{x \mid \varphi(r, x)\}$ is a ground (possibly $W_{r}=V$ ), and (ii) for every ground $W$ there is $r$ such that $W=W_{r}$. Therefore we can discuss grounds uniformly, and $\mathscr{M}$ and $\mathscr{M}_{\kappa}$ are transitive classes which are definable ( $\mathscr{M}_{\kappa}$ from parameter $\kappa$ ).

In $\S 2$ we will give the proof of ground definability, but from somewhat less than ZFC: we show that it holds under a certain theory $T_{1}$ (see 2.2), which is true in $\mathcal{H}_{\kappa}$ whenever $\kappa$ is a strong limit cardinal (assuming ZFC). The proof is essentially the usual ZFC proof, however.

From now on, we take $W_{r}$ to be defined as in §2, by which $r=\left(\mathcal{H}_{\gamma^{+}}\right)^{W}$ for some $\gamma \geq \omega$ for which there is $\mathbb{P} \in r$ and a $(W, \mathbb{P})$-generic $G$ with $W[G]=V$.

Let $\theta$ be a strong limit cardinal. By Usuba [10], the grounds are set-directed. By [10] and [8, this is moreover reasonably local, and in particular if $X \in \mathcal{H}_{\theta}$, then there

[^0]is $s \in \mathcal{H}_{\theta}$ with $W_{s} \subseteq \bigcap_{r \in X} W_{r}$. (For following Usuba's proof of [10, Proposition 5.1], note that we can take the regular cardinal $\kappa$ of that proof with $\kappa<\theta$, and then the model $W$ constructed there satisfies the $\kappa^{++}$-uniform covering property for $V$. Usuba then uses Bukovsky's theorem, [10, Fact 3.9], to deduce that $W$ is a ground of $V$. But by [8, Theorem 3.11], the forcing for this can be taken of size $2^{\kappa^{++}}$in $W[g]=V$.)

Also by [10, $\mathscr{M} \models$ ZFC, and by [11, §2], $\mathscr{M}_{\theta} \models \mathrm{ZF}$ (so note $\mathscr{M}_{\theta} \models$ " $\theta$ is a strong limit cardinal", in the ZF sense that $\mathscr{M}_{\theta}$ has no surjection $\pi: V_{\alpha} \rightarrow \theta$ with $\alpha<\theta$ ). If $V_{\theta} \preccurlyeq_{n} V$ with $n$ large enough, then $V_{\theta}^{\mathscr{M}_{\theta}}=V_{\theta}^{\mathscr{M}}$, and hence $V_{\theta}^{\mathscr{M}_{\theta}} \models \mathrm{AC} 3^{3}$ Usuba showed in 11 that if $\kappa$ is an extendible cardinal then $\mathscr{M}_{\kappa}=\mathscr{M}$, so in this case, $\mathscr{M}_{\kappa} \models$ ZFC. Hence Usuba asked in [11] about whether $\mathscr{M}_{\kappa} \models$ ZFC in general. We consider related questions in this paper. Let us first sketch some further history.

Suppose now $\kappa$ is inaccessible. Then $V_{\kappa}^{\mathscr{M}_{\kappa}} \models$ ZFC. For note that by inaccessibility and the remarks above, for each $\alpha<\kappa$ there is some $r \in V_{\kappa}$ such that $V_{\alpha}^{W_{r}}=V_{\alpha}^{\mathcal{M}_{\kappa}}$. Since each $W_{r} \models$ ZFC, it follows that $V_{\kappa}^{\mathscr{M}_{\kappa}} \models$ ZFC. Clearly $\mathscr{M}_{\kappa} \models$ " $\kappa$ is inaccessible", and if $\kappa$ is Mahlo then $\mathscr{M}_{\kappa} \models$ " $\kappa$ is Mahlo".

However, A. Lietz (5) answered Usuba's question above negatively (assuming large cardinals), showing that in fact it is consistent relative to a Mahlo cardinal that $\kappa$ is Mahlo but $\mathscr{M}_{\kappa} \models " \kappa$-AC fails". In fact, Lietz constructs a forcing extension $L[G]$ of $L$ in which $\kappa$ is Mahlo and $\mathscr{M}_{\kappa}^{L[G]}$ satisfies "there is a function $f: \kappa \rightarrow \mathcal{H}_{\kappa^{+}}$for which there is no choice function". He also proved other related things.

In the last few years, the theory of Varsovian models has also been developed by Fuchs, Schindler, Sargsyan and more recently the author. Here, among other things, full mantles $\mathscr{M}$ of certain fully iterable mice have been analyzed, and shown to be strategy mice, hence satisfying ZFC. Analysis of natural $\kappa$-mantles of those mice was, however, missing. But using Varsovian model techniques, the author then analyzed the $\kappa_{0}$-mantle of the mouse $M_{\text {swsw }}$ (Definition 3.1), showing that it is a strategy mouse, modelling ZFC + GCH. A very brief outline is given in $\$ 3$ (but the other results in the note do not rely on this, and no inner model theory appears elsewhere in the paper). The argument has elements in common with Usuba's extendibility proof.

Schindler then showed that if $\kappa$ is measurable then $\mathscr{M}_{\kappa} \models \mathrm{AC}$, hence ZFC; see 8. In this note we adapt this argument, deducing that fragments of choice hold in $\mathscr{M}_{\kappa}$ from the weak compactness and inaccessibility of $\kappa$ respectively.
1.1 Definition. Given an ordinal $\alpha$ and set $X$, let $(\alpha, X)$-Choice be the assertion that for every function $f: \alpha \rightarrow X$, there is a choice function for $f$. And $(<\alpha, X)$ Choice is the assertion that ( $\beta, X$ )-Choice holds for all $\beta<\alpha$.

Part 4 of the following theorem applies to the kind of functions involved in the failure of $\kappa$-AC in Lietz' example, but now with domain $<\kappa$. Note that we assume ZFC except where otherwise stated; $\kappa$-amenable-closure is defined in 2.18 ,
Theorem (3.15). If $\kappa$ be inaccessible then:

1. $\mathscr{M}_{\kappa} \models$ " $\kappa$ is inaccessible" and $\mathcal{H}_{\kappa}^{\mathscr{M}_{\kappa}}=V_{\kappa}^{M_{\kappa}} \models$ ZFC.
2. $\mathscr{M}_{\kappa}$ is $\kappa$-amenably-closed.
3. $\mathscr{M}_{\kappa} \models "\left(\kappa, \mathcal{H}_{\kappa}\right)$-Choice" $\Longleftrightarrow \mathscr{M}_{\kappa} \models$ " $\mathcal{H}_{\kappa}$ is wellordered".
4. $\mathscr{M} \models$ " $\left(<\kappa, \mathcal{H}_{\kappa^{+}}\right)$-Choice holds, so $\left(\mathcal{H}_{\kappa^{+}}\right)^{<\kappa} \subseteq \mathcal{H}_{\kappa^{+}}$".

[^1]1.2 Remark. In part 3 the " $\kappa^{+}$" and " $\mathcal{H}_{\kappa^{+}}$" are both in the sense of $\mathscr{M}_{\kappa}$. However, it can be that $\kappa$ is Mahlo and $\mathscr{M}_{\kappa} \models^{"}\left(\kappa, \mathcal{H}_{\kappa^{+}}\right)$-Choice fails, and $\left(\mathcal{H}_{\kappa^{+}}\right)^{\kappa} \nsubseteq \mathcal{H}_{\kappa^{+}}$"; indeed, note that this occurs in Lietz' example $L[G]$ mentioned above.

In the following theorem, the initial observation that $\mathscr{M}_{\kappa} \models$ " $\mathcal{H}_{\kappa}$ is wellordered" was due to Lietz:
Theorem (3.14). $4^{4}$ Let $\kappa$ be weakly compact. Then:

1. $\mathscr{M}_{\kappa} \models \kappa$-DC $+" \kappa$ is weakly compact" 5
2. for each $A \in \mathscr{M}_{\kappa} \cap \mathcal{H}_{\kappa^{+}}, \mathscr{M}_{\kappa} \models$ " $A$ is wellordered". 6]
3. if $\mathcal{P}(\kappa)^{\mathscr{M}_{\kappa}}$ has cardinality $\kappa$ then (i) $\kappa$ is measurable in $\mathscr{M}_{\kappa}$, and (ii) $x^{\#}$ exists for every $x \in \mathcal{P}(\kappa)^{\mathscr{M}_{\kappa}}$, and $x^{\#} \in \mathscr{M}_{\kappa}$.
4. If $\mathscr{M}_{\kappa} \models$ " $\mu$ is a countably complete ultrafilter over $\gamma \leq \kappa$ ", then the ultrapower $\operatorname{Ult}\left(\mathscr{M}_{\kappa}, \mu\right)$ is wellfounded and the ultrapower embedding

$$
i_{\mu}^{\mathscr{M}_{\kappa}}: \mathscr{M}_{\kappa} \rightarrow \operatorname{Ult}\left(\mathscr{M}_{\kappa}, \mu\right)
$$

is fully elementary.
As a corollary to Schindler's proof, one easily gets:
Fact (3.3). Let $\kappa$ be measurable and $\mu$ be a normal measure on $\kappa$. Then for $\mu$ measure one many $\gamma<\kappa, \mathscr{M}_{\gamma} \models$ " $V_{\gamma+1}$ is wellorderable".

As mentioned above, Usuba showed that $\mathscr{M}=\mathscr{M}_{\kappa}$ assuming $\kappa$ is extendible. The next result indicates that there are signs of this in the leadup to an extendible cardinal (for the definition of a $\Sigma_{2}$-strong cardinal, see 3.5):
Theorem (3.9). Suppose $\kappa$ is $\Sigma_{2}$-strong. Then $V_{\kappa+1}^{\mathscr{M}_{\kappa}}=V_{\kappa+1}^{\mathscr{M}}$.
Analogously, down lower:
Theorem (3.4). Let $A$ be a set such that $A^{\#}$ exists. Let $\kappa$ be an $A$-indiscernible. Then $V_{\kappa+1}^{\mathscr{M}_{\kappa}^{L(A)}}=V_{\kappa+1}^{\mathscr{M}^{L(A)}}$ and this set is wellordered in $\mathscr{M}_{\kappa}^{L(A)}$.

For further related results, which involve some inner model theory, see 9.
Before beginning our discussion of these results, we go through some background set-theoretic geology, including a proof of the the definability of grounds from a theory modelled by $\mathcal{H}_{\kappa}$ whenever $\kappa$ is a strong limit cardinal.

## 2 Grounds and mantles

We discuss here some background, starting with the key fact of the definability of set-forcing grounds under ZFC, proved by some combination of Laver, Woodin and Hamkins:
2.1 Fact. Let $M, N$ be proper class transitive inner models of ZFC and $\gamma \in \mathrm{OR}$ with $\mathcal{P}(\gamma) \cap M=\mathcal{P}(\gamma) \cap N$. Let $\mathbb{P} \in M$ and $\mathbb{Q} \in N$, with $\mathbb{P}, \mathbb{Q} \subseteq \gamma$, and let $G$ be $(M, \mathbb{P})$-generic and $H$ be $(N, \mathbb{Q})$-generic and suppose $M[G]=N[H]=V$. Then $M=N$.

[^2]We will discuss the proof of the result above, for two purposes. The fact is central to our concerns, and the proof contains elements which will come up in various places later, so it is natural to collect all these things together. Second, we wish to prove a version which assumes less background theory (than ZFC). The authors of [3] make use of an analysis of the complexity of the definability of grounds. As shown there, each ground $W$ is, in particular, $\Sigma_{2}$ in a parameter $r$. However, the $\Sigma_{2}$ definition given there is not particularly local: to compute $V_{\alpha}^{W}$, they work in $V_{\beta}$, for a significantly larger ordinal $\beta$. So for [3, Theorem 4], they adopt the background theory $\mathrm{ZFC}_{\delta}$. We show here that the ground definability can be done much more locally (though still requiring $\Sigma_{2}$ complexity), hence requiring significantly less than $\mathrm{ZFC}_{\delta}$.
2.2 Definition. Let $T_{1}^{-}$be the following theory in the language of set theory. The axioms are Extensionality, Foundation, Pairing, Union, Infinity, "Every set is bijectable with an ordinal", $\Sigma_{1}$-Separation and $\Sigma_{1}$-Collection. Now let

$$
T_{1}=T_{1}^{-}+\text {Powerset. }
$$

Note that $T_{1}^{-} \models$ AC. We will show that models of $T_{1}$ can uniformly define their grounds from parameters. First we give some lemmas.
2.3 Lemma. Assume ZFC. Then for every cardinal $\kappa \geq \omega$, (i) $\mathcal{H}_{\kappa} \models T_{1}^{-}$, and (ii) $\mathcal{H}_{\kappa} \models T_{1}$ iff $\kappa$ is a strong limit cardinal.

The usual proofs from ZFC easily adapt to give:
2.4 Lemma. Assume $T_{1}$. Then (i) for each ordinal $\xi$, $\mathcal{H}_{\xi}$ exists, (ii) $V=\bigcup_{\xi \in \mathrm{OR}} \mathcal{H}_{\xi}$, (iii) $\mathcal{H}_{\xi} \preccurlyeq_{1} V$, (iv) $\mathcal{H}_{\xi} \models T_{1}^{-}$, (iv) the Lowenheim-Skolem theorem holds.

