# On approximate data reduction for the Rural Postman Problem: Theory and experiments* 

René van Bevern<br>Department of Mathematics and Mechanics, Novosibirsk State University, Novosibirsk, Russian Federation, rvb@nsu.ru

Till Fluschnik<br>Technische Universität Berlin, Faculty IV, Algorithmics and Computational Complexity, Berlin, Germany, till.fluschnik@tu-berlin.de

Oxana Yu. Tsidulko<br>Department of Mathematics and Mechanics, Novosibirsk State University, Novosibirsk, Russian Federation, Sobolev Institute of Mathematics of the Siberian Branch of the Russian Academy of Sciences, Novosibirsk, Russian Federation, tsidulko@math.nsc.ru


#### Abstract

Given an undirected graph with edge weights and a subset $R$ of its edges, the Rural Postman Problem (RPP) is to find a closed walk of minimum total weight containing all edges of $R$. We prove that RPP is WK[1]-complete parameterized by the number and weight $d$ of edges traversed additionally to the required ones. Thus RPP instances cannot be polynomial-time compressed to instances of size polynomial in $d$ unless the polynomial-time hierarchy collapses. In contrast, denoting by $b \leq 2 d$ the number of vertices incident to an odd number of edges of $R$ and by $c \leq d$ the number of connected components formed by the edges in $R$, we show how to reduce any RPP instance $I$ to an RPP instance $I^{\prime}$ with $2 b+O(c / \varepsilon)$ vertices in $O\left(n^{3}\right)$ time so that any $\alpha$-approximate solution for $I^{\prime}$ gives an $\alpha(1+\varepsilon)$-approximate solution for $I$, for any $\alpha \geq 1$ and $\varepsilon>0$. That is, we provide a polynomial-size approximate kernelization scheme (PSAKS). We experimentally evaluate it on wide-spread benchmark data sets as well as on two real snow plowing instances from Berlin. We also make first steps towards a PSAKS for the parameter $c$.


Keywords: Eulerian extension; capacitated arc routing; lossy kernelization; above-guarantee parameterization; NP-hard problem; parameterized complexity

## 1 Introduction

In the framework of lossy kernelization [24, 45], we study trade-offs between the provable effect of data reduction and the provably achievable solution quality for the following classical vehicle routing problem [47].
Problem 1.1 (Rural Postman Problem, RPP).
Instance: An undirected graph $G=(V, E)$ with $n$ vertices, edge weights $\omega: E \rightarrow \mathbb{N} \cup\{0\}$, and a multiset $R$ of required edges of $G$.
Task: Find a closed walk $W^{*}$ in $G$ containing each edge of $R$ and minimizing the total weight $\omega\left(W^{*}\right)$ of the edges on $W^{*}$.

[^0]We call any closed walk containing each edge of $R$ an $R P P$ tour. By RPP we will also refer to the decision problem where one additionally gets a non-negative integer $k \in \mathbb{N}$ in the input and the task is to decide whether there is an RPP tour $W$ of weight $\omega(W) \leq k$.

RPP has direct applications in snow plowing, street sweeping, meter reading [16, 22], vehicle depot location [33], drilling, and plotting [32,35]. The undirected version occurs especially in rural areas, where service vehicles can operate in both directions even on one-way roads [19]. Moreover, RPP is a special case of the Capacitated Arc Routing Problem (CARP) [34] and used in all "route first, cluster second" algorithms for CARP [1, 11, 54], which are notably the only ones with proven approximation guarantees [7, 40, 55]. Improved approximations for RPP automatically lead to better approximations for CARP.

There is a folklore polynomial-time 3/2-approximation for RPP based on the Christofides-Serdyukov algorithm for the metric Traveling Salesman Problem [9, 13, 51] (we refer to Eiselt et al. [22] or van Bevern et al. [8] for a detailed algorithm description). We aim for $(1+\varepsilon)$-approximations for all $\varepsilon>0$. Unfortunately, RPP contains the metric Traveling Salesman Problem as a special case, which cannot be polynomial-time approximated within any factor smaller than $123 / 122$ unless $\mathrm{P}=\mathrm{NP}$ [41]. Thus, finding $(1+\varepsilon)$-approximations even for constant small $\varepsilon>0$ typically requires exponential time. We present data reduction rules for this task. Their effectivity depends on the desired $\varepsilon$.

### 1.1 Our contributions and outline of this paper

In Section 2, we introduce basic notation of graph theory, approximation algorithms, parameterized complexity, problem kernelization, and WK[1]-completeness. In Section 3, we prove basic properties of optimal RPP tours.

Using the recently introduced concept of WK[1]-hardness [37], in Section 4, we prove that it is hard to reduce exactly solving RPP to solving instances of size polynomial in $\omega\left(W^{*}\right)-\omega(R)+\left|W^{*}\right|-|R|$, which is the weight and number of the deadheading edges traversed additionally to the required ones:

Theorem 1.2. RPP is WK[1]-complete parameterized by $\Theta\left(\omega\left(W^{*}\right)-\omega(R)+\left|W^{*}\right|-|R|\right)$, where $W^{*}$ is an optimal RPP tour with a minimum amount of edges and WK[1]-hardness holds even in complete graphs with metric edge weights 1 and 2.

In contrast to Theorem 1.2, in Section 5, we show that effective data reduction for RPP is possible if one is interested in $(1+\varepsilon)$-approximations.

Theorem 1.3. There is an algorithm that, given $\varepsilon>0$ and an RPP instance $(G, R, \omega)$ with $b$ vertices incident to an odd number of edges in $R$ and whose edges in $R$ form $c$ connected components, reduces $(G, R, \omega)$ to an RPP instance $\left(G^{\prime}, R^{\prime}, \omega^{\prime}\right)$ in $O\left(n^{3}+|R|\right)$ time such that
(i) the number of vertices in $G^{\prime}$ is $2 b+O(c / \varepsilon)$,
(ii) the number of required edges is $\left|R^{\prime}\right| \leq 4 b+O(c / \varepsilon)$,
(iii) the maximum edge weight with respect to $\omega^{\prime}$ is $O\left((b+c) / \varepsilon^{2}\right)$, and
(iv) any $\alpha$-approximate solution for $I^{\prime}$ for some $\alpha \geq 1$ can be transformed into an $\alpha(1+\varepsilon)$-approximate solution for $I$ in polynomial time.

Finally, in Section 6, we experimentally evaluate the data reduction algorithm from Theorem 1.3.

Discussion of our results. Theorems 1.2 and 1.3 complement each other since the number $\left|W^{*}\right|-|R|$ of deadheading arcs is at least $\max \{b / 2, c\}$ (see Section 3.3). Thus, Theorem 1.2 shows that it is hard to polynomialtime reduce RPP to instances of size poly $(b+c)$ without loss in the solution quality, whereas Theorem 1.3 allows to do so with arbitrarily small loss. In Section 5.5, we will also discuss difficulties of getting rid of $b$ or $c$ in the size of the reduced instance $I^{\prime}$.

Notably, the $\alpha$-approximate solution for $I^{\prime}$ in Theorem 1.3 may be obtained by any means, for example exact algorithms or heuristics. Thus, Theorem 1.3 can be used to speed up expensive heuristics without much loss in the solution quality. In terms of the recently introduced concept of lossy kernelization [45], Theorem 1.3 yields a polynomial-size approximate kernelization scheme (PSAKS).

In experiments, on instances with few connected components, the number of vertices and required edges is reduced to about $50 \%$ at an $1 \%$ loss in the solution quality. On real-world snow plowing data from Berlin, the number of vertices is reduced to about $20 \%$ without loss in the solution quality.

### 1.2 Related work

Classical complexity. RPP is strongly NP-hard [30, 44], its special case with $R=E$ is the polynomial-time solvable Chinese Postman problem [12, 18]. Containing the metric Traveling Salesman Problem as a special case, RPP is APX-hard [41]. There is a folklore polynomial-time 3/2-approximation based on the Christofides-Serdyukov algorithm $[9,13,51]$ for the metric Traveling Salesman Problem (we refer to arc routing surveys [8, 22] for a detailed algorithmic description). The Chinese Postman Problem is equivalent to finding a minimum-weight set of edges whose addition makes a connected graph Eulerian [12, 18, 50]. For a disconnected graph, this is exactly RPP [8, 17, 53].

Parameterized complexity. Dorn et al. [17] showed an $O\left(4^{d} \cdot n^{3}\right)$-time algorithm for the directed RPP, where $d=\left|W^{*}\right|-|R|$ is the minimum number of deadheading arcs in an optimal solution $W^{*}$. It can be easily adapted to the undirected RPP. Sorge et al. [52] showed an $O\left(4^{c \log b^{2}} \operatorname{poly}(n)\right)$-time algorithm for the directed RPP, where $c$ is the number of (weakly) connected components induced by the required arcs in $R$ and $b=\sum_{v \in V}|\operatorname{indeg}(v)-\operatorname{outdeg}(v)|$. It is not obvious whether this algorithm can be adapted to the undirected RPP maintaining its running time. Gutin et al. [36] showed a randomized algorithm that solves the directed and undirected RPP in $f(c)$ poly $(n)$ time if edge weights are bounded polynomially in $n$. It is based on the Schwartz-Zippel Lemma $[49,56]$ for randomized polynomial identity testing. The existence of a deterministic algorithm with this running time is open [8, 36, 53].

Exact kernelization. RPP can easily be reduced to an equivalent instance with $2|R|$ vertices [8]. By shrinking the weights using a theorem of Frank and Tardos [29] one gets a so-called problem kernel of size polynomial in the number of required edges (we refer to Bentert et al. [2, Section 5.3] for details). In contrast, Sorge et al. [52] showed that, unless the polynomial-time hierarchy collapses, the directed RPP has no problem kernel of size polynomial in the number of deadheading arcs. This result is strengthened by our Theorem 1.2, which shows even WK[1]-hardness, also of the directed RPP.

Lossy kernelization. Due to the kernelization hardness of many problems, recently the concept of approximate kernelization has gained increased interest [24, 45]. In this context, Eiben et al. [20] called for finding connectivity-constrained problems that do not have polynomial-size kernels but $\alpha$-approximate polynomial-size kernels. Our Theorems 1.2 and 1.3 exhibit that RPP is such a problem. Among the so far few known lossy kernels [20, 21, 42, 43, 45], our Theorem 1.3 stands out since it shows a time and size efficient PSAKS, which is a property previously observed only in results of Krithika et al. [42]. Moreover, Theorem 1.3 is apparently the first lossy kernelization result for parameters above lower bounds, which previously got attention in exact kernelization.

## 2 Preliminaries

### 2.1 Set and graph theory

Sets and multisets. By $\mathbb{N}$ we denote the set of natural numbers including zero. For two multisets $A$ and $B$, $A \uplus B$ is the multiset obtained by adding the multiplicities of elements in $A$ and $B$. By $A \backslash B$ we denote the multiset obtained by subtracting the multiplicities of elements in $B$ from the multiplicities of elements in $A$. Finally, given some weight function $\omega: A \rightarrow \mathbb{N}$, the weight of a multiset $A$ is $\omega(A):=\sum_{e \in A} v(e) \omega(e)$, where $v(e)$ is the multiplicity of $e$ in $A$.

Graph theory. Graphs in our work are allowed to have loops and parallel edges, so that they are actually multigraphs $G=(V, E)$ with a set $V(G):=V$ of vertices, a multiset $E(G):=E$ over $\{\{u, v\} \mid u, v \in V\}$ of (undirected) edges, and edge weights $\omega: E \rightarrow \mathbb{N}$. Parallel edges in our graphs are indistinguishable from each other and all have the same weight. For a multiset $R$ of edges, we denote by $V(R)$ the set of their incident vertices.

Paths and cycles. A walk from $v_{0}$ to $v_{\ell}$ in $G$ is a sequence $w=\left(v_{0}, e_{1}, v_{1}, e_{2}, v_{2}, \ldots, e_{\ell}, v_{\ell}\right)$ such that $e_{i}$ is an edge with end points $v_{i-1}$ and $v_{i}$ for each $i \in\{1, \ldots, \ell\}$. If $v_{0}=v_{\ell}$, then we call $w$ a closed walk. If all vertices on $w$ are pairwise distinct, then $w$ is a path. If only its first and last vertex coincide, then $w$ is a cycle. By $E(w)$ we denote the multiset of edges on $w$, that is, each edge appears on $w$ and in $E(w)$ equally often. The length of walk $w$ is its number
$|w|:=\ell=|E(w)|$ of edges. The weight of walk $w$ is $\omega(w):=\sum_{i=1}^{\ell} \omega\left(e_{\ell}\right)$. An Euler tour for $G$ is a closed walk that traverses each edge of $G$ exactly as often as it is present in $G$. A graph is Eulerian if it allows for an Euler tour.

Connectivity and blocks. Two vertices $u, v$ of $G$ are connected if there is a path from $u$ to $v$ in $G$. A connected component of $G$ is a maximal subgraph of $G$ in which the vertices are mutually connected. A vertex $v$ of $G$ is a cut vertex if removing $v$ and its incident edges increases the number of connected components of $G$. A biconnected component or block of $G$ is a maximal subgraph without cut vertices.

Edge- and vertex-induced subgraphs. For a subset $U \subseteq V$ of vertices, the subgraph $G[U]$ of $G=(V, E)$ induced by $U$ consists of the vertices of $U$ and all edges of $G$ between them (respecting multiplicities). For a multiset $R$ of edges of $G, G\langle R\rangle:=(V(R), R)$ is the graph induced by the edges in $R$. For a walk $w$, we also denote $G\langle w\rangle:=G\langle E(w)\rangle$. Note that $G\langle R\rangle$ and $G\langle w\rangle$ do not contain isolated vertices yet might contain edges with a higher multiplicity than $G$ and, therefore, are not necessarily sub(multi)graphs of $G$.

### 2.2 Decision problems, optimization problems, approximation

Definition 2.1. A decision problem is a subset $\Pi \subseteq \Sigma^{*}$ for some finite alphabet $\Sigma$. The task is, given an instance $x \in \Sigma^{*}$, determining whether $x \in \Pi$. If $x \in \Pi$, then $x$ is a yes-instance. Otherwise, it is a no-instance.

For optimization problems, we use the terminology of Garey and Johnson [31]. We will only consider minimization problems in our work.

Definition 2.2. An combinatorial optimization problem $\Pi$ is a triple $\Pi=\left(D_{\Pi}, S_{\Pi}, m_{\Pi}\right)$, where

1. $D_{\Pi}$ is a set of instances,
2. $S_{\Pi}$ is a function assigning to each instance $I \in D_{\Pi}$ a finite set $S_{\Pi}(I)$ of (feasible) solutions, and
3. $m_{\Pi}$ is a function that assigns a solution cost $m_{\Pi}(I, \sigma)$ to each feasible solution $\sigma \in S_{\Pi}(I)$ of an instance $I \in D_{\Pi}$. An optimal solution for an instance $I \in D_{\Pi}$ is a feasible solution $\sigma \in S_{\Pi}(I)$ minimizing $m_{\Pi}(I, \sigma)$. Its cost is denoted as $\operatorname{OPT}_{\Pi}(I)$, where we drop the subscript $\Pi$ when the optimization problem is clear from context.

If the set $S_{\Pi}(I)$ for any instance $I \in D_{\Pi}$ is polynomial-time recognizable, any solution $\sigma \in S_{\Pi}(I)$ has size $|\sigma| \leq$ poly $(|I|)$, and the function $m_{\Pi}$ is polynomial-time computable, then one also calls $\Pi$ an NP-optimization problem.

Definition 2.3. An $\alpha$-approximate solution for an instance $I$ of a combinatorial optimization problem $\Pi$ is a feasible solution of cost at most $\alpha \cdot \operatorname{OPT}_{\Pi}(I)$.

