PIECE SELECTION AND CARDINAL ARITHMETIC

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Abstract

We study the effects of piece selection principles on cardinal arithmetic (Shelah style). As an application, we discuss questions of Abe and Usuba. In particular, we show that if $\lambda \geq 2^\kappa$, then (a) $I_{\kappa,\lambda}$ is not $(\lambda, 2)$-distributive, and (b) $I^+_{\kappa,\lambda} \to (I^+_{\kappa,\lambda})^2_\omega$ does not hold.

1 Introduction

In [2] Abramson, Harrington, Kleinberg and Zwicker pointed out that many large cardinal properties can be reformulated as flipping properties, which are of the following type : One is given a family F of subsets of a set X. The property asserts that for some "flip" $h \in \prod_{A \in F}\{A, X \setminus A\}$, the $h(A)$'s satisfy some intersection property (for instance the finite intersection property). "Intersection" is taken in a wide sense so that e.g. *diagonal* intersections are allowed. Notice that with the family F is associated the family of all *partitions* of the form $\{A, X \setminus A\} \setminus \{\emptyset\}$ for $A \in F$. A flip of F can now be seen as a piece selection operation. Namely, for each partition $\{A, X \setminus A\} \setminus \{\emptyset\}$, we choose one piece, either A or its complement.
For a typical example, let κ be a measurable cardinal, and U be a (κ-complete) measure on κ. For F take the collection of all partitions of κ into one or two pieces. For each such partition, select the piece in U. Then the pieces chosen have the property that the intersection of any less than κ many of them is cofinal in κ. Notice that since U is κ-complete, it does not matter whether our partitions have one, two or more pieces, as long as the number of pieces is less than κ. By thus increasing the number of pieces, we obtain a natural generalization of the original flipping properties. In this extended framework, regularity of an infinite cardinal κ can be expressed as the property that for any partition of κ into less than κ many pieces, one of the pieces must be cofinal in κ. The setting can be generalized by allowing J-partitions, and not just partitions.

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(Recall that for an ideal J on a set X, a J-partition of X is a subset Q of J^+ such that

- $A \cap B \in J$ for any two distinct members A, B of Q.
- For any $C \in J^+$, there is $A \in Q$ with $A \cap C \in J^+$.)

This is a way to handle properties defined in terms of distributivity. For a further generalization we relax the requirement that a piece has to be selected in *each* partition. So for instance we're given κ many partitions of κ, and we might be happy to pick one piece in κ many partitions. Piece selection principles of this type have been introduced in our joint paper [11] with Laura Fontanella. Their study is continued in the present paper.
Our starting point is Solovay's celebrated result [33] on strongly compact cardinals and the Singular Cardinal Hypothesis. This theorem can be revisited in a number of ways. For instance it is shown in [22] that if $\mathrm{cf}(\lambda) < \kappa$ and there is a weakly λ^{++}-saturated, $(\mathrm{cf}(\lambda))^+$-complete ideal on $P_\kappa(\lambda)$, then $\mathrm{pp}(\lambda) = \lambda^+$. In another direction, Usuba [35] established that if κ is mildly λ-ineffable and $\mathrm{cf}(\lambda) \geq \kappa$, then $\lambda^{<\kappa} = \lambda$, or equivalently (since κ is inaccessible), there is a cofinal subset of $P_\kappa(\lambda)$ of size λ (i.e. $u(\kappa, \lambda) = \lambda$). Now it was noted [8] from the very beginning that mild ineffability can be reformulated as a piece selection principle. We consider various weakenings of this principle and attempt to compare their relative strengths. Some of these properties can be satisfied at a weakly, but not strongly, inaccessible cardinal, or even at a successor cardinal. So this part of the paper is a contribution to the age-old program of determining what's left of weak or strong compactness when inaccessibility is removed. As the program developed, an impressive list of properties emerged, especially in connection with *weak* compactness. Some of these properties can be tentatively classified as weak (the tree property), of medium strength (our PS^+) or strong (the weak compactness of the infinitary language $L_{\kappa\omega}$). For others (e.g. our PS), the situation is not so clear and further work is needed. Our version of Solovay's result reads as follows (see Proposition 3.4 and Observation 4.6).

THEOREM 1.1. *Suppose that* $\mathrm{cf}(\lambda) < \kappa$ *and* $PS^+((\mathrm{cf}(\lambda))^+, \kappa, \lambda)$ *holds. Then* $\mathrm{cov}(\lambda, \lambda, (\mathrm{cf}(\lambda))^+, 2) = \lambda^+$.

In the remainder of the paper, which is devoted to applications, we still deal with variants of mild ineffability, but this time inaccessibility of κ is implied. It is a central problem in the theory of $P_\kappa(\lambda)$ to determine how the infinite Ramsey theorem generalizes in this framework. (Note that the theorem comes in several versions, from the weak "$\{\omega\} \to (I_\omega^+)^2_2$" to the strong "$I_\omega^+ \to (I_\omega^+)^n_m$ for all finite n, m"). The study of partition relations on $P_\kappa(\lambda)$ is known to be tricky business. Carr [6] mentions that "repeated efforts to obtain"

- $\{P_\kappa(\lambda)\} \to (I^+_{\kappa,\lambda})^2_2$ implies κ is mildly λ-ineffable,
- κ is mildly λ-ineffable implies $\{P_\kappa(\lambda)\} \to (I^+_{\kappa,\lambda})^3_2$

"failed miserably". Johnson asked in [13] whether the $(\lambda, 2)$-distributivity of $I_{\kappa,\lambda}$ follows from the mild λ-ineffabilty of κ. This was answered in the negative by Abe [1] who showed that if $\lambda^{<\kappa} = 2^\lambda$, then (a) $I_{\kappa,\lambda}|A$ is not $(\lambda, 2)$-distributive for any stationary A, and (b) $I^+_{\kappa,\lambda} \to (I^+_{\kappa,\lambda})^2_2$ does not hold. This led him to ask whether $\lambda > \kappa$ implies that (a) $I_{\kappa,\lambda}$ is not $(\lambda, 2)$-distributive, and (b) $I^+_{\kappa,\lambda} \to (I^+_{\kappa,\lambda})^2_2$ fails. This was answered, again in the negative, by Shioya [32]. It should be noted that his model is obtained by adding many Cohen subsets of κ. In fact it was shown in [17] that if κ is mildly $\lambda^{<\kappa}$-ineffable and $\lambda^{<\kappa} < \mathbf{cov}(\mathbf{M}_{\kappa,\lambda})$, then $I^+_{\kappa,\lambda} \to (I^+_{\kappa,\lambda})^n_\eta$ holds for any $n < \omega$ and any $\eta < \kappa$. Since $\mathbf{cov}(\mathbf{M}_{\kappa,\lambda}) \leq \mathbf{cov}(\mathbf{M}_{\kappa,\kappa}) \leq \mathfrak{d}_\kappa \leq \mathbf{2}^\kappa$, it made us think that maybe it could be proved that if $\lambda \geq 2^\kappa$, then (a) $I_{\kappa,\lambda}$ is not $(\lambda, 2)$-distributive, and (b) $I^+_{\kappa,\lambda} \to (I^+_{\kappa,\lambda})^2_2$ fails.

Part of the difficulty with the ordinary partition relation on $P_\kappa(\lambda)$ stems from the fact that $\subset$ is not a *linear* ordering. To avoid this kind of pitfalls we chose to work with weaker partition relations (note that by negating them, we will obtain stronger results). Given an ideal J on $P_\kappa(\lambda)$ and a coloring of $P_\kappa(\lambda) \times P_\kappa(\lambda)$, we are looking for a color i and a set A in J^+ that is not necessarily i-homogeneous (all pairs (a, b) from $A \times A$ with $a \subset b$ have color i), but at least i-homogeneous mod J (meaning that for each a in A, the set of all b in A such that (a, b) does not have color i lies in J). We denote this particular partition relation by $\{P_\kappa(\lambda)\} \xrightarrow{J} (J^+)^2_\rho$, where ρ denotes the number of available colors. Still weaker partition relations are obtained in a similar fashion by coloring $\kappa \times P_\kappa(\lambda)$ (given (a, b), we look at the color of $(\sup(a \cap \kappa), b)$) or $\kappa^+ \times P_\kappa(\lambda)$ (replace $\sup(a \cap \kappa)$ with $\sup(a \cap \kappa^+)$). To denote the corresponding partition property, we use $\xrightarrow[\kappa]{J}$ (respectively, $\xrightarrow[\kappa^+]{J}$). Proofs involve the usual ingredients (κ-normality, covering numbers, etc., and indeed some proofs are slight modifications of proofs of earlier results. Progress is achieved via a broader appeal to Shelah's pcf theory.

Our efforts to prove the conjectures described above were only partially successful. By combining Observation 8.1 and Propositions 6.19, 8.4 and 9.6, one obtains the following.

THEOREM 1.2. *Suppose that $2^\kappa \leq \lambda$, and let $D \in NS^*_{\kappa,\lambda}$. Then setting $J = I_{\kappa,\lambda}|D$, the following hold :*

(i) $J^+ \xrightarrow{J} (J^+)^2_\omega$ *does not hold.*

(ii) $J^+ \xrightarrow{J} (J^+)^3_2$ *does not hold.*

(iii) J *is not* $(\lambda, 2)$-*distributive.*

There is a wide wide gap between this and what we can establish (see Proposition 5.32, Corollary 6.10, Observation 8.3 and Fact 9.5) under extra cardinal arithmetic assumptions such as Shelah's Strong Hypothesis (SSH).

THEOREM 1.3. *Assuming SSH, the following hold :*

(i) *If* $\overline{\mathfrak{d}}_\kappa \leq \lambda$ *and* $\mathrm{cf}(\lambda) \neq \kappa$*, then for any* $D \in NS^*_{\kappa,\lambda}$*,* $I_{\kappa,\lambda}|D$ *is not* $(\kappa, 2)$*-distributive and* $(I_{\kappa,\lambda}|D)^+ \xrightarrow[\kappa]{I_{\kappa,\lambda}} (I^+_{\kappa,\lambda}, \omega_1)^2$ *fails. If moreover* κ *is weakly Mahlo, then for any* $D \in NS^*_{\kappa,\lambda}$*,* $(I_{\kappa,\lambda}|D)^+ \xrightarrow[\kappa]{I_{\kappa,\lambda}} [I^+_{\kappa,\lambda}]^2_\lambda$ *fails.*

(ii) *If* $2^\kappa \leq \lambda$ *and* $\mathrm{cf}(\lambda) = \kappa$*, then for any* $D \in NS^*_{\kappa,\lambda}$*,* $I_{\kappa,\lambda}|D$ *is not* $(\kappa^+, 2)$*-distributive, and moreover* $(I_{\kappa,\lambda}|D)^+ \xrightarrow[\kappa^+]{I_{\kappa,\lambda}} [I^+_{\kappa,\lambda}]^2_\lambda$ *fails.*

On the positive side we have the following (see Corollary 9.2 and Fact 9.3).

THEOREM 1.4. *Suppose that SSH holds and either* $\mathrm{cf}(\lambda) = \kappa$*, or* $\mathrm{cf}(\lambda) < \kappa$ *and* $\lambda^+ < \mathfrak{d}_\kappa$*, or* $\mathrm{cf}(\lambda) > \kappa$ *and* $\lambda < \mathfrak{d}_\kappa$*. Then the following hold :*

(i) *Suppose that* κ *is weakly inaccessible. Then* $I^+_{\kappa,\lambda} \xrightarrow[\kappa]{I_{\kappa,\lambda}} [I^+_{\kappa,\lambda}]^2_{\kappa^+}$ *holds.*

(ii) *Suppose that* κ *is weakly compact. Then* $I_{\kappa,\lambda}$ *is* $(\kappa, 2)$*-distributive, and moreover* $I^+_{\kappa,\lambda} \xrightarrow[\kappa]{I_{\kappa,\lambda}} (I^+_{\kappa,\lambda})^2_\eta$ *holds whenever* $0 < \eta < \kappa$*.*

Concerning $I^+_{\kappa,\lambda} \to (I^+_{\kappa,\lambda})^2_2$, it remains open whether it fails whenever $2^\kappa \leq \lambda$. What we do know is that it fails if λ is large enough. In fact as shown in [18], $\{P_\kappa(\lambda)\} \xrightarrow{I_{\kappa,\lambda}} [I^+_{\kappa,\lambda}]^2_\lambda$ fails if λ is large enough.

The article is organized as follows. Section 2 is devoted to piece selection principles on $P_\kappa(\lambda)$, with emphasis on $PS^+(\tau, \kappa, \lambda)$. It is shown that if $\mathrm{cf}(\lambda) < \kappa$ and $PS^+((\mathrm{cf}(\lambda))^+, \kappa, \lambda)$ holds, then there is no remarkably good scale on λ. Section 3 is concerned with piece selection principles on κ. It is observed that the tree property $TP(\kappa)$ is one of them. We use scales to establish that if $\mathrm{cf}(\lambda) < \kappa$, then $PS^+((\mathrm{cf}(\lambda))^+, \kappa, \lambda)$ implies $PS^+((\mathrm{cf}(\lambda))^+, \lambda^+)$. Section 4 is devoted to Shelah's covering numbers. It is shown that if λ is singular and $PS^+((\mathrm{cf}(\lambda))^+, \lambda^+)$ holds, then $\mathrm{cov}(\lambda, \lambda, (\mathrm{cf}(\lambda))^+, 2) = \lambda^+$. In Section 5 we give cardinal arithmetic conditions under which for any club subset C of $P_\kappa(\lambda)$, the partition property $(I_{\kappa,\lambda}|C)^+ \xrightarrow[\kappa]{I_{\kappa,\lambda}} (I^+_{\kappa,\lambda}, \rho)^2$ fails. This is continued in Section 6 where we deal with the stronger partition relations $(I_{\kappa,\lambda}|C)^+ \xrightarrow[\kappa]{I_{\kappa,\lambda}} (I^+_{\kappa,\lambda})^2_2$ and $(I_{\kappa,\lambda}|C)^+ \xrightarrow{I_{\kappa,\lambda}} (I^+_{\kappa,\lambda})^2_\omega$. Mild ineffability is the subject of Section 7. We prove that if κ is mildly λ-ineffable and $\mathrm{cf}(\lambda) \neq \kappa$, then $\mathrm{cov}(\lambda, \kappa^+, \kappa^+, \kappa) = \lambda$. Section 8 contains results on the non-distributivity of $I_{\kappa,\lambda}|C$ for a club subset C of $P_\kappa(\lambda)$. Finally in Section 9, we deal with the remaining case, that is the case when $\mathrm{cf}(\lambda) = \kappa$, and explain why this case must be handled separately.

2 Piece selection

Throughout the paper κ will denote a regular uncountable cardinal, and λ a cardinal greater than or equal to κ. We start with some definitions.

DEFINITION 2.1. For a set A and a cardinal τ, we set $P_\tau(A) = \{a \subseteq A : |a| < \tau\}$ and $[A]^\tau = \{x \subseteq A : |x| = \tau\}$.

DEFINITION 2.2. By a *partition* of a set X we mean a subset Q of $P(X)\setminus\{\emptyset\}$ such that:

- $A \cap B = \emptyset$ for any two distinct members A, B of Q.
- $\bigcup Q = X$.

DEFINITION 2.3. An *ideal* on a set X is a nonempty collection J of subsets of X such that :

- $A \cup B \in J$ whenever $A, B \in J$.
- $P(A) \subseteq J$ for all $A \in J$.
- $X \notin J$.

Given an ideal J on X, we denote by J^+ the set $\{A \subseteq X : A \notin J\}$, while J^* denotes the set $\{A \subseteq X : X \setminus A \in J\}$. For any $A \in J^+$, we let $J|A = \{B \subseteq X : B \cap A \in J\}$.
We say that J is *κ-complete* if for any collection Z of less than κ many sets in J, one has $\bigcup Z \in J$.
An ideal K on X *extends* J if $J \subseteq K$.
We let $I_\kappa = \bigcup_{\alpha<\kappa} P(\alpha)$ and
$$I_{\kappa,\lambda} = \bigcup_{a\in P_\kappa(\lambda)} P(\{b \in P_\kappa(\lambda) : a \setminus b \neq \emptyset\}).$$
An ideal J on κ (respectively, $P_\kappa(\lambda)$) is *fine* if it extends I_κ (respectively, $I_{\kappa,\lambda}$). We let NS_κ (respectively, $NS_{\kappa,\lambda}$) denote the nonstationary ideal on κ (respectively, $P_\kappa(\lambda)$).

DEFINITION 2.4. Let τ be an infinite cardinal less than or equal to κ. The piece selection principle $PS^+(\tau,\kappa,\lambda)$ means that given a partition Q_a of $P_\kappa(\lambda)$ with $|Q_a| < \tau$ for each $a \in P_\kappa(\lambda)$, there is $B \in I^+_{\kappa,\lambda}$ and $h \in \prod_{a\in P_\kappa(\lambda)} Q_a$ such that for any $a, b \in B$, the set $\{c \in h(a) \cap h(b) : a \cup b \subseteq c\}$ is nonempty.
$PS^*(\tau,\kappa,\lambda))$ (respectively, $PS(\tau,\kappa,\lambda)$) means that given a partition Q_a of $P_\kappa(\lambda)$ with $|Q_a| < \tau$ for each $a \in P_\kappa(\lambda)$, we may find $B \in I^+_{\kappa,\lambda}$ and $h \in \prod_{a\in P_\kappa(\lambda)} Q_a$ such that for any $a, b \in B$, there is t in B (respectively, in $P_\kappa(\lambda)$) such that $a \cup b \subseteq t$ and the two sets $\{c \in h(a)\cap h(t) : t \subseteq c\}$ and $\{d \in h(b)\cap h(t) : t \subseteq d\}$ are nonempty.

OBSERVATION 2.5. $PS^+(\tau,\kappa,\lambda) \Rightarrow PS^*(\tau,\kappa,\lambda) \Rightarrow PS(\tau,\kappa,\lambda)$.

OBSERVATION 2.6. *Suppose that $PS^*(\kappa,\kappa,\lambda)$ holds. Then κ is weakly inaccessible.*

Proof. Suppose otherwise, and let $\kappa = \nu^+$. For $\nu \leq \gamma < \kappa$, select a bijection $j_\gamma : \gamma \to \nu$. Put $A = \{a \in P_\kappa(\lambda) : \nu \subseteq a\}$. For $a \in A$ and $i < \nu$, let Q_a^i denote the collection of all $c \in P_\kappa(\lambda)$ such that $a \subseteq c$ and $j_{(\sup(c\cap\kappa))+1}(\sup(a \cap \kappa)) = i$. We may find $B \in I^+_{\kappa,\lambda} \cap P(A)$ and $i < \nu$ such that for any $a, b \in B$, there is t in B such that $a \cup b \subseteq t$ and the two sets $\{c \in Q_a^i \cap Q_t^i : t \subseteq c\}$ and $\{d \in Q_b^i \cap Q_t^i : t \subseteq d\}$ are nonempty. Now pick $a, b \in B$ with $\sup(a \cap \kappa) < \sup(b \cap \kappa)$. There must be t in B with $a \cup b \subseteq t$, $c \in Q_a^i \cap Q_t^i$ with $t \subseteq c$ and $d \in Q_b^i \cap Q_t^i$ with $t \subseteq d$. But then

- $j_{(\sup(c\cap\kappa))+1}(\sup(a \cap \kappa)) = j_{(\sup(c\cap\kappa))+1}(\sup(t \cap \kappa))$.
- $j_{(\sup(d\cap\kappa))+1}(\sup(b \cap \kappa)) = j_{(\sup(d\cap\kappa))+1}(\sup(t \cap \kappa))$.