In the following lemma, the forcing relations $\|_{\Sigma_{i}}$ for $i \in\{0,1\}$, and $\|_{\Pi_{1}}$, are the relations defined in a first-order manner over $M$ in the usual manner, and the strong- $\Sigma_{i+1}$-forcing relation $\vdash_{\Sigma_{i+1}}^{*}$ is the relation for which, given a $\Pi_{i}$ formula $\psi(\vec{x}, \vec{y})$ with free variables $\vec{x}, \vec{y}$, and given $\vec{\tau} \in\left(M^{\mathbb{P}}\right)^{<\omega}$, we say $p \Vdash_{\Sigma_{i+1}}^{*} \exists \vec{y} \psi(\vec{\tau}, \vec{y})$ iff there is $\vec{\sigma} \in\left(M^{\mathbb{P}}\right)^{<\omega}$ such that $p \|_{\Pi_{i}} \psi(\vec{\tau}, \vec{\sigma})$.
2.5 Lemma (Forcing over $T_{1}^{-}$and $T_{1}$ ). Let $M \models T_{1}^{-}$. Let $\mathbb{P} \in M$ be a poset with $\mathbb{P} \subseteq \gamma \in \mathrm{OR}^{M}$ and $G$ be $(M, \mathbb{P})$-generic. Then:

1. We have:
(a) The $\Sigma_{0}$-forcing relation $\vdash_{\Sigma_{0}}$ for $(M, \mathbb{P})$ is $\Delta_{1}^{M}(\{\mathbb{P}\})$, uniformly.
(b) The $\Sigma_{1}$-forcing relation $\vdash_{\Sigma_{1}}$ for $(M, \mathbb{P})$ is $\Sigma_{1}^{M}(\{\mathbb{P}\})$, uniformly $]^{7}$
(c) The $\Pi_{1}$-forcing relation $\|{ }_{\Pi_{1}}$ for $(M, \mathbb{P})$ is $\Pi_{1}^{M}(\{\mathbb{P}\})$, uniformly.

Hence, $\| \vdash_{\Sigma_{0}}, \Vdash_{\Sigma_{1}}$ and $\Vdash_{\Pi_{1}}$ are absolute to $\mathcal{H}_{\kappa}^{M}$, for $M$-cardinals $\kappa>\gamma$.
2. The strong- $\Sigma_{2}$-forcing relation $\Vdash_{\Sigma_{2}}^{*}$ for $(M, \mathbb{P})$ is $\Sigma_{2}^{M}(\{\mathbb{P}\})$, uniformly.
3. The forcing theorem for $\Sigma_{0}, \Sigma_{1}, \Pi_{1}$ formulas holds for $M[G]$, with respect to $\|_{\Sigma_{0}}, \Vdash_{\Sigma_{1}}, \Vdash_{\Pi_{1}}$; likewise for $\Sigma_{2}$ and $\mapsto_{\Sigma_{2}}^{*}$. That is, if $\varphi$ is $\Sigma_{i}$, where $i \in\{0,1\}$, and $\vec{\tau} \in\left(M^{\mathbb{P}}\right)^{<\omega}$, then

$$
M[G] \models \varphi\left(\vec{\tau}_{G}\right) \Longleftrightarrow \exists p \in G\left[M \models " p \Vdash_{\Sigma_{i}} \varphi(\vec{\tau}) "\right] .
$$

[^3]Likewise for $\Pi_{1}$ with $\vdash_{\Pi_{1}}$, and for $\Sigma_{2}$ with $\Vdash_{\Sigma_{2}}^{*}$.
4. $M[G] \models T_{1}^{-}$, and if $M \models T_{1}$ then $M[G] \models T_{1}$.
5. $M$ and $M[G]$ have the same cardinals $\kappa>\gamma$,
6. for each $M$-cardinal $\kappa>\gamma$, we have $\mathcal{H}_{\kappa}^{M[G]}=\mathcal{H}_{\kappa}^{M}[G]$.

Such local forcing calculations are very common in the literature, in particular in fine structure theory, where much more local calculations are often used. But we include a proof in case the reader has not seen these before.

Proof. Parts 1 , 3 for $\Sigma_{0}$ : The usual internal definition of the $\Sigma_{0}$-forcing relation $\| \mapsto_{0}$ works locally; in fact, for each $\xi \in \mathrm{OR}^{M}$ with $\xi \geq \gamma$, the $\Sigma_{0}$-forcing relation for names in $\mathcal{H}_{\xi}$, is $\Delta_{1}^{\mathcal{H}_{\xi}}(\{\mathbb{P}\})$, uniformly in $\xi$. This gives the Forcing Theorem for $\Sigma_{0}$ formulas in the usual manner.

Parts 13 for $\Sigma_{1}$ : We defined the strong- $\Sigma_{1}$-forcing relation $\| \vdash_{1}^{*}$ over $M$ above. Using the $\Sigma_{0}$-Forcing Theorem, note that $M[G] \models \exists y \varphi\left(y, \tau_{G}\right)$ iff there is $p \in G$ such that $M \models " p \|_{1}^{*} \exists y \varphi(y, \tau)$ ". Moreover, $\Vdash^{*}{ }_{1}^{*}$ is uniformly $\Sigma_{1}^{M}(\{\mathbb{P}\})$-definable.

Note that we take $\|-_{1}$ defined over $M$ as follows: Working in $M$, for $\varphi$ being $\Sigma_{1}$ and $\tau \in M^{\mathbb{P}}$, set

$$
p \Vdash_{1} \varphi(\tau) \Longleftrightarrow \forall q \leq p \exists r \leq q\left[r \Vdash_{1}^{*} \varphi(\tau)\right] .
$$

We claim that $p \vdash_{1} \varphi(\tau)$ iff $p \Vdash_{1}^{*} \varphi(\tau)$. For the non-trivial direction, suppose $p \| \vdash_{1} \varphi(\tau)$. Then working in $M$, using $\Sigma_{1}$-Collection and AC, we can put together a name $\sigma \in M^{\mathbb{P}}$ showing that $p \vdash_{1}^{*} \varphi(\tau)$. This completes the calculation for $\Sigma_{1}$.

Parts [1] for $\Pi_{1}: \|_{\Pi_{1}}$ is defined as usual: Working in $M$, for $\varphi$ being $\Pi_{1}$ and $\tau \in M^{\mathbb{P}}$, say $p \Vdash_{\Pi_{1}} \varphi(\tau)$ iff there is no $q \leq p$ such that $q \|_{\Sigma_{1}} \neg \varphi(\tau)$. So $\Vdash_{\Pi_{1}}$ is $\Pi_{1}^{M}(\{\mathbb{P}\})$. If $p \in G$ and $p \Vdash_{\Pi_{1}} \varphi(\tau)$, then clearly $M[G] \models \varphi\left(\tau_{G}\right)$. So suppose $M[G] \models \varphi\left(\tau_{G}\right)$ where $\varphi$ is $\Pi_{1}$. Let

$$
D=\left\{p \in \mathbb{P} \mid p \|_{\left.\Sigma_{\Sigma_{1}} \neg \varphi(\tau)\right\} .}\right.
$$

By $\Sigma_{1}$-Separation, $D \in M$. Let $D^{\prime}=D \cup\{p \in \mathbb{P} \mid \neg \exists q \in D[q \leq p]\}$, then $D^{\prime} \in M$, and since $D^{\prime}$ is dense, this easily suffices.

Parts 13 for $\Sigma_{2}$ : Here we only consider the strong- $\Sigma_{2}$ forcing relation $\|_{\Sigma_{2}}^{*}$, and the claims regarding this follow immediately just like for $\|_{\Sigma_{1}}^{*}$.

Part 4: Most of the axioms are routine consequences of the previous parts. Let us verify that $M[G] \models \Sigma_{1}$-Collection. Fix a $\Sigma_{0}$ formula $\varphi$ and $\sigma, \tau \in M^{\mathbb{P}}$. Let $t \in M$ be the transitive closure of $\{\sigma, \tau\}$. Then there is $w \in M$ such that for all $p \in \mathbb{P}$ and $\varrho \in t$, if

$$
p \vdash_{\Sigma_{1}} " \varrho \in \sigma \text { and } \exists y \varphi(\varrho, \tau, y) ",
$$

then there is $y \in M^{\mathbb{P}} \cap w$ such that $p \Vdash_{\Sigma_{0}}$ " $\varrho \in \sigma$ and $\varphi(\varrho, \tau, y)$ ". But then using $w$, we easily get a bound on witnesses in $M[G]$, as desired. This and the $\Sigma_{0}$-Forcing Theorem easily yields $\Sigma_{1}$-Separation.

The remaining parts follow from routine calculations with nice names.
2.6 Definition. Let $(M, E) \models T_{1}^{-}$. A ground of $M$ is a $W \subseteq M$ such that:

1. $(W, E \upharpoonright W)$ is $M$-transitive; that is, for all $x \in W$ and all $y \in M$, if $y E x$ then $y \in W$,
2. $W \models T_{1}^{-}$,
3. there is $\mathbb{P} \in W$ and a $(W, \mathbb{P})$-generic $G \in M$ such that $M=W[G]$.
4. If $(M, E) \models T_{1}$ then $(W, E \upharpoonright W) \models T_{1}$.

We now prove that $T_{1}$ suffices for the definability of grounds (in the sense of the definition above). The proof is essentially that due to some combination of Laver, Woodin and Hamkins. In the proof we make implicit use of Lemma 2.5, to allow the forcing calculations:
2.7 Theorem (Ground definability under $T_{1}$ ). Assume $T_{1}$. Let $\gamma \in \mathrm{OR}, H \subseteq \mathcal{H}_{\gamma^{+}}$ and $\kappa \geq \gamma^{+}$a cardinal. Then there is at most one transitive $M \subseteq \mathcal{H}_{\kappa}$ such that $M \models T_{1}^{-},\left(\mathcal{H}_{\gamma^{+}}\right)^{M}=H$, and $M$ is a ground for $\mathcal{H}_{\kappa}$ via some $\mathbb{P} \in H$.

Proof. We proceed by induction on $\kappa$. For $\kappa=\gamma^{+}$it is trivial.
Suppose $\kappa$ is a limit cardinal, and that for each cardinal $\theta \in\left[\gamma^{+}, \kappa\right)$, there is a (unique) model $M_{\theta}$ of ordinal height $\theta$ with the stated properties. Then clearly $M=\bigcup_{\theta<\kappa} M_{\theta}$ is the unique candidate at $\kappa$. To see that $M$ works, we just need to verify that $M$ is indeed a set-ground of $\mathcal{H}_{\kappa}$ via some $\mathbb{P} \in H$; i.e. there is $\mathbb{P} \in H$ and an $(M, \mathbb{P})$-generic $G \subseteq \mathbb{P}$ such that $M[G]=\mathcal{H}_{\kappa}$. But we can use any $(\mathbb{P}, G)$ which worked at some earlier $\theta$. For let $\theta_{0} \leq \theta_{1}<\kappa$, and let $\left(\mathbb{P}_{0}, G_{0}\right),\left(\mathbb{P}_{1}, G_{1}\right)$ work for $M_{0}=M_{\theta_{0}}$ and $M_{1}=M_{\theta_{1}}$. So $G_{0}$ is also ( $M_{1}, \mathbb{P}_{0}$ )-generic, and vice versa. And since $\mathcal{H}_{\gamma^{+}}^{M_{0}}=H=\mathcal{H}_{\gamma^{+}}^{M_{1}}$, and $H\left[G_{0}\right]=\mathcal{H}_{\gamma^{+}}=H\left[G_{1}\right]$, it follows that $\mathcal{H}_{\kappa}=M_{0}\left[G_{0}\right]=M_{0}\left[G_{1}\right]$ and $M_{1}\left[G_{0}\right]=M_{1}\left[G_{1}\right]=\mathcal{H}_{\kappa}$, so the specific choice of $(\mathbb{P}, G)$ is irrelevant.

So consider $\kappa=\theta^{+}>\gamma^{+}$. Let $M, N$ be grounds of $\mathcal{H}_{\kappa}$ with the stated properties. By induction, $M \cap \mathcal{H}_{\theta}=N \cap \mathcal{H}_{\theta}$. It just remains to verify that $\mathcal{P}(\theta) \cap M=\mathcal{P}(\theta) \cap N$. The proof is, however, not by contradiction; we will not assume that $M \neq N$. Fix $(\mathbb{P}, G)$ such that $\mathbb{P} \in H$ and $G$ is $(M, \mathbb{P})$-generic and $M[G]=\mathcal{H}_{\kappa}$.

Suppose first that $\operatorname{cof}(\theta)>\gamma$, as this case is easier; however, it is in the end subsumed into the general case. Let $A \subseteq \theta$. Then:
Claim 1. $A \in M$ iff $A \cap \alpha \in M$ for all $\alpha<\theta$.
Proof. For the non-trivial direction, suppose $A \cap \alpha \in M$ for every $\alpha<\theta$. Let $f: \theta \rightarrow M$ be $f(\alpha)=A \cap \alpha$. Then $f \in \mathcal{H}_{\kappa}$. So there is a $\mathbb{P}$-name $\dot{f} \in M$ with $\dot{f}_{G}=f$. Working in $M$, for $p \in \mathbb{P}$, compute

$$
D_{p}=\{\alpha<\theta \mid \exists x[p \|-\dot{f}(\check{\alpha})=\check{x}]\},
$$

and let $f_{p}: D_{p} \rightarrow \theta$ be the function

$$
f_{p}(\alpha)=\text { unique } x \text { such that } p \|-\dot{f}(\check{\alpha})=\check{x}
$$

So $\left\langle D_{p}, f_{p}\right\rangle_{p \in \mathbb{P}} \in M$, and because $\operatorname{cof}(\theta)>\gamma$, there is $p \in G$ such that $D_{p}$ is cofinal in $\theta$. Then $f=\left(\bigcup_{\alpha \in D_{p}} f_{p}(\alpha)\right) \in M$.