### 2.3 Kernelization

Kernelization is the main notion of data reduction with provable performance guarantees [28]. Since proving that a polynomial-time algorithm always shrinks the input instance of an NP-hard problem, say from size $n$ to $n-1$, would imply $\mathrm{P}=\mathrm{NP}$, the size of the reduced instance is measured in dependence of a parameter of the input instance.

We formalize parameters using the terminology of Flum and Grohe [27], since it allows to parameterize decision and optimization problems in a uniform way:

Definition 2.4. A parameterization is a polynomial-time computable mapping $\kappa: \Sigma^{*} \rightarrow \mathbb{N}$ of instances (of decision or optimization problems) to a parameter. For a (decision or optimization) problem $\Pi$ and parameterization $\kappa$, $(\Pi, \kappa)$ is called a parameterized (decision or optimization) problem.

Definition 2.5. A kernelization for a parameterized decision problem $(\Pi, \kappa)$ is a polynomial-time algorithm that maps any instance $x \in \Sigma^{*}$ to an instance $x^{\prime} \in \Sigma^{*}$ such that
(i) $x \in \Pi \Longleftrightarrow x^{\prime} \in \Pi$, and
(ii) $\left|x^{\prime}\right| \leq g(\kappa(x))$ for some computable function $g$.

We call $x^{\prime}$ the problem kernel and $g$ its size.
A generalization of problem kernels are Turing kernels, where one is allowed to generate multiple reduced instances instead of a single one.

Definition 2.6. A Turing kernelization for a parameterized decision problem ( $\Pi, \kappa$ ) is an algorithm $A$ that decides $x \in \Pi$ in polynomial time given access to an oracle that answers $x^{\prime} \in \Pi$ in constant time for any $x^{\prime} \in \Sigma^{*}$ with $\left|x^{\prime}\right| \leq g(\kappa(x))$, where $g$ is an arbitrary function called the size of the Turing kernel.
Since Theorem 1.2 means that it is hard to obtain problem kernels for RPP even with size polynomial in a relatively large parameter, we will consider approximate problem kernels [45]:
Definition 2.7. A $\beta$-approximate kernelization for a parameterized optimization problem $(\Pi, \kappa)$ consists of two polynomial-time algorithms:
(i) The first algorithm reduces an instance $I$ of $\Pi$ to an instance $I^{\prime}$ of $\Pi$ such that $\left|I^{\prime}\right| \leq g(\kappa(I))$ for some computable function $g: \mathbb{N} \rightarrow \mathbb{N}$.
(ii) The second algorithm turns any $\alpha$-approximate solution for $I^{\prime}$ into an $\alpha \beta$-approximate solution for $I$.

We call $g$ the size of the approximate kernel $I^{\prime}$.
In fact, we show polynomial-size approximate kernelization schemes [45]:
Definition 2.8. A polynomial-size approximate kernelization scheme (PSAKS) is a family of $(1+\varepsilon)$-approximate kernelizations yielding approximate kernels of polynomial size for every fixed $\varepsilon>0$.

### 2.3.1 Kernelization hardness

WK[1]-complete parameterized decision problems do not have problem kernels of polynomial size unless the polynomial-time hierarchy collapses and are conjectured not to have Turing kernels of polynomial size either [37]. An archetypal WK[1]-complete problem is the following [37]:

Problem 2.9 (NDTM Halting).
Instance: A nondeterministic Turing machine $\mathcal{M}$ and an integer $t$.
Parameter: $t \log |\mathcal{M}|$.
Question: Does $\mathcal{M}$ halt in $t$ steps on the empty input string?
The class WK[1] can now be defined as the class of all parameterized problems reducible to NDTM Halting using the following type of reduction.

Definition 2.10. A polynomial parameter transformation (PPT) of a parameterized decision problem ( $\Pi, \kappa)$ into a parameterized decision problem $\left(\Pi^{\prime}, \kappa^{\prime}\right)$ is an algorithm that maps any instance $x \in \Sigma^{*}$ to an instance $x^{\prime} \in \Sigma^{*}$ in polynomial time so that
(i) $x \in \Pi \Longleftrightarrow x^{\prime} \in \Pi^{\prime}$ and
(ii) $\kappa^{\prime}\left(x^{\prime}\right) \in \operatorname{poly}(\kappa(x))$.

Definition 2.11. WK[1] is the class of parameterized decision problems PPT-reducible to NDTM Halting. A parameterized decision problem $(\Pi, \kappa)$ is $W K[1]$-hard if every parameterized decision problem in WK[1] is PPT-reducible to $(\Pi, \kappa)$. It is WK[1]-complete if it is WK[1]-hard and contained in WK[1].

Notably, since PPT-reducibility is a transitive relation, to prove WK[1]-hardness of a parameterized decision problem, it is enough to PPT-reduce any other WK[1]-hard parameterized decision problem to it.

### 2.4 Approximate weight reduction

We will use the following lemma to shrink edge weights so that their encoding length will be polynomial in the number of vertices and edges of the graph. It is a generalization of an idea implicitly used for weight reduction in a proof of Lokshtanov et al. [45, Theorem 4.2] and shrinks weights faster and more significantly than a theorem of Frank and Tardos [29] that is frequently used in the exact kernelization of weighted problems [2, 5, 23, 46]. We first state the lemma, and thereafter intuitively describe its application to RPP.

Lemma 2.12 (lossy weight reduction). Let $\mathcal{F} \subseteq \mathbb{Q}_{\geq 0}^{n}$ and $\omega \in \mathbb{Q}_{\geq 0}^{n}$ be such that

- $\|\omega\|_{\infty} \leq \beta$ for some $\beta \in \mathbb{Q}$ and
- $\|x\|_{1} \leq N$ for some $N \in \mathbb{N}$ and all $x \in \mathcal{F}$.

Let $x^{*} \in \arg \min \left\{\omega^{\top} x \mid x \in \mathcal{F}\right\}$ and $\bar{x}^{*} \in \arg \min \left\{\bar{\omega}^{\top} x \mid x \in \mathcal{F}\right\}$. Then, for any $\varepsilon>0$, using $O(n)$ arithmetic operations involving only the numbers $\varepsilon, \beta$, and the components of $\omega$, one can compute $\bar{\omega} \in \mathbb{N}^{n}$ such that
(i) $\|\bar{\omega}\|_{\infty} \leq N / \varepsilon$ and
(ii) for any $\alpha \in \mathbb{Q}$ and $x \in \mathcal{F}$ with $\bar{\omega}^{\top} x \leq \alpha \cdot \bar{\omega}^{\top} \bar{x}^{*}$, one has $\omega^{\top} x \leq \alpha \cdot \omega^{\top} x^{*}+\varepsilon \beta$.

Note that one can easily prove a version of Lemma 2.12 for maximization problems. To apply Lemma 2.12 to RPP, we will take $\mathcal{F}$ to be the set of inclusion-minimal RPP tours (encoded as vectors $x \in \mathcal{F}$ having an entry for each edge that specifies how often it is included in the RPP tour), and $\omega$ to be a vector having an entry for each edge specifying its weight. Then the linear forms $\omega^{\top} x$ and $\bar{\omega}^{\top} x$ give the weight of the RPP tour encoded by $x$ with respect to the initial weights $\omega$ and the reduced weights $\bar{\omega}$, respectively. The linear forms occurring in the lemma seem to limit it to problems with linear or additive goal functions, yet in fact are powerful enough to model many non-additive goal functions as well [2].

Proof of Lemma 2.12. Choose $M=(\varepsilon \beta) / N$ and $\bar{\omega}_{i}=\left\lfloor\omega_{i} / M\right\rfloor$ for each $i \in\{1, \ldots, n\}$. Since $\omega_{i} \geq 0$ for each $i \in\{1, \ldots, n\}$, we have $\bar{\omega} \in \mathbb{N}^{n}$. Moreover, due to $\|\omega\|_{\infty} \leq \beta$, for each $i \in\{1, \ldots, n\}$, we have $\bar{\omega}_{i} \leq \beta / M=N / \varepsilon$, proving (i).

To prove (ii), let $x \in \mathcal{F}$ be such that $\bar{\omega}^{\top} x \leq \alpha \cdot \bar{\omega}^{\top} \bar{x}^{*}$. By the choice of $\bar{\omega}$ for each $i \in\{1, \ldots, n\}$, we have $\omega_{i} \leq M \cdot\left(\bar{\omega}_{i}+1\right)$. Moreover, we have

$$
\begin{equation*}
M \cdot \bar{\omega}^{\top} x \leq \omega^{\top} x \quad \text { for all } x \in \mathcal{F} . \tag{2.1}
\end{equation*}
$$

It follows that $\omega^{\top} x \leq M \cdot\left(\bar{\omega}^{\top} x+\|x\|_{1}\right) \leq M \cdot \bar{\omega}^{\top} x+\varepsilon \beta \leq \alpha \cdot M \cdot \bar{\omega}^{\top} \bar{x}^{*}+\varepsilon \beta$. By (2.1) and the choice of $\bar{x}^{*}$, we have $\bar{\omega}^{\top} \bar{x}^{*} \leq \bar{\omega}^{\top} x^{*} \leq \omega^{\top} x^{*} / M$. Finally, $\omega^{\top} x \leq \alpha \cdot M \cdot \bar{\omega}^{\top} \bar{x}^{*}+\varepsilon \beta \leq \alpha \cdot \omega^{\top} x^{*}+\varepsilon \beta$.

## 3 Solution structure

In this section, we prove fundamental properties of optimal RPP tours. To make these hold, we first establish the triangle inequality in Section 3.1. In Section 3.2, we translate RPP to the problem of finding Eulerian extensions. In Section 3.3, we derive inequalities to bound parts of optimal RPP tours.

### 3.1 Triangle inequality

Without loss of generality, we will assume that the weight function satisfies the triangle inequality:
Proposition 3.1 ([6]). In $O\left(n^{3}\right)$ time, an RPP instance ( $G, R, \omega$ ) can be turned into an RPP instance ( $G^{\prime}, R, \omega^{\prime}$ ) such that

1. $G^{\prime}$ is a complete graph,
2. $\omega^{\prime}$ satisfies the triangle inequality, and
3. any $\alpha$-approximate solution for $\left(G, R, \omega^{\prime}\right)$ can be turned into an $\alpha$-approximate solution for $(G, R, \omega)$ in polynomial time.

Remark 3.2. Proposition 3.1 holds in particular for $\alpha=1$ and does not increase the number of connected components of $G\langle R\rangle$, the number of odd-degree vertices of $G\langle R\rangle$, the number and weight of deadheading edges of an optimal RPP tour. Thus, it is sufficient to prove Theorems 1.2 and 1.3 for RPP with triangle inequality. We will henceforth assume that the input graph is complete and satisfies the triangle inequality.

### 3.2 Edge-minimizing Eulerian extensions

Consider any RPP tour $W$ for an RPP instance $(G, R, \omega)$. Then $G\langle W\rangle$ is an Eulerian supergraph of $G\langle R\rangle$ whose total edge weight is $\omega(W)$. Moreover, any Eulerian supergraph $G\left\langle W^{\prime}\right\rangle$ of $G\langle R\rangle$ yields an RPP tour for ( $G, R, \omega$ ) of total weight $\omega\left(W^{\prime}\right)$. Thus, RPP tours one-to-one correspond to Eulerian extensions [17]: ${ }^{1}$

Definition 3.3. An Eulerian extension (EE) for an RPP instance ( $G, R, \omega$ ) is a multiset $S$ of edges such that $G\langle R \uplus S\rangle$ is Eulerian. We say that an Eulerian extension $S$ is edge-minimizing if there is no Eulerian extension $S^{\prime}$ with $\left|S^{\prime}\right|<|S|$ and $\omega\left(S^{\prime}\right) \leq \omega(S)$.

[^1]

Figure 3.1: Proof that the bound given in Lemma 3.8 is tight: $G\langle R\rangle$ has $c=4$ connected components and $2 c-2=6$ vertices are incident to the Eulerian extension.

In the following, we will concentrate on finding minimum-weight Eulerian extensions rather than RPP tours and exploit that a graph without isolated vertices is Eulerian if and only if it is connected and balanced:

Definition 3.4. A vertex is balanced if it has even degree. A graph is balanced if each of its vertices is balanced.
Thus, solving RPP reduces to finding a minimum-weight set $S$ of edges such that $G\langle R \uplus S\rangle$ is connected and balanced. Since an Euler tour in the Eulerian graph $G\langle R \uplus S\rangle$ is computable in linear time using Hierholzer’s algorithm [26, 38], we can easily recover an RPP tour from an Eulerian extension.
Proposition 3.5. Let $(G, R, \omega)$ be an RPP instance.
(i) From any RPP tour $W$ for $(G, R, \omega)$, one can compute an Eulerian extension $S$ of weight $\omega(S)=\omega(W)-\omega(R)$ in time linear in $|W|$.
(ii) From any Eulerian extension $S$ for $(G, R, \omega)$, one can compute an RPP tour $W$ of weight $\omega(W)=\omega(R)+\omega(S)$ in time linear in $|R|+|S|$.
Assuming the triangle inequality, any RPP tour can be shortcut to contain only vertices incident to required edges.
Observation 3.6. Any edge-minimizing Eulerian extension $S$ for an RPP instance $(G, R, \omega)$ satisfies $V(S) \subseteq V(R)$.
The following lemma, in particular, shows that no edge-minimizing Eulerian extension contains required edges between balanced vertices.

Lemma 3.7. An edge-minimizing Eulerian extension $S$ for an RPP instance $(G, R, \omega)$ does not contain any edge $\{u, v\}$ such that $u$ and $v$ belong to the same connected component of $G\langle R\rangle$ and such that $u$ is balanced in $G\langle R\rangle$.

Proof. Towards a contradiction, assume that $\{u, v\} \in S$. Since $u$ is balanced in $G\langle R\rangle$ and $G\langle R \uplus S\rangle, S$ additionally contains an edge $\{u, w\}$ (possibly, $v=w$ ). Then $\left(S^{\prime} \backslash\{\{u, v\},\{u, w\}\}\right) \uplus\{\{v, w\}\}$ satisfies $\left|S^{\prime}\right|<|S|$ and also is an Eulerian extension: the balance of $u, v$ and $w$ is the same in $G\langle R \uplus S\rangle$ and $G\left\langle R \uplus S^{\prime}\right\rangle$, and $u$ still is connected to $v$ in $G\left\langle R \uplus S^{\prime}\right\rangle$ since $u$ and $v$ belong to the same connected component of $G\langle R\rangle$. Finally, using the triangle inequality, $\omega\left(S^{\prime}\right) \leq \omega(S)$, contradicting the fact that $S$ is edge-minimizing.

Lemma 3.8. Let $(G, R, \omega)$ be an RPP instance and $c$ be the number of connected components of $G\langle R\rangle$. At most $2 c-2$ balanced vertices in $G\langle R\rangle$ are incident to edges of an edge-minimizing Eulerian extension and this bound is tight.
Proof. Let $S$ be an edge-minimizing Eulerian extension for $(G, R, \omega)$ and $T \subseteq S$ be an inclusion-minimal subset such that $G\langle R \uplus T\rangle$ is connected. Then $|T|=c-1$ and $S \backslash T$ is an edge-minimizing Eulerian extension for $(G, R \uplus T, \omega)$. Thus, by Observation 3.6, $V(S \backslash T) \subseteq V(R \uplus T)$. Combining this with Lemma 3.7, $S \backslash T$ does not contain any edges incident to balanced vertices of $G\langle R \uplus T\rangle$. The only vertices that might be balanced in $G\langle R\rangle$ but not in $G\langle R \uplus T\rangle$ are the at most $2 c-2$ end points of edges in $T$. In the worst case, all of them are incident to edges in $S$. Figure 3.1 shows that the bound is tight.

Remark 3.9. The following lemma shows that an edge-minimizing Eulerian extension contains exactly one edge incident to each unbalanced vertex of $G\langle R\rangle$ and either no or two edges incident to each balanced vertex of $G\langle R\rangle$.