It follows that $\sup(a \cap \kappa) = \sup(t \cap \kappa) = \sup(b \cap \kappa)$. Contradiction. $\square$

OBSERVATION 2.7. *Suppose that $PS^+(\tau, \kappa, \lambda)$ holds. Let $A \in I^+_{\kappa,\lambda}$, and for each $a \in A$, let Q_a be a partition of the set $\{c \in A : a \subseteq c\}$ with $|Q_a| < \tau$. Then there is $B \in I^+_{\kappa,\lambda} \cap P(A)$ and $h \in \prod_{a\in B} Q_a$ such that for any $a, b \in B$, $h(a) \cap h(b) \neq \emptyset$.*

Proof. Define $\psi : P_\kappa(\lambda) \to A$ so that $x \subseteq \psi(x)$ for all $x \in P_\kappa(\lambda)$. For $x \in P_\kappa(\lambda)$, let T_x denote the set of all $z \in P_\kappa(\lambda)$ such that $x \subseteq z$ but $\psi(x) \setminus z \neq \emptyset$, and set

$$Z_x = \{\{z \in P_\kappa(\lambda) : \psi(x) \subseteq z \text{ and } \psi(z) \in W\} : W \in Q_{\psi(x)}\}.$$

We may find $H \in I^+_{\kappa,\lambda}$ and $k \in \prod_{x\in P_\kappa(\lambda)}(Z_x \cup \{T_x\})$ such that $k(x) \cap k(y) \neq \emptyset$ for any $x, y \in H$.

Claim. Let $x \in H$. Then $k(x) \in Z_x$.

Proof of the claim. Suppose otherwise. Pick $y \in H$ with $\psi(x) \subseteq y$, and $z \in k(x) \cap k(y)$. Then $\psi(x) \subseteq y \subseteq z$. This contradiction completes the proof of the claim.

Now put $B = \psi``H$, and define $f : B \to H$ such that $\psi(f(a)) = a$ for all $a \in B$. Notice that $B \in I^+_{\kappa,\lambda} \cap P(A)$. Let $h \in \prod_{a\in B} Q_a$ be such that for any $a \in B$,

$$k(f(a)) = \{z \in P_\kappa(\lambda) : \psi(f(a)) \subseteq z \text{ and } \psi(z) \in h(a)\}.$$

Given $a, b \in B$, we may find z in $k(f(a)) \cap k(f(b))$. Then $a \cup b = \psi(f(a)) \cup \psi(f(b)) \subseteq z \subseteq \psi(z)$, and moreover $\psi(z) \in h(a) \cap h(b)$. $\square$

OBSERVATION 2.8. *Suppose that $PS^+(\tau, \kappa, \lambda)$ holds. Then for any cardinal χ with $\kappa \leq \chi < \lambda$, $PS^+(\tau, \kappa, \chi)$ holds.*

Proof. Let χ be a cardinal with $\kappa \leq \chi < \lambda$, and for each $y \in P_\kappa(\chi)$, let Q_y be a partition of $P_\kappa(\chi)$ with $|Q_y| < \tau$. For $a \in P_\kappa(\lambda)$, put

$$W_a = \{\{c \in P_\kappa(\lambda) : c \cap \chi \in Z\} : Z \in Q_{a\cap\chi}\}.$$

Note that $\{c \cap \chi : c \in S\} \in Q_{a\cap\chi}$ for all $S \in W_a$. We may find $B \in I^+_{\kappa,\lambda}$ and $h \in \prod_{a\in P_\kappa(\lambda)} W_a$ such that for any $a, b \in B$,

$$\{c \in h(a) \cap h(b) : a \cup b \subseteq c\} \neq \emptyset.$$

Set $Y = \{b \cap \chi : b \in B\}$. Notice that $Y \in I^+_{\kappa,\chi}$. Select $\psi : Y \to B$ so that for any $y \in Y$, $y = \psi(y) \cap \chi$, and define $k \in \prod_{y\in Y} Q_y$ by $k(y) = \{c \cap \chi : c \in h(\psi(y))\}$.

Now given $x, y \in Y$, pick $c \in h(\psi(x)) \cap h(\psi(y))$ with $\psi(x) \cup \psi(y) \subseteq c$. Then clearly, $x \cup y \subseteq c \cap \chi$, and moreover $c \cap \chi \in k(x) \cap k(y)$. $\square$

Let us next recall some material concerning scales in pcf theory.

DEFINITION 2.9. Let A be an infinite set of regular cardinals such that $|A| < \min A$, and I be an ideal on A such that $\{A \cap a : a \in A\} \subseteq I$.
We let $\prod A = \prod_{a \in A} a$. For $f, g \in \prod A$, we let $f <_I g$ if $\{a \in A : f(a) \geq g(a)\} \in I$.
Let π be a regular cardinal greater than $\sup A$. An increasing, cofinal sequence $\vec{f} = \langle f_\alpha : \alpha < \pi \rangle$ in $(\prod A, <_I)$ is said to be a *scale of length* π. If there is such a sequence, we set $\mathrm{tcf}(\prod A/I) = \pi$.

FACT 2.10. ([30, Theorem 1.5 p. 50]) *Suppose that λ is a singular cardinal. Then there is a set A of regular cardinals such that o.t.$(A) = \mathrm{cf}(\lambda) < \min A, \sup A = \lambda$ and $\mathrm{tcf}(\prod A/I) = \lambda^+$, where I is the noncofinal ideal on A.*

DEFINITION 2.11. Let $\vec{f} = \langle f_\alpha : \alpha < \pi \rangle$ be an increasing, cofinal sequence in $(\prod A, <_I)$. An infinite limit ordinal $\delta < \pi$ is a *good* (respectively, *remarkably good*) *point* for $\vec{f}$ if there is a cofinal (respectively, closed unbounded) subset $X \subseteq \delta$, and $Z_\xi \in I$ for $\xi \in X$ such that $f_\beta(a) < f_\xi(a)$ whenever $\beta < \xi$ are in X and $a \in A \setminus (Z_\beta \cup Z_\xi)$. δ is a *better point* for $\vec{f}$ if we may find a closed unbounded subset X of δ, and $Z_\xi \in I$ for $\xi \in X$ such that $f_\beta(a) < f_\xi(a)$ whenever $\beta < \xi$ are in X and $a \in A \setminus Z_\xi$. δ is a *very good point* for $\vec{f}$ if there is a closed unbounded subset X of δ, and $Z \in I$ such that $f_\beta(a) < f_\xi(a)$ whenever $\beta < \xi$ are in X and $a \in A \setminus Z$.
The scale $\vec{f} = \langle f_\alpha : \alpha < \pi \rangle$ is *good* (respectively, *remarkably good*, *better*, *very good*) if there is a closed unbounded subset C of π with the property that every infinite limit ordinal δ in C such that $\mathrm{cf}(\delta) < \sup A$ and I is not $\mathrm{cf}(\delta)$-complete is a good (respectively, remarkably good, better, very good) point for $\vec{f}$.

FACT 2.12. ([7], [20]) *Let $\delta < \pi$ be an infinite limit ordinal such that I is $\mathrm{cf}(\delta)$-complete (respectively, $(\mathrm{cf}(\delta))^+$-complete). Then δ is a better (respectively, very good) point for $\vec{f}$.*

We will show that if $\mathrm{cf}(\lambda) < \kappa$ and $PS^+((\mathrm{cf}(\lambda))^+, \kappa, \lambda^+)$ holds, then there is no remarkably good scale on λ.

DEFINITION 2.13. Given two infinite cardinals τ and χ such that $\tau \leq \chi = \mathrm{cf}(\chi)$, we let E^χ_τ (respectively, $E^\chi_{<\tau}$) denotes the set of all infinite limit ordinals $\alpha < \chi$ such that $\mathrm{cf}(\alpha) = \tau$ (respectively, $\mathrm{cf}(\alpha) < \tau$).

OBSERVATION 2.14. *Suppose that $\mathrm{tcf}(\prod A/I) = \pi$, where*

- *A is an infinite set of regular cardinals such that $|A| < \min A$,*
- *I is an ideal on A such that $\{A \cap a : a \in A\} \subseteq I$,*

and $\vec{f} = \langle f_\alpha : \alpha < \pi \rangle$ is an increasing, cofinal sequence in $(\prod A, <_I)$. Let χ be an infinite cardinal with $\chi \leq (\sup A)^+$. Then the following are equivalent :

(i) *There is a closed unbounded subset C of π such that for any regular infinite cardinal $\theta < \chi$, and any $\delta \in C \cap E^\pi_\theta$, δ is a remarkably good point for $\vec{f}$.*

(ii) *There is a closed unbounded subset D of π such that for any $e \in P_\chi(D)$, there is $g : e \to I$ such that $f_\alpha(a) < f_\beta(a)$ whenever $\alpha < \beta$ are in e and $a \in A \setminus (g(\alpha) \cup g(\beta))$.*

Proof. (i) $\to$ (ii) : By Proposition 8.11 of [21] (we can take $D = C$).

(ii) $\to$ (i) : Easy (take $C =$ the set of limit points of D). $\square$

PROPOSITION 2.15. *Suppose that* $\mathrm{tcf}(\prod A/I) = \lambda^+$, *where*

- *A is a set of regular cardinals such that $|A| < \min\{\kappa, \min A\}$ and $\sup A = \lambda$.*
- *I is an ideal on A such that $\{A \cap a : a \in A\} \subseteq I$.*
- *$PS^+(|A|^+, \kappa, \lambda^+)$ holds.*

Let $\vec{f} = \langle f_\alpha : \alpha < \lambda^+ \rangle$ be an increasing, cofinal sequence in $(\prod A, <_I)$, and let S denote the set of all infinite limit points $\delta < \lambda^+$ such that

- $\mathrm{cf}(\delta) < \kappa$.
- *δ is not a remarkably good point for $\vec{f}$.*

Then S is stationary in λ^+.

Proof. Suppose otherwise, and select a closed unbounded subset C of λ^+ such that any infinite limit ordinal δ in C of cofinality less than κ is a remarkably good point for $\vec{f}$. By Fact 2.14, for any $e \in P_\kappa(C)$, there is $g_e : e \to I$ such that $f_\alpha(a) < f_\beta(a)$ whenever $\alpha < \beta$ are in e and $a \in A \setminus (g_e(\alpha) \cup g_e(\beta))$. Pick a bijection $k : \lambda^+ \to C$ and for each nonempty $b \in P_\kappa(\lambda^+)$, $t_b \in \prod_{\alpha \in k``b}(A \setminus g_{k``b}(\alpha))$. Put $X = \{x \in P_\kappa(\lambda^+) : \sup x \in x\}$. For $x \in X$ and $a \in A$, set

$$Q^a_x = \{b \in X : x \subseteq b \text{ and } t_b(k(\sup x)) = a\}.$$

By Observation 2.7 we may find $B \in I^+_{\kappa,\lambda} \cap P(X)$ and $h : B \to A$ such that for any $x, y \in B$, $Q^{h(x)}_x \cap Q^{h(y)}_y \neq \emptyset$. There must be $H \in I^+_{\kappa,\lambda^+} \cap P(B)$ and $a \in A$ such that h takes the constant value a on H. Put $D = \{k(\sup x) : x \in H\}$.

Claim. Let $\alpha < \beta$ in D. Then $f_\alpha(a) < f_\beta(a)$.

Proof of the claim. Pick $x, y \in H$ with $k(\sup x) = \alpha$ and $k(\sup y) = \beta$, and $b \in Q^a_x \cap Q^a_y$. Then $t_b(\alpha) = a = t_b(\beta)$. Thus $a \in A \setminus (g_{k``b}(\alpha) \cup g_{k``b}(\beta))$, and therefore $f_\alpha(a) < f_\beta(a)$, which completes the proof of the claim.

By the claim, the function $u : D \to A$ defined by $u(\alpha) = f_\alpha(a)$ is one-to-one. Contradiction. $\square$

Let us observe that by replacing the hypothesis that $PS^+(|A|^+, \kappa, \lambda^+)$ holds with the stronger hypothesis that κ is mildly λ^+-ineffable, we can actually prove [20] that for *cofinally many* regular uncountable cardinals $\sigma < \kappa$, the set of all points $\delta \in E^{\lambda^+}_\sigma$ that are not remarkably good for $\vec{f}$ is stationary.
The conclusion of Proposition 2.15 entails the failure of a square principle of the following type.

DEFINITION 2.16. Let θ and χ be two infinite cardinals such that $\theta \leq \chi = \mathrm{cf}(\chi)$, and $S \subseteq E^\chi_{<\chi}$. Then $\mathrm{WWS}^\theta_\chi(S)$ asserts the existence of $\mathcal{C}_\gamma$ for $\gamma < \chi$ such that

- $|\mathcal{C}_\gamma| < \chi$;
- $\mathcal{C}_\gamma \subseteq P_\theta(\gamma)$;
- if $\alpha \in S$, then there is a closed unbounded subset C of α of order type $\mathrm{cf}(\alpha)$ such that $C \cap \gamma \in \bigcup_{D \in \mathcal{C}_\gamma} P(D)$ for every $\gamma \in C$.

FACT 2.17. ([21]) *Let A, I and π be such that*

- *A is an infinite set of regular cardinals ;*
- $|A| < \min A$ *;*
- $\sup A < \pi$ *;*
- *I is an ideal on A such that $\{A \cap a : a \in A\} \subseteq I$;*
- $\mathrm{tcf}(\prod A/I) = \pi$.

Further let T be a collection of regular cardinals such that for any $\sigma \in T$,

- $\sigma < \sup A$ *;*
- $\mathrm{WWS}^{\sup A}_\pi(E^\pi_\sigma)$ *holds.*

Then there is an increasing, cofinal sequence $\vec{f} = \langle f_\alpha : \alpha < \pi\rangle$ in $(\prod A, <_I)$ such that for any $\sigma \in T$, and any $\zeta \in E^\pi_\sigma$, ζ is a better point for $\vec{f}$.

PROPOSITION 2.18. *Suppose that* $\mathrm{cf}(\lambda) < \kappa$ *and* $PS^+((\mathrm{cf}(\lambda))^+, \kappa, \lambda^+)$ *holds. Then* $\mathrm{WWS}^\lambda_{\lambda^+}(E^{\lambda^+}_\sigma)$ *fails for some regular cardinal σ with* $\mathrm{cf}(\lambda) < \sigma < \kappa$.

Proof. By Proposition 2.15 and Facts 2.10, 2.12 and 2.17. □

We next consider the two-cardinal version of the tree property.

DEFINITION 2.19. $TP(\kappa, \lambda)$ asserts the following. Let $s_a \subseteq a$ for $a \in P_\kappa(\lambda)$ be such that for some $C \in NS^*_{\kappa,\lambda}$, we have $|\{s_a \cap c : c \subseteq a\}| < \kappa$ for all $c \in C$. Then there is $S \subseteq \lambda$ with the property that for every $b \in P_\kappa(\lambda)$, there is $a \in P_\kappa(\lambda)$ such that $b \subseteq a$ and $S \cap b = s_a \cap b$.

FACT 2.20. (([37]) *It is consistent relative to a supercompact cardinal that* $TP(\omega_2, \chi)$ *holds for every cardinal* $\chi \geq \omega_2$.

OBSERVATION 2.21. *Let* $s_a \subseteq a$ *for* $a \in P_\kappa(\lambda)$. *Suppose that there is* $C \in I^+_{\kappa,\lambda}$ *such that* $|\{s_a \cap c : c \subseteq a\}| < \kappa$ *for all* $c \in C$. *Then* $|\{s_a \cap d : d \subseteq a\}| < \kappa$ *for all* $d \in P_\kappa(\lambda)$.

Proof. Suppose otherwise. Then we may find $d \in P_\kappa(\lambda)$ and $a_i \in P_\kappa(\lambda)$ for $i < \kappa$ so that

- $d \subseteq a_i$ for all $i < \kappa$.
- $s_{a_i} \cap d \neq s_{a_j} \cap d$ whenever $i < j < \kappa$.

Pick $c \in C$ with $d \subseteq c$, and let $i < j < \kappa$. Then $s_{a_i} \cap c \neq s_{a_j} \cap c$, since otherwise we would have $s_{a_i} \cap d = s_{a_j} \cap d$. Contradiction. □

FACT 2.22. $PS(\kappa, \kappa, \lambda)$ *implies* $TP(\kappa, \lambda)$.

Proof. By Proposition 5.4 of [11] and Observation 2.20. □

We will now see that if in the definition of PS^+ we insist on selecting a piece from *every* partition, then what we obtain is an apparently much stronger principle.

DEFINITION 2.23. For a fine ideal J on $P_\kappa(\lambda)$, the *ideal extension principle* $IE(\tau, \kappa, \lambda, J)$ means that given a partition Q_a of $P_\kappa(\lambda)$ with $|Q_a| < \tau$ for each $a \in P_\kappa(\lambda)$, there is $h \in \prod_{a \in P_\kappa(\lambda)} Q_a$ and an ideal K on $P_\kappa(\lambda)$ extending J such that $ran(h) \subseteq K^*$.

OBSERVATION 2.24. $IE(\omega, \kappa, \lambda, J)$ *holds.*

Proof. Let Q_a be a finite partition of $P_\kappa(\lambda)$ for $a \in P_\kappa(\lambda)$. Select a prime ideal K extending J. For each $a \in P_\kappa(\lambda)$, there must be $W_a \in Q_a$ with $W_a \in K^*$. Now define $h \in \prod_{a \in P_\kappa(\lambda)} Q_a$ by $h(a) = W_a$. □

FACT 2.25. ([25] *There is a partition of* $P_\kappa(\lambda)$ *into* $\lambda^{<\kappa}$ *sets in* $I^+_{\kappa,\lambda}$.

OBSERVATION 2.26. *The following are equivalent :*

(i) $IE(\tau, \kappa, \lambda, I_{\kappa,\lambda})$ *holds.*

(ii) *Given a partition* Q_a *of* $P_\kappa(\lambda)$ *with* $|Q_a| < \tau$ *for each* $a \in P_\kappa(\lambda)$, *there is* $h \in \prod_{a \in P_\kappa(\lambda)} Q_a$ *such that for any* $a, b \in P_\kappa(\lambda)$, *there is* $c \in h(a) \cap h(b)$ *with* $a \cup b \subseteq c$.

Proof. (i) $\rightarrow$ (ii) : Trivial.

(ii) $\rightarrow$ (i) : Assume that (ii) holds, and let Q_a be a partition of $P_\kappa(\lambda)$ with $|Q_a| < \tau$ for each $a \in P_\kappa(\lambda)$. By Fact 2.25, we may find a partition T of $P_\kappa(\lambda)$ into $\lambda^{<\kappa}$ sets in $I^+_{\kappa,\lambda}$. Let $\langle T_w : w \in P_\kappa(\lambda)\rangle$ be a one-to-one enumeration of T. Pick a bijection $F : P_\kappa(\lambda) \to P_\omega(P_\kappa(\lambda)) \setminus \{\emptyset\}$. For $w \in P_\kappa(\lambda)$ and $x \in T_w$, put $W_x = \{\bigcap_{a \in F(w)} k(a) : k \in \prod_{a \in F(w)} Q_a\}$. We may find $g \in \prod_{x \in P_\kappa(\lambda)} W_x$ such that for any $x, y \in P_\kappa(\lambda)$, there is $z \in g(x) \cap g(y)$ with $x \cup y \subseteq z$. For $w \in P_\kappa(\lambda)$ and $x \in T_w$, let $k_x \in \prod_{a \in F(w)} Q_a$ be such that $g(x) = \bigcap_{a \in F(w)} k_x(a)$.