We now argue in general.
Claim 2. Let $A \subseteq \theta$. Then $A \in M$ iff for every $X \in \mathcal{P}(\theta) \cap M$ such that $\operatorname{card}(X)<$ $\left(\gamma^{+}\right)=\left(\gamma^{+}\right)^{M}$, we have $A \cap X \in M$.

Proof. The forward direction is trivial. So let $A \subseteq \theta$ with $A \notin M$. Let $\dot{A} \in M$ be a $\mathbb{P}$-name and $p_{0} \in G$ such that $p_{0} \| \vdash \dot{A} \subseteq \check{\theta}$. For each $q \leq p_{0}$, if there is $\alpha<\theta$ such that

$$
q \| \not \check{\alpha} \in \dot{A} \text { and } q \| \nvdash \check{\alpha} \notin \dot{A},
$$

then let $\alpha_{q}$ be the least such $\alpha$; otherwise $\alpha_{q}$ is undefined. Let $D$ be the set of all $q \leq p_{0}$ such that $\alpha_{q}$ exists. Then $G \subseteq D$, because otherwise $q$ decides all elements of $\dot{A}$, so $A \in M$.

In $M$, let $X=\left\{\alpha_{q} \mid q \in D\right\}$. Then $X \in M, \operatorname{card}^{M}(X) \leq \gamma$ and $X \cap A \notin M$, as desired. For given $Y \in \mathcal{P}(X) \cap M$, an easy density argument shows that $Y \neq$ $X \cap A$.

Claim 3. Let $X \subseteq \theta$ with $\operatorname{card}(X)<\gamma^{+}$. Then $X \in M$ iff $X \in N$.
Proof. Suppose $X_{0}=X \in N$. Let $\dot{X} \in M$ be a $\mathbb{P}$-name for $X$. Using the forcing relation and $\dot{X}$, there is a set $X_{1} \in \mathcal{P}(\theta) \cap M$ with $X_{0} \subseteq X_{1}$ and $\operatorname{card}\left(X_{1}\right)<\left(\gamma^{+}\right)^{V}$. Proceeding back-and-forth, construct (in $V$ ) a continuous sequence of sets $\left\langle X_{\alpha}\right\rangle_{\alpha<\gamma^{+}}$ such that (i) $X_{0}=X$, (ii) $X_{\omega \alpha+2 n+1} \in M$ and $X_{\omega \alpha+2 n+2} \in N$, and (iii) $\operatorname{card}\left(X_{\alpha}\right)<$ $\left(\gamma^{+}\right)^{V}$.

Now $\gamma^{+}<\kappa$, so $\left\langle X_{\alpha}\right\rangle_{\alpha<\gamma^{+}} \in \mathcal{H}_{\kappa}$, so $M, N$ have names for this sequence. So as in the $\operatorname{cof}(\theta)>\gamma$ case, we get a cofinal set $D_{M} \subseteq \gamma^{+}$such that $D_{M} \in M$ and $\left\langle X_{\alpha}\right\rangle_{\alpha \in D_{M}} \in M$. Likewise with a cofinal set $D_{N} \in N$. Let $D_{M}^{\prime}$ be the set of limit points of $D_{M}$, and $D_{N}^{\prime}$ likewise. So these are club in $\gamma^{+}$. Let $\alpha \in D_{M}^{\prime} \cap D_{N}^{\prime}$. Then note that

$$
X_{\alpha}=\left(\bigcup_{\beta \in D_{M} \cap \alpha} X_{\beta}\right)=\left(\bigcup_{\beta \in D_{N} \cap \alpha} X_{\beta}\right) \in M \cap N .
$$

Let $\pi: \xi \rightarrow X_{\alpha}$ be the increasing enumeration of $X_{\alpha}$. Then $\xi<\gamma^{+}$and $\pi \in M \cap N$. We have $X \subseteq \operatorname{rg}(\pi)$. Let $\bar{X}=\pi^{-1}(X)$. Then $\bar{X} \in N$. But $\mathcal{H}_{\gamma^{+}}^{M}=H=\mathcal{H}_{\gamma^{+}}^{N}$, so $\bar{X} \in M$. So $\pi " \bar{X}=X \in M$, as desired.

This completes the proof of ground definability under $T_{1}$.
2.8 Remark. If $M \models T_{1}^{-}+$"there is a largest cardinal $\kappa$, and $\kappa$ is regular", then grounds of $M$ via forcings $\mathbb{P}$ of $M$-cardinality $<\kappa$ are also definable from parameters over $M$, by arguing much as above.
2.9 Definition. Assume $T_{1}$. Let $\varphi_{\operatorname{grd}}(r, x)$ be the formula " there are $\gamma, \mathbb{P}, G, M$, $\kappa$ such that $\gamma<\kappa$ are cardinals, $M \subseteq \mathcal{H}_{\kappa}$ is transitive, $M \models T_{1}^{-}, \mathbb{P} \in r=\left(\mathcal{H}_{\gamma^{+}}\right)^{M}$, $G$ is $(M, \mathbb{P})$-generic, $\mathcal{H}_{\kappa}=M[G]$ and $x \in M^{\prime \prime}$.

We write $W_{r}^{\prime}=\left\{x \mid \varphi_{\text {grd }}(r, x)\right\}$. We say $r$ is a true index iff $W_{r}^{\prime}$ is proper class. We write $W_{r}=W_{r}^{\prime}$ for true indices $r$, and $W_{r}=V$ otherwise.
2.10 Corollary. Assume ZFC+GCH and let $\lambda$ be a limit cardinal. Then the grounds of $\mathcal{H}_{\lambda}$ are definable from parameters over $\mathcal{H}_{\lambda}$.
2.11 Remark. Assume ZFC +GCH . Then for each limit ordinal $\xi, V_{\omega+\xi}$ is equivalent in the codes to $\mathcal{H}_{\aleph_{\xi}}$. So one can correctly formulate "grounds" of $V_{\omega+\xi}$, and they are definable over $V_{\omega+\xi}$ from parameters.

So we have the standard uniform definability of grounds, just assuming $T_{1}$ :
2.12 Lemma. Let $M \models T_{1}$. Then $\left\{W_{r}^{M} \mid r \in M\right\}$ enumerates exactly the grounds of $M$ (with repetitions, including $M$ itself).
2.13 Remark. Assume $T_{1}$. Note that $\varphi_{\text {grd }}$ is $\Sigma_{2}$, and the assertion " $r$ is a true index" is $\Pi_{2}$. (In fact, there are fixed $\Sigma_{2}$ and $\Pi_{2}$ formulas, such that $T_{1}$ proves that these fixed formulas always work.) Moreover, letting $\xi=\operatorname{card}(\operatorname{trcl}(\{r, x\}))$, note that $\varphi_{\mathrm{grd}}(r, x)$ is absolute between $V$ and $\mathcal{H}_{\left(2^{\xi}\right)^{+}}$. (It is witnessed by some $\left(\mathcal{H}_{\xi^{+}}, M\right)$, a structure of size $2^{\xi}$.) Therefore:
2.14 Fact (Local definability of grounds). Assume $T_{1}+$ "There is a proper class of strong limit cardinals". Let $\lambda$ be a strong limit cardinal. Let $r \in \mathcal{H}_{\lambda}$ be a true index. Then $\mathcal{H}_{\lambda} \models$ " $r$ is a true index" and $W_{r}^{\mathcal{H}_{\lambda}}=W_{r} \cap \mathcal{H}_{\lambda}=\mathcal{H}_{\lambda}^{W_{r}}$.

It seems it might be possible, however, that $\mathcal{H}_{\lambda} \models$ " $r$ is a true index" while $r$ fails to be a true index in $V$.

The remaining facts in this section, and the rest of the paper, have a background theory of ZFC. We have not investigated to what extent things go through under $T_{1}$. By [10, Proposition 5.1] and an examination of its proof, we have:
2.15 Fact (Local set-directedness of grounds (Usuba)). (Assume ZFC.) Let $\theta$ be a strong limit cardinal and $R \in \mathcal{H}_{\theta}$. Then there is $t \in \mathcal{H}_{\theta}$ with $t \in \bigcap_{r \in R} W_{r}$ and $W_{t} \subseteq W_{r}$ and $W_{t}=W_{t}^{W_{r}}$ for each $r \in R$. In particular, $W_{t} \subseteq \bigcap_{r \in R} W_{r}$.

Proof. We refer here to the $\lambda$-uniform covering property for $V$; see [10, Definition 4.2] or [8, Definition 2.1]. Let us set up some of the notation from the proof of [10, Proposition 5.1]. Let $X=R$ (following the notation from [10). 8 We may assume that $X$ is a set of true indices $r$. For $r \in X$ let $\mathbb{P}_{r} \in W_{r}$ be a forcing witnessing that $r$ is a true index. Let $\kappa$ be a regular cardinal with $\kappa>\operatorname{card}(X)$ and $\kappa>\operatorname{card}\left(\mathbb{P}_{r}\right)$ for each $r$ (so it suffices if $\kappa>\operatorname{card}(\operatorname{trcl}(X))$ ). Then the proof of [10, Proposition 5.1] constructs a ground $W \subseteq \bigcap_{r \in X} W_{r}$ with the $\lambda=\kappa^{++}$-uniform covering property for $V$. Therefore by [7, Theorem 3.3], there is $\mathbb{P} \in W$ such that $W \models$ " $\operatorname{card}(\mathbb{P})=2^{2<\lambda}$ " and $W$ is a ground of $V$ via $\mathbb{P}$. Let $\gamma_{0}=\operatorname{card}^{W}(\mathbb{P})$ and $t_{0}=\left(\mathcal{H}_{\gamma_{0}^{+}}\right)^{W}$. So $\gamma_{0}<\theta$, $t_{0}$ is a true index and $W=W_{t_{0}}$. Let $\mathbb{B} \in W$ be such that $W \models$ " $\mathbb{B}$ is the complete Boolean algebra determined by $\mathbb{P} "$ (so $\mathbb{P}$ is a dense sub-order of $\mathbb{B})$. So card ${ }^{W}(\mathbb{B}) \leq$ $\left(2^{\gamma_{0}}\right)^{W}<\theta$. Then by [2, Lemma 15.43] (or [10, Fact 3.1]) for each $r \in X$ there is some $\mathbb{B}_{r} \in W$ with $\mathbb{B}_{r} \subseteq \mathbb{B}$ and there is a $\left(W, \mathbb{B}_{r}\right)$-generic $G_{r}$ such that $W\left[G_{r}\right]=W_{r}$. So letting $\gamma=\left(2^{\gamma_{0}}\right)^{W}$, then $t=\left(\mathcal{H}_{\gamma^{+}}\right)^{W}$ is as desired.

An easy corollary of local set-directedness is:
2.16 Fact (Invariance of $\mathscr{M}_{\kappa}$ ). Let $\kappa$ be a strong limit cardinal and $r \in \mathcal{H}_{\kappa}$. Then $\mathscr{M}_{\kappa}^{W_{r}}=\mathscr{M}_{\kappa}$.
2.17 Lemma (Absoluteness of $\mathscr{M}_{\kappa}$ ). Let $\kappa<\lambda$ be strong limit cardinals and suppose $\mathcal{H}_{\lambda}=V_{\lambda} \preccurlyeq_{2} V$. Then for each $r \in \mathcal{H}_{\kappa}$, we have:
(i) $<\kappa$-grounds and $\mathscr{M}_{\kappa}$ are absolute to $V_{\lambda}$ :

$$
W_{r}^{V_{\lambda}}=W_{r} \cap V_{\lambda}=V_{\lambda}^{W_{r}} \text { and } \mathscr{M}_{\kappa}^{V_{\lambda}}=\mathscr{M}_{\kappa} \cap V_{\lambda}=V_{\lambda}^{\mathscr{M}_{\kappa}},
$$

(ii) $V_{\lambda}^{W_{r}} \preccurlyeq 2 W_{r}$,
(iii) $\mathscr{M}_{\kappa}^{V_{\lambda}^{W_{r}}}=\mathscr{M}_{\kappa}^{W_{r}} \cap V_{\lambda}^{W_{r}}=\mathscr{M}_{\kappa} \cap V_{\lambda}=\mathscr{M}_{\kappa}^{V_{\lambda}}$.