Lemma 3.10. Each vertex $v \in V$ is incident to at most two edges of an edge-minimizing Eulerian extension $S$ for an RPP instance ( $G, R, \omega$ ).
Proof. Towards a contradiction, assume that $S$ contains $e_{i}=\left\{u_{i}, v\right\}$ for $i \in\{1,2,3\}$. Obviously, $S^{\prime}=\left(S \backslash\left\{e_{1}, e_{2}\right\}\right) \uplus$ $\left\{\left\{u_{1}, u_{2}\right\}\right\}$ satisfies $\left|S^{\prime}\right|<|S|$. Moreover, $\omega\left(S^{\prime}\right) \leq \omega(S)$ follows from the triangle inequality. We argue that $S^{\prime}$ is an Eulerian extension, contradicting the choice of $S$. The proof is illustrated in Figure 3.2.
The balance of $v, u_{1}, u_{2}$, and $u_{3}$ is the same in $G\langle R \uplus S\rangle$ and $G\left\langle R \uplus S^{\prime}\right\rangle$. It remains to show that the vertices $v, u_{1}, u_{2}, u_{3}$ are connected in $G\left\langle R \uplus S^{\prime}\right\rangle$. To this end, observe that $G\langle R \uplus S\rangle$ is Eulerian and thus contains two edge-disjoint paths between $u_{2}$ and $u_{3}$. At most one of these paths contains the edge $e_{2}$ and is lost in $G\left\langle R \uplus S^{\prime}\right\rangle$. Thus, $G\left\langle R \uplus S^{\prime}\right\rangle$ contains the edges $\left\{v, u_{3}\right\},\left\{u_{1}, u_{2}\right\}$, and a path between $u_{2}$ and $u_{3}$.


Figure 3.2: Illustration of the proof of Lemma 3.10. The wavy edge is a $u_{2}-u_{3}$ path.

### 3.3 Inequalities

Definition 3.11. In the context of an RPP instance $(G, R, \omega)$, we denote by
$R$ - the set of required edges,
$c$ - the number of connected components in $G\langle R\rangle$,
$b$ - the number of imbalanced vertices in $G\langle R\rangle$,
$W^{*}$ - a minimum-weight RPP tour with a minimum number of edges,
$D$ - a minimum-weight edge-minimizing Eulerian extension for $(G, R, \omega)$,
$T$ - a minimum-weight set of edges such that $G\langle R \uplus T\rangle$ is connected, of minimum cardinality, and $M$ - a minimum-weight set of edges such that $G\langle R \uplus M\rangle$ is balanced, of minimum cardinality.

Lemma 3.12. The following relations hold:

$$
\begin{align*}
\omega\left(W^{*}\right) & =\omega(R)+\omega(D)  \tag{3.1}\\
\omega(M) & \leq \omega(D)  \tag{3.2}\\
\omega(T) & \leq \omega(D)  \tag{3.3}\\
\omega(D) & \leq \omega(M)+2 \omega(T) \tag{3.4}
\end{align*}
$$

$$
\begin{align*}
\left|W^{*}\right| & =|R|+|D|,  \tag{3.5}\\
2 b=|M| & \leq|D|,  \tag{3.6}\\
c-1=|T| & \leq|D|,  \tag{3.7}\\
|D| & \leq|M|+2|T|, \tag{3.8}
\end{align*}
$$

where $|S| \leq|M|+2|T|$ holds for any edge-minimizing Eulerian extension $S$.
Proof. Equations (3.1) and (3.5) follow from Proposition 3.5. Inequalities (3.2) and (3.6) follow by choice of $M$ and the fact that, since we assume the triangle inequality, $M$ is simply a minimum-weight perfect matching on the $b$ imbalanced vertices in $G\langle R\rangle[12,18,50]$. Inequalities (3.3) and (3.7) follow by choice of $T$. Inequality (3.4) follows from the fact that $G\langle R \uplus M\rangle$ is balanced and adding each edge of $T$ twice to it does not change the balance of vertices, yet connects the graph. We now derive inequality (3.8). Consider any edge-minimizing Eulerian extension $S$. By Lemma 3.8, a set $X$ of at most $2 c-2$ balanced vertices in $G\langle R\rangle$ are incident to edges of $S$. By Remark 3.9, $S$ contains exactly one edge incident to each imbalanced vertex in $G\langle R\rangle$ and exactly two edges incident to each vertex in $X$. Thus, by the handshaking lemma, we get $2|X|+b=2|S|$. Therefore, $|S|=|X|+b / 2 \leq 2 c-2+|M|=2|T|+|M|$.

## 4 Hardness of kernelization

In this section, we prove Theorem 1.2. We first show WK[1]-hardness in Lemma 4.1, then we show containment in WK[1] in Lemma 4.3. Theorem 1.2 immediately follows from Lemmas 4.1 and 4.3 using (3.1) and (3.5).

Lemma 4.1 means that, according to the conjecture of Hermelin et al. [37], RPP has no Turing kernels with size polynomial in the number and weight of deadheading edges in an optimal RPP tour.

Lemma 4.1. RPP is WK[1]-hard parameterized by $|T|+|M|+\omega(T)+\omega(M) \in \Theta(|D|+\omega(D))$ even in complete graphs with metric edge weights one and two.

To prove Lemma 4.1, we provide a polynomial parameter transformation from the following known WK[1]-complete parameterized problem [37].

Problem 4.2 (Multicolored Cycle).
Instance: An undirected graph $G=(V, E)$ with a vertex coloring $c: V \rightarrow\{1, \ldots, k\}$.
Parameter: $k$.


Figure 4.1: Illustration for the proof of Lemma 4.1. Thick solid edges are the required edges $R$. Thin dashed edges are a colorful cycle and, at the same time, an Eulerian extension.

Question: Is there a cycle in $G$ containing exactly one vertex of each color?
Proof of Lemma 4.1. Let $I:=(G, c)$ with a graph $G=(V, E)$ and a vertex $k$-coloring $c: V \rightarrow\{1, \ldots, k\}$ be an instance of Multicolored Cycle. For $i \in\{1, \ldots, k\}$, we denote by $V_{i}:=\{v \in V \mid c(v)=i\}$ the vertices of color $i$. Now, consider the RPP instance $I^{\prime}=\left(G^{\prime}, R, \omega\right)$, illustrated in Figure 4.1: $G^{\prime}=\left(V, E^{\prime}\right)$ is a complete graph, the set $R$ contains a cycle on the vertices in $V_{i}$ for each $i \in\{1, \ldots, k\}$, and

$$
\omega: E^{\prime} \rightarrow \mathbb{N}, e \mapsto \begin{cases}1 & \text { if } e \in E \cup R \\ 2 & \text { otherwise }\end{cases}
$$

Note that, since all edge weights $\omega$ are one and two, $\omega$ is metric. Thus, by Lemma 3.12, $|T|+|M|+\omega(T)+\omega(M) \in$ $\Theta(|D|+\omega(D))$. Moreover, since $G^{\prime}\langle R\rangle$ is balanced, $|T|+|M|+\omega(T)+\omega(M)=|T|+\omega(T) \in O(k)$. We show that $I$ is a yes-instance if and only if $I^{\prime}$ has an RPP tour of weight $\omega(R)+k=|R|+k$, which, by Proposition 3.5 , is equivalent to having an Eulerian extension $S$ of weight $\omega(S) \leq k$.
$(\Rightarrow)$ Let $S$ be a multicolored cycle in $G$. Since $G^{\prime}\langle R\rangle$ is a disjoint union of cycles, $G^{\prime}\langle R\rangle$ is balanced. Since $S$ is a cycle, $G^{\prime}\langle R \uplus S\rangle$ is also balanced. Since $S$ contains one vertex of each color, $G^{\prime}\langle R \uplus S\rangle$ is additionally connected. Thus, $S$ is an Eulerian extension for $\left(G^{\prime}, R, \omega\right)$. Since $S$ consists of edges of $G$, we conclude $\omega(S)=|S|=k$.
$(\Leftarrow)$ Let $S$ be an edge-minimizing Eulerian extension with $\omega(S) \leq k$ for $\left(G^{\prime}, R, \omega\right)$. Since $G^{\prime}\langle R\rangle$ and $G^{\prime}\langle R \uplus S\rangle$ are balanced, so is $G^{\prime}\langle S\rangle$. Since $G^{\prime}\langle R \uplus S\rangle$ is connected and $G^{\prime}\langle S\rangle$ is balanced, $S$ contains at least two edges incident to a vertex in $V_{i}$ for each $i \in\{1, \ldots, k\}$. Thus, since $\omega(S) \leq k, G^{\prime}\langle S\rangle$ has to contain exactly $k$ edges, all of weight one, and exactly one vertex of $V_{i}$ for each $i \in\{1, \ldots, k\}$, that is, $k$ vertices. Since $G^{\prime}\langle S\rangle$ is balanced, it follows that $G^{\prime}\langle S\rangle$ is a collection of cycles whose color sets do not intersect. Thus, if $G^{\prime}\langle S\rangle$ was not connected, then $G^{\prime}\langle R \uplus S\rangle$ would not be either. We conclude that $G^{\prime}\langle S\rangle$ is connected, that is, a single cycle containing exactly one vertex of each color. By Lemma 3.7, none of the edges in $S$ are in $R$. Since all of them have weight one, they are in $G$. It follows that $S$ forms a multicolored cycle in $G$.

Having shown WK[1]-hardness in Lemma 4.1, we now show containment in WK[1], concluding the proof of Theorem 1.2. Note that we showed hardness for a parameter in $\Theta(|D|+\omega(D))$, whereas containment we show for an even smaller parameter in $O(|D|+\log (1+\omega(D))) \subseteq O(|D|+\omega(D))$. This means that, if any problem in WK[1] turns out to have a polynomial-size Turing kernel, then there will be a Turing kernel for RPP with size polynomial even in $|D|+\log (1+\omega(D))$.

Lemma 4.3. RPP parameterized by $|T|+|M|+\log (1+\omega(T)+\omega(M)) \in O(|D|+\log (1+\omega(D)))$ is in WK[1].
Proof. We prove a polynomial parameter transformation from RPP parameterized by $|T|+|M|+\log (1+\omega(T)+\omega(M))$ to NDTM Halting (Problem 2.9). By Remark 3.2, it is sufficient to reduce RPP with triangle inequality. To this end, we construct a number $t \in \mathbb{N}$ and a nondeterministic Turing machine $\mathcal{M}$ that, given an empty input string, has a computation path halting within $t$ steps if and only if a given RPP instance $I=(G, R, \omega)$ on a graph $G=(V, E)$ with $n$ vertices, $m$ edges, and triangle inequality has an RPP tour of weight at most $\omega(R)+k$, that is, an Eulerian extension of weight at most $k$. To this end, let

$$
d_{1}:=|M|+2|T| \quad \text { and } \quad d_{2}:=\omega(M)+2 \omega(T)
$$

By (3.8), there is an optimal Eulerian extension of at most $d_{1}$ edges for $(G, R, \omega)$. Thus, if $d_{1} \leq \log n$, or $n=0$, then we optimally solve $I$ in polynomial time [17] and return $t=1$ with a Turing machine of constant size that
immediately halts or never halts in dependence of whether $I$ is a yes-instance. Thus, we henceforth assume

$$
\begin{equation*}
0 \leq \log n<d_{1} . \tag{4.1}
\end{equation*}
$$

If $k \geq d_{2}$, then, by (3.4), $I$ is a yes-instance and we simply return $t=1$ and a Turing machine $\mathcal{M}$ of constant size that immediately halts. Thus, we henceforth assume

$$
\begin{equation*}
0 \leq k<d_{2} \tag{4.2}
\end{equation*}
$$

By (4.2), edges $e \in E$ with weight $\omega(e) \geq d_{2}$ will not be part of the sought Eulerian extension of weight $k$, thus we lower their weight to $d_{2}$ and henceforth assume

$$
\begin{equation*}
\omega(e) \leq d_{2} \quad \text { for all } e \in E \tag{4.3}
\end{equation*}
$$

We now construct a nondeterministic Turing machine $\mathcal{M}$. The Turing machine has a binary alphabet and $G$ is assumed to be encoded in binary. The state names of $\mathcal{M}$ encode the incidence matrix of $G$, the weight $\omega(e)$ of each edge $e \in E$ in binary, and, for each vertex $v \in V$, the number of its connected component in $G\langle R\rangle$ in binary. Turing machine $\mathcal{M}$ uses three tapes: on the edge tape, it guesses at most $d_{1}$ edges, on the connection tape, it records which of the initially $O\left(d_{1}\right)$ connected components of $G\langle R\rangle$ (by (3.7)) are connected by the guessed edges, and on the balancing tape, it records all imbalanced vertices, of which initially there are $O\left(d_{1}\right)$ by (3.6) and whose number will never exceed $O\left(d_{1}\right)$ by adding at most $d_{1}$ guessed edges. The program of Turing machine $\mathcal{M}$ is as follows. On the empty input, at most $d_{1}$ times:

1. Write the name $\{u, v\}$ of an arbitrary edge of $G$ (listed in the state names) onto the edge tape. This takes $O(\log n) \in O\left(d_{1}\right)$ steps $($ by $(4.1))$.
2. Flip the balance of $u$ and $v$ on the balancing tape in poly $\left(d_{1}\right)$ steps because there are only $O\left(d_{1}\right)$ vertices on it, each of which is encoded in $O(\log n) \subseteq O\left(d_{1}\right)$ bits (by (4.1)).
3. Record the connectivity of the components containing $u$ and $v$ on the connection tape in poly $\left(d_{1}\right)$ steps because there are only $O\left(d_{1}\right)$ component names on it.
If, after at most $d_{1}$ guessed edges, the computation does not reach a configuration where all vertices are balanced and all components of $G\langle R\rangle$ are connected, then $\mathcal{M}$ goes into an infinite loop. Otherwise, in poly $\left(d_{1}\right)$ steps, we reached such a configuration and it remains to check whether the guessed edges have weight at most $k$. To this end, $\mathcal{M}$ writes down the weights of the at most $d_{1}$ guessed edges in binary, sums them up, and compares them to $k$ in poly $\left(d_{1}+\log d_{2}\right)$ steps because of (4.2) and (4.3). If their weight is more than $k$, then $\mathcal{M}$ goes into an infinite loop. Otherwise, $\mathcal{M}$ stops. Observe that each computation path of $\mathcal{M}$, if it terminates, then it does so within $t$ steps for some $t \in \operatorname{poly}\left(d_{1}+\log d_{2}\right)$.

We have shown a correct reduction from RPP to NDTM Halting. To show that it is a polynomial parameter transformation, it remains to show $t \log |\mathcal{M}| \in \operatorname{poly}\left(d_{1}+\log d_{2}\right)=\operatorname{poly}(|T|+|M|+\log (1+\omega(T)+\omega(M))$. Since $t \in \operatorname{poly}\left(d_{1}+\log d_{2}\right)$, it remains to show that $\log |\mathcal{M}| \in \operatorname{poly}\left(d_{1}+\log d_{2}\right)$. The graph $G$ can be hard-coded in Turing machine $\mathcal{M}$ using poly $(n)$ symbols. The encoded edge weights have total size poly $\left(n+\log d_{2}\right)$ by (4.3). Its program therefore has size $|\mathcal{M}| \leq \operatorname{poly}\left(n+d_{1}+\log d_{2}\right)$. Thus,

$$
\begin{aligned}
\log |\mathcal{M}| & \leq \log \left(n+d_{1}+\log d_{2}\right)^{c}=c \log \left(n+d_{1}+\log d_{2}\right) \\
& =c\left(\log n+\log \left(1+\frac{d_{1}}{n}\right)+\log \left(1+\frac{\log d_{2}}{n+d_{1}}\right)\right) \\
& \leq c\left(d_{1}+\log \left(1+d_{1}\right)+\log \left(1+\log d_{2}\right)\right) \\
& \leq c\left(2 d_{1}+\log d_{2}\right) \in \operatorname{poly}\left(d_{1}+\log d_{2}\right)
\end{aligned}
$$

for some constant $c$
using $\log (a+b)=\log a+\log (1+a / b)$
using (4.1), $n \geq 1$, and $d_{1} \geq 1$
using $\log (1+x) \leq x$.