Claim 1. Let $v, w \in P_\kappa(\lambda)$, $x \in T_v$, $y \in T_w$ and $a \in F(v) \cap F(w)$. Then $k_x(a) = k_y(a)$.

Proof of Claim 1. Suppose otherwise. Then $k_x(a) \cap k_y(a) = \emptyset$. Since $g(x) \subseteq k_x(a)$ and $g(y) \subseteq k_y(a)$, it follows that $g(x) \cap g(y) = \emptyset$. This contradiction completes the proof of the claim.

Put $h = \bigcup_{x \in P_\kappa(\lambda)} k_x$. Using Claim 1, it is easy to see that $h \in \prod Q_a$.

Claim 2. Let $e \in P_\omega(P_\kappa(\lambda)) \setminus \{\emptyset\}$. Then $\bigcap_{a \in e} h(a) \in I^+_{\kappa,\lambda}$.

Proof of Claim 2. Let $e = F(w)$. Now given $s \in P_\kappa(\lambda)$, pick $x \in T_w$ with $s \subseteq x$. There must be $z \in \bigcap_{a \in F(w)} k_x(a)$ with $x \subseteq z$. Then clearly, $s \subseteq z$, and moreover $z \in \bigcap_{a \in e} h(a)$. This completes the proof of the claim and that of the observation. □

3 Piece selection at κ

In this section we concentrate on the case $\lambda = \kappa$.

DEFINITION 3.1. Given an infinite cardinal τ, we let $PS^+(\tau, \kappa)$ assert the following: For $\beta \in \kappa$, let Q_β be a partition of $\kappa \setminus \beta$ with $|Q_\beta| < \tau$. Then there is a cofinal subset B of κ and $h \in \prod_{\beta \in B} Q_\beta$ such that for any $\alpha, \beta \in B$, we have $h(\alpha) \cap h(\beta) \neq \emptyset$.
$PS^*(\tau, \kappa)$ (respectively $PS(\tau, \kappa)$) asserts the following: For $\beta \in \kappa$, let Q_β be a partition of $\kappa \setminus \beta$ with $|Q_\beta| < \tau$. Then we may find a cofinal subset B of κ and $h \in \prod_{\beta \in \kappa} Q_\beta$ such that for any $\alpha, \beta \in B$, there is ζ in B (respectively, in κ) such that $\max\{\alpha, \beta\} \leq \zeta$ and we have $h(\alpha) \cap h(\zeta) \neq \emptyset$ and $h(\beta) \cap h(\zeta) \neq \emptyset$.

OBSERVATION 3.2. *The following are equivalent :*

(i) $PS^+(\tau, \kappa)$.

(ii) $PS^+(\tau, \kappa, \kappa)$.

Proof. (i) $\to$ (ii) : Suppose that (i) holds. For $b \in P_\kappa(\kappa)$, let Q_b be a partition of the set $\{c \in P_\kappa(\kappa) : b \subseteq c\}$ into less than τ many pieces. For $b \in P_\kappa(\kappa)$, put $\rho_b = |Q_b|$ and let $\langle Q^i_b : i < \rho_b \rangle$ be a one-to-one enumeration of Q_b. Now for $\beta < \kappa$ and $i < \rho_\beta$, set $W^i_\beta = Q^i_\beta \cap \kappa$. We may find a cofinal subset B of κ and $h \in \prod_{\beta \in \kappa} \rho_\beta$ such that for any $\alpha, \beta \in B$, we have $W^{h(\alpha)}_\alpha \cap W^{h(\beta)}_\beta \neq \emptyset$. Then clearly, $B \in I^+_{\kappa,\kappa}$, and moreover $Q^{h(\alpha)}_\alpha \cap Q^{h(\beta)}_\beta \neq \emptyset$ for all $\alpha, \beta \in B$. □

(ii) $\to$ (i) : By Observation 2.7. □

OBSERVATION 3.3. (i) $PS^*(\tau, \kappa)$ *implies* $PS^*(\tau, \kappa, \kappa)$.

(ii) *$PS(\tau,\kappa)$ implies $PS(\tau,\kappa,\kappa)$.*

Proof. Argue as for Observation 3.2. $\square$

COROLLARY 3.4. *$PS(\kappa,\kappa)$ implies $TP(\kappa,\kappa)$.*

Proof. Use Fact 2.22. $\square$

PROPOSITION 3.5. (i) *Suppose that λ is regular and $PS^+(\tau,\kappa,\lambda)$ holds. Then $PS^+(\tau,\lambda)$ holds.*

(ii) *Suppose that $\mathrm{cf}(\lambda)<\kappa$ and $PS^+((\mathrm{cf}(\lambda))^+,\kappa,\lambda)$ holds. Then $PS^+((\mathrm{cf}(\lambda))^+,\lambda^+)$ holds.*

Proof. (i) : For $\alpha<\lambda$, let Q_α be a partition of $\lambda\setminus\alpha$ into less than τ many pieces. For $\alpha\in\lambda$, put $\rho_\alpha=|Q_\alpha|$ and let $\langle Q^i_\alpha : i<\rho_\alpha\rangle$ be a one-to-one enumeration of Q_α. Now for $a\in P_\kappa(\lambda)$ and $i<\rho_{\sup a}$, set

$$W^i_a=\{c\in P_\kappa(\lambda) : a\subseteq c \text{ and } \sup c\in Q^i_{\sup a}\}.$$

We may find $B\in I^+_{\kappa,\lambda}$ and $h\in\prod_{a\in B}\rho_{\sup a}$ such that for any $a,b\in B$, we have $W^{h(a)}_a\cap W^{h(b)}_b\neq\emptyset$. Put $A=\{\sup a : a\in B\}$, and pick $\psi : A\to B$ so that $\sup(\psi(\alpha))=\alpha$ for all $\alpha\in A$. Notice that $A\in I^+_\lambda$. Given $\alpha,\beta\in A$, pick $c\in W^{h(\psi(\alpha))}_{\psi(\alpha)}\cap W^{h(\psi(\beta))}_{\psi(\beta)}$. Then clearly, $\sup c\in Q^{h(\psi(\alpha))}_\alpha\cap Q^{h(\psi(\beta))}_\beta$.

(ii) : Put $\mathrm{cf}(\lambda)=\sigma$. For $\xi<\lambda^+$, let W_ξ be a partition of $\lambda^+\setminus\xi$ into at most σ many pieces. For $\xi\in\lambda^+$, put $\rho_\xi=|W_\xi|$ and let $\langle W^\delta_\xi : \delta<\rho_\xi\rangle$ be a one-to-one enumeration of W_ξ. By Fact 2.10, we may find an increasing sequence $\langle\lambda_i : i<\sigma\rangle$ of regular cardinals greater than κ with supremum λ, and an increasing cofinal sequence $\langle f_\alpha : \alpha<\lambda^+\rangle$ in $(\prod_{i<\sigma}\lambda_i,<^*)$, where $f<^* g$ just in case $|\{i<\sigma : f(i)\geq g(i)\}|<\sigma$. For $a\in P_\kappa(\lambda)$, define $\chi_a\in\prod_{i<\sigma}\lambda_i$ by $\chi_a(i)=\sup(a\cap\lambda_i)$. Define $s : P_\kappa(\lambda)\to\lambda^+$ by : $s(a)=$ the least α such that $\chi_a<^* f_\alpha$. For $a\in P_\kappa(\lambda)$ and $\delta<\sigma$, let Q^δ_a denote the collection of all $c\in P_\kappa(\lambda)$ such that $a\subseteq c$ and $s(c)\in W^\delta_{s(a)}$. We may find $B\in I^+_{\kappa,\lambda}$ and $h\in\prod_{a\in B}\rho_{s(a)}$ such that for any $a,b\in B$, we have $Q^{h(a)}_a\cap Q^{h(b)}_b\neq\emptyset$.

We inductively define $a_n\in B$ for $n<\lambda^+$ so that $s(a_m)<s(a_n)$ whenever $m<n<\lambda^+$. Suppose that a_m has been defined for each $m<n$. Putting $\gamma=\sup\{s(a_m) : m<n\}$, we select $a_n\in B$ so that $ran(f_{\gamma+1})\subseteq a_n$. Now let $m<n<\lambda^+$ be given. There must be some c in $Q^{h(a_m)}_{a_m}\cap Q^{h(a_n)}_{a_n}$. Then clearly, $s(c)\geq\max\{s(a_m),s(a_n)\}$, and moreover $s(c)\in W^{h(a_m)}_{s(a_m)}\cap W^{h(a_n)}_{s(a_n)}$. $\square$

DEFINITION 3.6. Given an infinite cardinal χ, the *Almost Disjoint Set principle ADS_χ* asserts the existence of a cofinal subset y_α of χ of order-type $\mathrm{cf}(\chi)$ for each $\alpha<\chi^+$ such that for each nonzero $\beta<\chi^+$, there is $k\in\prod_{\alpha<\beta}y_\alpha$ with the property that $(y_\delta\setminus(k(\delta))\cap(y_\alpha\setminus(k(\alpha))=\emptyset$ whenever $\delta<\alpha<\beta$.

It is known [21] that if there is a remarkably good scale on χ, then ADS_χ holds. Thus the following is closely related to Proposition 2.15.

PROPOSITION 3.7. *Let χ be a singular cardinal such that $PS^+((\mathrm{cf}(\chi))^+, \chi)$ holds. Then ADS_χ fails.*

Proof. Let y_α be a cofinal subset of χ of order-type $\mathrm{cf}(\chi)$ for each $\alpha < \chi^+$. Suppose that for each nonzero $\beta < \chi^+$, there is $k_\beta \in \prod_{\alpha<\beta} y_\alpha$ with the property that $(y_\delta \setminus (k_\beta(\delta)) \cap (y_\alpha \setminus (k_\beta(\alpha)) = \emptyset$ whenever $\delta < \alpha < \beta$. For $\alpha < \chi^+$ and $\xi \in y_\alpha$, let Q^ξ_α denote the set of all γ such that $\alpha \leq \gamma < \chi^+$ and $k_{\gamma+1}(\alpha) = \xi$. We may find $B \in [\chi^+]^{\chi^+}$ and $h \in \prod_{\alpha \in B} y_\alpha$ such that $Q^{h(\delta)}_\delta \cap Q^{h(\alpha)}_\alpha \neq \emptyset$ whenever $\delta, \alpha \in B$. Now given $\delta < \alpha < \chi^+$, pick $\gamma \in Q^{h(\delta)}_\delta \cap Q^{h(\alpha)}_\alpha$. Then clearly, $k_{\gamma+1}(\delta) = h(\delta)$ and $k_{\gamma+1}(\alpha) = h(\alpha)$, and consequently $(y_\delta \setminus (h(\delta)) \cap (y_\alpha \setminus (h(\alpha)) = \emptyset$. Contradiction. □

Let us now turn to the tree property.

DEFINITION 3.8. The *tree property $TP(\kappa)$* asserts that any tree of height κ each of whose levels has size less than κ has a κ-branch.

FACT 3.9. (i) ([37]) *$TP(\kappa)$ and $TP(\kappa, \kappa)$ are equivalent.*

(ii) ([34]) *Let τ be an infinite cardinal such that $\tau^{<\tau} = \tau$. Then $TP(\tau^+)$ fails.*

$TP(\kappa)$ can be recast as a piece selection principle.

OBSERVATION 3.10. *The following are equivalent:*

(i) *$TP(\kappa)$.*

(ii) *Let $\langle Q_\alpha : \alpha < \kappa \rangle$ be a sequence of partitions of κ into less than κ many pieces with the property that $Q_\beta \subseteq \bigcup_{W \in Q_\alpha} P(W)$ whenever $\alpha < \beta < \kappa$. Then there is $h \in \prod_{\alpha<\kappa} Q_\alpha$ such that $|h(\alpha) \cap h(\beta)| \geq 2$ whenever $\alpha < \beta < \kappa$.*

Proof. (i) $\to$ (ii) : Suppose that (i) holds, and let $\langle Q_\alpha : \alpha < \kappa \rangle$ be as in (ii). Consider the tree $(T, <_T)$, where

- $T = \bigcup_{\gamma<\kappa} L_\gamma$, where L_γ consists of all $g \in \prod_{\alpha<\gamma+1} \{A \in Q_\alpha : |A| \geq 2\}$ such that $g(\beta) \subseteq g(\alpha)$ whenever $\alpha < \beta < \gamma$.
- $f <_T g$ just in case $f \subset g$.

(ii) $\to$ (i) : Suppose that (ii) holds, and let $T = (\kappa, <_T)$ be a tree of height κ with each level L_α of size less than κ. Consider the sequence $\langle Q_\alpha : \alpha < \kappa \rangle$ of partitions of κ defined by : $Q_\alpha = \{Q^\xi_\alpha : \xi \in L_\alpha\}$, where

$$Q^\xi_\alpha = \{\zeta \in \kappa : \zeta <_T \xi\} \cup \{\xi\} \cup \{\eta \in \kappa : \xi <_T \eta\}.$$

□

This can be used to reformulate $TP(\kappa)$ in terms of partitions relations.

OBSERVATION 3.11. *The following are equivalent:*

(i) $TP(\kappa)$.

(ii) *Suppose that $F : \kappa \times \kappa \to \kappa$ has the following property : if $\beta < \gamma < \delta < \kappa$ are such that $F(\beta, \gamma) = F(\beta, \delta)$, then $F(\alpha, \gamma) = F(\alpha, \delta)$ for all $\alpha < \beta$. Then there is $A \in [\kappa]^\kappa$ such that one of the following holds :*

- *$F(\beta, \gamma) \neq F(\beta, \delta)$ whenever $\beta < \gamma < \delta$ are in A.*
- *$F(\beta, \gamma) = F(\beta, \delta)$ whenever $\beta < \gamma < \delta$ are in A.*

Proof. (i) $\to$ (ii) : Assume that (i) holds, and let F be as in (ii). For $\alpha < \kappa$, consider the equivalence relation $\sim_\alpha$ defined on κ by : $\beta \sim_\alpha \gamma$ if and only if either $\gamma = \beta \leq \alpha$, or $\beta, \gamma > \alpha$ and $F(\xi, \beta) = F(\xi, \gamma)$ for all $\xi \leq \alpha$. Let Q_α be the set of all equivalence classes with respect to $\sim_\alpha$.
Case 1 : There is $\eta < \kappa$ such that $|Q_\eta| = \kappa$. Pick $A \in [\kappa \setminus (\eta + 1))]^\kappa$ so that $|A \cap H| \leq 1$ for all $H \in Q_\eta$. Now if $\beta < \gamma < \delta$ are in A, we must have $F(\beta, \gamma) \neq F(\beta, \delta)$, since otherwise we would have $F(\eta, \gamma) = F(\eta, \delta)$.
Case 2 : $|Q_\alpha| < \kappa$ for all $\alpha < \kappa$. Then by Observation 3.10, we may find $h \in \prod_{\alpha<\kappa} Q_\alpha$ such that $|h(\alpha) \cap h(\beta)| \geq 2$ whenever $\alpha < \beta < \kappa$. It is simple to see that $\{h(\alpha) : \alpha < \kappa\} \subseteq [\kappa]^\kappa$. Furthermore, $h(\beta) \subseteq h(\alpha)$ whenever $\alpha < \beta < \kappa$. Now inductively define an increasing sequence $\langle \alpha_i : i < \kappa \rangle$ of elements of κ so that for any $j < \kappa$, $\alpha_j \in h((\sup\{\alpha_i : i < j\}) + 1)$. Then clearly, $F(\alpha_i, \alpha_j) = F(\alpha_i, \alpha_k)$ whenever $i < j < k < \kappa$.

(ii) $\to$ (i) : Assume that (ii) holds, and let $\langle Q_\alpha : \alpha < \kappa \rangle$ be as in (ii) of Observation 3.10. For $\alpha < \kappa$, let $\langle Q^i_\alpha : i < |Q_\alpha| \rangle$ be a one-to-one enumeration of Q_α. Define $F : \kappa \times \kappa \to \kappa$ by : $F(\alpha, \beta) = i$ just in case $\beta \in Q^i_\alpha$. There must be $A \in [\kappa]^\kappa$ and $f \in \prod_{\alpha\in A} |Q_\alpha|$ such that $F(\beta, \gamma) = f(\beta)$ whenever $\beta < \gamma$ are in A. Then $A \setminus (\beta + 1) \subseteq Q^{f(\beta)}_\beta$ for all $\beta \in A$. It easily follows that the conclusion of (ii) of Observation 3.10 holds. $\square$

QUESTION. Is it consistent that $TP(\kappa)$ holds, but $PS(\kappa, \kappa)$ fails ?

We return to ideal extension, but this time for ideals on κ.

DEFINITION 3.12. We let $L_{\kappa\omega}$ denote the infinitary language which allows conjunctions and disjunctions of less than κ many formulas, and universal and existential quantification over finitely many variables.
$L_{\kappa\omega}$ is *weakly compact* if any set of κ sentences from $L_{\kappa\omega}$ without a model has a subset of smaller size without a model.

DEFINITION 3.13. For a fine ideal J on κ, $IE(\tau, \kappa, J)$ means that given a partition Q_α of $\kappa \setminus \alpha$ with $|Q_\alpha| < \tau$ for each $\alpha \in \kappa$, there is $h \in \prod_{\alpha\in\kappa} Q_\alpha$ and an ideal K on κ extending J such that $ran(h) \subseteq K^*$.

OBSERVATION 3.14. *Suppose that $L_{\kappa\omega}$ is weakly compact. Then $IE(\kappa, \kappa, I_\kappa)$ holds.*

Proof. For $\alpha < \kappa$, let Q_α be a partition of $\kappa \setminus \alpha$ with $|Q_\alpha| < \kappa$. Consider the $L_{\kappa\omega}$ language with one unary predicate S and constant symbols c_A for $A \in \bigcup_{\alpha<\kappa} Q_\alpha$. Let Σ consist of the following sentences :

- $\bigvee_{A \in Q_\alpha} S(c_A)$ for each $\alpha < \kappa$.
- $\neg(S(c_{A_0}) \wedge S(c_{A_1}) \wedge \cdots \wedge S(c_{A_n}))$ whenever $0 < n < \omega$, $A_0, A_1, \cdots, A_n \in \bigcup_{\alpha<\kappa} Q_\alpha$ and $A_0 \cap A_1 \cap \cdots \cap A_n = \emptyset$.

Notice that for $0 < \beta < \kappa$, $\bigcap_{\alpha<\beta} k_\beta(\alpha) \neq \emptyset$, where $k_\beta : \beta \to \kappa$ is defined by $k_\beta(\alpha) =$ the unique $A \in Q_\alpha$ such that $\beta \in A$. It easily follows that any subset of Σ of size less than κ is satisfiable. Hence so is Σ itself, and there must be $h \in \prod_{\alpha<\kappa} Q_\alpha$ with the property that for each $e \in P_\omega(\kappa) \setminus \{\emptyset\}$, $\bigcap_{\alpha \in e} h(\alpha)$ is nonempty. In fact, $\bigcap_{\alpha\in e} h(\alpha) \in I_\kappa^+$. Suppose otherwise, and let $\delta < \kappa$ such that $\bigcap_{\alpha \in e} h(\alpha) \subseteq \delta$. Then $\bigcap_{\alpha\in d} h(\alpha) = \emptyset$, where $d = e \cup \{\delta\}$. Contradiction. $\square$

Boos [4] showed that if κ is weakly compact, then in the extension obtained by adding κ^+ many Cohen reals, $L_{\kappa\omega}$ is still weakly compact. Thus $IE(\kappa, \kappa, I_\kappa)$ (and hence $PS^+(\kappa, \kappa)$) may hold without κ being inaccessible.