[^4]Proof. Part (i) The absoluteness of $W_{r}$ is because the class true indices $r$ is $\Pi_{2}$, and each $W_{r}$ is $\Sigma_{2}(\{r\})$. But then clearly

$$
\mathscr{M}_{\kappa}^{V_{\lambda}}=\bigcap_{r \in V_{\kappa}} W_{r}^{V_{\lambda}}=\bigcap_{r \in V_{\kappa}} V_{\lambda}^{W_{r}}=V_{\lambda}^{\mathscr{M}_{\kappa}} .
$$

Part (ii) If $W_{r}=V$ then this is trivial. Suppose $W_{r} \subsetneq V$ and let $\varphi$ be $\Sigma_{2}$ and $x \in W_{r} \cap V_{\lambda}$ and suppose that $W_{r} \models \varphi(x)$. Then by Fact 2.14, $V \models \psi(x)$ where $\psi$ asserts "There is a strong limit cardinal $\xi$ such that $W_{r}^{\mathcal{H}_{\xi}} \models \varphi(x)$ ", but this is also $\Sigma_{2}$, so $V_{\lambda} \models \psi(x)$, so letting $\xi<\lambda$ witness this, again by Fact 2.14, we get $W_{r} \cap \mathcal{H}_{\xi} \models \varphi(x)$, so $W_{r} \cap V_{\lambda} \models \varphi(x)$.

Part (iii). This follows from the previous parts and Fact 2.16
2.18 Definition. Let $N$ be an inner model. Let $f: \kappa \rightarrow N$. Say that $f$ is amenable to $N$ iff $f \upharpoonright \alpha \in N$ for every $\alpha<\kappa$. Say that $N$ is $\kappa$-amenably-closed iff for every $f: \kappa \rightarrow N$, if $f$ is amenable to $N$ then $f \in N$. Say that $N$ is $\kappa$-stationarilycomputing ( $\kappa$-unboundedly-computing) iff for every $f: \kappa \rightarrow N$, there is a stationary (unbounded) $A \subseteq \kappa$ such that $f \upharpoonright A \in N$.
2.19 Lemma. Let $N$ be an inner model of ZF and $\kappa>\omega$ be regular. If $N$ is $\kappa$ -stationarily-computing then $N$ is $\kappa$-unboundedly-computing. If $N$ is $\kappa$-unboundedlycomputing then $N$ is $\kappa$-amenably-closed.
2.20 Lemma. Let $W$ be $a<\kappa$-ground of $V$, where $\kappa>\omega$ is regular. Then $W$ is $\kappa$-stationarily-computing.
2.21 Lemma. The intersection of any family of $\kappa$-amenably-closed structures is $\kappa$ -amenably-closed.
2.22 Lemma. If $\kappa$ is inaccessible then $\mathscr{M}_{\kappa}$ is $\kappa$-amenably-closed.

## 3 Choice principles in the $\kappa$-mantle

As mentioned above, from now on we have ZFC as background theory.
The first positive results along the lines of what we will prove here (regarding about $\kappa$-mantles when $\kappa<\infty$ ), consists in Usuba's work, including his extendibility result. This was followed by Lietz' negative results [5]. Some time after this, using the general theory of [6], the author showed that the $\kappa_{0}$-mantle $\mathscr{M}_{\kappa_{0}}^{M}$ of $M=M_{\text {swsw }}$ (see below) is a strategy mouse. We give an outline of this argument, but it is primarily intended for the reader familiar with inner model theory, and can be safely skipped over, as the remainder of the paper does not depend on it. We omit all specifics to do with Varsovian models, just mentioning enough to indicate what is relevant here. The full proof will appear in [6].
3.1 Definition. $M_{\text {swsw }}$ denotes the least iterable proper class mouse with ordinals $\delta_{0}<\kappa_{0}<\delta_{1}<\kappa_{1}$ satisfying "each $\delta_{i}$ is Woodin and each $\kappa_{i}$ is strong".

The Varsovian model analysis produces a mouse $M_{\infty}$, which is the direct limit of (pseudo-)iterates $P$ of $M$ via correct iteration trees $\mathcal{T}$ on $M$, with $\mathcal{T} \in M \mid \kappa_{0}$, and which are based on $M \mid \delta_{0}$. It also defines a certain fragment $\Sigma$ of the iteration strategy for $M_{\infty}$, yielding a strategy mouse $M_{\infty}[\Sigma]$. It turns out that $M_{\infty}[\Sigma]$ has universe $\mathscr{M}_{\kappa_{0}}^{M}$.

What is relevant here is the proof that $\mathscr{M}_{\kappa_{0}}^{M} \subseteq M_{\infty}[\Sigma]$. Let $X \in \mathscr{M}_{\kappa_{0}}^{M}$ be a set of ordinals. We must see that $X \in M_{\infty}[\Sigma]$. 9 Now $\kappa_{0}$ is measurable in $M$. Let $E$ be a normal measure on $\kappa_{0}$, in the extender sequence of $M$, and let

$$
j: M \rightarrow U=\operatorname{Ult}(M, E)
$$

be the ultrapower map. By elementarity, $j(X) \in \mathscr{M}_{j\left(\kappa_{0}\right)}^{U}$. With methods from the Varsovian model analysis, one can then construct a specific $<j\left(\kappa_{0}\right)$-ground $W$ of $U$, with $W \subseteq M_{\infty}[\Sigma]$. So

$$
j(X) \in \mathscr{M}_{j\left(\kappa_{0}\right)}^{U} \subseteq W \subseteq M_{\infty}[\Sigma]
$$

Other facts from Varsovian model analysis give $j\left\lceil\alpha \in M_{\infty}[\Sigma]\right.$ for each $\alpha \in$ OR. But then $X \in M_{\infty}[\Sigma]$, as desired, since

$$
\beta \in X \Longleftrightarrow j(\beta) \in j(X)
$$

The preceding argument has structural similarities to Usuba's extendibility proof (see [11). Schindler then found the following result (see [8]). We will use an adaptation of the proof for Theorem 3.14 later, so we present this one first as a warmup, and in order to note a simple corollary. We give essentially Schindler's proof, although the precise implementation might differ slightly.
3.2 Fact (Schindler). Let $\kappa$ be measurable. Then $\mathscr{M}_{\kappa} \models \mathrm{AC}$, so $\mathscr{M}_{\kappa} \models$ ZFC.

Proof. Let $A \in \mathscr{M}_{\kappa}$. We will find a wellorder $<_{A}$ of $A$ with $<_{A} \in \mathscr{M}_{\kappa}$.
Let $\mu$ be a normal measure on $\kappa, M=\operatorname{Ult}(V, \mu)$ and $j=i_{\mu}^{V}: V \rightarrow M$ the ultrapower map. So $\kappa=\operatorname{cr}(j)$ and $j(A) \in \mathscr{M}_{j(\kappa)}^{M}$.
Claim 1. We have:

1. $\mathscr{M}_{j(\kappa)}^{M} \subseteq \mathscr{M}_{\kappa}^{M} \subseteq \mathscr{M}_{\kappa}$, and
2. $j \upharpoonright \mathscr{M}_{\kappa}$ is amenable to $\mathscr{M}_{\kappa}$.

Proof. Part 1] The first $\subseteq$ is immediate. For the second, we have

$$
\mathscr{M}_{\kappa}=\bigcap_{r \in V_{\kappa}} W_{r} \text { and } \mathscr{M}_{\kappa}^{M}=\bigcap_{r \in V_{\kappa}} W_{r}^{M} .
$$

Let $\mu_{r}=\mu \cap W_{r}$. Then by standard forcing calculations and elementarity, we get $\mu_{r} \in W_{r}$ and

$$
W_{r}^{M}=j\left(W_{r}\right)=\operatorname{Ult}\left(W_{r}, \mu\right)^{V}=\operatorname{Ult}\left(W_{r}, \mu_{r}\right)^{W_{r}}
$$

so $W_{r}^{M} \subseteq W_{r}$, so $\mathscr{M}_{\kappa}^{M} \subseteq \mathscr{M}_{\kappa}$ as desired.
Part2 Let $r \in V_{\kappa}$. Then calculations as above give $i_{\mu_{r}}^{W_{r}} \upharpoonright W_{r} \subseteq j$. But $\mathscr{M}_{\kappa} \subseteq W_{r}$, and so $j \upharpoonright \mathscr{M}_{\kappa}$ is amenable to $W_{r}$. Therefore $j \upharpoonright \mathscr{M}_{\kappa}$ is amenable to $\mathscr{M}_{\kappa}$, as desired.

Since $\kappa$ is a strong limit, Fact 2.15 gives $s \in V_{j(\kappa)}^{M}$ such that

$$
\mathscr{M}_{j(\kappa)}^{M} \subseteq W=W_{s}^{M} \subseteq \mathscr{M}_{\kappa}^{M}
$$

So $j(A) \in W \models$ ZFC, so there is a wellorder $<^{*}$ of $j(A)$ with $<^{*} \in W$. But $W \subseteq \mathscr{M}_{\kappa}^{M}$, so $<^{*} \in \mathscr{M}_{\kappa}^{M} \subseteq \mathscr{M}_{\kappa}$.

Now working in $\mathscr{M}_{\kappa}$, where we have $k=j \upharpoonright A$ and $j(A)$ and $<^{*}$, we can define a wellorder $<_{A}$ of $A$ by setting, for $x, y \in A$ :

$$
x<_{A} y \Longleftrightarrow k(x)<^{*} k(y)
$$

This completes the proof.

[^5]As a corollary to the proof above, we observe:
3.3 Corollary. Let $\kappa$ be measurable and $\mu$ be a normal measure on $\kappa$. Then for $\mu$-measure one many $\gamma<\kappa, \mathscr{M}_{\gamma} \models$ " $V_{\gamma+1}$ is wellorderable".

Proof. Continue with the notation from the proof of Fact 3.2. We show $\mathscr{M}_{\kappa}^{M} \models$ " $V_{\kappa+1}$ is wellorderable".
Claim. $V_{\kappa+1} \cap \mathscr{M}_{j(\kappa)}^{M}=V_{\kappa+1} \cap \mathscr{M}_{\kappa}^{M}=V_{\kappa+1} \cap \mathscr{M}_{\kappa}$.
Proof. We have $V_{\kappa+1} \cap \mathscr{M}_{\kappa} \subseteq V_{\kappa+1} \cap \mathscr{M}_{j(\kappa)}^{M}$ since $j \upharpoonright \mathscr{M}_{\kappa}: \mathscr{M}_{\kappa} \rightarrow \mathscr{M}_{j(\kappa)}^{M}$ is elementary and $\kappa=\operatorname{cr}(j)$. By Claim 1 of the proof of Fact [3.2, this suffices.

By Fact [3.2, $\mathscr{M}_{\kappa} \models \mathrm{AC}$, so $\mathscr{M}_{j(\kappa)}^{M} \models \mathrm{AC}$ also. Let $<^{*} \in \mathscr{M}_{j(\kappa)}^{M}$ be a wellorder of $V_{\kappa+1} \cap \mathscr{M}_{j(\kappa)}^{M}$. Then $<^{*} \in \mathscr{M}_{\kappa}^{M}$ and $<^{*}$ is a wellorder of $V_{\kappa+1} \cap \mathscr{M}_{\kappa}^{M}$.

We next use the simple idea above to prove that certain cardinals are "stable" with respect to the mantle. The first observation is:
3.4 Theorem. Let $A$ be a set such that $A^{\#}$ exists. Let $\kappa$ be an $A$-indiscernible. Then $V_{\kappa+1}^{\mathcal{M}_{\kappa}^{L(A)}}=V_{\kappa+1}^{\mathcal{M}^{L(A)}}$ and this set is wellordered in $\mathscr{M}_{\kappa}^{L(A)}$.

Proof. Let $j: L(A) \rightarrow L(A)$ be elementary with $\operatorname{cr}(j)=\kappa$. We write $\mathscr{M}_{\kappa}$ for $\mathscr{M}_{\kappa}^{L(A)}$; likewise $\mathscr{M}_{j(\kappa)}$. Now $j \upharpoonright \mathscr{M}_{\kappa}: \mathscr{M}_{\kappa} \rightarrow \mathscr{M}_{j(\kappa)}$ is elementary. Clearly $\mathscr{M}_{j(\kappa)} \subseteq \mathscr{M}_{\kappa}$. But also, $B=V_{\kappa+1}^{\mathscr{M}_{\kappa}} \subseteq V_{\kappa+1}^{\mathscr{M}_{j(\kappa)}}$ as in the previous proof. So $V_{\kappa+1}^{\mathscr{M}_{j(\kappa)}}=B$. But $V_{j(\kappa)}^{\mathscr{M}_{j(\kappa)}} \models$ ZFC, so there is a wellorder of $B$ in $\mathscr{M}_{j(\kappa)} \subseteq \mathscr{M}_{\kappa}$.

It now follows that $V_{\kappa+1}^{\mathscr{M}_{\kappa}}=V_{\kappa+1}^{\mathscr{M}}$, because we can take $j(\kappa)$ as large as we like, hence past any true index.
3.5 Definition. A cardinal $\kappa$ is $\Sigma_{2}$-strong iff for every $\alpha \in$ OR there is an elementary embedding $j: V \rightarrow M$ with $\alpha<j(\kappa)$ and $V_{\alpha} \subseteq M$ and $\operatorname{Th}_{\Sigma_{2}}^{M}\left(V_{\alpha}\right)=\operatorname{Th}_{\Sigma_{2}}^{V}\left(V_{\alpha}\right) 10$

An embedding $j: V \rightarrow M$ is superstrong iff $V_{j(\kappa)} \subseteq M$. A cardinal $\kappa$ is $\infty$ superstrong iff for every $\alpha \in \mathrm{OR}$ there is a superstrong embedding $j$ with $\operatorname{cr}(j)=\kappa$ and $j(\kappa)>\alpha$.