## 5 Approximate kernelization schemes

In Section 4, we have seen that provably effective and efficient data reduction for RPP is hard when one requires exact solutions. In this section, we show effective data reduction rules that only slightly decrease the solution quality. Indeed, we will prove Theorem 1.3. To this end, in Sections 5.1 to 5.3, we present three data reduction rules. In Section 5.4, we then show how to apply these rules to obtain a polynomial-size approximate kernelization scheme (PSAKS) of size $2 b+O(c / \varepsilon)$, proving Theorem 1.3. Finally, in Section 5.5, we discuss some problems that one faces when trying to improve it to a PSAKS for RPP parameterized only by $b$ or only by $c$.

### 5.1 Removing vertices non-incident to required edges

Recall that, by Remark 3.2, we can assume that the input graph $G$ is complete and the edge weights satisfy the triangle inequality. Thus, we can simply delete vertices that are not incident to required edges [8].

Reduction Rule 5.1. Let $(G, R, \omega)$ be an RPP instance with triangle inequality. Delete all vertices that are not incident to edges in $R$.
Since, by Observation 3.6, no edge-minimizing Eulerian extension uses vertices outside of $V(R)$, the following proposition is immediate.

Proposition 5.2. Reduction Rule 5.1 turns an RPP instance ( $G, R, \omega$ ) into an RPP instance ( $G^{\prime}, R, \omega$ ) such that

- any edge-minimizing Eulerian extension for $(G, R, \omega)$ is one for $\left(G^{\prime}, R, \omega\right)$ and
- any Eulerian extension for $\left(G^{\prime}, R, \omega\right)$ is one for $(G, R, \omega)$.


### 5.2 Reducing the number of required edges

In this section, we present a data reduction rule to shrink the set of required edges. This will be crucial since other data reduction rules only reduce the number of vertices, yet may leave the multiset of required edges between them unbounded.

Reduction Rule 5.3. Let $(G, R, \omega)$ be an instance of RPP and $C$ be a cycle in $G\langle R\rangle$ such that $G\langle R \backslash C\rangle$ has the same number of connected components as $G\langle R\rangle$, then delete the edges of $C$ from $R$.

Lemma 5.4. Using Reduction Rule 5.3, one can in $O(|R|)$ time compute a set $R^{\prime} \subseteq R$ of required edges with the following properties.
(i) Any Eulerian extension for $\left(G, R^{\prime}, \omega\right)$ is one for $(G, R, \omega)$ and vice versa.
(ii) The number of edges in each connected component of $G\left\langle R^{\prime}\right\rangle$ with $k$ vertices is at most $\max \{1,2 k-2\}$.

Proof. We apply Reduction Rule 5.3 as follows. For $i \in\{1, \ldots, c\}$, let $R_{i} \subseteq R$ be the set of required edges in the $i$-th connected component of $G\langle R\rangle$. In $O\left(\left|R_{i}\right|\right)$ time, one can compute a depth-first search tree $T_{i}$ of $G\left\langle R_{i}\right\rangle$, which is a spanning tree of $G\left\langle R_{i}\right\rangle$. Now we remove all cycles from $G\left\langle R_{i} \backslash T_{i}\right\rangle$ as follows. We start a depth-first search on $G\left\langle R_{i} \backslash T_{i}\right\rangle$. Whenever we meet a vertex $v$ a second time, we backtrack to the previous occurrence of $v$, deleting all visited edges from the graph on the way. This procedure removes all cycles from $G\left\langle R_{i} \backslash T_{i}\right\rangle$ and looks at each edge in $R_{i} \backslash T_{i}$ at most twice, thus works in $O\left(\left|R_{i}\right|\right)$ time.
(i) Any two vertices are connected in $G\langle R\rangle$ if and only if they are connected in $G\left\langle R^{\prime}\right\rangle$. Moreover, the balance of each vertex is the same in $G\langle R\rangle$ and $G\left\langle R^{\prime}\right\rangle$.
(ii) Each component of $G\left\langle R^{\prime}\right\rangle$ with $k=1$ vertex has one edge (a loop). Each component of $G\left\langle R^{\prime}\right\rangle$ with $k>1$ vertices consists of $k-1$ edges of a spanning tree $T_{i}$ for some $i \in\{1, \ldots, c\}$ and at most $k-1$ additional edges, otherwise they would contain a cycle.

### 5.3 Reducing the number of balanced vertices

In this section, we present a data reduction rule that removes balanced vertices. To this end, we introduce an operation that allows us to remove balanced vertices while maintaining the balance of their neighbors.

First, the following lemma in particular shows that removing a balanced vertex with all its incident edges changes the balance of an even number of vertices. This allows us to restore their original balance by adding a matching to the set of required edges, not increasing the total weight of required edges. This will be crucial to prove that our reduction rules maintain approximation factors.

Lemma 5.5. Let $\Gamma=(V, E)$ be a graph, $\omega:\{\{u, v\} \mid u, v \in V\} \rightarrow \mathbb{N}$ satisfy the triangle inequality, and $F$ be an even-cardinality submultiset of edges incident to a common vertex $v \in V$. Then
(i) The set $U \subseteq V \backslash\{v\}$ of vertices incident to an odd number of edges of $F$ has even cardinality.
(ii) For any matching $M_{v}$ in the complete graph on $U, \omega\left(M_{v}\right) \leq \omega(F)$ and $\left|M_{v}\right| \leq|F|$.

Proof. (i) Any graph, in particular $\Gamma\langle F\rangle$, has an even number of odd-degree vertices. Since $|F|$ is even, $v$ is not one of them.


Figure 5.1: Illustration of Definition 5.6(a). Only required edges are shown. Thick edges on the right are the added matching $M_{v}$.
(ii) Let $e_{i}:=\left\{x_{i}, y_{i}\right\}$ for $i \in\left\{1, \ldots,\left|M_{v}\right|\right\}$ be the edges of $M_{v}$. Then there are pairwise edge-disjoint paths $p_{i}:=\left(x_{i}, v, y_{i}\right)$ for $i \in\left\{1, \ldots,\left|M_{\nu}\right|\right\}$ in $\Gamma\langle F\rangle$. Thus

$$
\omega\left(M_{v}\right)=\sum_{i=1}^{\left|M_{v}\right|} \omega\left(e_{i}\right) \leq \sum_{i=1}^{\left|M_{v}\right|} \omega\left(p_{i}\right) \leq \omega(F)
$$

Finally, $\left|M_{v}\right| \leq|U| \leq|F|$.
We now use Lemma 5.5 to define an operation that allows us to remove a balanced vertex from $G\langle R\rangle$. It is illustrated in Figure 5.1.

Definition 5.6 (vertex extraction). Let $(G, R, \omega)$ be an RPP instance with $\omega$ satisfying the triangle inequality, $v$ be a vertex that

- is balanced in a connected component of $G\langle R\rangle$ with at least three vertices and
- not a cut vertex of $G\langle R\rangle$ or contained in exactly two blocks of $G\langle R\rangle$,
and let $R_{v} \subseteq R$ be the required edges incident to $v$. The result of extracting $v$ from $G\langle R\rangle$ is defined as follows.
(a) If $v$ is not a cut vertex of $G\langle R\rangle$, then let $M_{v}$ be any perfect matching on the set of vertices incident to an odd number of edges of $R_{v}$. The result of extracting $v$ is $R^{\prime}=\left(R \backslash R_{v}\right) \uplus M_{v}$.
(b) If $v$ is a cut vertex of $G\langle R\rangle$ contained in exactly two blocks $A$ and $B$ of $G\langle R\rangle$, then let $a$ be a neighbor of $v$ in $A$, $b$ be a neighbor of $v$ in $B$, and $R^{\prime}=(R \backslash\{\{a, v\},\{b, v\}\}) \uplus\{a, b\}$.

1. If $v$ is not contained in $G\left\langle R^{\prime}\right\rangle$, then $R^{\prime}$ is the result of extracting $v$.
2. Otherwise, $v$ is not a cut vertex of $G\left\langle R^{\prime}\right\rangle$ and the result of extracting $v$ from $G\langle R\rangle$ is defined as the result of extracting $v$ from $G\left\langle R^{\prime}\right\rangle$.

Lemma 5.7. Let $(G, R, \omega)$ be an RPP instance and $R^{\prime}$ be the result of extracting a balanced vertex $v$ of $G\langle R\rangle$. Then the following properties hold.
(i) $V\left(R^{\prime}\right)=V(R) \backslash\{v\}$.
(ii) $\omega\left(R^{\prime}\right) \leq \omega(R)$ and $\left|R^{\prime}\right| \leq|R|$.
(iii) Each vertex of $G\left\langle R^{\prime}\right\rangle$ is balanced if and only if it is balanced in $G\langle R\rangle$.
(iv) Two vertices of $G\left\langle R^{\prime}\right\rangle$ are connected if and only if they are so in $G\langle R\rangle$.
(v) Any multiset $S$ of edges with $V(S) \subseteq V\left(R^{\prime}\right)$ is an Eulerian extension for $\left(G, R^{\prime}, \omega\right)$ if and only if it is one for $(G, R, \omega)$.

Proof. (i) First, assume that $R^{\prime}$ was obtained according to Definition 5.6(a). Let $R_{v} \subseteq R$ be the required edges incident to $v$ and $U \subseteq V$ be the set of vertices incident to edges of $R_{v}$. Obviously, $V\left(R^{\prime}\right) \subseteq V(R) \backslash\{v\}$ and $V\left(R^{\prime}\right) \supseteq V(R) \backslash U$. Moreover, $V\left(R^{\prime}\right) \supseteq U \backslash\{v\}$ : since $v$ is in a connected component of $G\langle R\rangle$ with at least three vertices but not a cut vertex, the vertices in $U \backslash\{v\}$ are incident to edges in $R \backslash R_{v}$. These are retained in $G\left\langle R^{\prime}\right\rangle$. If $R^{\prime}$ was obtained according to Definition 5.6(b), then $R^{\prime}$ is the same as if it were obtained from Definition 5.6(a) by extracting $v$ from $G\langle R \uplus\{\{a, b\}\}\rangle$, where it is not a cut vertex.
(ii)-(iv) If $R^{\prime}$ was obtained according to Definition 5.6(a), then (ii) and (iii) follow from Lemma 5.5 applied to $\Gamma=G\langle R\rangle$ and $F=R_{v}$, whereas (iv) is clear since $v$ is not a cut vertex of $G\langle R\rangle$ and $v$ is not in $G\left\langle R^{\prime}\right\rangle$. If $R^{\prime}$ was obtained according to Definition 5.6(b1), then (ii)-(iv) trivially hold since $\omega(\{a, b\}) \leq \omega(\{a, v\})+\omega(\{b, v\})$ and $v$ is
not in $G\left\langle R^{\prime}\right\rangle$. If $R^{\prime}$ was obtained according to Definition 5.6(b2), then (ii)-(iv) hold since $R^{\prime}$ is the same as extracting $v$ from $G\langle(R \backslash\{\{a, v\},\{b, v\}\}) \uplus\{\{a, b\}\}\rangle$, where it is not a cut vertex, and from $\omega(\{a, b\}) \leq \omega(\{a, v\})+\omega(\{b, v\})$.
(v) We show that $G\langle R \uplus S\rangle$ is connected and balanced if and only if $G\left\langle R^{\prime} \uplus S\right\rangle$ is.

Connectivity. By (iv), two vertices of $V\left(R^{\prime}\right)$ are connected in $G\left\langle R^{\prime}\right\rangle$ if and only if they are connected in $G\langle R\rangle$. Since $V(S) \subseteq V\left(R^{\prime}\right) \subseteq V(R)$ by (i), two vertices in $V\left(R^{\prime}\right)=V\left(R^{\prime} \uplus S\right)$ are connected in $G\left\langle R^{\prime} \uplus S\right\rangle$ if and only if they are connected in $G\langle R \uplus S\rangle$. By (i), the only vertex of $G\langle R \uplus S\rangle$ that is absent from $G\left\langle R^{\prime} \uplus S\right\rangle$ is $v$, which is not isolated in $G\langle R \uplus S\rangle$ since it is not isolated in $G\langle R\rangle$.

Balance. By (iii), each vertex in $V\left(R^{\prime}\right)$ is balanced in $G\left\langle R^{\prime}\right\rangle$ if and only if it is balanced in $G\langle R\rangle$. Since $V(S) \subseteq V\left(R^{\prime}\right) \subseteq V(R)$ by (i), each vertex in $V\left(R^{\prime}\right)=V\left(R^{\prime} \uplus S\right)$ is balanced in $G\left\langle R^{\prime} \uplus S\right\rangle$ if and only if it is balanced in $G\langle R \uplus S\rangle$. By (i), the only vertex in $G\langle R \uplus S\rangle$ that is absent from $G\left\langle R^{\prime} \uplus S\right\rangle$ is $v$. If so, then $v \notin V(S)$ and $v$ is balanced in $G\langle R \uplus S\rangle$ because it is balanced in $G\langle R\rangle$.

We can now turn Definition 5.6 into a data reduction rule. Its parameter $\gamma \in \mathbb{Q}$ allows a trade-off between its strength and the introduced error.

Reduction Rule 5.8. Let $(G, R, \omega)$ be an RPP instance with $G=(V, E), \omega$ satisfying the triangle inequality, and $\gamma \in \mathbb{Q}$. Let $C_{i}$ be the vertices in connected component $i \in\{1, \ldots, c\}$ of $G\langle R\rangle$ and $B_{i} \subseteq C_{i}$ be an inclusion-maximal set of vertices such that, for each $u, v \in B_{i}$ with $u \neq v$, one has $\omega(\{u, v\})>\gamma$. Finally, let

$$
B:=\bigcup_{i=1}^{c} B_{i} .
$$

Now, initially let $R^{\prime}:=R$ and, as long as $G\left\langle R^{\prime}\right\rangle$ contains a vertex $v \in V \backslash B$ that can be extracted using Definition 5.6, replace $R^{\prime}$ by the result of extracting $v$.

We now analyze the effectivity and error of Reduction Rule 5.8.
Lemma 5.9. Let ( $G, R, \omega$ ) be an RPP instance with $\omega$ satisfying the triangle inequality. Then, Reduction Rule 5.8 in $O\left(n^{3}\right)$ time yields a multiset $R^{\prime}$ of edges such that
(i) $\omega\left(R^{\prime}\right) \leq \omega(R)$ and $V\left(R^{\prime}\right) \subseteq V(R)$.
(ii) Any multiset $S$ of edges with $V(S) \subseteq V\left(R^{\prime}\right)$ is an Eulerian extension for $\left(G, R^{\prime}, \omega\right)$ if and only if it is one for $(G, R, \omega)$.
(iii) Any edge-minimizing Eulerian extension $S$ for $(G, R, \omega)$ can be turned into an Eulerian extension $S^{\prime}$ for $\left(G, R^{\prime}, \omega\right)$ such that $\omega\left(S^{\prime}\right) \leq \omega(S)+2 \gamma \cdot(2 c-2)$.
(iv) $G\left\langle R^{\prime}\right\rangle$ contains at most $2 b+2 c+4 \omega(R) / \gamma$ vertices.