QUESTION. Is it consistent that $PS^+(\kappa, \lambda')$ holds for every cardinal $\lambda' \geq \kappa$, but κ is not inaccessible ?

DEFINITION 3.15. For an infinite cardinal τ, the *transversal property* $PT(\kappa, \tau)$ means that for any size κ family of sets of size less than τ without a transversal (i.e. a one-to-one choice function), there exists a subfamily of size less than κ without a transversal.

FACT 3.16. ([27]) *It is consistent (relative to infinitely many supercompact cardinals) that $PT(k, \omega_1)$ holds for every regular infinite cardinal k greater than the least fixed point of the aleph function.*

OBSERVATION 3.17. *Suppose that $IE(\tau, \kappa, I_\kappa)$ holds. Then so does $PT(\kappa, \tau)$.*

Proof. Let $\langle X_\alpha : \alpha < \kappa\rangle$ be a sequence of sets of size less than τ with the property that for any nonzero $\beta < \kappa$, there is a one-to-one k_β in $\prod_{\alpha<\beta} X_\alpha$. Pick a one-to-one function $j : \bigcup_{\alpha<\kappa} X_\alpha \to \kappa$. For $\alpha < \kappa$ and $i \in j``X_\alpha$, set $A^i_\alpha = \{\gamma \in \kappa \setminus \alpha : j(k_{\gamma+1}(\alpha)) = i\}$. There must be $h \in \prod_{\alpha<\kappa} j``X_\alpha$ such that $A^{h(\alpha)}_\alpha \cap A^{h(\beta)}_\beta \neq \emptyset$ for all $\alpha, \beta < \kappa$. Define $g \in \prod_{\alpha<\kappa} X_\alpha$ so that $j(g(\alpha)) = h(\alpha)$. We will show that g is one-to-one. Thus let $\alpha < \beta < \kappa$. Select γ in $A^{h(\alpha)}_\alpha \cap A^{h(\beta)}_\beta$. Then $j(k_{\gamma+1}(\alpha)) = h(\alpha)$ and $j(k_{\gamma+1}(\beta)) = h(\beta)$, which gives $k_{\gamma+1}(\alpha) = g(\alpha)$ and $k_{\gamma+1}(\beta) = g(\beta)$. It follows that $g(\alpha) \neq g(\beta)$. $\square$

Let $PT^-(\kappa, \tau)$ mean that for any size κ family of sets of size less than τ with the property that any subfamily of size less than κ has a transversal, there exists a subfamily of size κ with a transversal. Then by the proof of Observation 3.17, $PS^+(\tau, \kappa, \tau)$ implies $PT^-(\kappa, \tau)$.

QUESTION. Is it consistent that $PS^+(\kappa, \kappa)$ holds, but $IE(\kappa, \kappa, I_\kappa)$ fails ?

QUESTION. What is the least possible value of κ at which $PS^+(\kappa, \kappa)$ (respectively, $PS^*(\kappa, \kappa)$, $PS(\kappa, \kappa)$) may hold ?

4 Covering numbers

In this section we study the consequences of PS^+ in terms of cardinal arithmetic (in the sense of Shelah).

DEFINITION 4.1. Given two infinite cardinals $\rho \leq \sigma$, $u(\rho, \sigma)$ denotes the cofinality of the poset $(P_\rho(\sigma), \subseteq)$.

FACT 4.2. (Folklore) *Let $\rho \leq \sigma$ be two infinite cardinals. Then $\sigma^{<\rho} = \max\{2^{<\rho}, u(\rho, \sigma)\}$.*

DEFINITION 4.3. Given four cardinals $\rho_1, \rho_2, \rho_3, \rho_4$ with $\rho_1 \geq \rho_2 \geq \rho_3 \geq \omega$ and $\rho_3 \geq \rho_4 \geq 2$, $\text{cov}(\rho_1, \rho_2, \rho_3, \rho_4)$ denotes the least cardinality of any $X \subseteq P_{\rho_2}(\rho_1)$ such that for any $a \in P_{\rho_3}(\rho_1)$, there is $Q \in P_{\rho_4}(X)$ with $a \subseteq \bigcup Q$.

Note that $u(\rho, \sigma) = \text{cov}(\sigma, \rho, \rho, 2)$.

FACT 4.4. ([30, pp. 85-86], [19]) *Let ρ_1, ρ_2, ρ_3 and ρ_4 be four cardinals such that $\rho_1 \geq \rho_2 \geq \rho_3 \geq \omega$ and $\rho_3 \geq \rho_4 \geq 2$. Then the following hold :*

(i) *If $\rho_1 = \rho_2$ and either $\text{cf}(\rho_1) < \rho_4$ or $\text{cf}(\rho_1) \geq \rho_3$, then $\text{cov}(\rho_1, \rho_2, \rho_3, \rho_4) = \text{cf}(\rho_1)$.*

(ii) *If either $\rho_1 > \rho_2$, or $\rho_1 = \rho_2$ and $\rho_4 \leq \text{cf}(\rho_1) < \rho_3$, then $\text{cov}(\rho_1, \rho_2, \rho_3, \rho_4) \geq \rho_1$.*

(iii) $\text{cov}(\rho_1, \rho_2, \rho_3, \rho_4) = \text{cov}(\rho_1, \rho_2, \rho_3, \max\{\omega, \rho_4\})$.

(iv) $\text{cov}(\rho_1^+, \rho_2, \rho_3, \rho_4) = \max\{\rho_1^+, \text{cov}(\rho_1, \rho_2, \rho_3, \rho_4)\}$.

(v) *If $\rho_1 > \rho_2$ and $\text{cf}(\rho_1) < \rho_4 = \text{cf}(\rho_4)$, then*
$$\text{cov}(\rho_1, \rho_2, \rho_3, \rho_4) = \sup\{\text{cov}(\rho, \rho_2, \rho_3, \rho_4) : \rho_2 \leq \rho < \rho_1\}.$$

(vi) *If ρ_1 is a limit cardinal such that $\rho_1 > \rho_2$ and $\text{cf}(\rho_1) \geq \rho_3$, then*
$$\text{cov}(\rho_1, \rho_2, \rho_3, \rho_4) = \sup\{\text{cov}(\rho, \rho_2, \rho_3, \rho_4) : \rho_2 \leq \rho < \rho_1\}.$$

(vii) *If $\rho_3 > \rho_4 \geq \omega$, then*
$$\text{cov}(\rho_1, \rho_2, \rho_3, \rho_4) = \sup\{\text{cov}(\rho_1, \rho_2, \rho^+, \rho_4) : \rho_4 \leq \rho < \rho_3\}.$$

(viii) *If $\rho_3 \leq \rho_2 = \text{cf}(\rho_2)$, $\omega \leq \rho_4 = \text{cf}(\rho_4)$ and $\rho_1 < \rho_2^{+\rho_4}$, then $\text{cov}(\rho_1, \rho_2, \rho_3, \rho_4) = \rho_1$.*

(ix) *If $\rho_3 = \text{cf}(\rho_3)$, then either $\text{cf}(\text{cov}(\rho_1, \rho_2, \rho_3, \rho_4)) < \rho_4$, or $\text{cf}(\text{cov}(\rho_1, \rho_2, \rho_3, \rho_4)) \geq \rho_3$.*

(x) *Suppose that $\rho_3 > \text{cf}(\rho_2) \geq \rho_4$ and $\text{cf}(\rho_3) \neq \text{cf}(\rho_2)$. Then $\text{cov}(\rho_1, \rho_2, \rho_3, \rho_4) = \text{cov}(\rho_1, \rho, \rho_3, \rho_4)$ for some cardinal ρ with $\rho_2 > \rho \geq \rho_3$.*

FACT 4.5. (i) ([30, Remark 6.6.A p. 101]) *Let χ be a singular cardinal. Then $\text{cov}(\chi, \chi, (\text{cf}(\chi))^+, \text{cf}(\chi)) > \chi^+$ if and only if $\text{cov}(\chi, \chi, (\text{cf}(\chi))^+, 2) > \chi^+$.*

(ii) ([30, p. 99], [19])) *Let χ be a singular cardinal. Suppose that* $\mathrm{cov}(\chi, \chi, (cf(\chi))^+, 2) > \chi^+$. *Then we may find $y_\alpha \in P_{(cf(\chi))^+}(\chi)$ for $\alpha < \chi^+$ such that for any nonzero $\beta < \chi^+$, there is a one-to-one $h \in \prod_{\alpha<\beta} y_\alpha$.*

Note that if $\vec{f} = \langle f_\alpha : \alpha < \pi \rangle$ is an increasing, cofinal sequence in $(\prod A, <_I)$, then by a result of Shelah [30, Theorem 5.4 pp. 87-88], $\pi \leq \mathrm{cov}(\sup A, \sup A, |A|^+, 2)$.

OBSERVATION 4.6. *Let χ be a singular cardinal such that $PS^+((\mathrm{cf}(\chi))^+, \chi^+)$ holds. Then* $\mathrm{cov}(\chi, \chi, (\mathrm{cf}(\chi))^+, 2) = \chi^+$.

Proof. Suppose otherwise. Put $\sigma = \mathrm{cf}(\chi)$. By Fact 4.5, we may find $y_\alpha \in P_{\sigma^+}(\chi)$ for $\alpha < \chi^+$ such that for any nonzero $\beta < \chi^+$, there is a one-to-one $h_\beta \in \prod_{\alpha<\beta} y_\alpha$. For $\alpha < \chi^+$, let $\langle y^i_\alpha : i < |y_\alpha| \rangle$ be a one-to-one enumeration of y_α, and set $Q^i_\alpha = \{\delta \in \chi^+ \setminus \alpha : h_{\delta+1}(\alpha) = y^i_\alpha\}$. There must be $B \in [\chi^+]^{\chi^+}$ and $g \in \prod_{\alpha \in B} |y_\alpha|$ with the property that $Q^{g(\alpha)}_\alpha \cap Q^{g(\gamma)}_\gamma \neq \emptyset$ whenever $\alpha, \gamma \in B$. Then clearly, the function $t : \chi^+ \to \chi$ defined by $t(\alpha) = y^{g(\alpha)}_\alpha$ is one-to-one. Contradiction. $\square$

Neeman [29] established the consistency relative to large cardinals of the existence of a singular strong limit cardinal χ of cofinality ω such that $TP(\chi^+)$ holds and $2^\chi > \chi^+$. Note that

$$2^\chi = \chi^{\mathrm{cf}(\chi)} \leq \max\{\mathrm{cov}(\chi, \chi, (\mathrm{cf}(\chi))^+, 2), 2^{<\chi}\} = \mathrm{cov}(\chi, \chi, (\mathrm{cf}(\chi))^+, 2) \leq 2^\chi,$$

and therefore by Observation 4.6, $PS^+((\mathrm{cf}(\chi))^+, \chi^+)$ fails. In Neeman's model, there is both a very good (and hence remarkably good) scale of length χ^+ on χ, and a scale of length χ^+ on χ that is not good (so that approachability fails, and in fact [21] there is a regular uncountable cardinal $\sigma < \chi$ such that $E^{\chi^+}_\sigma \notin I[\chi^+; \chi]$).

To get the most out of Observation 4.6 we will vary the value of λ. We will thus be able to use the following results of pcf theory.

FACT 4.7. ([19]) *Let σ, k, μ and ν be four infinite cardinals such that $\mathrm{cf}(\sigma) = \sigma \leq k$ and $\sigma < \mathrm{cf}(\mu) = \mu < \nu$. Suppose that*

- $\mathrm{cov}(k, \rho^+, \rho^+, \sigma) \leq k^{++}$ *for every cardinal ρ with $\sigma \leq \rho < \min\{k, \mu\}$.*
- $\mathrm{cov}(\chi, \chi, \sigma^+, \sigma) = \chi^+$ *for every cardinal χ with $k < \chi \leq \nu$ and $\mathrm{cf}(\chi) = \sigma$.*

Then $\mathrm{cov}(\nu, \mu, \mu, \sigma) \leq \nu^+$.

OBSERVATION 4.8. (i) *Let π and μ be two regular cardinals such that $\omega \leq \pi \leq \kappa \leq \mu \leq \lambda$. Suppose that*

- *either $\mathrm{cf}(\lambda) < \pi$, or $\mathrm{cf}(\lambda) \geq \mu$.*
- $u(\rho^+, \kappa) \leq \kappa^{++}$ *for every cardinal ρ with $\omega \leq \rho < \kappa$.*
- $\mathrm{cov}(\chi, \chi, \omega_1, 2) = \chi^+$ *for every cardinal χ with $\kappa < \chi < \lambda$ and $\mathrm{cf}(\chi) = \omega$.*

Then $\mathrm{cov}(\lambda,\mu,\mu,\pi)=\lambda$.

(ii) *Let π and μ be two regular cardinals such that $\omega_1\le\pi\le\kappa\le\mu\le\lambda$. Suppose that*

- *either* $\mathrm{cf}(\lambda)<\pi$, *or* $\mathrm{cf}(\lambda)\ge\mu$.
- $\mathrm{cov}(\kappa,\rho^+,\rho^+,\omega_1)\le\kappa^{++}$ *for every cardinal ρ with $\omega_1\le\rho<\kappa$.*
- $\mathrm{cov}(\chi,\chi,\omega_2,\omega_1)=\chi^+$ *for every cardinal χ with $\kappa<\chi<\lambda$ and* $\mathrm{cf}(\chi)=\omega_1$.

Then $\mathrm{cov}(\lambda,\mu,\mu,\pi)=\lambda$.

Proof. We prove (i) and leave the similar proof of (ii) to the reader. By Fact 4.4 ((i) and (ii)), $\mathrm{cov}(\tau,\mu,\mu,\pi)\ge\tau$ for every cardinal $\tau\ge\mu$. Furthermore by Fact 4.7, $\mathrm{cov}(\nu,\mu,\mu,\pi)\le u(\mu,\nu)\le\nu^+$ for any cardinal ν with $\mu<\nu<\lambda$.

Case 1 : $\lambda=\mu$. Then by Fact 4.4 (i),
$$\lambda\le\mathrm{cov}(\lambda,\mu,\mu,\pi)\le u(\mu,\lambda)=\lambda.$$
Case 2 : λ is the successor of some cardinal $\sigma\ge\mu$. Then by Fact 4.4 (i),
$$\lambda\le\mathrm{cov}(\lambda,\mu,\mu,\pi)\le u(\mu,\lambda)=\max\{\lambda,u(\mu,\sigma)\}=\lambda.$$
Case 3 : $\mathrm{cf}(\lambda)<\pi$. Then by Fact 4.4 (v),
$$\lambda\le\mathrm{cov}(\lambda,\mu,\mu,\pi)=\sup\{\mathrm{cov}(\nu,\mu,\mu,\pi):\mu\le\nu<\lambda\}\le\lambda.$$
Case 4 : λ is a limit cardinal with $\mu\le\mathrm{cf}(\lambda)$. Then by Fact 4.4 (vi),
$$\lambda\le\mathrm{cov}(\lambda,\mu,\mu,\pi)=\sup\{\mathrm{cov}(\nu,\mu,\mu,\pi):\mu\le\nu<\lambda\}\le\lambda.$$
□

FACT 4.9. ([9])

(i) *Let π be a regular cardinal such that $\kappa\le\pi\le\lambda$. Then* $u(\kappa,\lambda)\le\max\{u(\kappa,\pi),u(\pi,\lambda)\}$.

(ii) *Suppose that λ is a limit cardinal. Then*
$$u(\kappa,\lambda)=\max\{\mathrm{cov}(\lambda,\lambda,\kappa,2),\sup\{u(\kappa,\chi):\kappa\le\chi<\lambda\}\}.$$

OBSERVATION 4.10. *Suppose that*

- $\mathrm{cf}(\lambda)=\omega$.
- $u(\rho^+,\kappa)\le\kappa^{++}$ *for every cardinal ρ with $\omega\le\rho<\kappa$.*
- $\mathrm{cov}(\chi,\chi,\omega_1,2)=\chi^+$ *for every cardinal χ with $\kappa<\chi<\lambda$ and* $\mathrm{cf}(\chi)=\omega$.

Then $u(\kappa,\lambda)=\mathrm{cov}(\lambda,\lambda,\omega_1,2)$.

Proof.

Claim 1. Let χ be a cardinal with $\kappa^{++}\le\chi\le\lambda$. Then $u(\omega_1,\chi)\le u(\kappa,\chi)$.

Proof of Claim 1. By Facts 4.4 and 4.9 (i),
$$u(\omega_1,\chi)\le\max\{u(\omega_1,\kappa),u(\kappa,\chi)\}=u(\kappa,\chi),$$
which completes the proof of the claim.

Claim 2. $u(\kappa,\lambda)=u(\omega_1,\lambda)$.

Proof of Claim 2. By Observation 4.8 (i), $\mathrm{cov}(\lambda, \lambda, \omega_1, \omega_1) = \lambda$. Hence by Claim 1,

$$u(\kappa, \lambda) \leq u(\omega_1, \mathrm{cov}(\lambda, \lambda, \omega_1, \omega_1)) = u(\omega_1, \lambda) \leq u(\kappa, \lambda),$$

which completes the proof of the claim.

Claim 3. $u(\omega_1, \lambda) = \mathrm{cov}(\lambda, \lambda, \omega_1, 2)$.

Proof of Claim 3. By Claim 1 and Facts 4.4 and 4.9 (ii),

$$u(\omega_1, \lambda) = \max\{\mathrm{cov}(\lambda, \lambda, \omega_1, 2), \sup\{u(\omega_1, \tau) : \omega_1 \leq \tau < \lambda\}\} = \mathrm{cov}(\lambda, \lambda, \omega_1, 2),$$

which completes the proof of the claim and that of the observation. $\square$

5 Unbalanced partition properties

Let us start with partitions of $P_\kappa(\lambda) \times P_\kappa(\lambda)$.

DEFINITION 5.1. Given two collections X and Y of subsets of $P_\kappa(\lambda)$, an ideal J on $P_\kappa(\lambda)$, and a cardinal ρ with $0 < \rho \leq \kappa$, $X \xrightarrow{Y} (J^+, \rho)^2$ means that for any $F : P_\kappa(\lambda) \times P_\kappa(\lambda) \to 2$ and any $A \in X$, there is either $B \in J^+ \cap P(A)$ such that $\{b \in B : F(a, b) \neq 0\} \in Y$ for all $a \in B$, or an increasing sequence $\langle c_i : i < \rho \rangle$ in $(A, \subset)$ such that $F(c_i, c_j) = 1$ whenever $i < j < \rho$.

OBSERVATION 5.2. *Let J be a fine ideal on $P_\kappa(\lambda)$. Then $J^+ \xrightarrow{J} (J^+, \omega)^2$ holds.*

Proof. Fix $F : P_\kappa(\lambda) \times P_\kappa(\lambda) \to 2$ and $A \in J^+$. Define $\psi : P_\kappa(\lambda) \times P(A) \to P(A)$ by

$$\psi(a, X) = \{b \in X : a \subset b \text{ and } F(a, b) = 1\}.$$

Case 1 : There is $Y \in J^+ \cap P(A)$ such that $\psi(a, Y) \in J$ for all $a \in Y$. Then clearly, $F(a, b) = 0$ whenever a is in Y and b is in $Y \setminus \psi(a, Y)$ with $a \subset b$.
Case 2 : For each $X \in J^+ \cap P(A)$, there is a_X in X such that $\psi(a_X, X) \in J^+$. Inductively define X_n for $n < \omega$ by :

- $X_0 = A$.
- $X_{n+1} = \psi(a_{X_n}, X_n)$.