A superstrong extender is the $V_{\beta}$-extender derived from a superstrong embedding $j: V \rightarrow M$ where $\beta=j(\kappa)$ and $\kappa=\operatorname{cr}(j)$.

Note that:
3.6 Lemma. If $E$ is a superstrong extender and $W \models$ ZFC is a transitive proper class with $E \in W$, then $W \models$ " $E$ is a superstrong extender".
3.7 Remark. Say that a cardinal $\kappa$ is $\infty-1$-extendible iff for every $\alpha \in$ OR there is $\beta \in \mathrm{OR}$ with $\beta \geq \alpha$ and and an elementary $j: V_{\kappa+1} \rightarrow V_{\beta+1}$ (hence $j(\kappa)=\beta$ ) with $\operatorname{cr}(j)=\kappa$.
3.8 Theorem. We have:

1. Every extendible cardinal is $\infty$-1-extendible and carries a normal measure concentrating on $\infty$-1-extendible cardinals.
2. Every $\infty$-1-extendible cardinal is $\infty$-superstrong and carries a normal measure concentrating on $\infty$-superstrong cardinals.

[^6]3. Every $\infty$-superstrong cardinal is $\Sigma_{2}$-strong and carries a normal measure concentrating on $\Sigma_{2}$-strong cardinals.

Proof. Part 1: This is routine and left to the reader.
Part 2, Let $\kappa$ be $\infty$-1-extendible. Let $j: V_{\kappa+1} \rightarrow V_{\beta+1}$ be elementary with $\operatorname{cr}(j)=\kappa$. Let $E$ be the extender derived from $j$ with support $V_{\beta}$. Let $M=\operatorname{Ult}(V, E)$ and $k: V \rightarrow M$ be the ultrapower map. Then one can show that $k$ is a superstrong embedding with $k(\kappa)=\beta$ and that $M \models " \kappa$ is $\infty$-superstrong". (For the last clause, consider ultrapowers $\operatorname{Ult}(M, E \upharpoonright \alpha)$ where $\alpha<\beta$, and show that unboundedly many of these produce superstrong embeddings in $M$, and also use that $\beta$ is $1-\infty$-extendible in $M$.)

Part 3 Let $\kappa$ be $\infty$-superstrong. We show first that $\kappa$ is $\Sigma_{2}$-strong. So let $\alpha \in$ OR. We may assume that $V_{\alpha} \preccurlyeq_{2} V$. Let $j: V \rightarrow M$ be any superstrong embedding with $\operatorname{cr}(j)=\kappa$ and $\alpha<j(\kappa)$. It suffices to verify:
Claim 1. $\operatorname{Th}_{\Sigma_{2}}^{M}\left(V_{\alpha}\right)=\operatorname{Th}_{\Sigma_{2}}^{V}\left(V_{\alpha}\right)$.
Proof. Let $\varphi$ be $\Sigma_{2}$ and $\vec{x} \in\left(V_{\alpha}\right)^{<\omega}$. If $V \models \varphi(\vec{x})$ then $V_{\alpha} \models \varphi(\vec{x})$, which implies $M \models \varphi(\vec{x})$. Conversely, suppose $M \models \varphi(\vec{x})$. Because $\kappa$ is $\infty$-superstrong, it is clearly strong, which implies that $V_{\kappa} \preccurlyeq_{2} V$. Therefore $V_{j(\kappa)}^{M} \preccurlyeq_{2} M$. Therefore $V_{j(\kappa)}^{M} \models \varphi(\vec{x})$. But $V_{j(\kappa)}^{M}=V_{j(\kappa)}$, so $V_{j(\kappa)} \models \varphi(\vec{x})$, so $V \models \varphi(\vec{x})$, as desired.

Now let $j: V \rightarrow M$ be a superstrong embedding with $\operatorname{cr}(j)=\kappa$. We will show that $M \models$ " $\kappa$ is $\Sigma_{2}$-strong", which completes the proof.
Claim 2. $M \models$ " $\kappa$ is $<\beta$ - $\Sigma_{2}$-strong", where $\beta=j(\kappa)$. That is, for each $\alpha<\beta, M$ has an elementary $k: M \rightarrow N$ with $\operatorname{cr}(k)=\kappa$ and $V_{\alpha} \subseteq N$ and $\operatorname{Th}_{\Sigma_{2}}^{N}\left(V_{\alpha}\right)=\operatorname{Th}_{\Sigma_{2}}^{M}\left(V_{\alpha}\right)$.

Proof. Since $M \models$ " $\beta$ is strong", $V_{\beta}^{M} \preccurlyeq 2 M$ and there are club many $\alpha<\beta$ such that $V_{\alpha}^{M}=V_{\alpha} \preccurlyeq 2 M$. Fix some such $\alpha$. Let $E_{\alpha}$ be the extender derived from $j$ with support $V_{\alpha}$. Then $E_{\alpha} \in V_{\beta} \subseteq M$, and $M \models$ " $E_{\alpha}$ is an extender". Moreover, letting $N_{\alpha}=\operatorname{Ult}\left(M, E_{\alpha}\right)$, we have $V_{\alpha} \subseteq N_{\alpha}$ and

$$
\operatorname{Th}_{\Sigma_{2}}^{N_{\alpha}}\left(V_{\alpha}\right)=\operatorname{Th}_{\Sigma_{2}}^{M}\left(V_{\alpha}\right)
$$

For let $t=\operatorname{Th}_{\Sigma_{2}}^{V}\left(V_{\kappa}\right)=\operatorname{Th}_{\Sigma_{2}}^{M}\left(V_{\kappa}\right)$. Then letting $k_{\alpha}: M \rightarrow N_{\alpha}$ be the ultrapower map,

$$
j(t)=\operatorname{Th}_{\Sigma_{2}}^{M}\left(V_{\beta}\right) \text { and } k_{\alpha}(t)=\operatorname{Th}_{\Sigma_{2}}^{N}\left(V_{k_{\alpha}(\kappa)}^{N}\right) .
$$

So $\operatorname{Th}_{\Sigma_{2}}^{M}\left(V_{\alpha}\right)=j(t) \cap V_{\alpha}=k_{\alpha}(t) \cap V_{\alpha}=\operatorname{Th}_{\Sigma_{2}}^{N}\left(V_{\alpha}\right)$.
Now since $\kappa$ is $\Sigma_{2}$-strong, $M \models$ " $\beta=j(\kappa)$ is $\Sigma_{2}$-strong". So let $\alpha \in$ OR be a strong limit cardinal. Then $M$ has an embedding $\ell: M \rightarrow N$ with $\operatorname{cr}(\ell)=\beta$ and $V_{\alpha}^{M}=V_{\alpha}^{N}$ and $\operatorname{Th}_{\Sigma_{2}}^{M}\left(V_{\alpha}^{M}\right)=\operatorname{Th}_{\Sigma_{2}}^{N}\left(V_{\alpha}^{M}\right)$. By the claim and elementarity, $N \models " \kappa$ is $<\ell(\beta)-\Sigma_{2}$-strong". But then extenders in $N$ which witness $<\alpha$ - $\Sigma_{2}$-strength in $N$ also witness this in $M$. Since $\alpha$ was arbitrary, we are done.

We now prove an analogue of Usuba's extendibility result down lower:
3.9 Theorem. Suppose $\kappa$ is $\Sigma_{2}$-strong. Then $V_{\kappa+1}^{\mathcal{M}_{\kappa}}=V_{\kappa+1}^{\mathscr{M}}$.

Proof. Suppose not and let $r$ be such that $V_{\kappa+1}^{W_{r}} \subsetneq V_{\kappa+1}^{\mathcal{M}_{\kappa}}$. Let $\lambda \in$ OR be such that $\beth_{\lambda}=\lambda$ and $r \in V_{\lambda}$. Let $j: V \rightarrow M$ witness $\Sigma_{2}$-strength with respect to $\lambda$.

Since the class of true indices is $\Pi_{2}, M \models$ " $r$ is a true index". Also, by the local definability of grounds,

$$
W_{r}^{M} \cap V_{\lambda}=W_{r}^{V_{\lambda}^{M}}=W_{r}^{V_{\lambda}}=W_{r} \cap V_{\lambda} .
$$

In particular, $V_{\kappa+1}^{W_{r}^{M}}=V_{\kappa+1}^{W_{r}} \subsetneq V_{\kappa+1}^{\mathcal{M}_{\kappa}}$.
Since $r \in V_{\lambda} \subseteq V_{j(\kappa)}^{M}$, therefore $\mathscr{M}_{j(\kappa)}^{M} \cap V_{\kappa+1} \subsetneq \mathscr{M}_{\kappa} \cap V_{\kappa+1}$. But since $\operatorname{cr}(j)=\kappa$, as in the proof of Theorem 3.2 we have

$$
\mathscr{M}_{\kappa} \cap V_{\kappa+1} \subseteq \mathscr{M}_{j(\kappa)}^{M} \cap V_{\kappa+1}
$$

a contradiction.
3.10 Question. Suppose $\kappa$ is strong. Is $V_{\kappa+1}^{\mathcal{M}}=V_{\kappa+1}^{M_{\kappa}}$ ?

We now move toward the positive results in the cases that $\kappa$ is inaccessible and/or weakly compact. Toward these we first prove a couple of lemmas.
3.11 Lemma ( $\kappa$-uniform hulls). Let $\kappa$ be inaccessible. For true indices $r \in V_{\kappa}$, let $\left(\mathbb{P}_{r}, G_{r}\right)$ witness this, and otherwise let $\mathbb{P}_{r}=G_{r}=\emptyset$. Let $\lambda=\beth_{\lambda}$ with $\operatorname{cof}(\lambda)>\kappa$ and $V_{\lambda} \preccurlyeq_{2} V$. Let $S \in V_{\lambda}$. Then there is $\widetilde{X}$ such that, letting $\widetilde{X}_{r}=\widetilde{X} \cap V_{\lambda}^{W_{r}}$ for $r \in V_{\kappa}$, we have:

1. $V_{\kappa} \cup\{S, \kappa\} \subseteq \widetilde{X} \preccurlyeq V_{\lambda}$ and $\widetilde{X}^{<\kappa} \subseteq \widetilde{X}$ and $|\widetilde{X}|=\kappa$,
2. $\widetilde{X}_{r} \in W_{r}$ and $\widetilde{X}_{r} \preccurlyeq V_{\lambda}^{W_{r}} \preccurlyeq 2 W_{r}$,
and letting $X$ be the transitive collapse of $\widetilde{X}$ and $\sigma: X \rightarrow \widetilde{X}$ the uncollapse and $X_{r}, \sigma_{r}$ likewise, then:
3. $X_{r} \subseteq X$ and in fact, $X_{r}=W_{r}^{X}$,
4. $\sigma: X \rightarrow V_{\lambda}$ is fully elementary with $\operatorname{cr}(\sigma)>\kappa$,
5. $\sigma_{r}: X_{r} \rightarrow V_{\lambda}^{W_{r}}$ is fully elementary and $\sigma_{r} \subseteq \sigma$,
6. $G_{r}$ is $\left(X_{r}, \mathbb{P}_{r}\right)$-generic and $X=X_{r}\left[G_{r}\right]$,
7. $\mathscr{M}_{\kappa}^{X}=\mathscr{M}_{\kappa}^{X_{r}}=\bigcap_{s \in V_{\kappa}} X_{s}$; hence $\mathscr{M}_{\kappa}^{X} \in \mathscr{M}_{\kappa}$,
8. $X^{<\kappa} \subseteq X$ and $X_{r}^{<\kappa} \cap W_{r} \subseteq X_{r}$ and $\left(\mathscr{M}_{\kappa}^{X}\right)^{<\kappa} \cap \mathscr{M}_{\kappa} \subseteq \mathscr{M}_{\kappa}^{X}$,
9. $\sigma \upharpoonright \mathscr{M}_{\kappa}^{X}=\sigma_{r} \upharpoonright \mathscr{M}_{\kappa}^{\tilde{\tilde{X}}_{r}}$; hence $\sigma \upharpoonright \mathscr{M}_{\kappa}^{X} \in \mathscr{M}_{\kappa}$,
10. $\sigma \upharpoonright \mathscr{M}_{\kappa}^{X}: \mathscr{M}_{\kappa}^{X} \rightarrow \mathscr{M}_{\kappa}^{V_{\lambda}}$ is fully elementary.
11. $V_{\lambda}, \widetilde{X}, X, \widetilde{X}_{r}, X_{r}$ each satisfy $T_{1}$ and the following statements:
(a) "There are unboundedly many $\eta$ such that $\eta=\beth_{\eta}$ ",
(b) "Fact 2.15",
(c) "There is $\xi=\beth_{\xi}$ such that for each $r \in V_{\kappa}$ and $s \in V_{\kappa}^{W_{r}}$, we have $W_{r} \models$ " $s$ is a index" iff $V_{\xi}^{W_{r}} \models$ " $s$ is a true index".

