Stackelberg Network Pricing is Hard to Approximate

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Abstract

In the Stackelberg network pricing problem, one has to assign tariffs to a certain subset of the arcs of a given transportation network. The aim is to maximize the amount paid by the user of the network, knowing that the user will take a shortest *st*-path once the tariffs are fixed. Roch, Savard, and Marcotte (*Networks*, Vol. 46(1), 57–67, 2005) proved that this problem is NP-hard, and gave an $O(\log m)$ -approximation algorithm, where *m* denote the number of arcs to be priced. In this note, we show that the problem is also APX-hard.

Keywords: Combinatorial optimization; APX-hardness; Network pricing; Stackelberg games

1 Introduction

We consider a network pricing problem involving two non-cooperative players, a *leader* and a *follower*. The leader owns a subset of the arcs of a given transportation network. She has to set tariffs on these arcs, knowing that the follower will compute a shortest st-path once the tariffs are fixed. The goal is to maximize the revenue of the leader, which depends on the path chosen by the follower.

This problem is known as *Stackelberg network pricing*; it is formally described as follows:

INSTANCE:	- a directed graph $D = (V, A)$
	$-$ a cost function $c: A \to \mathbb{R}_+$
	- a pair (s,t) of distinct nodes $s,t \in V$
	$-$ a subset $T \subseteq A$ of tariff arcs
SOLUTION:	– an assignment $d: T \to \mathbb{R}_+$ of tariffs to the arcs in T
	– an st-path P of D minimizing its total cost $\sum_{a \in P} c(a) + \sum_{a \in P \cap T} d(a)$
OBJECTIVE:	- maximize the revenue $\sum_{a \in P \cap T} d(a)$

Before going further, let us make two remarks on the above formulation: First, it is usually assumed that there exists an *st*-path in D that uses only arcs of A - T, since otherwise the optimum is unbounded. Second, once the tariffs are fixed, we can easily choose among all *st*-paths of minimum (total) cost one path P that maximizes the revenue. In other words, we assume that the follower always makes the best choice for the leader. This is a standard assumption, justified by the fact that decreasing the prices of every arc in $P \cap T$ by an arbitrarily small amount ensures that P is the unique *st*-path of minimum cost.

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The Stackelberg network pricing problem has recently been studied by Roch, Savard, and Marcotte [11], motivated by applications in transportation and telecommunications. They proved that the problem is NP-hard and described a polynomial-time algorithm approximating the optimum within a ratio of $\frac{1}{2} \log_2 |T| + 1$.

The purpose of this note is to show that the Stackelberg network pricing problem is also APX-hard:

Theorem 1. For some $\epsilon > 0$, it is NP-hard to approximate Stackelberg network pricing within a ratio of $1 - \epsilon$.

We conclude this introduction by mentioning other related works. A generalization of Stackelberg network pricing, the *toll setting problem*, has been considered by Labbé, Marcotte, and Savard [10]. It involves multiple, weighted followers: the *i*th follower computes a shortest $s_i t_i$ path in the network once the tariffs are fixed, and has a demand of d_i . The objective is then to maximize the sum of the revenues obtained from each individual follower, weighted by their respective demands. As observed by Roch *et al.* [11], the approximation algorithm mentioned above for Stackelberg network pricing directly gives a $O(k \log m)$ -approximation algorithm in the case of unit demands, where k denotes the number of followers.

A special case of the toll setting problem occurs when every $s_i t_i$ -path in D uses at most one arc in T, and is known under the name river tariff pricing. It has been considered by Bouhtou, Grigoriev, van Hoesel, van der Kraaij, Spieksma, and Uetz [1], and is closely related to the profitmaximizing envy-free pricing problem studied by Guruswami, Hartline, Karlin, Kempe, Kenyon, and McSherry [7]. Among others, both sets of authors proved (independently) that the river tariff pricing problem is APX-hard, even in the case of unit demands. Briest, Hoefer, and Krysta [2] later derived stronger inapproximability results, relying on a recent inapproximability result of Demaine, Feige, Hajiaghayi, and Salavatipour [5] for profit-maximizing envy-free pricing. We note that each of these inapproximability results uses in a crucial way the fact that there are multiple followers.

Finally, we mention that some other combinatorial optimization problems similar to Stackelberg network pricing have been considered recently. This includes pricing edges of an undirected graph knowing that the follower will compute a minimum spanning tree [3], and pricing vertices of a bipartite undirected graph when the follower buys a minimum cost vertex cover [2]. Other kinds of Stackelberg games in networks have been studied by Cole, Dodis, and Roughgarden [4], Roughgarden [12], and Swamy [13].

2 The Proof

In order to prove Theorem 1, we need the following lemma on bounded-degree graphs. Here and throughout the text, a *linear ordering* of the vertices of a graph G = (V, E) is a bijective mapping $\ell: V \to \{1, \ldots, |V|\}$.