Then clearly, $\{a_{X_n} : n < \omega\} \subseteq A$. Moreover, if $m < n < \omega$, then $a_{X_m} \subset a_{X_n}$ and $F(a_{X_m}, a_{X_n}) = 1$. $\square$

DEFINITION 5.3. For an ideal J on a set X, $MAD(J)$ (respectively, $MAD_d(J)$) denotes the collection of all $Q \subseteq J^+$ such that

- $A \cap B \in J$ (respectively, $A \cap B = \emptyset$) for any two distinct members A, B of Q.
- For any $C \in J^+$, there is $A \in Q$ with $A \cap C \in J^+$.

Let ρ and ν be two nonzero cardinals. J is (ρ,ν)-*distributive* (respectively, *disjointly* (ρ,ν)-*distributive*) if given $A \in J^+$, and Q_α in $MAD(J)$ (respectively, $MAD_d(J)$) with $|Q_\alpha| \leq \nu$ for $\alpha < \rho$, there is $B \in J^+ \cap P(A)$ and $h \in \prod_{\alpha<\rho} Q_\alpha$ such that $B \setminus h(\alpha) \in J$ for every $\alpha < \rho$.

OBSERVATION 5.4. *Suppose that J is a $(\theta, 2)$-distributive ideal on a set X, where θ is an infinite cardinal. Then J is (θ,θ)-distributive.*

Proof. Let $A \in J^+$, and $Q_\alpha \in MAD(J)$ with $|Q_\alpha| \leq \theta$ for $\alpha < \theta$. Put $Z = \bigcup_{\alpha<\theta} Q_\alpha$. There must be $B \in J^+ \cap P(A)$ and $h \in \prod_{W \in Z}\{W, X \setminus W\}$ such that $B \setminus h(W) \in J$ for every $W \in Z$. Now given $\alpha < \theta$, we have $Q_\alpha \in MAD(J)$, so there is a (unique) $W \in Q_\alpha$ such that $h(W) = W$. $\square$

OBSERVATION 5.5. *Suppose that $J^+ \xrightarrow{J} (J^+, \sigma^+)^2$ holds, where J is a κ-complete, fine ideal on $P_\kappa(\lambda)$, and σ is an infinite cardinal. Then J is disjointly $(\sigma, \lambda^{<\kappa})$-distributive.*

Proof. Let $A \in J^+$, and Q_α in $MAD_d(J)$ for $\alpha < \sigma$. For $\alpha < \sigma$, put $X^0_\alpha = P_\kappa(\lambda) \setminus \bigcup Q_\alpha$, and let $\langle X^i_\alpha : 0 < i < |Q_\alpha|\rangle$ be a one-to-one enumeration of Q_α. Define $g : \sigma \times P_\kappa(\lambda) \to \lambda^{<\kappa}$ so that for any $a \in P_\kappa(\lambda)$ and any $\alpha < \sigma$, $a \in X^{g(\alpha,a)}_\alpha$. Now define $F : P_\kappa(\lambda) \times P_\kappa(\lambda) \to 2$ by : $F(a,b) = 1$ if and only if there is $\alpha < \sigma$ such that $g(\alpha,a) \neq g(\alpha,b)$, and for the least such α, $g(\alpha,a) > g(\alpha,b)$.
<u>Case 1</u> : There is $B \in J^+$, and $Z_a \in J$ for $a \in B$ such that $F(a,b) = 0$ whenever $a, b \in B$ and $b \notin Z_a$. Assume toward a contradiction that there is $\gamma < \sigma$ such that $\{B \setminus X^i_\gamma : 0 < i < |Q_\gamma|\} \subseteq J^+$, and let α denote the least such γ. Let $h \in \prod_{\beta<\alpha}\{k : 0 < k < |Q_\beta|\}$ such that $\{B \setminus X^{h(\beta)}_\beta : \beta < \alpha\} \subseteq J$. There must be $0 < i < j < |Q_\alpha|$ such that $\{B \cap X^i_\alpha, B \cap X^j_\alpha\} \subseteq J^+$. Pick a in $(B \cap X^j_\alpha) \setminus \bigcup_{\beta<\alpha}(B \setminus X^{h(\beta)}_\beta)$, and b in $(B \cap X^i_\alpha) \setminus (Z_a \cup \bigcup_{\beta<\alpha}(B \setminus X^{h(\beta)}_\beta))$. Then $F(a,b) = 1$. Contradiction.
<u>Case 2</u> : There is an increasing sequence $\langle a_\delta : \delta < \sigma^+\rangle$ in $(A, \subset)$ such that $F(a_\gamma, a_\delta) = 1$ whenever $\gamma < \delta < \sigma^+$. We will show that this is contradictory. For this we inductively define $\xi_\alpha < \sigma^+$ for $\alpha < \sigma$ so that $g(\alpha, a_{\xi_\alpha}) = g(\alpha, a_\delta)$ whenever $\xi_\alpha < \delta < \sigma^+$. Thus suppose that ξ_β has been defined for each $\beta < \alpha$.

Claim. There is $\xi < \sigma^+$ such that $g(\alpha, a_\xi) = g(\alpha, a_\delta)$ whenever $\xi < \delta < \sigma^+$.

Proof of the claim. Suppose otherwise. Inductively define γ_n for $n < \omega$ so that

- $\gamma_0 = \sup\{\xi_\beta : \beta < \alpha\}$.
- $\gamma_{n+1} > \gamma_n$ and $g(\alpha, a_{\gamma_{n+1}}) \neq g(\alpha, a_{\gamma_n})$.

Then $g(\alpha, a_{\gamma_0}) > g(\alpha, a_{\gamma_1}) > g(\alpha, a_{\gamma_2}) \cdots$. This contradiction completes the proof of the claim.

Using the claim, we let $\xi_\alpha =$ the least $\xi < \sigma^+$ such that $g(\alpha, a_\xi) = g(\alpha, a_\delta)$ whenever $\xi < \delta < \sigma^+$.

Finally pick γ, δ so that $\sup\{\xi_\eta : \eta < \sigma\} \leq \gamma < \delta < \sigma^+$. Then $g(\eta, a_\xi) = g(\eta, a_\delta)$ for all $\eta < \sigma$. Hence $F(a_\gamma, a_\delta) = 0$, which yields the desired contradiction. □

The remainder of the section is devoted to partitions of $\kappa \times P_\kappa(\lambda)$.

DEFINITION 5.6. Given two collections X and Y of subsets of $P_\kappa(\lambda)$, an ideal J on $P_\kappa(\lambda)$, and a cardinal ρ with $0 < \rho \leq \kappa$, $X \underset{\kappa}{\xrightarrow{Y}} (J^+, \rho)^2$ means that for any $F : \kappa \times P_\kappa(\lambda) \to 2$ and any $A \in X$, there is either $B \in J^+ \cap P(A)$ such that $\{b \in B : F(\sup(a \cap \kappa), b) \neq 0\} \in Y$ for all $a \in B$, or an increasing sequence $\langle c_i : i < \rho \rangle$ in $(A, \subset)$ such that $F(\sup(c_i \cap \kappa), c_j) = 1$ whenever $i < j < \rho$.

OBSERVATION 5.7. *$X \xrightarrow{J} (J^+, \rho)^2$ implies $X \underset{\kappa}{\xrightarrow{J}} (J^+, \rho)^2$.*

OBSERVATION 5.8. *Assuming J is fine, the following are equivalent :*

(i) $X \underset{\kappa}{\xrightarrow{J}} (J^+, \rho)^2$ *holds.*

(ii) For any $G : \kappa \times P_\kappa(\lambda) \to 2$ and any $A \in X$, there is either $B \in J^+ \cap P(A)$ such that $\{b \in B : G(\sup(a \cap \kappa), b) \neq 0\} \in J$ for all $a \in B$, or an increasing sequence $\langle c_i : i < \rho \rangle$ in $(A, \subset)$ such that $\sup(c_i \cap \kappa) < \sup(c_j \cap \kappa)$ *and* $G(\sup(c_i \cap \kappa), c_j) = 1$ *whenever $i < j < \rho$.*

Proof. (i) → (ii) : Given $G : \kappa \times P_\kappa(\lambda) \to 2$ and $A \in X$, define $F : \kappa \times P_\kappa(\lambda) \to 2$ by : $F(\alpha, b) = 1$ if and only if $G(\alpha, b) = 1$ and $\alpha < \sup(a \cap \kappa)$.
(ii) → (i) : Trivial. □

DEFINITION 5.9. An ideal J on $P_\kappa(\lambda)$ is *κ-normal* if for any $A \in J^+$ and any $f : A \to \kappa$ with the property that $f(a) \in a$ for all $a \in A$, there is $B \in J^+ \cap P(A)$ such that f is constant on B.
We let $NS^\kappa_{\kappa,\lambda}$ denote the smallest κ-normal, fine ideal on $P_\kappa(\lambda)$.

DEFINITION 5.10. We let $\Omega_{\kappa,\lambda}$ denote the set of all $a \in P_\kappa(\lambda)$ such that $a \cap \kappa$ is an infinite limit ordinal.

FACT 5.11. ([25]) $\Omega_{\kappa,\lambda} \in (NS^\kappa_{\kappa,\lambda})^*$.

DEFINITION 5.12. An ideal J on $P_\kappa(\lambda)$ is a *weak π-point* if for any $A \in J^+$ and any $f : \kappa \to J$, there is $B \in J^+ \cap P(A)$ such that $B \cap f(\alpha) \in I_{\kappa,\lambda}$ for every $\alpha \in \kappa$.
J is a *weak χ-point* if for any $A \in J^+$ and any $g : \kappa \to P_\kappa(\lambda)$, there is $B \in J^+ \cap P(A)$ such that $g(\sup(a \cap \kappa) \subseteq b$ for all $a, b \in B$ with $\sup(a \cap \kappa) < \sup(b \cap \kappa)$.

OBSERVATION 5.13. *(i) $I_{\kappa,\lambda}$ is a weak π-point.*

(ii) Any weak χ-point is fine.

(iii) If J is fine and κ-normal, then it is both a weak π-point and a weak χ-point.

DEFINITION 5.14. Given a collection X of subsets of $P_\kappa(\lambda)$, an ideal J on $P_\kappa(\lambda)$, and a cardinal ρ with $0 < \rho \leq \kappa$, $X \overset{\prec}{\underset{\kappa}{\longrightarrow}} (J^+, \rho)^2$ means that for any $F : \kappa \times P_\kappa(\lambda) \to 2$ and any $A \in X$, there is either $B \in J^+ \cap P(A)$ such that $F(\sup(a \cap \kappa), b) = 0$ whenever $a, b \in B$ are such that $a \subset b$ and $\sup(a \cap \kappa) < \sup(b \cap \kappa)$, or an increasing sequence $\langle c_i : i < \rho \rangle$ in $(A, \subset)$ such that $\sup(c_i \cap \kappa) < \sup(c_j \cap \kappa)$ and $F(\sup(c_i \cap \kappa), c_j) = 1$ whenever $i < j < \rho$.

FACT 5.15. ([16]) *Let ρ be an uncountable cardinal less than or equal to κ, and $X = \{a \in \Omega_{\kappa,\lambda} : \mathrm{cf}(a \cap \kappa) < \rho\}$. Then $X \overset{\prec}{\underset{\kappa}{\longrightarrow}} (J^+, \rho)^2$ fails.*

OBSERVATION 5.16. *Suppose that $J^+ \overset{J}{\underset{\kappa}{\longrightarrow}} (J^+, \rho)^2$ holds, where J is both a weak π-point and a weak χ-point, and ρ is a cardinal with $0 < \rho \leq \kappa$. Then $J^+ \overset{\prec}{\underset{\kappa}{\longrightarrow}} (J^+, \rho)^2$ holds.*

Proof. Use Observation 5.8. □

OBSERVATION 5.17. *Suppose that $J^+ \overset{J}{\underset{\kappa}{\longrightarrow}} (J^+, \rho)^2$ holds, where J is a κ-normal, fine ideal on $P_\kappa(\lambda)$, and ρ is a cardinal with $\omega < \rho < \kappa$. Then $\{a \in \Omega_{\kappa,\lambda} : \mathrm{cf}(a \cap \kappa) < \rho\} \in J$.*

Proof. By Fact 5.15 and Observations 5.13 and 5.16. □

Let us now concentrate on $J^+ \overset{J}{\underset{\kappa}{\longrightarrow}} (J^+, \kappa)^2$.

OBSERVATION 5.18. *Suppose that $\{P_\kappa(\lambda)\} \overset{J}{\underset{\kappa}{\longrightarrow}} (J^+, \kappa)^2$ holds, where J is a κ-complete, fine ideal on $P_\kappa(\lambda)$. Then κ is weakly compact.*

Proof. Given $f : \kappa \times \kappa \to 2$, define $F : \kappa \times P_\kappa(\lambda) \to 2$ by : $F(\alpha, b) = 1$ if and only if $\alpha < \sup(b \cap \kappa)$ and $f(\alpha, \sup(b \cap \kappa)) = 1$.
<u>Case 1</u> : There is $B \in J^+$ such that $T_a \in J$ for all $a \in B$, where $T_a = \{b \in B : F(\sup(a \cap \kappa), b) \neq 0\}$. Proceeding by induction, define $a_\alpha \in A$ for $\alpha < \kappa$ so that for any $\beta < \alpha$, $a_\alpha \notin T_{a_\beta}$ and $\sup(a_\beta \cap \kappa) < \sup(a_\alpha \cap \kappa)$. Then clearly, $f(\sup(a_\beta \cap \kappa), \sup(a_\alpha \cap \kappa)) = 0$ whenever $\beta < \alpha < \kappa$.
<u>Case 2</u> : There is an increasing sequence $\langle c_i : i < \kappa \rangle$ in $(A, \subset)$ such that $F(\sup(c_i \cap \kappa), c_j) = 1$ whenever $i < j < \kappa$. Then given $i < j < \kappa$, we have $\sup(c_i \cap \kappa) < \sup(c_j \cap \kappa)$ and $f(\sup(c_i \cap \kappa), \sup(c_j \cap \kappa)) = 1$. □

DEFINITION 5.19. Given an ideal J on $P_\kappa(\lambda)$, we let $J \restriction \kappa$ denote the set of all $X \subseteq \kappa$ such that $\{a \in P_\kappa(\lambda) : \sup(a \cap \kappa) \in X\}$ lies in J.

DEFINITION 5.20. An ideal J on κ is *weakly selective* if for any $A \in J^+$ and any partition Q of A into sets in J, there is $B \in J^+ \cap P(A)$ such that $|B \cap W| \leq 1$ for every $W \in Q$.

OBSERVATION 5.21. *(i) $J \restriction \kappa$ is an ideal on κ.*

(ii) If J is fine, then so is $J \restriction \kappa$.

(iii) If J is κ-complete, then so is $J \restriction \kappa$.

(iv) If J is both a weak π-point and a weak χ-point, then $J \restriction \kappa$ is weakly selective.

(v) If J is fine and κ-normal, then $J \restriction \kappa$ is normal.

(vi) $J \restriction \kappa \subseteq K \restriction \kappa$ for every ideal K on $P_\kappa(\lambda)$ extending J.

FACT 5.22. ([28]) $NS_{\kappa,\lambda} \restriction \kappa = NS_\kappa$.

DEFINITION 5.23. Given an ideal J on a set X, we let $cof(J)$ denote the least size of any $\mathcal{B} \subseteq J$ such that $J = \bigcup_{B \in \mathcal{B}} P(B)$.
Assuming that J is κ-complete, but not κ^+-complete, we let $\overline{cof}(J)$ denote the least size of any $\mathcal{B} \subseteq J$ such that for any $A \in J$, there is $b \in P_\kappa(\mathcal{B})$ with $A \subseteq \bigcup b$.

DEFINITION 5.24. The *dominating number* $\mathfrak{d}_\kappa$ denotes the least size of any $F \subseteq {}^\kappa\kappa$ with the property that for any $g \in {}^\kappa\kappa$, there is $f \in F$ such that $g(\alpha) < f(\alpha)$ for every $\alpha \in \kappa$.
We let $\overline{\mathfrak{d}}_\kappa$ denote the least size of any $F \subseteq {}^\kappa\kappa$ with the property that for any $g \in {}^\kappa\kappa$, there is $x \in P_\kappa(F)$ such that $g(\alpha) < \sup\{f(\alpha) : f \in x\}$ for every $\alpha \in \kappa$.

FACT 5.25. *(i)* ([14]) $cof(NS_\kappa) = \mathfrak{d}_\kappa$.

(ii) ([26]) $\overline{cof}(NS_\kappa) = \overline{\mathfrak{d}}_\kappa$.

OBSERVATION 5.26. *(i) Let J be a κ-complete, fine ideal on κ such that $\overline{cof}(J) \leq \lambda$. Then $J = (I_{\kappa,\lambda}|A) \restriction \kappa$ for some $A \in I^+_{\kappa,\lambda} \cap P(\Omega_{\kappa,\lambda})$.*

(ii) Suppose $\overline{\mathfrak{d}}_\kappa \leq \lambda$. Then $NS_\kappa = (I_{\kappa,\lambda}|C) \restriction \kappa$ for some $C \in NS^*_{\kappa,\lambda}$.

(iii) Let J be a κ-complete, fine ideal on κ such that $\overline{cof}(J) \leq \lambda$. Then $J = (I_{\kappa,\lambda}|A) \restriction \kappa$ for some $A \in I^+_{\kappa,\lambda}$ such that $(A, \subset)$ and $(P_\kappa(\lambda), \subset)$ are isomorphic.

Proof. (i) : Pick $\mathcal{B} \subseteq J$ with $|\mathcal{B}| = \overline{cof}(J)$ such that for any $B \in J$, there is $b \in P_\kappa(\mathcal{B})$ with $B \subseteq \bigcup b$. Select $v \subseteq \lambda \setminus \kappa$ with $|v| = |\mathcal{B}|$, and a bijection $h : v \to \mathcal{B}$. Let A be the set of all $a \in \Omega_{\kappa,\lambda}$ such that $a \cap \kappa \notin h(\alpha)$ for all $\alpha \in v \cap a$.

Claim. Let $X \in J^+$. Then $\{a \in A : a \cap \kappa \in X\} \in I^+_{\kappa,\lambda}$.
Proof of the claim. Fix $c \in P_\kappa(\lambda)$. Pick $\delta \in X \setminus (\bigcup_{\alpha \in v \cap c} h(\alpha))$ with $c \cap \kappa \subseteq \delta$. Set $a = \delta \cup (c \setminus \kappa)$. Then clearly, $v \cap c = v \cap a$. Moreover, $c \subseteq a$ and $a \in A$, which completes the proof of the claim.

By the claim, $A \in I^+_{\kappa,\lambda}$, and moreover $(I_{\kappa,\lambda}|A) \restriction \kappa \subseteq J$. For the reverse inclusion, fix $X \in J$. We may find $e \in P_\kappa(v)$ such that $X \subseteq \bigcup_{\alpha \in e} h(\alpha)$. Then $a \cap \kappa \notin X$ for all $a \in A$ with $e \subseteq a$. Hence $X \in (I_{\kappa,\lambda}|A) \restriction \kappa$.
(ii) : By Fact 5.11, 5.22 and 5.25 and the proof of (i).
(iii) : Let $\mathcal{B}$, v and h be as in the proof of (i), and let A be the set of all $a \in P_\kappa(\lambda)$ such that

- $\sup(a \cap \kappa) \in a$.
- $\sup(a \cap \kappa) \notin h(\alpha)$ for all $\alpha \in v \cap a$.