Proof. The fact that $V_{\lambda}^{W_{r}} \preccurlyeq 2 W_{r}$ is by Lemma 2.17.
Construct an increasing sequence $\left\langle\widetilde{X}_{\alpha}\right\rangle_{\alpha<\kappa}$ such that $\widetilde{X}_{\alpha} \preccurlyeq V_{\lambda}$ and $V_{\kappa} \cup\{x\} \subseteq \widetilde{X}_{\alpha}$ and $\widetilde{X}_{\alpha}^{<\kappa} \subseteq \widetilde{X}_{\alpha}$ and $\left|\widetilde{X}_{\alpha}\right|=\kappa$, and such that for each $r \in V_{\kappa}$ there are cofinally many $\alpha<\kappa$ such that $\widetilde{X}_{\alpha} \cap W_{r} \in W_{r}$.

To construct this sequence, suppose we have constructed $\widetilde{X}_{\alpha}$, and let $r \in V_{\kappa}$. Let $X=\widetilde{X}_{\alpha} \cap W_{r}$. By elementarity, $X \preccurlyeq V_{\lambda}^{W_{r}}$ and

$$
\widetilde{X}_{\alpha}=X\left[G_{r}\right]=\left\{\tau_{G_{r}} \mid \tau \in \overline{\tilde{X}}\right\}
$$

Since $|X|=\kappa$, there is some $\widetilde{X}^{\prime} \in W_{r}$ with $\left|\tilde{X}^{\prime}\right|=\kappa$ (hence $W_{r} \models$ " $\left|\tilde{X}^{\prime}\right|=\kappa$ "), and $X \subseteq \widetilde{X}^{\prime}$, so there is also $\widetilde{X}^{\prime \prime} \in W_{r}$ with $\widetilde{X}^{\prime \prime} \preccurlyeq V_{\lambda}^{W_{r}}$ and $\widetilde{X}^{\prime} \subseteq \widetilde{X}^{\prime \prime}$ and $\left|\widetilde{X}^{\prime \prime}\right|=\kappa$ (in $V$ and $\left.W_{r}\right)$ and such that $W_{r} \models "\left(\widetilde{X}^{\prime \prime}\right)^{<\kappa} \subseteq\left(\widetilde{X}^{\prime \prime}\right)$ ". It easily follows that

$$
\widetilde{X}_{\alpha} \subseteq \widetilde{X}^{\prime \prime}\left[G_{r}\right]=\left\{\tau_{G_{r}} \mid \tau \in \widetilde{X}^{\prime \prime}\right\} \preccurlyeq V_{\lambda}
$$

and $\widetilde{X}^{\prime \prime}\left[G_{r}\right] \cap W_{r}=\widetilde{X}^{\prime \prime}$. We set $\widetilde{X}_{\alpha+1}=\widetilde{X}^{\prime \prime}\left[G_{r}\right]$. Then everything is clear except for the requirement that $\widetilde{X}_{\alpha+1}^{<\kappa} \subseteq \widetilde{X}_{\alpha+1}$. So let $f: \gamma \rightarrow \widetilde{X}_{\alpha+1}$ where $\gamma<\kappa$ (with $f \in V$ ); we claim that $f \in \widetilde{X}_{\alpha+1}$. Let $g: \gamma \rightarrow \widetilde{X}^{\prime \prime}$ be such that $g(\alpha)_{G_{r}}=f(\alpha)$ for each $\alpha<\gamma$. So $g \in V$, but we don't know that $g \in W_{r}$. But there is a $\mathbb{P}_{r}$-name $\dot{g} \in V_{\lambda}^{W_{r}}$ such that $\dot{g}_{G_{r}}=g$. And $\widetilde{X}^{\prime \prime} \in W_{r}$, so there is $p_{0} \in G_{r}$ forcing that $\operatorname{rg}(\dot{g}) \subseteq \widetilde{X}^{\prime \prime}$. Working in $W_{r}$ then, we may fix for each $\alpha<\gamma$ an antichain $A_{\alpha} \subseteq \mathbb{P}_{r}$ maximal below $p_{0}$ and for each $p \in A_{\alpha}$ some $\tau_{\alpha p} \in \widetilde{X}^{\prime \prime}$ such that $p$ forces that $\dot{g}(\alpha)=\tau_{\alpha p}$. Then the sequence $\left\langle\tau_{\alpha p}\right\rangle_{(\alpha, p) \in I}$, where

$$
I=\left\{(\alpha, p) \mid \alpha<\gamma \text { and } p \in A_{\alpha}\right\}
$$

is $\subseteq \widetilde{X}^{\prime \prime}$, and hence in $\widetilde{X}^{\prime \prime}$. Since $W_{r} \models$ " $\left(\widetilde{X}^{\prime \prime}\right)^{<\kappa} \subseteq\left(\widetilde{X}^{\prime \prime}\right)^{\prime}$, this gives a name $\dot{g}^{\prime \prime} \in \widetilde{X}^{\prime \prime}$ such that $p_{0}$ forces $\dot{g}^{\prime \prime}=\dot{g}$, and therefore

$$
g=\dot{g}_{G_{r}}=\dot{g}_{G_{r}}^{\prime \prime} \in \widetilde{X}^{\prime \prime}\left[G_{r}\right]=\widetilde{X}_{\alpha+1}
$$

But since $G_{r} \in \widetilde{X}_{\alpha+1}$, therefore $f \in \widetilde{X}_{\alpha+1}$, so $\widetilde{X}_{\alpha+1}^{<\kappa} \subseteq \widetilde{X}_{\alpha+1}$ as desired. With some simple bookkeeping then, we get an appropriate sequence.

Let now $\widetilde{X}=\bigcup_{\alpha<\kappa} \widetilde{X}_{\alpha}$. We claim that $\widetilde{X}$ is as desired. The only thing we need to verify is that for each $r \in V_{\kappa}$, we have

$$
\tilde{X}_{r}=\widetilde{X} \cap W_{r} \in W_{r} .
$$

Fix $r$. There is a $\mathbb{P}_{r}$-name $\tau \in W_{r}$ such that $\tau_{G_{r}}=\left\langle\widetilde{X}_{\alpha}\right\rangle_{\alpha<\kappa}$, and for cofinally many $\alpha<\kappa$ there is $p_{\alpha} \in G_{r}$ and $\widetilde{X}_{\alpha}^{r} \in W_{r}$ such that

$$
p_{\alpha} \Perp-\tau_{\alpha} \cap W_{r}=\check{\widetilde{X}}_{\alpha}^{r}
$$

(hence $\widetilde{X}_{\alpha}^{r}=\widetilde{X}_{\alpha} \cap W_{r}$ ). Since $\mathbb{P}_{r} \in V_{\kappa}$, there is therefore a fixed $p \in \mathbb{P}_{r}$ such that $p_{\alpha}=p$ for cofinally many $\alpha$. So $\widetilde{X}_{r}=\bigcup_{\alpha \in I} \widetilde{X}_{\alpha}^{r}$ where

$$
I=\left\{\alpha<\kappa \mid \exists x\left[p \|-\tau_{\alpha}=\check{x}\right]\right\}
$$

so $\widetilde{X}_{r} \in W_{r}$.

This completes the construction. The verification of the remaining properties is now straightforward. We omit discussing them, other than two remarks. In part 8 the third statement follows directly from the first two together with part 7 , the first two follow readily from the construction. And in part 11 note that $\xi$ exists because $\operatorname{cof}(\lambda)>\kappa=\left|V_{\kappa}\right|$.
3.12 Fact. Let $\kappa$ be weakly compact. Then $X$ be transitive with $\kappa \in X$ and $X^{<\kappa} \subseteq X$ and $|X|=\kappa{ }^{11}$ Then there is a non-principal $X$ - $\kappa$-complete $X$-norma ${ }^{12}$ ultrafilter $\mu$ over $\kappa$ such that letting $Y=\operatorname{Ult}(X, \mu)$ and $i_{\mu}^{X}$ the ultrapower embedding, then $Y$ is wellfounded. Moreover, $i_{\mu}^{X}$ is $\Sigma_{1}$-elementary and cofinal and $\operatorname{cr}\left(i_{\mu}^{X}\right)=\kappa$.

Proof. Let $\pi: X \rightarrow Z$ be any elementary embedding with $Z$ transitive and $\operatorname{cr}(\pi)=\kappa$. Let $\mu$ be the normal measure derived from $\pi$. Note that $\mu$ works.

We now extend the situation above, adding the assumption that $\kappa$ is weakly compact.
3.13 Lemma ( $\kappa$-uniform weak compactness embedding). Adopt the assumptions and notation from the statement and proof of Lemma 3.11. Assume further that $\kappa$ is weakly compact. Let $\pi: X \rightarrow Y$ witness the weak compactness of $\kappa$ in $V$, with $Y=\operatorname{Ult}(X, \mu)$ for an $X$ - $\kappa$-complete $X$-normal ultrafilter $\mu$ over $\kappa$, and $\pi=i_{\mu}^{X}$. For $r \in V_{\kappa}$, let $\mu_{r}=\mu \cap X_{r}$. Then:

1. $\mu_{r} \in W_{r}$ and $\mu_{r}$ is an $X_{r}-\kappa$-complete ultrafilter over $\kappa$; let

$$
Y_{r}=\operatorname{Ult}\left(X_{r}, \mu_{r}\right) \text { and } \pi_{r}: X_{r} \rightarrow Y_{r}
$$

the ultrapower map; so $Y_{r}, \pi_{r} \in W_{r}$,
2. $\mu$ is the $X$-ultrafilter generated by $\mu_{r}$ ( $\mu_{r}$ is dense in $\mu$ ).
3. For each $f: \kappa \rightarrow X_{r}$ with $f \in X$, there is $f_{r} \in X_{r}$ with $f_{r}: \kappa \rightarrow X_{r}$ and $f_{r}(\alpha)=f(\alpha)$ for $\mu$-measure one many $\alpha<\kappa 13$
4. The ultrapowers satisfy Los' theorem for $\Sigma_{1}$ formulas, and $\pi_{r}, \pi$ are $\Sigma_{2}$-elementary.
5. $Y, Y_{r} \models T_{1}$ and $Y_{r}$ is transitive, $Y_{r}=W_{r}^{Y}$, and $Y=Y_{r}\left[G_{r}\right]$.
6. $\pi_{r} \subseteq \pi$.
7. $\mathscr{M}_{\pi(\kappa)}^{Y}=\mathscr{M}_{\pi_{r}(\kappa)}^{Y_{r}} \in W_{r}$; hence this belongs to $\mathscr{M}_{\kappa}$.
8. $\pi \upharpoonright \mathscr{M}_{\kappa}^{X}: \mathscr{M}_{\kappa}^{X} \rightarrow \mathscr{M}_{\pi(\kappa)}^{Y}$ is cofinal $\Sigma_{1}$-elementary; this map belongs to $\mathscr{M}_{\kappa}$.
9. $\mathscr{M}_{\kappa}^{Y}=\bigcap_{s \in V_{\kappa}} W_{s}^{Y}=\mathscr{M}_{\kappa}^{Y_{r}} \in W_{r}$; hence this belongs to $\mathscr{M}_{\kappa}$.
10. $Y, Y_{r}$ each satisfy $T_{1}$ and the following statements:
(a) "There are unboundedly many $\eta$ such that $\eta=\beth_{\eta}$ ",
(b) "Fact 2.15 holds at $\theta=\pi(\kappa)=\beth_{\pi(\kappa)} "$,

[^7](c) "There is $\xi=\beth_{\xi}$ such that for each $r \in V_{\pi(\kappa)}$ and $s \in V_{\pi(\kappa)}^{W_{r}}$, we have $W_{r} \models$ "s is true" iff $V_{\xi}^{W_{r}} \models$ "s is true".

Therefore there is $t \in V_{\pi(\kappa)}^{Y}$ with $W_{t}^{Y} \subseteq \mathscr{M}_{\kappa}^{Y}$.
Proof. Parts 13: These are simple variants of the version for measurable cardinals $\kappa$ of $V[G]$ via small forcing $\mathbb{P} \in V_{\kappa}$; one uses especially, however, the fact that $X_{r}$ is $<\kappa$-closed in $W_{r}$. We leave the details to the reader.

Part 4 Note that $V_{\lambda}$ satisfies $\Sigma_{1}$-Collection and "For all $\alpha \in \mathrm{OR}, V_{\alpha}$ exists and $\beth_{\alpha} \in \mathrm{OR}$ exists, and $\mathrm{OR}=\beth_{\mathrm{OR}} "$, so $X_{r}, X$ do also. Therefore if $\varphi$ is $\Sigma_{0}$ and $x \in X$ and

$$
X \models \forall \alpha<\kappa \exists y \varphi(x, y, \alpha)
$$

then some $V_{\xi}^{X} \in X$ satisfies the same statement, and hence there is $f \in X$ picking witnesses $y$. This gives Los' theorem for $\Sigma_{1}$ formulas. The $\Sigma_{2}$-elementarity of $\pi$ : $X \rightarrow Y$ follows. Likewise for $X_{r}, \pi_{r}$.

Parts [5. 6] The fact that $Y, Y_{r} \models T_{1}$ follows from $\Sigma_{2}$-elementarity and cofinality of $\pi, \pi_{r}$, and (for $\Sigma_{1}$-Collection) that for each $\xi \in \mathrm{OR}^{X}$, we have $\mathcal{H}_{\xi}^{X} \preccurlyeq 1 X$ and $\mathcal{H}_{\xi}^{X_{r}} \preccurlyeq_{1} X_{r}$. The rest follows as usual from the fact that functions in $X$ with codomain $X_{r}$ are represented in $X_{r}$ (part 3), and again the $\Sigma_{2}$-elementarity of $\pi, \pi_{r}$.