Proof. (i) and (ii) follow from Lemma 5.7 since $R^{\prime}$ is the result of a sequence of vertex extractions.
(iii) We turn $S$ into an Eulerian extension $S^{\prime}$ with $V\left(S^{\prime}\right) \subseteq V\left(R^{\prime}\right)$ and then apply (ii). First, since $S$ is edgeminimizing and $\omega$ satisfies the triangle inequality, by Observation 3.6, $V(S) \subseteq V(R)$. By Reduction Rule 5.8, the vertices in $X:=V(R) \backslash V\left(R^{\prime}\right)$ are not in $B$ and, thus, for each $v \in X \cap C_{i}$, we find a vertex $v^{\prime} \in B_{i}$ such that $\omega\left(\left\{v, v^{\prime}\right\}\right) \leq \gamma$. Note that $v^{\prime} \in V\left(R^{\prime}\right)$. Since each vertex in $X$ is balanced in $G\langle R\rangle$, by Remark 3.9, each vertex $v \in X \cap V(S)$ is incident to exactly two edges $\{v, u\}$ and $\{v, w\}$ of $S$ (possibly, $u=w$ ). Since $\left\{v, v^{\prime}\right\} \subseteq C_{i}, S^{\prime}:=(S \backslash\{\{v, u\},\{v, w\}\}) \uplus\left\{v^{\prime}, u\right\} \uplus\left\{v^{\prime}, w\right\}$ is also an Eulerian extension for $(G, R, \omega)$. Moreover, $\omega\left(S^{\prime}\right) \leq \omega(S)+2 \gamma$. Doing this replacement for each $v \in X \cap V(S)$, we finally obtain an Eulerian extension $S^{\prime}$ for $(G, R, \omega)$ with $V\left(S^{\prime}\right) \subseteq V\left(R^{\prime}\right)$ and $\omega\left(S^{\prime}\right) \leq \omega(S)+2 \gamma \cdot|X \cap V(S)|$. Since each vertex in $X$ is balanced in $G\langle R\rangle$, by Lemma 3.8, $|X \cap V(S)| \leq 2 c-2$. Finally, by (ii), $S^{\prime}$ is an Eulerian extension for ( $G, R^{\prime}, \omega$ ).
(iv) The vertices of $G\left\langle R^{\prime}\right\rangle$ can be partitioned into $X \uplus Y \uplus Z$, where $X$ are imbalanced in $G\left\langle R^{\prime}\right\rangle, Y$ are balanced and in $B$, and $Z$ are balanced but not in $B$.

By Lemma 5.7(iii), the vertices in $X$ are imbalanced in $G\langle R\rangle$ also. Thus,

$$
\begin{equation*}
|X| \leq b \tag{5.1}
\end{equation*}
$$

We next analyze $|Y|$. For $i \in\{1, \ldots, c\}$, let $R_{i} \subseteq R$ be the edges between vertices in $C_{i}, T_{i}^{*}$ be the edge set of a tree of least weight in $G\left\langle R_{i}\right\rangle$ connecting all vertices in $B_{i}, T_{i}$ be the edge set of a minimum-weight spanning tree in $G\left[B_{i}\right]$, and $H_{i}$ be the edge set of a minimum-weight Hamiltonian cycle in $G\left[B_{i}\right]$. Doubling all edges of $T_{i}^{*}$ yields
a closed walk in $G\left\langle R_{i}\right\rangle$ containing the vertices in $B_{i}$. Using the triangle inequality of $\omega$, it can be shortcut to a Hamiltonian cycle in $G\left[B_{i}\right]$. Thus, $\omega\left(T_{i}\right) \leq \omega\left(H_{i}\right) \leq 2 \omega\left(T_{i}^{*}\right) .{ }^{2}$ We thus get

$$
\begin{align*}
\left(\left|B_{i}\right|-1\right) \gamma & =\sum_{e \in T_{i}} \gamma<\sum_{e \in T_{i}} \omega(e)=\omega\left(T_{i}\right) \leq 2 \omega\left(T_{i}^{*}\right) \leq 2 \omega\left(R_{i}\right) \quad \text { and thus } \\
|Y| \leq|B| & =\sum_{i=1}^{c}\left|B_{i}\right|<\sum_{i=1}^{c}\left(\frac{2 \omega\left(R_{i}\right)}{\gamma}+1\right)=2 \omega(R) / \gamma+c . \tag{5.2}
\end{align*}
$$

Finally, we analyze $|Z|$. Definition 5.6 is not applicable to any vertex $v \in Z$, since it would have been removed by Reduction Rule 5.8. Thus, $v$ is a cut vertex contained in at least three blocks of $G\left\langle R^{\prime}\right\rangle$ or its connected component of $G\left\langle R^{\prime}\right\rangle$ consists of only two vertices. To analyze $|Z|$, for each $i \in\{1, \ldots, c\}$, consider $X_{i}:=X \cap C_{i}, Z_{i}:=Z \cap C_{i}$, the set $R_{i}^{\prime} \subseteq R^{\prime}$ of edges between vertices in $C_{i}$, and the block-cut tree $T_{i}$ of $G\left\langle R_{i}^{\prime}\right\rangle$ : the vertices of $T_{i}$ are the cut vertices and the blocks of $G\left\langle R_{i}^{\prime}\right\rangle$ and there is an edge between a cut vertex $v$ and a block $A$ of $G\left\langle R_{i}^{\prime}\right\rangle$ in $T_{i}$ if $v$ is contained in $A$. Then either $\left|Z_{i}\right| \leq 2$ or the vertices in $Z_{i}$ have degree at least three in $T_{i}$. Therefore, $T_{i}$ has at most $\left|X_{i}\right|+\left|Y_{i}\right|$ leaves. Since a tree has at least two leaves, we get $\left|X_{i}\right|+\left|Y_{i}\right| \geq 2$. Moreover, since a tree with $\ell$ leaves has at most $\ell-1$ vertices of degree three, $\left|Z_{i}\right| \leq \max \left\{2,\left|X_{i}\right|+\left|Y_{i}\right|-1\right\} \leq\left|X_{i}\right|+\left|Y_{i}\right|$. Thus,

$$
\begin{equation*}
|Z|=\sum_{i=1}^{c}\left|Z_{i}\right| \leq|X|+|Y| . \tag{5.3}
\end{equation*}
$$

Combining (5.1), (5.2), (5.3), and that $\left|V\left(R^{\prime}\right)\right|=|X|+|Y|+|Z|$, (iv) follows.
We finally analyze the running time of Reduction Rule 5.8. For $i \in\{1, \ldots, c\}$, all sets $C_{i}$ and $B_{i}$ can be computed in $O\left(n^{2}\right)$ time. Also the blocks of $G\left\langle R^{\prime}\right\rangle$ required by Definition 5.6 are computable in $O\left(n^{2}\right)$ time using depth-first search. Thus, in $O(n)$ time, we can find a vertex $v$ to which Definition 5.6 applies. Vertex $v$ can then be extracted in $O(n)$ time since the matchings $M_{v}$ in Definition 5.6 can be chosen arbitrarily, that is, in particular greedily in $O(n)$ time, and the blocks can be recomputed in $O\left(n^{2}\right)$ time. Finally, we extract at most $n$ vertices.

### 5.4 A polynomial-size approximate kernelization scheme for the parameter $\boldsymbol{b}+\boldsymbol{c}$ (proof of Theorem 1.3)

This section proves Theorem 1.3. We describe how to transform a given RPP instance $I$ and $\varepsilon>0$ into an RPP instance $I^{\prime}$ such that any $\alpha$-approximate solution for $I^{\prime}$ can be transformed into an $\alpha(1+\varepsilon)$-approximate solution for $I$. Due to Proposition 3.1, we assume that $I=(G, R, \omega)$ has been preprocessed in $O\left(n^{3}\right)$ time so as to satisfy the triangle inequality.

### 5.4.1 Shrinking the graph

Choose $\varepsilon_{1}+\varepsilon_{2}=\varepsilon$. Apply Reduction Rule 5.8 with

$$
\begin{equation*}
\gamma=\frac{\varepsilon_{1} \cdot \omega(R)}{4 c-4} \tag{5.4}
\end{equation*}
$$

which, by Lemma 5.9 , in $O\left(n^{3}\right)$ time gives an instance $\left(G, R_{1}, \omega\right)$ with

$$
\begin{equation*}
\left|V\left(R_{1}\right)\right| \leq 2 b+2 c+\frac{16 c-16}{\varepsilon_{1}} \tag{5.5}
\end{equation*}
$$

To $\left(G, R_{1}, \omega\right)$, we apply Reduction Rule 5.3 , which, by Lemma 5.4, in $O(|R|)$ time gives an instance $\left(G, R_{2}, \omega\right)$ with

$$
\begin{equation*}
R_{2} \subseteq R_{1} \quad \text { and } \quad\left|R_{2}\right| \leq 4 b+4 c+\frac{32 c-32}{\varepsilon_{1}} . \tag{5.6}
\end{equation*}
$$

Finally, applying Reduction Rule 5.1 to $\left(G, R_{2}, \omega\right)$ in linear time yields an instance $\left(G_{2}, R_{2}, \omega\right)$ such that

$$
\begin{equation*}
\left|V\left(G_{2}\right)\right| \leq\left|V\left(R_{2}\right)\right| \leq\left|V\left(R_{1}\right)\right| . \tag{5.7}
\end{equation*}
$$

${ }^{2}$ That is, $T_{i}$ is the folklore 2-approximation of a Steiner tree with terminals $B_{i}$ in $G\left\langle R_{i}\right\rangle$.

### 5.4.2 Shrinking edge weights

Since $G\langle R \uplus T\rangle$ is connected, due to the triangle inequality of $\omega$, each edge $e=\{u, v\}$ of $G$, and thus of its subgraph $G_{2}$, satisfies $\omega(e) \leq \omega(R)+\omega(T)$. Moreover, by Lemma 3.12, any edge-minimizing Eulerian extension for $\left(G_{2}, R_{2}, \omega\right)$ has at most $|M|+2|T|=b / 2+2 c-2$ edges. Thus, we can apply Lemma 2.12 with $\beta=\omega(R)+\omega(T)$ and $N=\left|R_{2}\right|+b / 2+2 c-2$ to $\left(G_{2}, R_{2}, \omega\right)$ to get an instance $\left(G_{2}, R_{2}, \omega_{2}\right)$ such that for all edges $e$,

$$
\begin{equation*}
\omega(e) \leq \frac{\left|R_{2}\right|+b / 2+2 c-2}{\varepsilon_{2}} \in O\left((b+c) /\left(\varepsilon_{1} \varepsilon_{2}\right)\right) . \tag{5.8}
\end{equation*}
$$

In Lemma 2.12, set $\mathcal{F}$ just contains all vectors $x$ that encode RPP tours $W$ induced by edge-minimizing Eulerian extensions for $\left(G_{2}, R_{2}, \omega\right)$ (its entries describe how often each edge is in $W$ ). We finally return $\left(G_{2}, R_{2}, \omega_{2}\right)$, whose construction takes $O\left(n^{3}+|R|\right)$ time, as required by Theorem 1.3.

### 5.4.3 Kernel size analysis

The returned instance satisfies Theorem 1.3(i) due to (5.5) and (5.7), (ii) due to (5.6), and (iii) due to (5.8).

### 5.4.4 Approximation factor analysis

It remains to prove Theorem 1.3(iv), that is, that we can lift an $\alpha$-approximate solution for ( $G_{2}, R_{2}, \omega_{2}$ ) to an $\alpha(1+\varepsilon)$-approximate solution for $(G, R, \omega)$.

An optimal RPP tour for $(G, R, \omega)$ has weight $\omega\left(W^{*}\right)=\omega(R)+\omega(D)$ by (3.1), where $D$ is a minimum-weight Eulerian extension. By Lemma 5.9(iii) and (5.4), there is an Eulerian extension $D^{\prime}$ for ( $G, R_{1}, \omega$ ) with

$$
\begin{equation*}
\omega\left(D^{\prime}\right) \leq \omega(D)+2 \gamma(2 c-2)=\omega(D)+\varepsilon_{1} \cdot \omega(R) . \tag{5.9}
\end{equation*}
$$

By Lemma 5.4, $D^{\prime}$ is an Eulerian extension for $\left(G, R_{2}, \omega\right)$ and, by Proposition 5.2, for $\left(G_{2}, R_{2}, \omega\right)$. Then $D^{\prime}$ is also an Eulerian extension for $\left(G_{2}, R_{2}, \omega_{2}\right)$. Thus, an optimal RPP tour for $\left(G_{2}, R_{2}, \omega_{2}\right)$ has weight at most $\omega_{2}\left(R_{2}\right)+\omega_{2}\left(D^{\prime}\right)$. By Proposition 3.5, an $\alpha$-approximate solution for ( $G_{2}, R_{2}, \omega_{2}$ ), can be turned into an Eulerian extension $S$ such that

$$
\begin{equation*}
\omega_{2}\left(R_{2}\right)+\omega_{2}(S) \leq \alpha\left(\omega_{2}\left(R_{2}\right)+\omega_{2}\left(D^{\prime}\right)\right) \tag{5.10}
\end{equation*}
$$

By Proposition 5.2, $S$ is an Eulerian extension for $\left(G, R_{2}, \omega\right)$. By Lemma 5.4, $S$ is an Eulerian extension for $\left(G, R_{1}, \omega\right)$, and by Lemma 5.9, it is one for $(G, R, \omega)$, since $V(S) \subseteq V\left(G_{2}\right)=V\left(R_{2}\right) \subseteq V\left(R_{1}\right) \subseteq V(R)$. Thus, by Proposition 3.5, $S$ can be turned into an RPP tour of weight $\omega(R)+\omega(S)$ for $(G, R, \omega)$. We analyze this weight. By (5.10) and Lemma 2.12 with $\beta=\omega(R)+\omega(T)$,

$$
\omega\left(R_{2}\right)+\omega(S) \leq \alpha\left(\omega\left(R_{2}\right)+\omega\left(D^{\prime}\right)\right)+\varepsilon_{2}(\omega(R)+\omega(T))
$$

Using $\omega\left(R_{2}\right) \leq \omega\left(R_{1}\right) \leq \omega(R)$ from Lemmas 5.4 and 5.9 , and $\alpha \geq 1$, we get

$$
\begin{aligned}
\omega(R)+\omega(S) & \leq \alpha\left(\omega(R)+\omega\left(D^{\prime}\right)\right)+\varepsilon_{2}(\omega(R)+\omega(T)) & & \\
& \leq \alpha\left(\omega(R)+\omega\left(D^{\prime}\right)\right)+\varepsilon_{2}(\omega(R)+\omega(D)) & & \operatorname{using}(3.7) \\
& \leq \alpha\left(\omega(R)+\omega(D)+\varepsilon_{1} \omega(R)\right)+\varepsilon_{2}(\omega(R)+\omega(D)) & & \operatorname{using}(5.9) \\
& \leq \alpha\left(1+\varepsilon_{1}+\varepsilon_{2}\right)(\omega(R)+\omega(D))=\alpha(1+\varepsilon) \omega\left(W^{*}\right) & & \operatorname{using}(3.1) .
\end{aligned}
$$

Thus, we got an $\alpha(1+\varepsilon)$-approximation for $(G, R, c)$.

### 5.5 On polynomial-size approximate kernelization schemes for the parameters $\boldsymbol{b}$ and $\boldsymbol{c}$

In the previous section, we have shown a polynomial-size approximate kernelization scheme (PSAKS) for RPP parameterized by $b+c$. An obvious question is whether there is a PSAKS for the parameters $b$ or $c$. For the parameter $b$, we can easily answer this question.

Proposition 5.10. If RPP parameterized by $b$ has a $(1+\varepsilon)$-approximate kernel for any $\varepsilon<1 / 122$, then $\mathrm{P}=\mathrm{NP}$.


- required edges $R$
- added matching edges $M^{*}$
---- optimal Eulerian extension $D$

Figure 5.2: Example showing that the bound given in Observation 5.12(iii) is tight: adding the edges in $M^{*}$ to $R$ breaks the only optimal Eulerian extension $D$ (dashed). To fix it, one either has to double all edges of $D$ or add all edges of $M^{*}$ to $D$. Note that the star can be arbitrarily enlarged.