Claim. Let $X \in J^+$. Then $\{a \in A : \sup(a \cap \kappa) \in X\} \in I^+_{\kappa,\lambda}$.
Proof of the claim. Given $c \in P_\kappa(\lambda)$, pick $\delta \in X \setminus (\bigcup_{\alpha \in v \cap c} h(\alpha))$ with $\sup(c \cap \kappa) < \delta$. Set $a = c \cup \{\delta\}$. Then clearly, $c \subseteq a$, $a \in A$ and $\sup(a \cap \kappa) = \delta$, which completes the proof of the claim.

By the claim, $A \in I^+_{\kappa,\lambda}$, and moreover $(I_{\kappa,\lambda}|A) \restriction \kappa \subseteq J$. For the reverse inclusion, proceed as in the proof of (i). Finally, define $\varphi : P_\kappa(\lambda) \to \kappa$ and $\psi : P_\kappa(\lambda) \to P_\kappa(\lambda)$ by $\varphi(a) =$ the least $\delta > \sup(a \cap \kappa)$ such that $\delta \notin \bigcup_{\alpha \in v \cap a} h(\alpha)$, and $\psi(a) = a \cup \{\varphi(a)\}$. It is easy to see that ψ is an isomorphism from $(P_\kappa(\lambda), \subset)$ onto $(A, \subset)$. □

Notice that if $(A, \subset)$ and $(P_\kappa(\lambda), \subset)$ are isomorphic and, say, $\{P_\kappa(\lambda)\} \xrightarrow[\kappa]{I_{\kappa,\lambda}} (I^+_{\kappa,\lambda}, \kappa)^2$ holds, then so does $\{A\} \xrightarrow[\kappa]{I_{\kappa,\lambda}} (I^+_{\kappa,\lambda}, \kappa)^2$.

DEFINITION 5.27. Given a collection W of subsets of κ, an ideal K on κ, and a cardinal ρ with $0 < \rho \leq \kappa$, $W \to (K^+, \rho)^2$ means that for any $f : \kappa \times \kappa \to 2$ and any $A \in W$, there is either $B \in K^+ \cap P(A)$ such that $f(\alpha, \beta) = 0$ for any $\alpha < \beta$ in B, or an increasing sequence $\langle \gamma_i : i < \rho \rangle$ in $(A, <)$ such that $f(\gamma_i, \gamma_j) = 1$ whenever $i < j < \rho$.

PROPOSITION 5.28. *Suppose that $J^+ \xrightarrow[\kappa]{J} (J^+, \kappa)^2$ holds, where J is both a weak π-point and a weak χ-point. Then $(J \restriction \kappa)^+ \to ((J \restriction \kappa)^+, \kappa)^2$ holds.*

Proof. Let $X \in (J \restriction \kappa)^+$ and $f : \kappa \times \kappa \to \kappa$ be given. Put $A = \{a \in P_\kappa(\lambda) : \sup(a \cap \kappa) \in X\}$, and define $F : \kappa \times A \to 2$ by : $F(\alpha, b) = 1$ if and only if $\alpha < \sup(b \cap \kappa)$ and $f(\alpha, \sup(b \cap \kappa)) = 1$.
<u>Case 1</u> : There is $B \in J^+ \cap P(A)$, and $Z_a \in J$ for $a \in B$ such that $F(\sup(a \cap \kappa), b) = 0$ whenever $a, b \in B$ and $b \notin Z_a$. Set $T = \{\sup(a \cap \kappa) : a \in B\}$. For $\alpha \in T$, pick $a_\alpha \in B$ with $\sup(a_\alpha \cap \kappa) = \alpha$. There must be $S \in J^+ \cap P(B)$ such that for any $\alpha \in T$, and any $a, b \in S$ with $\sup(a \cap \kappa) = \alpha < \sup(b \cap \kappa)$, $b \notin Z_{a_\alpha}$. Now put $Y = \{\sup(b \cap \kappa) : b \in S\}$. Then $Y \in (J \restriction \kappa)^+ \cap P(X)$. Given $\alpha < \beta$ in Y, we may find $a, b \in S$ such that $\sup(a \cap \kappa) = \alpha$ and $\sup(b \cap \kappa) = \beta$. Then $b \notin Z_{a_\alpha}$. Since $a_\alpha, b \in B$, it follows that

$$0 = F(\sup(a_\alpha \cap \kappa), b) = f(\alpha, \sup(b \cap \kappa)) = f(\alpha, \beta).$$

<u>Case 2</u> : There is an increasing sequence $\langle a_\delta : \delta < \kappa \rangle$ in $(A, \subset)$ such that $F(\sup(a_\gamma \cap \kappa), a_\delta) = 1$ whenever $\gamma < \delta < \kappa$. Then clearly by definition of F, given $\gamma < \delta < \kappa$, we have $\sup(a_\gamma \cap \kappa) < \sup(a_\delta \cap \kappa)$, and moreover $f(\sup(a_\gamma \cap \kappa), \sup(a_\delta \cap \kappa)) = 1$. □

DEFINITION 5.29. We let NWC_κ denote the set of all $A \subseteq \kappa$ such that $\{A\} \to (NS^+_\kappa, \kappa)^2$ does not hold.

FACT 5.30. *(i)* ([3]) *Assuming κ is weakly compact, NWC_κ is the smallest normal ideal K on κ such that $K^+ \to (K^+, \kappa)^2$.*

(ii) ([3]) *The set of all those cardinals less than κ that are not inaccessible belongs to NWC_κ.*

(iii) ([16]) *Let J be a κ-normal, fine ideal on $P_\kappa(\lambda)$. Then $J^+ \overset{J}{\underset{\kappa}{\to}} (J^+, \kappa)^2$ holds if and only if $J \restriction \kappa$ extends NWC_κ.*

FACT 5.31. ([26], [19]) *The following are equivalent :*

(i) $\overline{\mathfrak{d}}_\kappa \leq \lambda = \mathrm{cov}(\lambda, \kappa^+, \kappa^+, \kappa)$.

(ii) *$I_{\kappa,\lambda}|C$ is κ-normal for some $C \in NS^*_{\kappa,\lambda}$.*

Notice that by Shelah's Revised GCH theorem [31, Conclusion 1.2], for any uncountable strong limit cardinal τ, there is $\theta < \tau$ with the property that if $\theta \leq \kappa < \tau < \lambda$, then $\mathrm{cov}(\lambda, \kappa^+, \kappa^+, \kappa) = \lambda$.

PROPOSITION 5.32. *Suppose $\overline{\mathfrak{d}}_\kappa \leq \lambda = \mathrm{cov}(\lambda, \kappa^+, \kappa^+, \kappa)$. Then the following hold.*

(i) *Let σ be an infinite cardinal. Then for any $D \in NS^*_{\kappa,\lambda}$, $J^+ \overset{J}{\underset{\kappa}{\to}} (J^+, \sigma^+)^2$ does not hold, where $J = I_{\kappa,\lambda}|\{a \in D \cap \Omega_{\kappa,\lambda} : \mathrm{cf}(a \cap \kappa) \leq \sigma\}$.*

(ii) *For any $D \in NS^*_{\kappa,\lambda}$ and any $X \in NS^+_\kappa \cap NWC_\kappa$, $J^+ \overset{J}{\underset{\kappa}{\to}} (J^+, \kappa)^2$ does not hold, where $J = I_{\kappa,\lambda}|\{a \in D : \sup(a \cap \kappa) \in X\}$.*

Proof. (ii) : Assume toward a contradiction that $J^+ \overset{J}{\underset{\kappa}{\to}} (J^+, \kappa)^2$ holds. By Fact 5.31, there is $C \in NS^*_{\kappa,\lambda}$ such that $I_{\kappa,\lambda}|C$ is κ-normal. Put $K = J|(C \cap \Omega_{\kappa,\lambda})$. Then clearly, K is κ-normal and $K^+ \overset{K}{\underset{\kappa}{\to}} (K^+, \kappa)^2$ holds. Hence by Observation 5.21 and Proposition 5.28, $K \restriction \kappa$ is a normal, fine ideal on κ such that $(K \restriction \kappa)^+ \to ((K \restriction \kappa)^+, \kappa)^2$ holds. By Fact 5.30, it follows that $X \in K \restriction \kappa$. Contradiction.
(i) : Proceed as in the proof of (ii), but this time appeal to Observation 5.17. $\square$

6 Balanced partition properties

We now turn to stronger partition properties that (in the case when $\mathrm{cf}(\lambda) \neq \kappa$) will imply that $\mathrm{cov}(\lambda, \kappa^+, \kappa^+, \kappa) = \lambda$.

DEFINITION 6.1. Given $2 \leq n < \omega$, two collections X and Y of subsets of $P_\kappa(\lambda)$, an ideal J on $P_\kappa(\lambda)$, and a cardinal ρ, $X \xrightarrow{Y} (J^+)^n_\rho$ means that for any $F : (P_\kappa(\lambda))^n \to \rho$ and any $A \in X$, there is $i < \rho$, $B \in J^+ \cap P(A)$, and $Z_a \in Y$ for $a \in B$ such that $F(a_1, a_2, \cdots, a_n) = i$ whenever $a_1, a_2, \cdots, a_n \in B$ and $a_{m+1} \notin Z_{a_1} \cup Z_{a_2} \cup \cdots \cup Z_{a_m}$ for $1 \leq m < n$.
$X \underset{\kappa}{\xrightarrow{Y}} (J^+)^2_\rho$ means that for any $F : \kappa \times P_\kappa(\lambda) \to \rho$ and any $A \in X$, there is $i < \rho$ and $B \in J^+ \cap P(A)$ such that $\{b \in B : F(\sup(a \cap \kappa), b) \neq i\} \in Y$ for all $a \in B$.

OBSERVATION 6.2. (i) *Suppose that* $J^+ \xrightarrow{J} (J^+)^2_2$ *holds. Then* $J^+ \xrightarrow{J} (J^+)^2_n$ *holds for every* n *with* $0 < n < \omega$.

(ii) $X \xrightarrow{J} (J^+)^2_\rho$ *implies* $X \underset{\kappa}{\xrightarrow{J}} (J^+)^2_\rho$.

(iii) *Suppose that* J *is fine and* $(\kappa, 2)$*-distributive. Then* $J^+ \underset{\kappa}{\xrightarrow{J}} (J^+)^2_\rho$ *whenever* $0 < \rho < \kappa$.

OBSERVATION 6.3. *Suppose that* $\{P_\kappa(\lambda)\} \xrightarrow{J} (J^+)^2_\rho$ *holds, where* ρ *is an infinite cardinal and* J *is a fine ideal on* $P_\kappa(\lambda)$. *Then* $PS^+(\rho^+, \kappa, \nu)$ *holds for any cardinal* ν *with* $\kappa \leq \nu \leq \lambda$.

Proof. Let ν be a cardinal with $\kappa \leq \nu \leq \lambda$, and for $x \in P_\kappa(\nu)$, Q_x be a partition of $P_\kappa(\nu)$ with $|Q_x| \leq \rho$. For $x \in P_\kappa(\nu)$, let $\langle Q^\xi_x : \xi < |Q_x|\rangle$ be a one-to-one enumeration of Q_x. Define $F : P_\kappa(\lambda) \times P_\kappa(\lambda) \to \rho$ by : $F(a, c) = \xi$ if and only if $c \cap \nu \in Q^\xi_{a\cap\nu}$. There must be $\xi < \rho$ and $B \in J^+$ such that $T_a \in J$ for all $a \in B$, where $T_a = \{c \in B : F(a, c) \neq \xi\}$. Set $X = \{a \cap \nu : a \in B\}$. Notice that $X \in I^+_{\kappa,\nu}$. Given $x, y \in X$, select $a, b \in B$ such that $x = a \cap \nu$ and $y = b \cap \nu$. Pick $c \in B \setminus (T_a \cup T_b)$ such that $a \cup b \subseteq c$. Then obviously, $x \cup y \subseteq c \cap \nu$. Moreover, $c \cap \nu \in Q^\xi_x \cap Q^\xi_y$. □

OBSERVATION 6.4. *Suppose that* $\{P_\kappa(\lambda)\} \xrightarrow{J} (J^+)^2_\omega$ *holds, where* J *is a* κ*-complete, fine ideal on* $P_\kappa(\lambda)$. *Then the following hold :*

(i) *Assume* $\mathrm{cf}(\lambda) \neq \kappa$. *Then* $\mathrm{cov}(\lambda, \kappa^+, \kappa^+, \kappa) = \lambda$.

(ii) *Let* τ *be a cardinal with* $\kappa^+ \leq \tau < \lambda$. *Then* $u(\kappa^+, \tau)$ *equals* τ *if* $\mathrm{cf}(\tau) > \kappa$, *and* τ^+ *otherwise.*

Proof. For any cardinal ν such that $\kappa < \nu < \lambda$ and $\mathrm{cf}(\nu) = \omega$, we have that $PS^+(\omega_1, \nu^+)$ holds by Proposition 3.5 and Observation 6.3, and hence that $\mathrm{cov}(\nu, \nu, \omega_1, 2) = \nu^+$ by Observation 4.6. Furthermore by Observation 5.18, κ is inaccessible. Now for (i), apply Observation 4.8 (i). To obtain (ii), appeal to Facts 4.4 and 4.7 if $\mathrm{cf}(\tau) \leq \kappa$, and to Observation 4.8 (i) otherwise. □

The remainder of the section is, just like the end of the previous section, devoted to negative partition relations.

DEFINITION 6.5. Given two collections X and Y of subsets of $P_\kappa(\lambda)$, an ideal J on $P_\kappa(\lambda)$ and two nonzero cardinals σ and ρ, the *square bracket partition relation* $X \xrightarrow[\sigma]{Y} [J^+]^2_\rho$ means that for any $F : \sigma \times P_\kappa(\lambda) \to \rho$ and any $A \in X$, there is $B \in J^+ \cap P(A)$ and $\xi \in \rho$ such that $\{b \in B : F(\sup(a \cap \sigma), b) = \xi\} \in Y$ for all $a \in B$.

DEFINITION 6.6. For any cardinal χ with $\kappa \leq \chi \leq \lambda$, we let $S^\chi_{\kappa,\lambda} = \{a \in P_\kappa(\lambda) : |a \cap \chi| = |a \cap \kappa|\}$.

FACT 6.7. ([18])

(i) *Suppose that* κ *is weakly inaccessible and* $2^\kappa \leq \lambda = \mathrm{cov}(\lambda, \kappa^+, \kappa^+, \kappa)$. *Then* $\{C \cap S^\lambda_{\kappa,\lambda}\} \xrightarrow[\kappa]{I_{\kappa,\lambda}} [I^+_{\kappa,\lambda}]^2_\lambda$ *fails for some* $C \in NS^*_{\kappa,\lambda}$.

(ii) *Suppose that* κ *is weakly Mahlo and* $\overline{\mathfrak{d}}_\kappa \leq \lambda$, *and let* A *be the set of all* $a \in S^{\overline{\mathfrak{d}}_\kappa}_{\kappa,\lambda}$ *such that* $a \cap \kappa$ *is a regular infinite cardinal. Then there is* $D \in NS^*_{\kappa,\lambda}$ *and* $F : \kappa \times P_\kappa(\lambda) \to \lambda$ *with the property that for any* $B \in (NS^\kappa_{\kappa,\lambda})^+ \cap P(A \cap D)$ *and any* $\xi \in \lambda$, *one may find* $a, b \in B$ *with* $a \cap \kappa < b \cap \kappa$ *and* $F(a \cap \kappa, b) = \xi$.

OBSERVATION 6.8. *Suppose that* κ *is weakly Mahlo,* $\overline{\mathfrak{d}}_\kappa \leq \lambda$, *and* J *is a* κ*-normal, fine ideal on* $P_\kappa(\lambda)$ *such that* $J^+ \xrightarrow[\kappa]{J} [J^+]^2_\lambda$ *holds. Then* $A \cap C \in J$ *for some* $C \in NS^*_{\kappa,\lambda}$, *where* A *denotes the set of all* $a \in S^{\overline{\mathfrak{d}}_\kappa}_{\kappa,\lambda}$ *such that* $a \cap \kappa$ *is a regular infinite cardinal.*

Proof. Suppose otherwise. Then clearly, $A \in NS^+_{\kappa,\lambda}$. Now let $D \in NS^*_{\kappa,\lambda}$ and $F : \kappa \times P_\kappa(\lambda) \to \lambda$. Since $\{A \cap D\} \xrightarrow[\kappa]{J} [J^+]^2_\lambda$ and J is κ-normal, there must be $B \in J^+ \cap P(A \cap D \cap \Omega_{\kappa,\lambda})$ and $\xi \in \lambda$ such that $F(a \cap \kappa, b) \neq \xi$ whenever $a, b \in B$ are such that $a \cap \kappa \in b$. This contradicts Fact 6.7. □

COROLLARY 6.9. *Suppose that* $\overline{\mathfrak{d}}_\kappa \leq \lambda$ *and* $J^+ \xrightarrow[\kappa]{J} (J^+)^2_2$ *holds, where* J *is a* κ*-normal, fine ideal on* $P_\kappa(\lambda)$. *Then* $S^{\overline{\mathfrak{d}}_\kappa}_{\kappa,\lambda} \cap C \in J$ *for some* $C \in NS^*_{\kappa,\lambda}$.

Proof. By Observation 5.18 and Fact 5.30, κ is weakly compact, and moreover the set of all $a \in P_\kappa(\lambda)$ such that $a \cap \kappa$ is an inaccessible cardinal lies in J^*. □

COROLLARY 6.10. *Suppose that* κ *is weakly Mahlo and* $\overline{\mathfrak{d}}_\kappa \leq \lambda = \mathrm{cov}(\lambda, \kappa^+, \kappa^+, \kappa)$. *Then for any* $D \in NS^*_{\kappa,\lambda}$, $J^+ \xrightarrow[\kappa]{J} [J^+]^2_\lambda$ *does not hold, where* $J = I_{\kappa,\lambda}|(D \cap S^{\overline{\mathfrak{d}}_\kappa}_{\kappa,\lambda})$.

Proof. Assume toward a contradiction that there exists $D \in NS^*_{\kappa,\lambda}$ such that $J^+ \xrightarrow[\kappa]{J} [J^+]^2_\lambda$ holds, where $J = I_{\kappa,\lambda}|(D \cap S^{\overline{\mathfrak{d}}_\kappa}_{\kappa,\lambda})$. By Fact 5.31, we may find $C \in NS^*_{\kappa,\lambda}$ such that $I_{\kappa,\lambda}|C$ is κ-normal. Put $K = J|C$. Then clearly, K is κ-normal, and moreover $K^+ \xrightarrow[\kappa]{K} (K^+)^2_2$ holds. This contradicts Observation 6.8. □

A similar result will be obtained as a variant of Proposition 5.32 (ii).