Lemma 1. Let G = (V, E) be an undirected graph with maximum degree $\Delta \ge 1$ and $n \ge c_{\Delta}$ vertices, where $c_{\Delta} := 4\Delta(\Delta + 1)$. Then a linear ordering ℓ of V with

$$|\ell(u) - \ell(v)| \ge \frac{n}{c_\Delta}$$

for every edge $uv \in E$ can be found in polynomial time.

A (proper, vertex) coloring of a graph G is equitable if every two color classes differ in size by at most 1. The proof of Lemma 1 relies on the following result on equitable colorings.

Theorem 2 (Hajnal and Szemerédi [8]). Every graph G with maximum degree Δ can be equitably colored with $\Delta + 1$ colors.

Kierstead and Kostochka [9] recently obtained a short proof of Theorem 2. Their proof yields also a polynomial-time algorithm finding such a coloring.

Proof of Lemma 1. The lemma is easily seen to hold if $\Delta = 1$, hence we assume $\Delta \geq 2$. Using the algorithmic version of Theorem 2 given by Kierstead and Kostochka [9], we first find in polynomial time an equitable coloring $S_1, \ldots, S_{\Delta+1}$ of the vertices of G. Let

$$s:=\left\lfloor \frac{n}{2\Delta(\Delta+1)}\right\rfloor$$

We have

$$s \geq \frac{n}{2\Delta(\Delta+1)} - 1 \geq \frac{n}{2\Delta(\Delta+1)} - \frac{n}{4\Delta(\Delta+1)} = \frac{n}{c_{\Delta}}$$

We show that a linear ordering ℓ of the vertices of G with $|\ell(u) - \ell(v)| \ge s$ for every edge $uv \in E$ can be found in polynomial time, which implies the claim.

Observe that

$$|S_i| > s(\Delta + 1) \qquad \text{for every } i \in \{1, \dots, \Delta + 1\}.$$
(1)

Indeed, using $\Delta \geq 2$ and $n \geq 4\Delta(\Delta + 1)$, Inequality (1) can be derived as follows:

$$|S_i| > \frac{n}{\Delta+1} - \Delta = \frac{4n - 4\Delta(\Delta+1)}{4(\Delta+1)} \ge \frac{3}{4} \cdot \frac{n}{(\Delta+1)} \ge \frac{\Delta+1}{2\Delta} \cdot \frac{n}{(\Delta+1)} \ge s(\Delta+1)$$

We partition each set S_i into three subsets S_i^1, S_i^2, S_i^3 as follows: First, the partition of S_1 is chosen arbitrarily, ensuring only $|S_1^1| = |S_1^3| = s$. Then, for $i = 2, 3, \ldots, \Delta + 1$, define S_i^1 as a subset of s vertices of S_i having no neighbor in S_{i-1}^3 . Such a subset always exists because, by (1), there are most $|S_{i-1}^3| \cdot \Delta = s\Delta < |S_i| - s$ vertices in S_i with a neighbor in S_{i-1}^3 . Let then S_i^3 be any subset of $S_i - S_i^1$ with cardinality s, and set finally $S_i^2 := S_i - (S_i^1 \cup S_i^3)$.

Consider the partial order \prec on V where, for $u, v \in V$, we have $u \prec v$ if $u \in S_i^p$, $v \in S_j^q$ with i < j, or with i = j and p < q. Define then ℓ as any linear ordering of V compatible with \prec .

By construction, the set $S_{i-1}^3 \cup S_i^1$ is a stable set for every $i \in \{2, \ldots, \Delta + 1\}$. It follows $|\ell(u) - \ell(v)| \ge s$ for every edge $uv \in E$. Since the linear ordering ℓ can clearly be computed in polynomial time, this completes the proof of the lemma.

We turn now to the proof of Theorem 1. We note that the gadgets used in the reduction are essentially the same as the one used by Roch *et al.* [11] in their NP-hardness proof.

Proof of Theorem 1. A 3SAT-5 formula is a CNF formula in which every clause contains exactly three literals, every variable appears in exactly five clauses, and a variable does not appear in a clause more than once. Such a formula is said to be δ -satisfiable if at most a δ -fraction of its clauses are satisfiable simultaneously. Our reduction is from the problem of distinguishing between satisfiable 3SAT-5 formulae and those which are δ -satisfiable. It is known that this problem is NP-hard for some constant δ with $0 < \delta < 1$; see Feige [6].

Suppose thus that we are given a 3SAT-5 formula φ with *n* clauses which is either satisfiable or δ -satisfiable. Two literals of φ are opposite if one is positive, the other negative, and they both correspond to the same variable. Let G_{φ} be the (simple, undirected) graph having one vertex per clause of φ , and where two distinct vertices are adjacent if there exists two opposite literals in the union of the corresponding two clauses. Notice that G_{φ} has maximum degree $\Delta \leq 12$.



Figure 1: Gadget for clause C_i . Tariff arcs are represented with dashed lines and only non-zero fixed costs are indicated.