Proof. Assume that RPP has an $(1+\varepsilon)$-approximate kernel of any size $g(b)$ for $\varepsilon<1 / 122$. We show how to find an $(1+\varepsilon)$-approximate solution for the metric Traveling Salesman problem in polynomial time, which implies $\mathrm{P}=\mathrm{NP}$ [41]. Given an instance $I$ of the metric Traveling Salesman problem, create an instance $I^{\prime}$ of RPP by adding a required zero-weight loop to each vertex. Compute an $(1+\varepsilon)$-approximate kernel $I^{\prime \prime}$ for $I^{\prime}$ in polynomial time. Since $I^{\prime}$ has no imbalanced vertices, the kernel $I^{\prime \prime}$ has size $g(b)=g(0) \in O(1)$. Computing an optimal solution in $I^{\prime \prime}$ thus takes constant time, can be lifted to an $(1+\varepsilon)$-approximate solution for $I^{\prime}$ in polynomial time and, after removing the loops, is an $(1+\varepsilon)$-approximation for $I$.

Answering the question about the existence of a PSAKS for the parameter $c$ is not quite as simple. In the following, we discuss the difficulties in resolving this question and make some first steps towards its resolution. In particular, we will show a PSAKS for the parameter $\omega(T)$.

To get the PSAKS for $c$, one has to reduce the number of imbalanced vertices in $G\langle R\rangle$. An obvious idea to do so is adding to $R$ cheap edges of a minimum-weight perfect matching $M$ on imbalanced vertices, since this is optimal if it happens to connect $G\langle R\rangle$.

Reduction Rule 5.11. Let $(G, R, \omega)$ be an RPP instance with triangle inequality and $\delta \in \mathbb{Q}$. Add to $R$ a subset $M^{*} \subseteq M$ of edges with

$$
\sum_{e \in M^{*}} \omega(e) \leq \delta
$$

Observation 5.12. Let $R^{\prime}=R \uplus M^{*}$ be obtained by applying Reduction Rule 5.11 to $R$.
(i) There are at most $2\left(|M|-\left|M^{*}\right|\right)$ imbalanced vertices in $G\left\langle R^{\prime}\right\rangle$.
(ii) For any Eulerian extension $S^{\prime}$ for $\left(G, R^{\prime}, \omega\right), S=S^{\prime} \uplus M^{*}$ is an Eulerian extension for $(G, R, \omega)$ and $\omega(R)+\omega(S)=\omega\left(R^{\prime}\right)+\omega\left(S^{\prime}\right)$.
(iii) For any Eulerian extension $S$ for $(G, R, \omega), S^{\prime}=S \uplus M^{*}$ is an Eulerian extension for $\left(G, R^{\prime}, \omega\right)$ with $\omega\left(S^{\prime}\right) \leq \omega(S)+\delta$.

To show a PSAKS with respect to the parameter $c$, this reduction rule is unsuitable for two reasons:

1. To reduce the number of imbalanced vertices in $G\langle R\rangle$ to some constant, we have to add all but a constant number of edges of $M$ to $R$, yet, by Observation 5.12(iii), each added edge potentially contributes to the error and thus would merely retain a 2-approximation. Unfortunately, Figure 5.2 shows that the bound given by Observation 5.12(iii) is tight.
2. Reduction Rule 5.11 increases the total weight of required edges. This makes it unusable for a PSAKS, since, in the resulting instance, a solution might be $(1+\varepsilon)$-approximate merely due to the fact that the lower bound $\omega(R)$ on the solution is sufficiently large (we will use this fact below).
Given the difficulties of showing a PSAKS for $c$, it is tempting to disprove its existence. However, we can easily build a PSAKS with size polynomial in $\omega(T)$, which gives a PSAKS of size polynomial in $c$ in case that the edge weights are bounded by poly $(c)$. More specifically, we prove the following.

Proposition 5.13. Let $(G, R, \omega)$ be an instance of RPP with triangle inequality.
(i) If $\omega(T) \leq \varepsilon(\omega(R)+\omega(M))$, then a (1+2 $\varepsilon$ )-approximate RPP tour for $(G, R, \omega)$ can be found in polynomial time.
(ii) If $\omega(M) \leq \varepsilon(\omega(R)+\omega(T))$, then $(G, R, \omega)$ has a $\left(1+3 \varepsilon+2 \varepsilon^{2}\right)$-approximate kernel with $O(c / \varepsilon)$ vertices.
(iii) Otherwise, $(G, R, \omega)$ has an (exact) problem kernel with respect to the parameter $\min \{\omega(T) / \varepsilon-\omega(M), \omega(M) / \varepsilon-$ $\omega(T)\}$.

Proposition 5.13 shows that, in order to exclude PSAKSes for RPP parameterized by $c$, a reduction must use unbounded edge weights, the weights of $T, M$, and $R$ may not differ too much (by (i) and (ii)), yet the weights of $T$ and $M$ must not be too close either (by (iii)). Given these restrictions, we conjecture:

Conjecture 5.14. RPP has a PSAKS with respect to the parameter $c$.
We finally prove Proposition 5.13.
Proof of Proposition 5.13. (i) Observe that the multiset $T \uplus T \uplus M$ is an Eulerian extension for ( $G, R, \omega$ ). Using Proposition 3.5, it yields an RPP tour of weight

$$
\begin{aligned}
\omega(R)+\omega(M)+2 \omega(T) & \leq \omega(R)+\omega(M)+2 \varepsilon(\omega(R)+\omega(M)) & & \\
& \leq(1+2 \varepsilon)(\omega(R)+\omega(D)) & & \text { using }(3.2) \\
& =(1+2 \varepsilon) \omega\left(W^{*}\right) & & \text { using (3.1). }
\end{aligned}
$$

(ii) Let $R^{\prime}$ be obtained from $R$ using Reduction Rule 5.11 with $\delta=\omega(M)$, that is, $R^{\prime}=R \uplus M$. In $G\left\langle R^{\prime}\right\rangle$, all vertices are balanced. Thus, applying Theorem 1.3 to $\left(G, R^{\prime}, \omega\right)$ gives an instance $\left(G_{2}, R_{2}, \omega_{2}\right)$ with $O(c / \varepsilon)$ vertices.

Let $D$ be an optimal Eulerian extension for $(G, R, \omega)$. Then, by Observation 5.12, an optimal Eulerian extension $D^{\prime}$ for $\left(G, R^{\prime}, \omega\right)$ has weight $\omega\left(D^{\prime}\right) \leq \omega(D)+\delta=\omega(D)+\omega(M)$ and, by Proposition 3.5, an optimal RPP tour for $\left(G, R^{\prime}, \omega\right)$ has weight $\omega\left(R^{\prime}\right)+\omega\left(D^{\prime}\right)$. Moreover, by Theorem 1.3, any $\alpha$-approximate RPP tour for $\left(G_{2}, R_{2}, \omega_{2}\right)$ can be lifted to an $\alpha(1+\varepsilon)$-approximate RPP tour $W$ for $\left(G, R^{\prime}, \omega\right)$. By Observation 5.12(ii), it yields a RPP tour for $(G, R, \omega)$ of weight

$$
\begin{array}{rlrl}
\omega(W) \leq \alpha(1+\varepsilon)\left(\omega\left(R^{\prime}\right)+\omega\left(D^{\prime}\right)\right) & \leq \alpha(1+\varepsilon)(\omega(R)+2 \omega(M)+\omega(D)) & \\
& \leq \alpha(1+\varepsilon)(\omega(R)+2 \varepsilon(\omega(R)+\omega(T))+\omega(D)) & & \text { using (ii) } \\
& \leq \alpha(1+\varepsilon)((1+2 \varepsilon) \omega(R)+(1+2 \varepsilon) \omega(D)) & & \text { using (3.3) } \\
& =\alpha(1+\varepsilon)(1+2 \varepsilon)(\omega(R)+\omega(D)) & \\
& =\alpha\left(1+3 \varepsilon+2 \varepsilon^{2}\right) \omega\left(W^{*}\right) & \text { using (3.1). }
\end{array}
$$

(iii) Otherwise, one has

$$
\omega(R) \leq \omega(M) / \varepsilon-\omega(T) \quad \text { and } \quad \omega(R) \leq \omega(T) / \varepsilon-\omega(M)
$$

and thus the known $2|R|$-vertex problem kernel [8] for RPP will be a kernel for both of these parameters.

## 6 Experiments

In this section, we experimentally evaluate the polynomial-size approximate kernelization scheme presented in Section 5.4.

Data instances. We evaluate the data reduction effect of our algorithm on three data sets generated by Corberán et al. [14, 15]:
alba- $p-i$ for each $p \in\{0.3,0.5,0.7\}$ and $i \in\{1, \ldots, 5\}$ : based on the street network of the Spanish town Albaida, where each edge is required with probability $p$ and $i$ is just a running index.
madr- $p-i$ for each $p \in\{0.3,0.5,0.7\}$ and $i \in\{1, \ldots, 5\}$ : based on the street network of the Spanish town Madrigueras, where each edge is required with probability $p$ and $i$ is just a running index.
ur- $n$ - $d-p$ for each $n \in\{500,750,1000\}, d \in\{3,4,5,6\}$, and $p \in\{0.25,0.5,0.75\}: n$ vertices are selected randomly from an $(1000 \times 1000)$-grid, distances are Euclidean, each vertex is connected to its $d$ closest neighbors, and each edge is required with probability $p$.
These data sets are widely used in the literature $[14,15,25,39,48] .{ }^{3}$ We also test our algorithm on two instances provided to us by Berliner Stadtreinigung, the company responsible for snow plowing and garbage collection in

[^2]

Figure 6.1: Data reduction effect of our PSAKS relative to the total number of input vertices (left) and to the number of vertices incident to required edges (right). Each dot represents an instance. The boxes show the first quartile, the median, and the third quartile. The whiskers extend up to 1.5 times the interquartile range.

Berlin. ${ }^{4}$ In the Berlin instances, both the street network as well as the required edges arise from a real snow plowing application, as opposed to generating the required edges randomly like in the other instances.

Characteristics of these instances and the weight $\omega(W)$ of a solution obtained via the 3/2-approximation can be seen in the "input instance" columns of Tables 1 to 4.

Experimental setup. Since our main goal is evaluating the effect of our data reduction rather than the running time of our algorithm, we sacrificed speed for simplicity and implemented the part of our PSAKS described in Section 5.4.1 in approximately 200 lines of Python (not counting the testing environment) using the NetworkX library for finding minimum-weight perfect matchings, (bi)connected components, cut vertices, and spanning trees. ${ }^{5}$ These routines are also contained in highly optimized C++ libraries like LEMON ${ }^{6}$ and we expect that one could achieve a speedup by orders of magnitude by implementing our PSAKS in C++. We did not implement the weight reduction step described in Section 5.4.2, since it is mainly of theoretical interest (to prove a polynomial size of the kernel rather than just a polynomial number of vertices and edges).

We kernelized each of the instances listed above for $\varepsilon=1 / 10$, that is, we require that a $11 \alpha / 10$-approximation be recoverable from an $\alpha$-approximate solution in the kernel. Since we do not reduce weights, this means we apply Reduction Rule 5.8 with $\varepsilon_{1}=\varepsilon=1 / 10$ in (5.4).

We also apply the folklore 3/2-approximation algorithm based on the Christofides-Serdyukov algorithm for the metric Traveling Salesman Problem [9, 13, 51] to compute a solution in the original and kernelized instance and compare their weights.

Experimental results. Figure 6.1 gives a rough idea of the data reduction effect of our PSAKS on the alba- $p-i$, madr- $p-i$, and ur- $n-d-p$ instances. The complete experimental results, including the two Berlin instances, are shown in Tables 1 to 4 , where additionally to the notation in Definition 3.11, we denote by
$\omega(W)$ - the weight of a $3 / 2$-approximation computed in the input graph,
$\omega\left(W^{\prime}\right)$ - the weight of a 3/2-approximation computed in the kernel and lifted to the input graph,
$\left|V^{\prime}\right|,\left|R^{\prime}\right|$ - the number of vertices and required edges in the kernel, respectively, and by
ms - the number of milliseconds it took to compute the kernel (not counting the time for computing pairwise shortest path lengths for establishing the triangle inequality using Proposition 3.1).
Since for the ur- $n-d-p$ instances, the weight of an optimal solution is known, we compare $\omega\left(W^{\prime}\right)$ to the optimum in Table 4. ${ }^{7}$ Remarkably, the best compression results are achieved on the Berlin instances, the only instances consisting purely out of real-world data: only $22 \%$ of the vertices incident to required edges remain. Figure 6.2 visualizes the kernelization effect on two strongly compressed instances.

[^3]Table 1: Results on the alba-p-i instances. Highlighted are rows where the weight of an approximate solution for the input instance differs from the weight of an approximate solution lifted from the kernel.

| $p$ | input instance |  |  |  |  |  | kernel |  |  |  | comparison |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\|V\|$ | $\|V(R)\|$ | $\|R\|$ |  | c | $\omega(W)$ | $\left\|V^{\prime}\right\|$ | $\left\|R^{\prime}\right\|$ | $\omega\left(W^{\prime}\right)$ | ms | $\frac{\left\|V^{\prime}\right\|}{\|V\|}$ | $\frac{\left\|V^{\prime}\right\|}{\|V(R)\|}$ | $\frac{\left\|R^{\prime}\right\|}{\|R\|}$ | $\frac{\omega\left(W^{\prime}\right)}{\omega(W)}$ |
| 0.3 | 116 | 72 | 51 | 54 | 22 | 7987 | 72 | 51 | 7987 | 2 | 0.62 | 1.00 | 1.00 | 1 |
| 0.3 | 116 | 68 | 46 | 54 | 23 | 6950 | 68 | 46 | 6950 | 2 | 0.59 | 1.00 | 1.00 | 1 |
| 0.3 | 116 | 59 | 44 | 36 | 15 | 7587 | 59 | 44 | 7587 | 2 | 0.51 | 1.00 | 1.00 | 1 |
| 0.3 | 116 | 70 | 49 | 48 | 21 | 7464 | 70 | 49 | 7464 | 2 | 0.60 | 1.00 | 1.00 | 1 |
| 0.3 | 116 | 73 | 57 | 48 | 19 | 7972 | 73 | 57 | 7972 | 2 | 0.63 | 1.00 | 1.00 | 1 |
| 0.5 | 116 | 101 | 88 | 68 | 18 | 11387 | 101 | 88 | 11387 | 3 | 0.87 | 1.00 | 1.00 | 1 |
| 0.5 | 116 | 100 | 92 | 58 | 14 | 10796 | 100 | 92 | 10796 | 4 | 0.86 | 1.00 | 1.00 | 1 |
| 0.5 | 116 | 99 | 92 | 50 | 11 | 9469 | 98 | 91 | 9469 | 4 | 0.84 | 0.99 | 0.99 | 1 |
| 0.5 | 116 | 91 | 88 | 50 | 8 | 9050 | 88 | 85 | 9050 | 5 | 0.76 | 0.97 | 0.97 | 1 |
| 0.5 | 116 | 102 | 91 | 60 | 16 | 10137 | 102 | 91 | 10137 | 3 | 0.88 | 1.00 | 1.00 | 1 |
| 0.7 | 116 | 104 | 118 | 64 | 6 | 11521 | 89 | 95 | 11641 | 12 | 0.77 | 0.86 | 0.81 | 1.01 |
| 0.7 | 116 | 108 | 122 | 56 | 2 | 11155 | 58 | 65 | 11155 | 30 | 0.50 | 0.54 | 0.53 | 1 |
| 0.7 | 116 | 110 | 113 | 60 | 9 | 11895 | 104 | 107 | 11895 | 8 | 0.90 | 0.95 | 0.95 | 1 |
| 0.7 | 116 | 110 | 119 | 66 | 4 | 11761 | 83 | 88 | 11761 | 25 | 0.72 | 0.75 | 0.74 | 1 |
| 0.7 | 116 | 110 | 116 | 58 | 7 | 11414 | 96 | 102 | 11414 | 13 | 0.83 | 0.87 | 0.88 | 1 |