PROPOSITION 6.11. *Suppose that $J^+ \overset{J}{\underset{\kappa}{\longrightarrow}} (J^+)^2_2$ holds, where J is a κ-normal, fine ideal on $P_\kappa(\lambda)$. Then $(J \upharpoonright \kappa)^+ \to ((J \upharpoonright \kappa)^+)^2$ holds.*

Proof. By the proof of Proposition 5.28. $\square$

OBSERVATION 6.12. *Given a κ-complete, fine ideal J on κ, the following are equivalent:*

(i) *J is $(\kappa, 2)$-distributive.*

(ii) *Let $0 < \eta \leq \kappa$, and $Q_\alpha \in MAD_d(J)$ for $\alpha < \eta$ be such that $Q_\beta \subseteq \bigcup_{W \in Q_\alpha} P(W)$ whenever $\alpha < \beta < \eta$. Then there is $B \in J^+ \cap P(A)$ and $h \in \prod_{\alpha<\eta} Q_\alpha$ such that $B \setminus h(\alpha) \in J$ for all $\alpha < \kappa$.*

Proof. (i) $\to$ (ii) : By Observation 5.4.
(ii) $\to$ (iii) : Suppose that (ii) holds. We claim that κ is inaccessible. Suppose otherwise, and let ν be the least cardinal such that $2^\nu \geq \kappa$. Let $\langle X_\xi : \xi < \kappa \rangle$ be a sequence of pairwise distinct subsets of ν. For $\delta < \nu$, put $A^0_\delta = \{\xi < \kappa : \delta \in X_\xi\}$ and $A^1_\delta = \kappa \setminus A^0_\delta$. Now let $Q_0 = \{\kappa\}$, and for $0 < \alpha < \nu$, $Q_\alpha = \{\bigcap_{\delta<\alpha} A^{k(\delta)}_\delta : k \in {}^\alpha 2\} \cap J^+$. Then $|\bigcap_{\alpha<\nu} h(\alpha)| \leq 1$ for all $h \in \prod_{\alpha<\nu} Q_\alpha$, which yields the desired contradiction.
Now suppose that $A \in J^+$ and for $\alpha < \kappa$, $W_\alpha \in MAD_d(J)$ with $|W_\alpha| \leq 2$. For $\alpha < \kappa$, let

$$T_\alpha = \{\bigcap_{\beta \leq \alpha} g(\beta) : g \in \prod_{\beta \leq \alpha} W_\beta\} \cap J^+.$$

There must be $B \in J^+ \cap P(A)$ and $f \in \prod_{\alpha<\kappa} T_\alpha$ such that $B \setminus f(\alpha) \in J$ for all $\alpha < \kappa$. For $\alpha < \kappa$, let $g_\alpha \in \prod_{\beta \leq \alpha} W_\beta$ be such that $f(\alpha) = \bigcap_{\beta \leq \alpha} g_\alpha(\beta)$. Then it is easy to see that $\bigcup_{\alpha<\kappa} g_\alpha \in \prod_{\beta \leq \kappa} W_\beta$. Moreover, $B \setminus (\bigcup_{\alpha<\kappa} g_\alpha)(\beta) \in J$ for all $\beta < \kappa$. $\square$

DEFINITION 6.13. Given an ideal J on κ, $2 \leq n < \omega$, a collection X of subsets of κ and an ordinal ρ, $X \to (J^+)^n_\rho$ means that for any $F : \kappa^n \to \rho$ and any $A \in X$, there is $i < \rho$ and $B \in J^+ \cap P(A)$ such that $F(\alpha_1, \alpha_2, \cdots, \alpha_n) = i$ whenever $\alpha_1 < a_2 < \cdots < \alpha_n$ are in B.

FACT 6.14. *Given a κ-complete, fine ideal J on κ, the following are equivalent :*

(i) $J^+ \to (J^+)^2_2$.

(ii) $J^+ \to (J^+)^n_\rho$ *whenever $0 < n < \omega$ and $0 < \rho < \kappa$.*

(iii) *J is $(\kappa, 2)$-distributive and weakly selective.*

Proof. By Theorem 9 in [12] and Observation 6.12. $\square$

DEFINITION 6.15. κ is *completely ineffable* if there exists a normal, $(\kappa, 2)$-distributive, fine ideal on κ.

FACT 6.16. *Suppose that κ is completely ineffable. Then there exists a smallest normal, $(\kappa, 2)$-distributive, fine ideal on κ.*

Proof. By the proof of Corollary 3 in [12] and Observation 6.12. $\square$

DEFINITION 6.17. We let NCI_κ denote the smallest normal, $(\kappa, 2)$-distributive, fine ideal on κ if κ is completely ineffable, and $P(\kappa)$ otherwise.

PROPOSITION 6.18. *Suppose* $\overline{\mathfrak{d}}_\kappa \leq \lambda = \text{cov}(\lambda, \kappa^+, \kappa^+, \kappa)$. *Then for any* $D \in NS^*_{\kappa,\lambda}$ *and any* $X \in NS^+_\kappa \cap NCI_\kappa$, $J^+ \underset{\kappa}{\xrightarrow{J}} (J^+)^2_2$ *does not hold, where* $J = I_{\kappa,\lambda}|\{a \in D : \sup(a \cap \kappa) \in X\}$.

Proof. Assume toward a contradiction that there are $D \in NS^*_{\kappa,\lambda}$ and $X \in NS^+_\kappa \cap NCI_\kappa$ such that $J^+ \underset{\kappa}{\xrightarrow{J}} (J^+)^2_2$ holds, where $J = I_{\kappa,\lambda}|\{a \in D : \sup(a \cap \kappa) \in X\}$. By Fact 5.31, there is $C \in NS^*_{\kappa,\lambda}$ such that $I_{\kappa,\lambda}|C$ is κ-normal. Put $K = J|C$. Then clearly, K is κ-normal, and moreover $K^+ \underset{\kappa}{\xrightarrow{K}} (K^+)^2_2$ holds. Hence by Observation 5.21, Proposition 6.11 and Fact 6.14, $K \restriction \kappa$ is a normal, $(\kappa, 2)$-distributive, fine ideal on κ. It follows that $X \in J \restriction \kappa$. Contradiction. $\square$

PROPOSITION 6.19. *Suppose that* $\overline{\mathfrak{d}}_\kappa \leq \lambda$ *and* cf$\lambda) \neq \kappa$. *Then the following hold :*

(i) For any $D \in NS^*_{\kappa,\lambda}$, $J^+ \xrightarrow{J} (J^+)^2_\omega$ *does not hold, where* $J = I_{\kappa,\lambda}|(D \cap S^{\overline{\mathfrak{d}}_\kappa}_{\kappa,\lambda})$.

(ii) For any $D \in NS^*_{\kappa,\lambda}$ *and any* $X \in NS^+_\kappa \cap NCI_\kappa$, $J^+ \xrightarrow{J} (J^+)^2_\omega$ *does not hold, where* $J = I_{\kappa,\lambda}|\{a \in D : \sup(a \cap \kappa) \in X\}$.

Proof. By Observation 6.4, Corollary 6.10 and Proposition 6.18. $\square$

In contrast to this, by a result of Usuba (see the proof of Theorem 1.9 in [36]), it is consistent relative to a large cardinal that "κ is not subtle, but $I^+_{\kappa,\lambda} \to (I^+_{\kappa,\lambda})^n_\eta$ holds for any $n < \omega$ and any $\eta < \kappa$".

7 Mild ineffability

DEFINITION 7.1. κ is *mildly λ-ineffable* if, given $s_a \subseteq a$ for $a \in P_\kappa(\lambda)$, there exists $S \subseteq \lambda$ with the property that for any $b \in P_\kappa(\lambda)$, there is $a \in P_\kappa(\lambda)$ such that $b \subseteq a$ and $S \cap b = s_a \cap b$.

We will establish that if κ is mildly λ-ineffable and cf$(\lambda) \neq \kappa$, then $\text{cov}(\lambda, \kappa^+, \kappa^+, \kappa) = \lambda$. We need some preparation.

FACT 7.2. ([5])

(i) If κ is mildly λ-ineffable, then it is mildly ν-ineffable for any cardinal ν with $\kappa \leq \nu \leq \lambda$.

(ii) κ is mildly κ-ineffable if and only if it is weakly compact.

FACT 7.3. ([11]) *Suppose that κ is inaccessible. Then the following hold :*

(i) κ is mildly λ-ineffable iff $TP(\kappa,\lambda)$ holds iff $TP^-(\kappa,\lambda)$ holds.

(ii) κ is mildly $\lambda^{<\kappa}$-ineffable iff $PS^+(\kappa,\kappa,\lambda)$ holds iff $PS^*(\kappa,\kappa,\lambda)$ holds iff $PS(\kappa,\kappa,\lambda)$ holds.

FACT 7.4. ([8]) *The following are equivalent :*

(i) κ is mildly λ-ineffable.

(ii) Given $W_\alpha \subseteq P_\kappa(\lambda)$ for $\alpha<\lambda$, there is $h \in \prod_{\alpha<\lambda}\{W_\alpha, P_\kappa(\lambda)\setminus W_\alpha\}$ such that $\bigcap_{\alpha\in e} h(\alpha) \in I^+_{\kappa,\lambda}$ for every nonempty $e \in P_\kappa(\lambda)$.

OBSERVATION 7.5. *Suppose that κ is mildly λ-ineffable, and for each $\alpha<\lambda$, let Q_α be a partition of $P_\kappa(\lambda)$ into less than κ many pieces. Then there is $h \in \prod_{\alpha<\lambda} Q_\alpha$ such that $\bigcap_{\alpha\in e} h(\alpha) \in I^+_{\kappa,\lambda}$ for every nonempty $e \in P_\kappa(\lambda)$.*

Proof. By Fact 7.4, we may find $h \in \prod_{W\in\bigcup_{\alpha<\lambda} Q_\alpha}\{W, P_\kappa(\lambda)\setminus W\}$ such that $\bigcap_{W\in x} h(W) \in I^+_{\kappa,\lambda}$ for every nonempty $x \in P_\kappa(\bigcup_{\alpha<\lambda} Q_\alpha)$. Now given $\alpha<\lambda$, we have $\bigcap_{W\in Q_\alpha}(P_\kappa(\lambda)\setminus W)=\emptyset$, and consequently $Q_\alpha \cap ran(h) \neq \emptyset$. $\square$

$TP(\kappa,\lambda)$ may be reformulated in the same way.

PROPOSITION 7.6. *The following are equivalent :*

(i) $TP(\kappa,\lambda)$.

(ii) For each $\alpha<\lambda$, let Q_α be a partition of $\{x \in P_\kappa(\lambda) : \alpha \in x\}$ into less than κ many pieces. Suppose that

$$|\{\bigcap_{\alpha\in d} g(\alpha) : g \in \prod_{\alpha\in d} Q_\alpha\}| < \kappa$$

for any nonempty $d \in P_\kappa(\lambda)$. Then there is $h \in \prod_{\alpha<\lambda} Q_\alpha$ such that $\bigcap_{\alpha\in e} h(\alpha) \in I^+_{\kappa,\lambda}$ for every nonempty $e \in P_\kappa(\lambda)$.

(iii) For each $\alpha<\lambda$, let Q_α be a partition of $\{x \in P_\kappa(\lambda) : \alpha \in x\}$ with $|Q_\alpha| \leq 2$. Suppose that

$$|\{\bigcap_{\alpha\in d} g(\alpha) : g \in \prod_{\alpha\in d} Q_\alpha\}| < \kappa$$

for any nonempty $d \in P_\kappa(\lambda)$. Then there is $h \in \prod_{\alpha<\lambda} Q_\alpha$ such that $\bigcap_{\alpha\in e} h(\alpha) \in I^+_{\kappa,\lambda}$ for every nonempty $e \in P_\kappa(\lambda)$.

Proof. (i) $\to$ (ii) : Assume that $TP(\kappa,\lambda)$ holds. Let Q_α be a partition of $\{x \in P_\kappa(\lambda) : \alpha \in x\}$ into less than κ many pieces for $\alpha<\kappa$ such that

$$|\{\bigcap_{\alpha\in d} g(\alpha) : g \in \prod_{\alpha\in d} Q_\alpha\}| < \kappa$$

for any nonempty $d \in P_\kappa(\lambda)$. For $\alpha<\lambda$, let $\langle Q^\xi_\alpha : \xi < |Q_\alpha|\rangle$ be a one-to-one enumeration of Q_α. Select a bijection $f : \kappa\times\lambda \to \lambda$. For $a \in P_\kappa(\lambda)$, define $t_a : a \to 2$ as follows. Given $\xi<\kappa$ and $\alpha<\lambda$ such that $f(\xi,\alpha) \in a$, we let $t_a(j(\xi,\alpha))=1$ just in case $\alpha \in a$ and $a \in Q^\xi_\alpha$. For $c \in P_\kappa(\lambda)$, let A_c denote the collection of all $\alpha < \lambda$ such that $f(\xi, \alpha) \in c$ for some $\xi < \kappa$. Let C be the set of all $c \in P_\kappa(\lambda)$ such that $A_c \subseteq c$. Note that $C \in NS^*_{\kappa,\lambda}$.

Claim 1. Let $c \in C$. Then $|\{t_a|c : c \subseteq a\}| < \kappa$.

Proof of Claim 1. Suppose otherwise, and let $a_i \in P_\kappa(\lambda)$ for $i < \kappa$ be such that

- $c \subseteq a_i$ for all $i < \kappa$.
- $t_{a_i}|c \neq t_{a_j}|c$ whenever $i < j < \kappa$.

For $i < \kappa$, define $k_i : A_c \to \kappa$ so that $a_i \in Q_\alpha^{k_i(\alpha)}$ for all $\alpha \in A_c$. Now given $i < j < \kappa$, we may find $\alpha \in A_c$ and $\xi < \kappa$ such that $f(\xi, \alpha) \in c$ and $t_{a_i}(f(\xi, \alpha)) \neq t_{a_i}(f(\xi, \alpha))$. Then it is easy to see that $k_i(\alpha) \neq k_j(\alpha)$. Hence, $k_i \neq k_j$. This contradiction completes the proof of the claim.

By Claim 1, we may find $T : \lambda \to 2$ such that for any $v \in P_\kappa(\lambda)$, there is $a \in P_\kappa(\lambda)$ with $v \subseteq a$ and $T|v = t_a|v$.

Claim 2. Let $\alpha < \lambda$. Then $T(f(\xi, \alpha)) = 1$ for some $\xi < |Q_\alpha|$.

Proof of Claim 2. Suppose otherwise. Pick $v \in P_\kappa(\lambda)$ such that $\{\alpha\} \cup \{f(\xi, \alpha) : \xi < |Q_\alpha|\} \subseteq v$. There must be $a \in P_\kappa(\lambda)$ such that $v \subseteq a$ and $T|v = t_a|v$. Then $a \notin Q_\alpha^\xi$ for all $\xi < |Q_\alpha|$. This contradiction completes the proof of the claim.

Claim 3. Let $\alpha < \lambda$. Then $|\{\xi < \lambda : T(f(\xi, \alpha)) = 1\} \leq 1$.

Proof of Claim 3. Let $\xi_1, \xi_2 < \lambda$ be such that $T(f(\xi_1, \alpha)) = T(f(\xi_2, \alpha)) = 1$. Pick $v \in P_\kappa(\lambda)$ such that $\{\alpha, f(\xi_1, \alpha), f(\xi_2, \alpha)\} \subseteq v$. There must be $a \in P_\kappa(\lambda)$ such that $v \subseteq a$ and $T|v = t_a|v$. Then $a \in Q_\alpha^{\xi_1} \cap Q_\alpha^{\xi_2}$. Hence $\xi_1 = \xi_2$, which completes the proof of the claim.

Using Claims 2 and 3, define $H \in \prod_{\alpha<\lambda} |Q_\alpha|$ by $H(\alpha) =$ the unique $\xi < |Q_\alpha|$ such that $T(f(\xi, \alpha)) = 1$. Now given $e, w \in P_\kappa(\lambda) \setminus \{\emptyset\}$, set $v = e \cup w \cup \{f(H(\alpha), \alpha) : \alpha \in e\}$. We may find $a \in P_\kappa(\lambda)$ such that $v \subseteq a$ and $T|v = t_a|v$. Then clearly,

- $w \subseteq a$.
- $a \in \bigcap_{\alpha \in e} Q_\alpha^{H(\alpha)}$.

(ii) $\to$ (iii) : Trivial.

(iii) $\to$ (i) : Assume that (iii) holds. Let $t_a : a \to 2$ for $a \in P_\kappa(\lambda)$ be such that $|\{t_a|d : d \subseteq a\}| < \kappa$ for all $d \in P_\kappa(\lambda)$. For $\alpha < \lambda$ and $i < 2$, let Q_α^i be the set of all $a \in P_\kappa(\lambda)$ such that $\alpha \in a$ and $t_a(\alpha) = i$.

Claim. Let $d \in P_\kappa(\lambda) \setminus \{\emptyset\}$. Then

$$|\{\bigcap_{\alpha \in d} u(\alpha) : u \in \prod_{\alpha \in d} \{Q_\alpha^0, Q_\alpha^1\}\}| < \kappa$$

.

Proof of the claim. Suppose otherwise. Pick $g_\xi : d \to 2$ for $\xi < \kappa$ so that $\bigcap_{\alpha\in d} Q_\alpha^{g_\eta(\alpha)} \neq \bigcap_{\alpha\in d} Q_\alpha^{g_\xi(\alpha)}$ whenever $\eta < \xi < \kappa$. For $\xi < \kappa$ with $\bigcap_{\alpha\in d} Q_\alpha^{g_\xi(\alpha)} \neq \emptyset$, pick $a_\xi \in \bigcap_{\alpha\in d} Q_\alpha^{g_\xi(\alpha)}$. Notice that $t_{a_\xi}|d = g_\xi$. Thus $|\{t_{a_\xi}|d : \xi < \kappa\}| = \kappa$. This contradiction completes the proof of the claim.

By the claim, we may find $T : \lambda \to 2$ such that $\bigcap_{\alpha\in e} Q_\alpha^{T(\alpha)} \in I^+_{\kappa,\lambda}$ for every nonempty $e \in P_\kappa(\lambda)$. It remains to observe that for any $e \in P_\kappa(\lambda) \setminus \{\emptyset\}$ and any $a \in \bigcap_{\alpha\in e} Q_\alpha^{T(\alpha)}$, we have $t_a|e = T|e$.

$\square$

DEFINITION 7.7. Given a set P and a κ-complete ideal J on P, we denote by $IE^2_\kappa(J)$ the following statement : Suppose that for each $p \in P$, there is a partition Q_p of P with $|Q_p| < \kappa$. Then there is $h \in \prod_{p\in P} Q_p$ and a κ-complete ideal K on P extending J such that $\mathrm{ran}(h) \subseteq K^*$.

OBSERVATION 7.8. *Suppose that κ is mildly λ-ineffable and λ is regular. Then $IE^2_\kappa(I_\lambda)$ holds.*

Proof. For each $\alpha < \lambda$, let W_α be a partition of λ with $|W_\alpha| < \kappa$. For $\alpha < \lambda$, put
$$Q_\alpha = \{\{a \in P_\kappa(\lambda) : \sup a \in T\} : T \in W_\alpha\}.$$
By Observation 7.5, we may find $h \in \prod_{\alpha<\lambda} W_\alpha$ such that
$$\{a \in P_\kappa(\lambda) : \sup a \in \textstyle\bigcap_{\alpha\in e} h(\alpha)\} \in I^+_{\kappa,\lambda}$$
for every nonempty $e \in P_\kappa(\lambda)$. It is easy to see that $\bigcap_{\alpha\in e} h(\alpha) \in I^+_\lambda$ for all $e \in P_\kappa(\lambda) \setminus \{\emptyset\}$. $\square$

Note that if λ is weakly compact, then by Fact 7.2 and Observation 7.8, $IE^2_\kappa(I_\lambda)$ (in fact $IE^2_\lambda(I_\lambda)$) holds. Thus the converse of Observation 7.8 does not hold.