Parts 7. 8 Basically by invariance of $\mathscr{M}_{\kappa}$ (Fact 2.16), we have $\mathscr{M}_{\kappa}^{X}=\mathscr{M}_{\kappa}^{X_{r}}$, and by part 11 of 3.11 there is $\xi<\mathrm{OR}^{X}$ such that for each $r \in V_{\kappa}$ and $s \in V_{\kappa}^{W_{r}}$, we have $X_{r} \models$ " $s$ is true" iff $V_{\xi}^{X_{r}} \models$ " $s$ is true". Let

$$
T_{r}=\left\{s \in V_{\kappa}^{W_{r}} \mid W_{r} \models \text { " } s \text { is true" }\right\} .
$$

So $T_{r} \in X_{r}$ and has the same definition there; likewise for $T_{r} \in Y_{r}$, since $\pi_{r}$ is $\Sigma_{2}$-elementary. And because of the existence of $\xi$,

$$
\pi\left(T_{r}\right)=\left\{s \in V_{\pi_{r}(\kappa)}^{Y_{r}} \mid Y_{r} \models " s \text { is true" }\right\}
$$

and it follows (in the case of $r=\emptyset$, but similarly in general),

$$
\mathscr{M}_{\pi(\kappa)}^{Y}=\left(\bigcap_{s \in V_{\pi(\kappa)}^{Y}} W_{s}^{Y}\right)=\left(\bigcup_{\zeta \in I} \pi\left(\mathscr{M}_{\kappa}^{V_{\zeta}}\right)\right)
$$

where $I$ is the set of all $\zeta \in\left[\xi, \mathrm{OR}^{X}\right)$ such that $\beth_{\zeta}^{X}=\zeta$. But $\mathscr{M}_{\kappa}^{X}=\mathscr{M}_{\kappa}^{X_{r}}$ and $\pi_{r} \subseteq \pi$, so $\mathscr{M}_{\pi(\kappa)}^{Y}=\mathscr{M}_{\pi_{r}(\kappa)}^{Y_{r}}$. The calculations above also show that

$$
\pi \upharpoonright \mathscr{M}_{\kappa}^{X}: \mathscr{M}_{\kappa}^{X} \rightarrow \mathscr{M}_{\pi(\kappa)}^{Y}
$$

is cofinal $\Sigma_{1}$-elementary, and likewise for $\pi_{r} \subseteq \pi$.
Part 9. By part 5, $W_{s}^{Y}=Y_{s}$, so $\mathscr{M}_{\kappa}^{Y}=\widehat{\bigcap}_{s \in V_{\kappa}} W_{s}^{Y}$. And note that the density of the grounds of $X_{r}$ in the grounds of $X$ is lifted to that for those of $Y_{r}$ in those of $Y$. (That is, for example, if $r, s$ are such that $X_{r} \subseteq X_{s}$, then $Y_{r} \subseteq Y_{s}$, as this is preserved by $\pi$.) So $\mathscr{M}_{\kappa}^{Y_{r}}=\mathscr{M}_{\kappa}^{Y}$, as desired.

Part 10a For each $\zeta \in X$ with $\zeta=\beth_{\zeta}^{X}$, we have $\pi(\zeta)=\beth_{\pi(\zeta)}^{Y}$.
Part 10c If $\xi$ witnesses the corresponding statement in $X$, note that $\pi(\xi)$ works in $Y$.

Part 10b; We consider literally $Y$, but the same proof works for $Y_{r}$. Note that there is a function $f: V_{\kappa} \rightarrow V_{\kappa}$ with $f \in X$, such that for each $R \in V_{\kappa}, X \models " t=f(R)$
is a true index and $t$ witnesses Fact 2.15 for $R "$ ( $f$ exists by the elementarity of $\sigma$ ). We claim that $\pi(f)$ has the same property for $Y$. For by $\Pi_{2}$-elementarity, $Y \models$ "Every $t \in \operatorname{rg}(\pi(f))$ is a true index". Moreover, let $\xi$ be as before. Then for each $\zeta$ such that $\xi<\zeta<\mathrm{OR}^{X}$ and $\zeta=\beth_{\zeta}^{X}, V_{\zeta}^{X}$ satisfies " $W_{f(R)} \subseteq W_{r}$ for each $R \in V_{\kappa}$ and $r \in R$ ". This lifts to $Y$ under $\pi$, and since $\pi$ is cofinal, this suffices.

We are now ready to prove the main theorem for weakly compact $\kappa$. The first proof that, under this assumption, $\mathscr{M}_{\kappa} \models$ " $V_{\kappa}$ is wellordered" is due to Lietz:
3.14 Theorem. Let $\kappa$ be weakly compact. Then:

1. $\mathscr{M}_{\kappa} \models \kappa$-DC + " $\kappa$ is weakly compact" 14
2. for each $A \in \mathscr{M}_{\kappa} \cap \mathcal{H}_{\kappa^{+}}, \mathscr{M}_{\kappa} \models$ " $A$ is wellordered". 15
3. if $\mathcal{P}(\kappa)^{\mathscr{M}_{\kappa}}$ has cardinality $\kappa$ then (i) $\kappa$ is measurable in $\mathscr{M}_{\kappa}$, and (ii) $x^{\#}$ exists for every $x \in \mathcal{P}(\kappa)^{\mathscr{M}_{\kappa}}$.
4. If $\mathscr{M}_{\kappa}=$ " $\mu$ is a countably complete ultrafilter over $\gamma \leq \kappa$ ", then the ultrapower $\operatorname{Ult}\left(\mathscr{M}_{\kappa}, \mu\right)$ is wellfounded and the ultrapower embedding

$$
i_{\mu}^{\mathscr{M}_{\kappa}}: \mathscr{M}_{\kappa} \rightarrow \operatorname{Ult}\left(\mathscr{M}_{\kappa}, \mu\right)
$$

is fully elementary.
Proof. Part 4 follows directly from part [1 as the wellfoundedness of $\operatorname{Ult}\left(\mathscr{M}_{\kappa}, \mu\right)$ requires only $\omega$-DC, and the proof of Los' theorem here only uses $\kappa$-choice. The conclusion that $x^{\#}$ exists in part 3 follows easily from the rest, using the elementarity of $i_{\mu}$ and that $\operatorname{Ult}\left(\mathscr{M}_{\kappa}, \mu\right)$ is wellfounded. To see that $\mathscr{M}_{\kappa} \models$ " $\kappa$ is weakly compact", let $T \subseteq{ }^{<\kappa} 2$ be a tree in $\mathscr{M}_{\kappa}$. Then $T$ has a cofinal branch $b$ in $V$, by weak compactness in $V$. But $b \cap V_{\alpha} \in \mathscr{M}_{\kappa}$ for each $\alpha<\kappa$. Therefore by 2.20, $b \in \mathscr{M}_{\kappa}$.

Here is Lietz' argument that $\mathscr{M}_{\kappa} \models$ " $V_{\kappa}$ is wellordered" ${ }^{16}$ Working in $\mathscr{M}_{\kappa}$, let $T$ be the tree of all attempts to build a wellorder of $V_{\kappa}$. (For example, let $T \subseteq{ }^{<\kappa} V_{\kappa}$ be the set of all functions $f: \alpha \rightarrow V_{\kappa}$ where $\alpha<\kappa$, such that for each $\beta<\alpha, f(\beta)$ is a wellorder of $V_{\beta}$, and for all $\beta_{1}<\beta_{2}<\alpha, f\left(\beta_{2}\right)$ is an end extension of $f\left(\beta_{1}\right)$.) Since $V_{\kappa}^{M_{\kappa}} \models$ ZFC, $T$ is unbounded in $V_{\kappa}$, and clearly $T \upharpoonright \alpha \in V_{\kappa}$ for each $\alpha<\kappa$. Therefore by weakly compactness in $\mathscr{M}_{\kappa}, \mathscr{M}_{\kappa}$ has a $T$-cofinal branch, and clearly this gives a wellorder of $V_{\kappa} \cap \mathscr{M}_{\kappa}$.

We proceed now to the proof that $\mathscr{M}_{\kappa} \models \kappa$-DC, and that every set $A \in \mathscr{M}_{\kappa} \cap \mathcal{H}_{\kappa^{+}}$ is wellordered in $\mathscr{M}_{\kappa}$. Let $\mathscr{T} \in \mathscr{M}$ be a $\kappa$-DC-tree 17 and let $A \in \mathscr{M}_{\kappa} \cap \mathcal{H}_{\kappa^{+}}$. Let $S=(\mathscr{T}, A) \in V_{\lambda}$ and $X$ be a $\kappa$-uniform hull, etc, with $S \in X$ and everything as in Lemma 3.11 and let $\pi: X \rightarrow Y$, etc, be as in Lemma 3.13. So $\sigma: X \rightarrow V_{\lambda}$ is fully elementary with $\kappa<\operatorname{cr}(\sigma)$. Let $\sigma(\overline{\mathscr{T}})=\mathscr{T}$ and $\sigma(A)=A$.

By 3.13, $\pi^{\prime}=\pi \upharpoonright \mathscr{M}_{\kappa}^{X}: \mathscr{M}_{\kappa}^{X} \rightarrow \mathscr{M}_{\pi(\kappa)}^{Y}$ is cofinal $\Sigma_{1}$-elementary, and these models and map belong to $\mathscr{M}_{\kappa}$. We have $A, \overline{\mathscr{T}} \in \mathscr{M}_{\kappa}^{X}$.

[^8]We first find a wellorder of $A$ in $\mathscr{M}_{\kappa}$, by arguing as in Schindler's proof of Fact 3.2, but using the weak compactness embedding. We have $\pi^{\prime}(A) \in \mathscr{M}_{\pi(\kappa)}^{Y}$. By 3.13, there is a ground $W$ of $\mathscr{M}_{\pi(\kappa)}^{Y}$ such that

$$
\mathscr{M}_{\pi(\kappa)}^{Y} \subseteq W \subseteq \mathscr{M}_{\kappa}^{Y} \in \mathscr{M}_{\kappa}
$$

So $W \models \mathrm{AC}$ and $\pi^{\prime}(A) \in W$. Let $<^{*} \in W$ be a wellorder of $\pi^{\prime}(A)$. So $<^{*} \in \mathscr{M}_{\kappa}$. Working in $\mathscr{M}_{\kappa}$, we can therefore wellorder $A$ by setting, for $x, y \in A$ :

$$
x<_{A} y \Longleftrightarrow \pi^{\prime}(x)<^{*} \pi^{\prime}(y)
$$

We now find a branch through $\overline{\mathscr{T}}$ in $\mathscr{M}_{\kappa}$, with length $\kappa$. Let $B \in \mathscr{M}_{\kappa}^{X}$ be the field of $\overline{\mathscr{T}}$. As above, there is a wellorder $<^{*}$ of $B$ in $\mathscr{M}_{\kappa}$. Working in $\mathscr{M}_{\kappa}$, we recursively construct a sequence $\left\langle x_{\alpha}\right\rangle_{\alpha<\kappa}$ constituting a branch through $\overline{\mathscr{T}}$, using $<^{*}$ to pick next elements, and noting that at limit stages $\eta<\kappa$, we get $\left\langle x_{\alpha}\right\rangle_{\alpha<\eta} \in \mathscr{M}_{\kappa}^{X}$, because by 3.13 part 8 we have $\left(\mathscr{M}_{\kappa}^{X}\right)^{<\kappa} \cap \mathscr{M}_{\kappa} \subseteq \mathscr{M}_{\kappa}^{X}$. By 3.11] $\sigma^{\prime}=\sigma \upharpoonright \mathscr{M}_{\kappa}^{X} \in \mathscr{M}_{\kappa}$, and note that $\left\langle\sigma^{\prime}\left(x_{\alpha}\right)\right\rangle_{\alpha<\kappa}$ is a cofinal branch through $\mathscr{T}$, as desired.

Part 3 Now suppose $\mathcal{P}(\kappa) \cap \mathscr{M}_{\kappa} \in \mathcal{H}_{\kappa^{+}}$. Then we may assume that $A=\mathcal{P}(\kappa) \cap$ $\mathscr{M}_{\kappa}$ above. Therefore $\pi^{\prime}: \mathscr{M}_{\kappa}^{X} \rightarrow \mathscr{M}_{\pi(\kappa)}^{Y}$ is $\mathscr{M}_{\kappa}$-total. Therefore $\kappa$ is measurable in $\mathscr{M}_{\kappa}$. Since $\mathscr{M}_{\kappa} \models \kappa$-DC, the rest now follows, as discussed in the first paragraph of the proof.