Table 2: Results on the madr- $p-i$ instances. Highlighted are rows where the weight of an approximate solution for the input instances differs from the weight of an approximate solution lifted from the kernel.

| $p$ | input instance |  |  |  |  |  | kernel |  |  |  | comparison |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\|V\|$ | $\|V(R)\|$ | $\|R\|$ | $b$ | c | $\omega(W)$ | $\left\|V^{\prime}\right\|$ | $\left\|R^{\prime}\right\|$ | $\omega\left(W^{\prime}\right)$ | ms | $\frac{\left\|V^{\prime}\right\|}{\|V\|}$ | $\frac{\left\|V^{\prime}\right\|}{\|V(R)\|}$ | $\frac{\left\|R^{\prime}\right\|}{\|R\|}$ | $\frac{\omega\left(W^{\prime}\right)}{\omega(W)}$ |
| 0.3 | 196 | 127 | 86 | 96 | 42 | 13090 | 127 | 86 | 13090 | 4 | 0.65 | 1.00 | 1.00 | 1 |
| 0.3 | 196 | 142 | 108 | 86 | 34 | 14220 | 142 | 108 | 14220 | 5 | 0.72 | 1.00 | 1.00 | 1 |
| 0.3 | 196 | 137 | 102 | 96 | 36 | 13510 | 137 | 102 | 13510 | 4 | 0.70 | 1.00 | 1.00 | 1 |
| 0.3 | 196 | 140 | 101 | 98 | 39 | 13765 | 140 | 101 | 13765 | 4 | 0.71 | 1.00 | 1.00 | 1 |
| 0.3 | 196 | 131 | 95 | 88 | 38 | 13275 | 131 | 95 | 13275 | 4 | 0.67 | 1.00 | 1.00 | 1 |
| 0.5 | 196 | 176 | 163 | 108 | 21 | 15780 | 176 | 163 | 15780 | 8 | 0.90 | 1.00 | 1.00 | 1 |
| 0.5 | 196 | 174 | 156 | 100 | 25 | 17120 | 174 | 156 | 17120 | 9 | 0.89 | 1.00 | 1.00 | 1 |
| 0.5 | 196 | 165 | 148 | 94 | 22 | 15465 | 165 | 148 | 15465 | 9 | 0.84 | 1.00 | 1.00 | 1 |
| 0.5 | 196 | 166 | 152 | 92 | 23 | 16920 | 166 | 152 | 16920 | 7 | 0.85 | 1.00 | 1.00 | 1 |
| 0.5 | 196 | 169 | 147 | 96 | 26 | 15835 | 169 | 147 | 15835 | 6 | 0.86 | 1.00 | 1.00 | 1 |
| 0.7 | 196 | 188 | 211 | 96 | 7 | 20660 | 124 | 134 | 20560 | 67 | 0.63 | 0.66 | 0.64 | 0.995 |
| 0.7 | 196 | 192 | 238 | 120 | 2 | 22220 | 123 | 151 | 22220 | 81 | 0.63 | 0.64 | 0.63 | 1 |
| 0.7 | 196 | 191 | 219 | 92 | 6 | 20785 | 118 | 132 | 20785 | 86 | 0.60 | 0.62 | 0.60 | 1 |
| 0.7 | 196 | 192 | 225 | 98 | 3 | 20815 | 103 | 123 | 20815 | 88 | 0.53 | 0.54 | 0.55 | 1 |
| 0.7 | 196 | 191 | 223 | 106 | 3 | 21150 | 110 | 124 | 21250 | 87 | 0.56 | 0.58 | 0.56 | 1.005 |

Table 3: Results on the instances from Berliner Stadtreinigung.

| input instance |  |  |  |  | kernel |  |  |  |  | comparison |  |  |  |
| ---: | ---: | ---: | ---: | :---: | :---: | ---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\|V\|$ | $\|V(R)\|$ | $\|R\|$ | $b$ | $c$ | $\omega(W)$ | $\left\|V^{\prime}\right\|$ | $\left\|R^{\prime}\right\|$ | $\omega\left(W^{\prime}\right)$ | ms | $\frac{\left\|V^{\prime}\right\|}{\|V\|}$ | $\frac{\left\|V^{\prime}\right\|}{\|V(R)\|}$ | $\frac{\left\|R^{\prime}\right\|}{\|R\|}$ | $\frac{\omega\left(W^{\prime}\right)}{\omega(W)}$ |
| 2593 | 285 | 289 | 34 | 3 | 21911 | 62 | 66 | 21911 | 263 | 0.02 | 0.22 | 0.23 | 1 |
| 5097 | 369 | 408 | 56 | 3 | 31694 | 70 | 82 | 31694 | 435 | 0.01 | 0.19 | 0.20 | 1 |

Table 4: Results on the ur- $n-d-p$ instances. In these instances, $|V|=|V(R)|$. Highlighted are rows where the weight of an approximate solution for the input instances differs from the weight of an approximate solution lifted from the kernel.

| parameters |  |  | input instance |  |  |  |  | kernel |  |  |  | comparison |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $n$ | $d$ | $p$ | $\|V(R)\|$ | $\|R\|$ |  | $c$ | $\omega(W)$ | $\left\|V^{\prime}\right\|$ | $\left\|R^{\prime}\right\|$ | $\omega\left(W^{\prime}\right)$ | ms | $\frac{\left\|V^{\prime}\right\|}{\|V(R)\|}$ | $\frac{\left\|R^{\prime}\right\|}{\|R\|}$ | $\frac{\omega\left(W^{\prime}\right)}{\omega\left(W^{\prime}\right)}$ | $\frac{\omega\left(W^{\prime}\right)}{\mathrm{opt}}$ |
| 500 | 3 | 0.25 | 298 | 206 | 218 | 99 | 18004 | 298 | 206 | 18004 | 9 | 1.00 | 1.00 | 1 | 1.0421 |
| 500 | 3 | 0.50 | 458 | 464 | 246 | 58 | 24249 | 449 | 454 | 24249 | 38 | 0.98 | 0.98 | 1 | 1.0260 |
| 500 | 3 | 0.75 | 493 | 671 | 246 | 19 | 30141 | 338 | 438 | 30161 | 336 | 0.69 | 0.65 | 1.0007 | 1.0021 |
| 500 | 4 | 0.25 | 343 | 268 | 216 | 85 | 19152 | 343 | 268 | 19152 | 12 | 1.00 | 1.00 | 1 | 1.0741 |
| 500 | 4 | 0.50 | 476 | 582 | 242 | 19 | 29865 | 346 | 400 | 29845 | 337 | 0.73 | 0.69 | 0.9993 | 1.0066 |
| 500 | 4 | 0.75 | 498 | 848 | 242 | 2 | 38692 | 244 | 339 | 38692 | 644 | 0.49 | 0.40 | 1 | 1 |
| 500 | 5 | 0.25 | 388 | 322 | 238 | 80 | 21124 | 387 | 321 | 21124 | 30 | 1.00 | 1.00 | 1 | 1.0511 |
| 500 | 5 | 0.50 | 490 | 672 | 242 | 5 | 34560 | 265 | 334 | 34524 | 650 | 0.54 | 0.50 | 0.9990 | 1.0010 |
| 500 | 5 | 0.75 | 498 | 1001 | 252 | 1 | 48307 | 255 | 377 | 48307 | 543 | 0.51 | 0.38 | 1 | 1 |
| 500 | 6 | 0.25 | 416 | 405 | 232 | 53 | 25214 | 406 | 392 | 25214 | 39 | 0.98 | 0.97 | 1 | 1.0268 |
| 500 | 6 | 0.50 | 496 | 793 | 248 | 2 | 42845 | 256 | 357 | 42853 | 648 | 0.52 | 0.45 | 1.0002 | 1.0006 |
| 500 | 6 | 0.75 | 499 | 1157 | 250 | 1 | 58971 | 250 | 396 | 58971 | 570 | 0.50 | 0.34 | 1 | 1 |
| 700 | 3 | 0.25 | 452 | 321 | 328 | 140 | 22114 | 451 | 320 | 22114 | 15 | 1.00 | 1.00 | 1 | 1.0474 |
| 700 | 3 | 0.50 | 662 | 648 | 378 | 100 | 29289 | 651 | 635 | 29288 | 63 | 0.98 | 0.98 | 1 | 1.0218 |
| 700 | 3 | 0.75 | 744 | 979 | 390 | 16 | 36588 | 423 | 540 | 36732 | 971 | 0.57 | 0.55 | 1.0039 | 1.0039 |
| 700 | 4 | 0.25 | 538 | 439 | 340 | 122 | 24084 | 536 | 437 | 24084 | 22 | 1.00 | 1.00 | 1 | 1.0677 |
| 700 | 4 | 0.50 | 713 | 808 | 378 | 57 | 32830 | 655 | 733 | 32857 | 229 | 0.92 | 0.91 | 1.0008 | 1.0112 |
| 700 | 4 | 0.75 | 745 | 1261 | 356 | 3 | 47769 | 366 | 498 | 47774 | 1486 | 0.49 | 0.39 | 1.0001 | 1.0002 |
| 700 | 5 | 0.25 | 580 | 506 | 344 | 108 | 26315 | 577 | 503 | 26317 | 27 | 0.99 | 0.99 | 1.0001 | 1.0472 |
| 700 | 5 | 0.50 | 724 | 1003 | 398 | 15 | 41897 | 418 | 521 | 41946 | 1012 | 0.58 | 0.52 | 1.0012 | 1.0041 |
| 700 | 5 | 0.75 | 748 | 1459 | 380 | 1 | 58416 | 388 | 592 | 58416 | 1145 | 0.52 | 0.41 | 1 | 1 |
| 700 | 6 | 0.25 | 593 | 530 | 360 | 103 | 28920 | 591 | 528 | 28920 | 27 | 1.00 | 1.00 | 1 | 1.0373 |
| 700 | 6 | 0.50 | 741 | 1179 | 376 | 2 | 50492 | 385 | 528 | 50508 | 1391 | 0.52 | 0.45 | 1.0003 | 1.0003 |
| 700 | 6 | 0.75 | 749 | 1747 | 396 | 1 | 72950 | 397 | 615 | 72950 | 1255 | 0.53 | 0.35 | 1 | 1 |
| 1000 | 3 | 0.25 | 605 | 411 | 442 | 204 | 25460 | 605 | 411 | 25460 | 19 | 1.00 | 1.00 | 1 | 1.0647 |
| 1000 | 3 | 0.50 | 892 | 892 | 502 | 124 | 33981 | 865 | 862 | 33981 | 146 | 0.97 | 0.97 | 1 | 1.0270 |
| 1000 | 3 | 0.75 | 980 | 1308 | 514 | 24 | 42894 | 585 | 739 | 43176 | 1809 | 0.60 | 0.56 | 1.0066 | 1.0089 |
| 1000 | 4 | 0.25 | 709 | 564 | 466 | 167 | 27290 | 703 | 558 | 27290 | 40 | 0.99 | 0.99 | 1 | 1.0682 |
| 1000 | 4 | 0.50 | 929 | 1086 | 506 | 71 | 39607 | 856 | 983 | 39609 | 344 | 0.92 | 0.91 | 1.0001 | 1.0154 |
| 1000 | 4 | 0.75 | 996 | 1684 | 496 | 4 | 55967 | 514 | 694 | 56010 | 2689 | 0.52 | 0.41 | 1.0008 | 1.0009 |
| 1000 | 5 | 0.25 | 766 | 661 | 488 | 149 | 30464 | 765 | 660 | 30467 | 30 | 1.00 | 1.00 | 1.0001 | 1.0515 |
| 1000 | 5 | 0.50 | 975 | 1352 | 494 | 5 | 49197 | 524 | 667 | 49216 | 2544 | 0.54 | 0.49 | 1.0004 | 1.0012 |
| 1000 | 5 | 0.75 | 1000 | 2029 | 516 | 2 | 70231 | 521 | 763 | 70231 | 2650 | 0.52 | 0.38 | 1 | 1 |
| 1000 | 6 | 0.25 | 802 | 728 | 486 | 138 | 33688 | 792 | 717 | 33690 | 60 | 0.99 | 0.98 | 1.0001 | 1.0417 |
| 1000 | 6 | 0.50 | 980 | 1563 | 510 | 9 | 58854 | 523 | 702 | 58951 | 2262 | 0.53 | 0.45 | 1.0016 | 1.0026 |
| 1000 | 6 | 0.75 | 1000 | 2304 | 502 | 1 | 82481 | 504 | 781 | 82481 | 2265 | 0.50 | 0.34 | 1 | 1 |

In the following, we discuss the data reduction effect in more detail with the help of the plots in Figure 6.3. Since some of the instances in the literature (concretely, the ur- $n-d-p$ instances) are already preprocessed with respect to Reduction Rule 5.1, we analyze the data reduction effect with respect to the number $|V(R)|$ of those input vertices incident to required edges. For the sake of completeness, the data reduction effect with respect to the number of all input vertices is shown in Tables 1 to 4.

In Figures 6.3 a and 6.3b, we see that the effectivity of our PSAKS grows with the number of input vertices incident to required edges and with the number of required edges themselves. In Figure 6.3c, we see that, as expected, the effectivity decreases as the number of connected components of $G\langle R\rangle$ grows. However, in all three plots, we see a clear clustering of different instance types. It is Figure 6.3d that shows the determining feature for the effectivity of our PSAKS: in all instances types, it uniformly grows with the average size of the connected components of $G\langle R\rangle$. This comes at no surprise, since the main action of our PSAKS is shrinking these connected components.

Regarding the solution quality, we point out that, despite kernelizing all instances with $\varepsilon=1 / 10$ and thus allowing a weight increase by a factor of 1.1 when lifting a solution from the kernel to the original instance, the maximal such weight increase observed is 1.01 (for the alba- $p-i$ instances in Table 1), whereas often no weight increase is observed. In some cases, kernelization leads to better solutions (for the instances in Tables 2 and 4). Also, in Table 4, the $3 / 2$-approximate solution lifted from the kernel turns out to be worse than the optimum by a factor not larger than 1.075 and is thus way below the allowance.

Possible improvements. The effectivity of our data reduction can be increased replacing $\omega(R)$ by

$$
\max \left\{\omega(R)+\omega(M), \omega(R)+\omega(T), \omega(R)+\frac{\omega(M)+\omega(T)}{2}\right\}
$$

in the choice of $\gamma$ in (5.4) for the application of Reduction Rule 5.8. Since this also is a lower bound for $\omega\left(W^{*}\right)$ (recall Lemma 3.12), such a replacement will still guarantee that a $\alpha(1+\varepsilon)$-approximation can be lifted from a $\alpha$-approximation on the kernel. However, this replacement removes about one or two percents of vertices more, whereas computing $\omega(M)$ in the larger instances took between 11 and 40 seconds, so the pay-off is very limited.

## 7 Conclusion

Our main algorithmic contribution is a polynomial-size approximate kernelization scheme (PSAKS) for the Rural Postman Problem parameterized by $b+c$, where $b$ is the number of vertices incident to an odd number of required edges and $c$ is the number of connected components formed by the required edges. Experiments show that the data reduction algorithm efficiently shrinks problem instances with few connected components without largely sacrificing solution quality. We also showed a PSAKS for the parameter $\omega(T)$, which gives a PSAKS for the parameter $c$ when edge weights are bounded polynomially in $c$. These results together naturally lead to the question whether a PSAKS for the parameter $c$ exists (we conjecture "yes")

We think that the approach taken by Reduction Rule 5.8, namely reducing all vertices that do not belong to some inclusion-maximal set $B$ of mutually sufficiently distant vertices, might be applicable to other metric graph problems: it ensures that, for each deleted vertex, some nearby representative in $B$ is retained. In preliminary research, for example, we also found it to applicable to a metric variant of the Min-Power Symmetric Connectivity problem where it is required to connect $c$ disconnected parts of a wireless sensor network [3] and to the Location Rural Postman Problem [10]. Notably, this approach does not generalize well to asymmetric distances, so that another vexing question besides proving Conjecture 5.14 is whether the scheme for the parameter $b+c$ presented in this work can be generalized to the directed Rural Postman Problem. We point out that, using known ideas [7], one can reduce any instance $I$ of the directed or undirected RPP to an instance $I^{\prime}$ with $c$ vertices in $O\left(n^{3} \log n\right)$ time such that any $\alpha$-approximation for $I^{\prime}$ yields an $(\alpha+1)$-approximation for $I$. Given that undirected RPP is $3 / 2$-approximable, this is interesting only for the directed RPP.