OBSERVATION 7.9. *Suppose that κ is mildly λ-ineffable. Then* $\mathrm{cov}(\nu, \nu, (\mathrm{cf}(\nu)^+, 2) = \nu^+$ *for each singular cardinal ν with $\kappa < \nu < \lambda$.*

Proof. Given a singular cardinal ν with $\kappa < \nu < \lambda$, κ is mildly ν^+-ineffable by Fact 7.2, so $PS^+((\mathrm{cf}(\nu)^+, \nu^+)$ holds by Observation 7.7, and therefore $\mathrm{cov}(\nu, \nu, (\mathrm{cf}(\nu))^+, 2) = \nu^+$ by Observation 4.6. $\square$

FACT 7.10. ([35]) *Suppose that $cf(\lambda) \geq \kappa$ and κ is mildly λ-ineffable. Then $\lambda^{<\kappa} = \lambda$.*

Proof. Since κ is inaccessible by Fact 7.2, it follows from Fact 4.2 and Observations 4.8 (i) and 7.9 that $\lambda^{<\kappa} = u(\kappa, \lambda) = \lambda$. $\square$

Usuba [35] asked whether $\lambda^{<\kappa} = \lambda^+$ whenever κ is mildly λ-ineffable and $\mathrm{cf}(\lambda) < \kappa$. The following provides a partial answer to this question.

PROPOSITION 7.11. *(i) Suppose that $\omega < cf(\lambda) < \kappa$, and κ is mildly ν-ineffable for every cardinal ν with $\kappa \leq \nu < \lambda$. Then $\lambda^{<\kappa} = \lambda^+$.*

(ii) Suppose that $\omega = cf(\lambda)$, and κ is mildly ν-ineffable for every cardinal ν with $\kappa \leq \nu < \lambda$. Then $\lambda^{<\kappa} = \mathrm{cov}(\lambda, \lambda, \omega_1, 2)$.

Proof. (i) : κ is inaccessible by Fact 7.2, so by Facts 4.2 and 4.7 and Observation 7.9, $\lambda^+ \leq \lambda^{<\kappa} = u(\kappa, \lambda) \leq \lambda^+$.
(ii) : Use Observation 4.10. $\square$

PROPOSITION 7.12. *Suppose that κ is mildly λ-ineffable and $\mathrm{cf}(\lambda) \neq \kappa$. Then $\mathrm{cov}(\lambda, \kappa^+, \kappa^+, \kappa) = \lambda$.*

Proof. Since κ is inaccessible by Fact 7.2, the result follows from Observations 7.9 and 4.8 (use (i) if $\mathrm{cf}(\lambda) \neq \omega$, and (ii) otherwise). $\square$

FACT 7.13. ([11]) *The following are equivalent:*

(i) κ is mildly $\lambda^{<\kappa}$-ineffable.

(ii) For any set P of size $\lambda^{<\kappa}$ and any κ-complete ideal J on P, $IE^2_\kappa(J)$ holds.

Suppose that κ is the successor of a singular limit of λ-compact cardinals. Then by a result of Magidor and Shelah [27], κ has the tree property, and in fact, as shown in [10], $TP(\kappa, \lambda')$ holds for every cardinal $\lambda' \geq \kappa$. We modify the proof so as to obtain the following which improves a result of [11].

PROPOSITION 7.14. *Suppose that $\kappa = \nu^+$, where ν is a singular limit of mildly $\lambda^{<\kappa}$-ineffable cardinals. Then $PS(\kappa, \lambda)$ holds.*

Proof. Set $\sigma = cf(\nu)$, and select an increasing sequence $\langle \nu_i : i < \sigma \rangle$ of mildly $\lambda^{<\kappa}$-ineffable cardinals with $\sigma < \nu_0$ and $\sup\{\nu_i : i < \sigma\} = \nu$. Suppose that for each $a \in P_\kappa(\lambda)$, Q_a is a partition of $P_\kappa(\lambda)$ with $|Q_a| \leq \nu$. For $a \in P_\kappa(\lambda)$, pick an onto function ψ_a from Q_a to ν, and let W^p_a denote the set of all $c \in P_\kappa(\lambda)$ such that $p =$ the least r such that $c \in \bigcup \psi_a``r$. By Fact 7.13, we may find $t : P_\kappa(\lambda) \to \sigma$ and a ν_0-complete ideal K on $P_\kappa(\lambda)$ extending $I_{\kappa,\lambda}$ such that $\{W^{t(a)}_a : a \in P_\kappa(\lambda)\} \subseteq K^*$. There must be $S \in K^+$ and $p < \sigma$ such that t takes the constant value p on S. Thus $W^p_a \cap W^p_x \in I^+_{\kappa,\lambda}$ for all $a, x \in S$. Define $g : S \times S \to P_\kappa(\lambda)$ so that $a \cup x \subseteq g(a, x)$ and $g(a, x) \in W^p_a \cap W^p_x$. Further define $h : S \times S \to \nu_p \times \nu_p$ so that $g(a, x) \in \psi_a(\alpha) \cap \psi_x(\beta)$, where $h(a, x) = (\alpha, \beta)$. For $a \in S$ and $(\alpha, \beta) \in \nu_p \times \nu_p$, put $T^{(\alpha,\beta)}_a = \{x \in S : h(a, x) = (\alpha, \beta)\}$. By Fact 7.13, we may find $u : S \to \nu_p \times \nu_p$ and a ν_{p+1}-complete ideal G on $P_\kappa(\lambda)$ extending $I_{\kappa,\lambda}|S$ such that $\{T^{u(a)}_a : a \in S\} \subseteq G^*$. There must be $A \in G^+$ and $(\alpha, \beta) \in \nu_p \times \nu_p$ such that u is constantly (α, β) on A. Pick $X \in I^+_{\kappa,\lambda} \cap P(A)$ so that $A \setminus X \in I^+_{\kappa,\lambda}$. Set $B = X$ if $A \setminus X \in G^+$, and $B = A \setminus X$ otherwise. Now given $a, b \in B$, pick $x \in T^{(\alpha,\beta)}_a \cap T^{(\alpha,\beta)}_b \cap (A \setminus B)$ with $a \cup b \subseteq x$. Then clearly, $g(a, x) \in \psi_a(\alpha) \cap \psi_x(\beta)$ and $g(b, x) \in \psi_b(\alpha) \cap \psi_x(\beta)$. $\square$

8 Distributivity

OBSERVATION 8.1. *Given a κ-complete, fine ideal J on $P_\kappa(\lambda)$, the following are equivalent :*

(i) *J is $(\lambda^{<\kappa}, 2)$-distributive.*

(ii) *Given $A \in J^+$ and $F : A \times A \to \lambda^{<\kappa}$ with the property that $|\{F(a,b) : b \in A\}| < \kappa$ for all $a \in A$, there is $B \in J^+ \cap P(A)$ and $h : B \to \lambda^{<\kappa}$ such that $\{b \in B : F(a,b) \neq h(a)\} \in J$ for all $a \in B$.*

(iii) *$J^+ \xrightarrow{J} (J^+)^n_\rho$ holds whenever $2 \leq n < \omega$ and $0 < \rho < \kappa$.*

(iv) *$J^+ \xrightarrow{J} (J^+)^3_2$ holds.*

Proof. (i) $\to$ (ii) : Use Observation 5.4.

(ii) $\to$ (iii) : Assume that (ii) holds. It is readily seen that $J^+ \xrightarrow{J} (J^+)^2$ (and hence $J^+ \xrightarrow{J} (J^+, \kappa)^2$) holds. By Observation 5.18, it follows that κ is weakly compact (and therefore inaccessible). Now to prove (iii), we proceed by induction on n. For $n = 2$, the assertion easily follows from (ii). Suppose now that the assertion has been verified for a certain n. Fix $A \in J^+$ and $F : A^{n+1} \to \rho$, where $2 \leq \rho < \kappa$. For $b \in A$, let T_b denote the collection of all functions t from $(A \cap P(b))^{n-1}$ to ρ. Notice that $|T_b| < \kappa$. Define $G : A \times A \to \bigcup_{b \in A} T_b$ as follows. Given $(b, c) \in A \times A$, let $G(b, c)$ be the element t of T_b defined by $t(a_1, \cdots, a_{n-1}) = F(a_1, \cdots, a_{n-1}, b, c)$. We may find $B \in J^+ \cap P(A)$, $h \in \prod_{b \in B} T_b$, and $X_b \in J$ for $b \in B$ such that $G(b, c) = h(b)$ whenever $b, c \in B$ and $c \notin X_b$. Define $H : \bigcup_{b \in B}((A \cap P(b))^{n-1} \times \{b\}) \to \rho$ by $H(a_1, \cdots, a_{n-1}, b) = h(b)(a_1, \cdots, a_{n-1})$. There must $C \in J^+ \cap P(B)$, $i < 2$, and $Y_a \in J$ for $a \in C$ with

$$\{b \in P_\kappa(\lambda) : a \setminus b \neq \emptyset\} \subseteq Y_a$$

such that $H(a_1, \cdots, a_{n-1}, a_n) = i$ whenever $a_1, \cdots, a_n \in C$ and $a_{m+1} \notin Y_{a_1} \cup \cdots \cup Y_{a_m}$ for $1 \leq m < n$. For $a \in C$, put $Z_a = X_a \cup Y_a$. Then clearly, $F(a_1, \cdots, a_{n+1}) = G(a_n, a_{n+1})(a_1, \cdots, a_{n-1}) = h(a_n)(a_1, \cdots, a_{n-1}) = H(a_1, \cdots, a_{n-1}, a_n) = i$ whenever $a_1, \cdots, a_{n+1} \in C$ and $a_{m+1} \notin Z_{a_1} \cup \cdots \cup Z_{a_m}$ for $1 \leq m \leq n$.

(iii) $\to$ (iv) : Trivial.

(iv) $\to$ (i) : Assume that (iv) holds. Then by Observation 5.18, κ is inaccessible. Now let $A \in J^+$, and $W_a \subseteq P_\kappa(\lambda)$ for $a \in P_\kappa(\lambda)$. Define $F : A \times A \times A \to 2$ by : $F(a, b, c) = 0$ if and only if $\{d \subseteq a : b \in W_d\} = \{d \subseteq a : c \in W_d\}$. There must be $B \in J^+ \cap P(A)$, $i < 2$, and $Z_a \in J$ for $a \in B$ such that $F(a, b, c) = i$ whenever $a, b, c \in B$, $b \notin Z_a$ and $c \notin Z_a \cup Z_b$. Given $a \in B$, we may find $C \in J^+ \cap P(B)$ such that $\{d \subseteq a : b \in W_d\} = \{d \subseteq a : c \in W_d\}$ whenever $d \subseteq a$ and $b, c \in C$. It easily follows that $i = 0$. Now fix $d \in P_\kappa(\lambda)$. Pick $a \in B$ with $d \subseteq a$, and $b \in B \setminus Z_a$. Then either $B \setminus (Z_a \cup Z_b) \subseteq W_d$, or $B \setminus (Z_a \cup Z_b) \subseteq P_\kappa(\lambda) \setminus W_d$. $\square$

Note the similarity with Fact 7.3 (ii) or Fact 7.12. One could indeed argue that $(\lambda^{<\kappa}, 2)$-distributivity of J (respectively, mild $\lambda^{<\kappa}$-ineffability of κ) makes more sense (or is more natural) than $(\lambda, 2)$-distributivity (respectively, mild λ-ineffability). However the question of Abe mentioned in the introduction concerns $(\lambda, 2)$-distributivity (and not $(\lambda^{<\kappa}, 2)$-distributivity) of $I_{\kappa,\lambda}$, so let us focus on $(\lambda, 2)$-distributivity.

PROPOSITION 8.2. *Suppose that J is a $(\kappa, 2)$-distributive, κ-normal, fine ideal on $P_\kappa(\lambda)$. Then $J \restriction \kappa$ is $(\kappa, 2)$-distributive.*

Proof. Let $X \in (J \restriction \kappa)^+$, and $W_\alpha \subseteq \kappa$ for $\alpha < \kappa$. For $\alpha < \kappa$, set $T_\alpha = \{a \in \Omega_{\kappa,\lambda} : a \cap \kappa \in W_\alpha\}$. We may find $h \in \prod_{\alpha<\kappa}\{T_\alpha, P_\kappa(\lambda) \setminus T_\alpha\}$, $B \in J^+ \cap P(\{a \in \Omega_{\kappa,\lambda} : a \cap \kappa \in X\})$, and $Z_\alpha \in J$ for $\alpha < \kappa$ such that $B \setminus Z_\alpha \subseteq h(\alpha)$ for all $\alpha < \kappa$. Define $k \in \prod_{\alpha<\kappa}\{W_\alpha, \kappa \setminus W_\alpha\}$ by : $k(\alpha) = W_\alpha$ if and only if $h(\alpha) = T_\alpha$. Put $S = \{a \in B : \forall \alpha \in a \cap \kappa (a \notin Z_\alpha)\}$ and $Y = \{a \cap \kappa : a \in S\}$. It is easy to see that $Y \in (J \restriction \kappa)^+ \cap P(X)$. Furthermore $Y \setminus (\alpha + 1) \subseteq k(\alpha)$ for all $\alpha < \kappa$. □

OBSERVATION 8.3. *Suppose that $\overline{\mathfrak{d}}_\kappa \leq \lambda = \mathrm{cov}(\lambda, \kappa^+, \kappa^+, \kappa)$. Then for any $D \in NS^*_{\kappa,\lambda}$ and any $X \in NS^+_\kappa \cap NCI_\kappa$, $I_{\kappa,\lambda}|\{a \in D : \sup(a \cap \kappa) \in X\}$ is not $(\kappa, 2)$-distributive.*

Proof. Given $D \in NS^*_{\kappa,\lambda}$ and $X \in NS^+_\kappa \cap NCI_\kappa$, set $K = I_{\kappa,\lambda}|\{a \in D : \sup(a \cap \kappa) \in X\}$. Assume toward a contradiction that K is $(\kappa, 2)$-distributive. By Fact 5.31, there is $C \in NS^*_{\kappa,\lambda}$ such that $I_{\kappa,\lambda}|C$ is κ-normal. Put $J = K|C$. Then clearly, J is κ-normal and $(\kappa, 2)$-distributive. Hence by Observation 5.21 and Proposition 8.2, $J \restriction \kappa$ is a normal, $(\kappa, 2)$-distributive, fine ideal on κ. It follows that $X \in J \restriction \kappa$. Contradiction. □

PROPOSITION 8.4. *Suppose that $\overline{\mathfrak{d}}_\kappa \leq \lambda$ and $\mathrm{cf}(\lambda) \neq \kappa$. Then for any $D \in NS^*_{\kappa,\lambda}$ and any $X \in NS^+_\kappa \cap NCI_\kappa$, $I_{\kappa,\lambda}|\{a \in D : \sup(a \cap \kappa) \in X\}$ is not $(\lambda, 2)$-distributive.*

Proof. By Fact 7.4, Proposition 7.12 and Observation 8.3. □

9 The case $\mathrm{cf}(\lambda) = \kappa$

By Fact 4.4, if $\mathrm{cf}(\lambda) = \kappa$, then $\mathrm{cov}(\lambda, \kappa^+, \kappa^+, \kappa) > \lambda$. Thus a different approach is needed in case $\mathrm{cf}(\lambda) = \kappa$.

PROPOSITION 9.1. *Let J be a κ-complete, fine ideal on $P_\kappa(\lambda)$ such that $cof(J) < \max\{\mathfrak{d}_\kappa, u(\kappa^+, \lambda)\}$. Then the following hold :*

(i) *Let $A_\alpha \in J^+$ for $\alpha < \kappa$ be given such that $A_\alpha \subseteq A_\beta$ whenever $\beta < \alpha < \kappa$. Then there is $C \in J^+$ such that $C \setminus A_\alpha \in I_{\kappa,\lambda}$ for all $\alpha < \kappa$.*

(ii) *Suppose that κ is weakly compact. Then given $W_\alpha \subseteq P_\kappa(\lambda)$ for $\alpha < \kappa$, there is $C \in J^+$ and $h \in \prod_{\alpha<\kappa}\{W_\alpha, P_\kappa(\lambda) \setminus W_\alpha\}$ such that $C \setminus h(\alpha) \in I_{\kappa,\lambda}$ for all $\alpha < \kappa$.*

Proof. (i) : The proof is a straightforward modification of that of Proposition 2.7 in [23].
(ii) : Assume that κ is weakly compact, and let $W_\alpha \subseteq P_\kappa(\lambda)$ for $\alpha < \kappa$. There must be $h \in \prod_{\alpha<\kappa}\{W_\alpha, P_\kappa(\lambda) \setminus W_\alpha\}$ such that $A_\alpha \in J^+$ for all $\alpha < \kappa$, where $A_\alpha = \bigcap_{\beta\leq\alpha} h(\beta)$. By (i) we may find $C \in J^+$ such that $C \setminus A_\alpha \in I_{\kappa,\lambda}$ for all $\alpha < \kappa$. Then clearly, $C \setminus h(\alpha) \in I_{\kappa,\lambda}$ for every $\alpha < \kappa$. □

COROLLARY 9.2. *Suppose that κ is weakly compact, and let J be a κ-complete, fine ideal on $P_\kappa(\lambda)$ such that $cof(J) < \max\{\mathfrak{d}_\kappa, u(\kappa^+, \lambda)\}$. Then the following hold :*

(i) *J is $(\kappa, 2)$-distributive.*

(ii) *$J^+ \xrightarrow[\kappa]{I_{\kappa,\lambda}} (J^+)^2_\rho$ holds for any nonzero cardinal $\rho < \kappa$.*

Proof. Use Observation 5.4. □

FACT 9.3. ([18], [27]) *Suppose that κ is weakly inaccessible, and let J be a κ-complete, fine ideal on $P_\kappa(\lambda)$ such that $cof(J) < \max\{\mathfrak{d}_\kappa, u(\kappa^+, \lambda)\}$. Then $J^+ \xrightarrow[\kappa]{I_{\kappa,\lambda}} [J^+]^2_{\kappa^+}$ holds.*

To obtain a negative partition relation, we will go one cardinal up and work with partitions of $\kappa^+ \times P_\kappa(\lambda)$.

DEFINITION 9.4. Given $2 \leq n < \omega$, two collections X and Y of subsets of $P_\kappa(\lambda)$, an ideal J on $P_\kappa(\lambda)$, and a cardinal ρ, $X \xrightarrow[\kappa^+]{Y} (J^+)^2_\rho$ means that for any $F : \kappa^+ \times P_\kappa(\lambda) \to \rho$ and any $A \in X$, there is $i < \rho$ and $B \in J^+ \cap P(A)$ such that $\{b \in B : F(\sup(a \cap \kappa^+), b) \neq i\} \in Y$ for all $a \in B$.

FACT 9.5. ([18]) *Suppose that the following hold :*

- *κ is weakly inaccessible.*
- *$\mathrm{cf}(\lambda) = \kappa$.*
- *$2^\kappa \leq \lambda$.*
- *$u(\kappa^+, \tau) \leq \lambda$ for any cardinal τ with $\kappa < \tau < \lambda$.*

Then $\{C \cap S^\lambda_{\kappa,\lambda}\} \xrightarrow[\kappa^+]{I_{\kappa,\lambda}} [I^+_{\kappa,\lambda}]^2_\lambda$ fails for some $C \in NS^*_{\kappa,\lambda}$.

PROPOSITION 9.6. *Suppose that $\mathrm{cf}(\lambda) = \kappa$ and $2^\kappa \leq \lambda$. Let $D \in NS^*_{\kappa,\lambda}$ and $J = I_{\kappa,\lambda}|(D \cap S^\lambda_{\kappa,\lambda})$. Then the following hold :*

(i) $J^+ \xrightarrow{J} (J^+)^2_\omega$ *does not hold.*

(ii) J *is not* $(\lambda, 2)$*-distributive.*

Proof. (i) : By Observations 5.18 and 6.4 and Fact 9.5.
(ii) : Suppose otherwise. Then $\lambda^{<\kappa} = \lambda$ by Fact 7.10, since κ is mildly λ-ineffable by Fact 7.4. Hence $J^+ \xrightarrow{J} (J^+)^2_\omega$ holds by Observation 8.1, which contradicts (i). $\square$

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