Using Lemma 1, we obtain in polynomial time a linear ordering ℓ of the vertices of G_{φ} such that

$$|\ell(u) - \ell(v)| \ge \frac{n}{4\Delta(\Delta+1)} \ge \frac{n}{624}$$

for every edge $uv \in E$. Denote by C_1, \ldots, C_n the clauses of φ , in the order given by ℓ . Denote also by C_i^1, C_i^2, C_i^3 the three literals of C_i , for every $i \in \{1, \ldots, n\}$.

We define an instance of the Stackelberg network pricing problem as follows. Each clause C_i has a corresponding *clause-gadget*, described in Figure 1. Take first the union of all these clause-gadgets (notice that node s_i appears in the gadgets of both C_i and C_{i+1} , for $i \in \{1, \ldots, n-1\}$). Then, for every $i, j \in \{2, \ldots, n\}$ with i < j and every $p, q \in \{1, 2, 3\}$, add the arc $(w_{i,p}, v_{j,q})$ with cost j - i - 1 if literals C_i^p and C_j^q are opposite. The latter arcs are said to be *jump arcs*. This defines the directed graph D = (V, A) and the cost function $c(\cdot)$. The set of tariff arcs is

$$T := \{ (v_{i,j}, w_{i,j}) : 1 \le i \le n, 1 \le j \le 3 \},\$$

and the origin-destination pair of the follower is (s, t), with $s := s_0$ and $t := s_n$.

If n < 1248, we simply use brute force to decide whether φ is satisfiable or only δ -satisfiable. Hence, we may assume $n \ge 1248$, and thus

$$j - i \ge \frac{n}{624} \ge 1 + \frac{n}{1248} \tag{2}$$

for every jump arc $(w_{i,p}, v_{j,q})$. Let

$$\lambda := \max\left\{\delta, 1 - \frac{1}{2496}\right\},\,$$

and denote by OPT the maximum revenue achievable on this instance of Stackelberg network pricing.

Claim 1. The following holds:

- (a) if φ is satisfiable then OPT = 2n;
- (b) if φ is δ -satisfiable then $OPT \leq \lambda \cdot 2n$.

Proof. Suppose first that φ is satisfiable and consider a truth assignment of the variables satisfying all clauses of φ . For every $i \in \{1, \ldots, n\}$ and $p \in \{1, 2, 3\}$, set the tariff of arc $(v_{i,p}, w_{i,p})$ to 2 if the literal C_i^p is true in the truth assignment, to 2n + 1 otherwise. Observe that, with these tariffs, any *st*-path that includes a jump arc has (total) cost at least 2n + 1. Also, the cost of every *st*-path is at least 2n, and there exists one such path with cost exactly 2n that uses one tariff arc per clause-gadget. Hence, $OPT \ge 2n$ in this case. On the other hand, $OPT \le 2n$ always holds since there exists an *st*-path with cost 2n in (V, A - T), that is, which avoids all tariff arcs. This proves part (a) of the claim.

Assume now that φ is δ -satisfiable and let $d(\cdot)$ be an optimal assignment of tariffs to arcs in T. Let P be any *st*-path of minimum (total) cost giving a revenue of OPT, and denote by z its cost. (Thus $z = \sum_{a \in P} c(a) + \sum_{a \in P \cap T} d(a) \leq 2n$.) If the path P includes a jump arc $(w_{i,p}, v_{j,q})$, then using (2) we obtain

OPT
$$\leq z - (j - i - 1) \leq 2n - (j - i - 1) \leq 2n - \frac{n}{1248} \leq \lambda \cdot 2n.$$

Hence, without loss of generality P includes no jump arc. It follows $d(a) \leq 2$ for every tariff arc $a \in P \cap T$, because of the arcs (s_{i-1}, s_i) with fixed cost 2.

Suppose P includes two tariff arcs corresponding to opposite literals, say arcs $(v_{i,p}, w_{i,p})$ and $(v_{j,q}, w_{j,q})$ with i < j. Then the revenue

$$\sum_{a \in P' \cap T} d(a)$$

given by the subpath P' of P going from $w_{i,p}$ to $v_{j,q}$ is at most j-i-1. This is because there exists a jump arc $(w_{i,p}, v_{j,q})$ in D, with fixed cost j-i-1. Hence, we deduce

$$OPT = \sum_{a \in (P - P') \cap T} d(a) + \sum_{a \in P' \cap T} d(a) \le 2(n - (j - i - 1)) + (j - i - 1) \le \lambda \cdot 2n,$$

using again (2). We may thus assume that P does not contain two tariff arcs corresponding to opposite literals.

Now, the set $P \cap T$ of tariff arcs included in P directly defines a truth assignment that satisfies at least $|P \cap T|$ clauses of φ (variables not appearing in $P \cap T$ are set arbitrarily). Since φ is only δ -satisfiable, we have $|P \cap T| \leq \delta$, and thus

$$OPT \le 2|P \cap T| \le \delta \cdot 2n \le \lambda \cdot 2n.$$

Part (b) of the claim follows.

By Claim 1, any polynomial-time algorithm approximating Stackelberg network pricing within a ratio strictly better than λ could be used to decide, in polynomial time, whether φ is satisfiable or δ -satisfiable. This completes the proof of Theorem 1.

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