Acknowledgments. We thank the anonymous referees of Networks for their constructive feedback.

Funding. R. van Bevern and O. Yu. Tsidulko are supported by the Russian Foundation for Basic Research, project 18-501-12031 NNIO_a. T. Fluschnik is supported by the German Research Foundation, project TORE (NI 369/18).


$$
\bigcirc \text { alba- } p-i \quad \Delta \text { madr- } p-i \quad \bullet \text { ur-500- } d-p \quad * \text { ur-700- } d-p \quad+\text { ur-1000- } d-p
$$



Figure 6.3: Effect of data reduction of our PSAKS.

## References

[1] J.-M. Belenguer, E. Benavent, P. Lacomme, and C. Prins, Lower and upper bounds for the mixed capacitated arc routing problem, Computers \& Operations Research 33 (2006), 3363-3383, doi:10.1016/j.cor.2005.02.009.
[2] M. Bentert, R. van Bevern, T. Fluschnik, A. Nichterlein, and R. Niedermeier, Polynomial-time data reduction for weighted problems beyond additive goal functions, 2020, URL https://arxiv.org/abs/1910.00277.
[3] M. Bentert, R. van Bevern, A. Nichterlein, and R. Niedermeier, Parameterized algorithms for power-efficient connected symmetric wireless sensor networks, A. F. Anta, T. Jurdzinski, M. A. Mosteiro, and Y. Zhang (eds.), ALGOSENSORS 2017, Springer, 2017, Lecture Notes in Computer Science, vol. 10718, pp. 26-40, doi:10.1007/978-3-319-72751-6_3.
[4] R. van Bevern, T. Fluschnik, and O. Yu. Tsidulko, On $(1+\varepsilon)$-approximate data reduction for the Rural Postman Problem, M. Khachay, Y. Kochetov, and P. Pardalos (eds.), MOTOR 2019, Springer, Lecture Notes in Computer Science, vol. 11548, 2019, pp. 279-294, doi:10.1007/978-3-030-22629-9_20.
[5] R. van Bevern, T. Fluschnik, and O. Yu. Tsidulko, Parameterized algorithms and data reduction for the short secluded $s$-t-path problem, Networks 75 (2020), 34-63, doi:10.1002/net.21904.
[6] R. van Bevern, S. Hartung, A. Nichterlein, and M. Sorge, Constant-factor approximations for capacitated arc routing without triangle inequality, Operations Research Letters 42 (2014), 290-292, doi:10.1016/j.orl.2014.05.002.
[7] R. van Bevern, C. Komusiewicz, and M. Sorge, A parameterized approximation algorithm for the mixed and windy capacitated arc routing problem: Theory and experiments, Networks 70 (2017), 262-278, doi:10.1002/net.21742.
[8] R. van Bevern, R. Niedermeier, M. Sorge, and M. Weller, Complexity of arc routing problems, Á. Corberán and G. Laporte (eds.), Arc Routing: Problems, Methods, and Applications, SIAM, MOS-SIAM Series on Optimization, vol. 20, 2015, pp. 19-52, doi:10.1137/1.9781611973679.ch2.
[9] R. van Bevern and V. A. Slugina, A historical note on the 3/2-approximation algorithm for the metric traveling salesman problem, Historia Mathematica (in press), doi:10.1016/j.hm.2020.04.003.
[10] R. van Bevern and O. Yu. Tsidulko, Data reduction for the location rural postman problem, Abstracts of the 33rd Annual Conference of the Belgian Operations Research Society (ORBEL 33), February 7-8, 2019, Hasselt University, Belgium, 2019, pp. 41-43.
[11] J. Brandão and R. Eglese, A deterministic tabu search algorithm for the capacitated arc routing problem, Computers \& Operations Research 35 (2008), 1112-1126, doi:10.1016/j.cor.2006.07.007.
[12] N. Christofides, The optimum traversal of a graph, Omega 1 (1973), 719-732, doi:10.1016/0305-0483(73)90089-3.
[13] N. Christofides, Worst-case analysis of a new heuristic for the traveling salesman problem, Tech. Rep. 388, Carnegie-Mellon University, Pittsburgh, Pennsylvania, USA, 1976.
[14] A. Corberán, A. N. Letchford, and J. M. Sanchis, A cutting plane algorithm for the general routing problem, Mathematical Programming 90 (2001), 291-316, doi:10.1007/PL00011426.
[15] A. Corberán, I. Plana, and J. M. Sanchis, A branch \& cut algorithm for the windy general routing problem and special cases, Networks 49 (2007), 245-257, doi:10.1002/net. 20176.
[16] Á. Corberán and G. Laporte (eds.), Arc Routing: Problems, Methods, and Applications, MOS-SIAM Series on Optimization, vol. 20, SIAM, 2015, doi:10.1137/1.9781611973679.
[17] F. Dorn, H. Moser, R. Niedermeier, and M. Weller, Efficient algorithms for Eulerian Extension and Rural Postman, SIAM Journal on Discrete Mathematics 27 (2013), 75-94, doi:10.1137/110834810.
[18] J. Edmonds and E. L. Johnson, Matching, Euler tours and the Chinese postman, Mathematical Programming 5 (1973), 88-124, doi:10.1007/BF01580113.
[19] R. W. Eglese and H. Murdock, Routeing road sweepers in a rural area, Journal of the Operational Research Society 42 (1991), 281-288, doi:10.1057/jors.1991.66.
[20] E. Eiben, D. Hermelin, and M. S. Ramanujan, Lossy kernels for hitting subgraphs, K. G. Larsen, H. L. Bodlaender, and J.-F. Raskin (eds.), MFCS 2017, Schloss Dagstuhl-Leibniz-Zentrum für Informatik, Dagstuhl, Germany, Leibniz International Proceedings in Informatics (LIPIcs), vol. 83, 2017, pp. 67:1-67:14, doi:10.4230/LIPIcs.MFCS.2017.67.
[21] E. Eiben, M. Kumar, A. E. Mouawad, F. Panolan, and S. Siebertz, Lossy kernels for connected dominating set on sparse graphs, R. Niedermeier and B. Vallée (eds.), 35th Symposium on Theoretical Aspects of Computer Science (STACS 2018), Schloss Dagstuhl-Leibniz-Zentrum für Informatik, Leibniz International Proceedings in Informatics (LIPIcs), vol. 96, 2018, pp. 29:1-29:15, doi:10.4230/LIPIcs.STACS.2018.29.
[22] H. A. Eiselt, M. Gendreau, and G. Laporte, Arc routing problems, part II: The Rural Postman Problem, Operations Research 43 (1995), 399-414, doi:10.1287/opre.43.3.399.
[23] M. Etscheid, S. Kratsch, M. Mnich, and H. Röglin, Polynomial kernels for weighted problems, Journal of Computer and System Sciences 84 (2017), 1-10, doi:10.1016/j.jcss.2016.06.004.
[24] M. R. Fellows, A. Kulik, F. A. Rosamond, and H. Shachnai, Parameterized approximation via fidelity preserving transformations, Journal of Computer and System Sciences 93 (2018), 30-40, doi:10.1016/j.jcss.2017.11.001.
[25] E. Fernández, O. Meza, R. Garfinkel, and M. Ortega, On the undirected rural postman problem: Tight bounds based on a new formulation, Operations Research 51 (2003), 281-291, doi:10.1287/opre.51.2.281.12790.
[26] H. Fleischner, Eulerian Graphs and Related Topics: Part 1, Volume 2, North-Holland, Amsterdam, The Netherlands, Annals of Discrete Mathematics, vol. 50, 1991, pp. X.1-X.14.
[27] J. Flum and M. Grohe, Parameterized Complexity Theory, Texts in Theoretical Computer Science, An EATCS Series, Springer, 2006, doi:10.1007/3-540-29953-X.
[28] F. V. Fomin, D. Lokshtanov, S. Saurabh, and M. Zehavi, Kernelization, Cambridge University Press, 2019, doi:10.1017/9781107415157.
[29] A. Frank and É. Tardos, An application of simultaneous diophantine approximation in combinatorial optimization, Combinatorica 7 (1987), 49-65, doi:10.1007/BF02579200.
[30] G. N. Frederickson, Approximation Algorithms for NP-hard Routing Problems, Ph.D. thesis, University of Maryland Graduate School, College Park, Maryland, USA, 1977.
[31] M. R. Garey and D. S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, Freeman, New York, USA, 1979.
[32] G. Ghiani and G. Improta, The laser-plotter beam routing problem, Journal of the Operational Research Society 52 (2001), 945-951, doi:10.1057/palgrave.jors. 2601161.
[33] G. Ghiani and G. Laporte, Eulerian location problems, Networks 34 (1999), 291-302, doi:10.1002/(SICI)1097-0037(199912)34:4<291::AID-NET9>3.0.CO;2-4.
[34] B. L. Golden and R. T. Wong, Capacitated arc routing problems, Networks 11 (1981), 305-315, doi:10.1002/net. 3230110308.
[35] M. Grötschel, M. Jünger, and G. Reinelt, Optimal control of plotting and drilling machines: A case study, Zeitschrift für Operations Research 35 (1991), 61-84, doi:10.1007/BF01415960.
[36] G. Gutin, M. Wahlström, and A. Yeo, Rural Postman parameterized by the number of components of required edges, Journal of Computer and System Sciences 83 (2017), 121-131, doi:10.1016/j.jcss.2016.06.001.
[37] D. Hermelin, S. Kratsch, K. Sołtys, M. Wahlström, and X. Wu, A completeness theory for polynomial (Turing) kernelization, Algorithmica 71 (2015), 702-730, doi:10.1007/s00453-014-9910-8.
[38] C. Hierholzer and C. Wiener, Ueber die Möglichkeit, einen Linienzug ohne Wiederholung und ohne Unterbrechung zu umfahren, Mathematische Annalen 6 (1873), 30-32, doi:10.1007/BF01442866.
[39] M. H. Hà, N. Bostel, A. Langevin, and L.-M. Rousseau, Solving the close-enough arc routing problem, Networks 63 (2014), 107-118, doi:10.1002/net. 21525.
[40] K. Jansen, Bounds for the general capacitated routing problem, Networks 23 (1993), 165-173, doi:10.1002/net. 3230230304.
[41] M. Karpinski, M. Lampis, and R. Schmied, New inapproximability bounds for TSP, Journal of Computer and System Sciences 81 (2015), 1665-1677, doi:10.1016/j.jcss.2015.06.003.
[42] R. Krithika, D. Majumdar, and V. Raman, Revisiting connected vertex cover: FPT algorithms and lossy kernels, Theory of Computing Systems 62 (2018), 1690-1714, doi:10.1007/s00224-017-9837-y.
[43] R. Krithika, P. Misra, A. Rai, and P. Tale, Lossy kernels for graph contraction problems, A. Lal, S. Akshay, S. Saurabh, and S. Sen (eds.), FSTTCS 2016, Schloss Dagstuhl-Leibniz-Zentrum für Informatik, Dagstuhl, Germany, Leibniz International Proceedings in Informatics (LIPIcs), vol. 65, 2016, pp. 23:1-23:14, doi:10.4230/LIPIcs.FSTTCS.2016.23.
[44] J. K. Lenstra and A. H. G. Rinnooy Kan, On general routing problems, Networks 6 (1976), 273-280, doi:10.1002/net. 3230060305.
[45] D. Lokshtanov, F. Panolan, M. S. Ramanujan, and S. Saurabh, Lossy kernelization, H. Hatami, P. McKenzie, and V. King (eds.), STOC 2017, ACM, 2017, pp. 224-237, doi:10.1145/3055399.3055456.
[46] D. Marx and L. A. Végh, Fixed-parameter algorithms for minimum-cost edge-connectivity augmentation, ACM Transactions on Algorithms 11 (2015), 27:1-27:24, doi:10.1145/2700210.
[47] C. S. Orloff, A fundamental problem in vehicle routing, Networks 4 (1974), 35-64, doi:10.1002/net. 3230040105.
[48] G. Reinelt, D. O. Theis, and K. M. Wenger, Computing finest mincut partitions of a graph and application to routing problems, Discrete Applied Mathematics 156 (2008), 385-396, doi:10.1016/j.dam.2007.03.022.
[49] J. T. Schwartz, Fast probabilistic algorithms for verification of polynomial identities, Journal of the ACM 27 (1980), 701-717, doi:10.1145/322217.322225.
[50] A. I. Serdyukov, O zadache nakhozhdeniya minimal'nogo Eilerova mul'tigrafa dlya svyaznogo grafa so vzveshennymi rebrami, Upravlyaemye sistemy 12 (1974), 61-67, URL http://nas1.math.nsc.ru/aim/journals/us/us12/us12_ 008.pdf.
[51] A. I. Serdyukov, O nekotorykh ekstremal'nykh obkhodakh v grafakh, Upravlyaemye sistemy 17 (1978), 76-79, URL http://nas1.math.nsc.ru/aim/journals/us/us17/us17_007.pdf.
[52] M. Sorge, R. van Bevern, R. Niedermeier, and M. Weller, From few components to an Eulerian graph by adding arcs, P. Kolman and J. Kratochvíl (eds.), WG 2011, Springer, Lecture Notes in Computer Science, vol. 6986, 2011, pp. 307-318, doi:10.1007/978-3-642-25870-1_28.
[53] M. Sorge, R. van Bevern, R. Niedermeier, and M. Weller, A new view on Rural Postman based on Eulerian Extension and Matching, Journal of Discrete Algorithms 16 (2012), 12-33, doi:10.1016/j.jda.2012.04.007.
[54] G. Ulusoy, The fleet size and mix problem for capacitated arc routing, European Journal of Operational Research 22 (1985), 329-337, doi:10.1016/0377-2217(85)90252-8.
[55] S. Wøhlk, An approximation algorithm for the Capacitated Arc Routing Problem, The Open Operational Research Journal 2 (2008), 8-12, doi:10.2174/1874243200802010008.
[56] R. Zippel, Probabilistic algorithms for sparse polynomials, E. W. Ng (ed.), EUROSAM 1979, Springer, Lecture Notes in Computer Science, vol. 72, 1979, pp. 216-226, doi:10.1007/3-540-09519-5_73.


[^0]:    *A preliminary version of this work appeared in the Proceedings of the 18th International Conference on Mathematical Optimization Theory and Operations Research (MOTOR 2019), Ekaterinburg, Russian Federation, July 8-12, 2019 [4]. This work provides all proofs of the theorems stated in the conference version, a stronger version of Proposition 5.13, WK[1]-completeness results (Section 4), and an experimental evaluation of our data reduction algorithm (Section 6).

[^1]:    ${ }^{1}$ Before, this correspondence was observed for the Chinese Postman Problem independently by Christofides [12], Edmonds and Johnson [18], and Serdyukov [50].

[^2]:    ${ }^{3}$ Available at https://www.uv.es/corberan/instancias.htm

[^3]:    ${ }^{4}$ Available at https://gitlab.com/rvb/rpp-psaks
    ${ }^{5}$ https://networkx.github.io/
    ${ }^{6}$ https://lemon.cs.elte.hu/trac/lemon
    ${ }^{7}$ For the alba- $p-i$ and madr- $p-i$ instances, the optimum is only known when considered as instances of the General Routing Problem, where also vertices have to be visited.