Recall $(\alpha, X)$-Choice from Definition 1.1
3.15 Theorem. Let $\kappa$ be inaccessible (so $\mathscr{M}_{\kappa} \models$ " $\kappa$ is inaccessible"). Then:

1. $\mathscr{M}_{\kappa}$ is $\kappa$-amenably-closed.
2. $\mathscr{M}_{\kappa} \models$ " $\left(\kappa, \mathcal{H}_{\kappa}\right)$-Choice" iff $\mathscr{M}_{\kappa} \models$ " $V_{\kappa}$ is wellordered".
3. $\mathscr{M} \models$ " $\left(<\kappa, \mathcal{H}_{\kappa^{+}}\right)$-Choice holds, and hence, $\left(\mathcal{H}_{\kappa^{+}}\right)^{<\kappa} \subseteq \mathcal{H}_{\kappa^{+}}$".
3.16 Remark. Note that in part 3 the " $\kappa^{+}$" and " $\mathcal{H}_{\kappa^{+}}$" are both in the sense of $\mathscr{M}_{\kappa}$. Note that also, as $\kappa$ is inaccessible, $V_{\kappa}^{\mathscr{M}_{\kappa}} \models$ ZFC, $\mathscr{M}_{\kappa} \models$ " $\kappa$ is inaccessible", and $\mathscr{M}_{\kappa}$ is $\kappa$-amenable closed, by Lemma 2.22,

Proof. Part 1 was Lemma 2.22, and since $V_{\kappa}^{\mathscr{M}_{\kappa}}=\mathcal{H}_{\kappa}^{\mathscr{M}_{\kappa}} \models$ ZFC, part 2 is easy.
Part 3. Let $\gamma<\kappa$ and $f \in \mathscr{M}_{\kappa}$ with $f: \gamma \rightarrow\left(\mathcal{H}_{\kappa^{+}}\right)^{\mathscr{M}_{\kappa}}$. We find a choice function for $f$ in $\mathscr{M}_{\kappa}$. Write $f_{\alpha}=f(\alpha)$. Fix a function $g: \gamma \rightarrow \mathscr{M}_{\kappa}$ with

$$
g_{\alpha}=g(\alpha): \kappa \rightarrow \operatorname{trancl}\left(f_{\alpha}\right)
$$

surjective for each $\alpha<\gamma$. Let $c: \gamma \rightarrow \mathscr{M}_{\kappa}$ be $c_{\alpha}=c(\alpha) \subseteq \kappa$ the induced code for $g_{\alpha}$ (so $c_{\alpha}, g_{\alpha} \in \mathscr{M}_{\kappa}$, but note we don't know that $c, g \in \mathscr{M}_{\kappa}$ ). Fix $\lambda$ and a $\kappa$-uniform hull $\widetilde{X} \preccurlyeq V_{\lambda}$ with $f, c, g \in \widetilde{X}$ and everything else as in 3.11. So $\sigma(f, c, g)=(f, c, g)$. Fix a club $C$ of $\bar{\kappa}<\kappa$ such that $\gamma<\bar{\kappa}$ and $V_{\bar{\kappa}} \preccurlyeq V_{\kappa}$ and such that we get a corresponding system of structures $X_{r}^{\bar{\kappa}}$ and elementary embeddings $\pi_{r}^{\bar{\kappa}}: X_{r}^{\bar{\kappa}} \rightarrow X_{r}$, for $r \in V_{\bar{\kappa}}$, with $X_{r}^{\bar{\kappa}}, \pi_{r}^{\bar{\kappa}} \in W_{r}, X_{r}^{\bar{\kappa}}$ of cardinality $\bar{\kappa}$ in $W_{r}, \operatorname{cr}\left(\pi_{r}^{\bar{\kappa}}\right)=\bar{\kappa}$ and $\pi_{r}^{\bar{\kappa}}(\bar{\kappa})=\kappa$, and each $X_{r}^{\bar{\kappa}}\left[G_{r}\right]=X_{\emptyset \bar{\kappa}}$ and $\pi_{r}^{\bar{\kappa}} \subseteq \pi_{\emptyset \bar{\kappa}}$, and with $f, c_{\alpha}, g_{\alpha} \in \operatorname{rg}\left(\pi_{r}^{\bar{\kappa}}\right)$ for each $\alpha<\gamma$. Write $\pi_{r}^{\bar{\kappa}}\left(f^{\bar{\kappa}}, c_{\alpha}^{\bar{\kappa}}, g_{\alpha}^{\bar{\kappa}}\right)=\left(f, c_{\alpha}, g_{\alpha}\right)$. So $c_{\alpha}^{\bar{\kappa}}=c_{\alpha} \cap \bar{\kappa}$, so $c_{\alpha}^{\bar{\kappa}}, g_{\alpha}^{\bar{\kappa}} \in\left(\mathcal{H}_{\bar{\kappa}^{+}}\right)^{\mathscr{M}_{\kappa}}$, and $f^{\bar{\kappa}}: \gamma \rightarrow\left(\mathcal{H}_{\bar{\kappa}^{+}}\right)^{\mathscr{M}_{\kappa}}$ with $f_{\alpha}^{\bar{\kappa}} \subseteq \operatorname{rg}\left(g_{\alpha}^{\bar{\kappa}}\right)$. Let $c^{\bar{\kappa}}: \gamma \rightarrow \mathscr{M}_{\kappa}$ be $c^{\bar{\kappa}}(\alpha)=c_{\alpha}^{\bar{\kappa}}$ and likewise for $g^{\bar{\kappa}}$.

In $V$, pick a sequence $\left\langle{<_{\bar{\kappa}}}_{\rangle_{\bar{\kappa} \in C}}\right.$ of wellorders ${<_{\bar{\kappa}}}$ of $\left(\mathcal{H}_{\bar{\kappa}^{+}}\right)^{\mathscr{M}_{\kappa}}$ with ${<_{\bar{\kappa}}} \in \mathscr{M}_{\kappa}$. Let $z_{\alpha}^{\bar{\kappa}}$ be the $<_{\bar{\kappa}}$-least element of $f_{\alpha}^{\kappa}$, and let $\xi_{\alpha}^{\bar{\kappa}}<\bar{\kappa}$ be the least $\xi$ with $g_{\alpha}^{\bar{\kappa}}(\xi)=z_{\alpha}^{\bar{\kappa}}$.

Let $S$ be the stationary set of all strong limit cardinals $\bar{\kappa} \in C$ of cofinality $\gamma^{+}$. Enumerate ${ }^{\gamma} \kappa$ as $\left\{s_{\beta}\right\}_{\beta<\kappa}$, with ${ }^{\gamma} \bar{\kappa}=\left\{s_{\beta}\right\}_{\beta<\bar{\kappa}}$ for each $\bar{\kappa} \in S$. For $\bar{\kappa} \in S$, let $\beta_{\bar{\kappa}}$ be the $\beta<\bar{\kappa}$ such that $s_{\beta}=\left\langle\xi_{\alpha}^{\bar{\kappa}}\right\rangle_{\alpha<\gamma}$. Let $S^{\prime} \subseteq S$ be stationary and such that $\beta_{\bar{\kappa}}$ is constant for $\bar{\kappa} \in S^{\prime}$.

Let $d: \gamma \rightarrow \mathscr{M}_{\kappa}$ be the choice function for $f$ given by $d(\alpha)=\pi_{\emptyset \bar{\kappa}}\left(z_{\alpha}^{\bar{\kappa}}\right)$, whenever $\bar{\kappa} \in S^{\prime}$. This is independent of $\bar{\kappa} \in S^{\prime}$. For if $\bar{\kappa}_{0}, \bar{\kappa}_{1} \in S^{\prime}$ with $\bar{\kappa}_{0}<\bar{\kappa}_{1}$, then for each $\alpha<\gamma$, we have $\xi=\xi_{\alpha}^{\bar{\kappa}_{0}}=\xi_{\alpha}^{\bar{\kappa}_{1}}$, so

$$
\pi_{\emptyset \bar{k}_{0}}\left(z_{\alpha}^{\bar{\kappa}_{0}}\right)=\pi_{\emptyset \bar{k}_{0}}\left(g_{\alpha}^{\bar{\kappa}_{0}}(\xi)\right)=g_{\alpha}(\xi)=\pi_{\emptyset \bar{\kappa}_{1}}\left(g_{\alpha}^{\bar{\kappa}_{1}}(\xi)\right)=\pi_{\emptyset \bar{\kappa}_{1}}\left(z_{\alpha}^{\bar{\kappa}_{1}}\right)
$$

But $d \in \mathscr{M}_{\kappa}$. For given $r \in V_{\kappa}$, let $\bar{\kappa} \in S^{\prime}$ with $r \in V_{\bar{\kappa}}$. Then $f^{\bar{\kappa}} \in X_{r}^{\bar{\kappa}}$ and $\pi_{r}^{\bar{\kappa}}\left(f^{\bar{\kappa}}\right)=f$, since $\pi_{r}^{\bar{\kappa}} \subseteq \pi_{\emptyset \bar{\kappa}}$. And $X_{r}^{\bar{\kappa}} \in W_{r}$, so $f^{\bar{\kappa}} \in W_{r}$. But $<_{\bar{\kappa}}$ is in $W_{r}$, and so $d^{\bar{\kappa}}=\left\langle z_{\alpha}^{\bar{\kappa}}\right\rangle_{\alpha<\gamma} \in W_{r}$. And since $\pi_{r}^{\bar{\kappa}} \subseteq \pi_{\emptyset \bar{\kappa}}, \pi_{r}^{\bar{\kappa}}\left(d^{\bar{\kappa}}\right)=d$. Since $\pi_{r}^{\bar{\kappa}} \in W_{r}$, therefore $d \in W_{r}$. So $d \in \mathscr{M}_{\kappa} \models$ "d is a choice function for $f$ ", so we are done.

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    ${ }^{1}$ Here we are not specific about exactly what formalization of classes we use. We could work in some class set theory, which allows quantification over such classes $W$, or we could with more care restrict to classes definable from parameters; for us a class must have the property that the structure $(V, \in, W)$ satisfies ZFC in the language with symbols $\dot{\in}, \dot{W}$ which interpret $\in$ and $W$.
    ${ }^{2}$ Throughout, we consider only set-forcing, no class-forcing.

[^1]:    ${ }^{3}$ An earlier draft asserted here that ([10] shows) $\mathcal{H}_{\theta}^{\mathscr{M}_{\theta}}=\mathcal{H}_{\theta}^{\mathscr{M}}$ for every strong limit cardinal $\theta$. This was however not used anywhere, and it is false, at least if $M_{1}$ exists; see 9 .

[^2]:    ${ }^{4}$ Regarding part [2] the author initially observed that a variant of Schindler's argument gives that $\mathscr{M}_{\kappa} \models \kappa$-DC, and then Lietz and the author independently noticed that the argument can be adjusted to show that every set in $\mathcal{H}_{\kappa}+\cap \mathscr{M}_{\kappa}$ is wellordered in $\mathscr{M}_{\kappa}$.
    ${ }^{5}$ So also $\mathscr{M}_{\kappa} \models$ " $\kappa^{+}$is regular and $\mathcal{H}_{\kappa}+\models \mathrm{ZFC}^{-}$".
    ${ }^{6}$ Note that the " $\kappa^{+}$" and " $\mathcal{H}_{\kappa^{+}}$" here are computed in $V$, not $\mathscr{M}_{\kappa}$.

[^3]:    ${ }^{7}$ In a previous draft of this document, it mistakenly said that the $\Sigma_{1}$-forcing relation is $\Delta_{1}^{M}$ definable, which is clearly false, since in the case of trivial forcing, it would imply that $\Sigma_{1}^{M}=\Delta_{1}^{M}$.

[^4]:    ${ }^{8}$ We wrote $R$ in the statement of the fact for consistency with later notation.

[^5]:    ${ }^{9}$ What blocks the more obvious attempt to prove this is that it is not clear that the iteration maps $i_{P Q}$ between the iterates $P, Q$ of the direct limit system eventually fix $X$.

[^6]:    ${ }^{10}$ That is, for each $\Sigma_{2}$ formula $\varphi$ and all $\vec{x} \in\left(V_{\alpha}\right)^{<\omega}$, we have $M \models \varphi(\vec{x})$ iff $V \models \varphi(\vec{x})$.

[^7]:    ${ }^{11}$ In an earlier draft, the hypothesis " $|X|=\kappa$ " was accidentally omitted, which obviously makes the statement equivalent to measurability.
    ${ }^{12}$ That is, $\kappa$-completeness and normality with respect to sequences in $X$.
    ${ }^{13}$ A draft assumed only $f: \kappa \rightarrow X$, not $f: \kappa \rightarrow X_{r}$, which obviously makes the statement false when $X \neq X_{r}$.

[^8]:    ${ }^{14}$ So also $\mathscr{M}_{\kappa} \models$ " $\kappa^{+}$is regular and $\mathcal{H}_{\kappa^{+}} \models$ ZFC $^{-"}$.
    ${ }^{15}$ Note that the " $\kappa^{+}$" and " $\mathcal{H}_{\kappa}+$ " here are computed in $V$, not $\mathscr{M}_{\kappa}$.
    ${ }^{16}$ The author first mistakenly thought that a similar argument worked with $\kappa$ only inaccessible, but Lietz noted that one seems to need weak compactness for this.
    ${ }^{17}$ That is, a set $\mathscr{F}$ of functions $f$ such that $\operatorname{dom}(f)<\kappa$, with $\mathscr{F}$ closed under initial segment, and no maximal elements; that is, for every $f \in \mathscr{F}$ there is $g \in \mathscr{F}$ with $\operatorname{dom}(f)<\operatorname{dom}(g)$ and $f=g \upharpoonright \operatorname{dom}(f)$. Note that $\kappa$-DC is just the assertion that for every $\kappa$-DC tree $\mathscr{T}$, there is a $\mathscr{T}$-maximal branch; that is, a function $f \notin \mathscr{T}$ such that $f \upharpoonright \alpha \in \mathscr{T}$ for all $\alpha<\operatorname{dom}(f)$.

