# Iterative representing set selection for nested cross approximation 

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#### Abstract

SUMMARY A new fast algebraic method for obtaining an $\mathcal{H}^{2}$-approximation of a matrix from its entries is presented. The main idea behind the method is based on the nested representation and the maximum-volume principle to select submatrices in low-rank matrices. A special iterative approach for the computation of so-called representing sets is established. The main advantage of the method is that it uses only the hierarchical partitioning of the matrix and does not require special "proxy surfaces" to be selected in advance. The numerical experiments for the electrostatic problem and for the boundary integral operator confirm the effectiveness and robustness of the approach. The complexity is linear in the matrix size and polynomial in the ranks. The algorithm is implemented as an open-source Python package that is available online. Copyright © 0000 John Wiley \& Sons, Ltd.


Received ..

KEY WORDS: FMM, $\mathcal{H}^{2}$-matrices, electrostatic problem, BEM, mosaic partitioning

## 1. INTRODUCTION

Cross approximation [1, 2] is a basic reduction technique for the approximation of large low-rank matrices. These matrices often appear as blocks of dense matrices, coming from the discretization of non-local operators (in particular, in FEM/BEM applications). The whole matrix is split into large blocks related to geometrically separated sets of sources and receivers, and those blocks are approximated by low-rank matrices. This partitioning forms the basis for the mosaic-skeleton method [3] or the $\mathcal{H}$-matrix format [4, 5]. The approximation can be computed only from the entries of the matrix and from the additional geometrical information that induces the partititioning of the matrix into blocks. The simplest partitioning scheme corresponding to the so-called weak admissibility condition [6] is presented on Figure 1, and is usually used for one-dimensional problems.

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Figure 1. Simplest mosaic partitioning. The blocks denoted in white correspond to low-rank matrices.

In the $\mathcal{H}$-matrix format, the blocks are approximated independently of each other, which leads to the logarithmic factor in storage and complexity. The $\mathcal{H}^{2}$-matrix format [7] provides a more effective representation of a matrix based on the analogy with the well-known fast multipole method (FMM) [8]. The approximation is performed by taking into account common information about the column and row bases in the low-rank blocks. This leads to a "nested" representation: we do not need to store the factors for each block, but only the coefficients that allow us to express the bases on the upper level from the previous level bases (so-called transfer matrices) and coefficients of interactions between bases of sources and receivers (so-called interaction matrices). This structure may lead to significant reduction in memory and complexity [9].

If the partitioning of a matrix into blocks is fixed, and the subroutine that allows to compute any prescribed element of a matrix is given, the main problem is to compute the approximation of such a matrix without computing all of its entries (that would give $\mathcal{O}\left(N^{2}\right)$ complexity since the matrix is dense). For the $\mathcal{H}$-matrix format, the problem reduces to the approximation of a low-rank matrix from its entries. A rank- $r$ matrix can be exactly recovered from $r$ columns and rows using skeleton decomposition [10] and in the approximate case, the existence of a quasioptimal skeleton decomposition can be proved based on the maximal volume principle [11]. In practice, such approximations are computed using cross approximation techniques $[1,2]$.

It is natural to consider a question, whether the $\mathcal{H}^{2}$-matrix representation can be computed directly from the entries of a matrix in a quasi-optimal cost (i.e. the number of sampled elements is close to the number of parameters in the representation). This construction can be done efficiently using interpolation [12], but it is not a purely algebraic approach. In [13] a nested cross approximation was proposed that uses only the entries of the matrix. Two methods were described, ACAGEO and ACAMERGE, which rely on different sampling strategies for the far zone (in other words, selection of the representing sets). This selection was not adaptive for ACAGEO, and for ACAMERGE, as we will see in numerical experiments, it may lead to significant loss of accuracy. We propose a new purely algebraic method for the adaptive selection of the representing sets to compute the $\mathcal{H}^{2}$-matrix approximant from the entries. The algorithm was motivated by two completely different approaches. Its first step comes as a generalization of the classical Barnes-Hut algorithm [14], which can be considered as one of the first algorithms for the fast approximate computation of dense matrix-by-vector products. The obtained approximation is satisfactory
for small accuracies (say, $\varepsilon=10^{-3}$ ) but does not perform well when the approximation threshold goes to zero. To make the method more robust, we establish the analogy between $\mathcal{H}^{2}$-matrix approximation and multidimensional tensors, and apply the ideas from the cross approximation of tensors [15] to our problem. The top-to-bottom and bottom-to-top tree traversal algorithm, proposed in this paper, is similar to the left-right sweeps in the TTcross algorithm of [15]. By numerical experiments we show that the resulting approach is robust in the sense that it is able to achieve better accuracy in comparison to the ACAGEO and ACAMERGE algorithms of [13].

Related work. The problem of computing the $\mathcal{H}^{2}$-matrix approximation from the entries was first considered in this setting in [13]. Our work follows a similar framework but has several important differences, and the proposed algorithm is more robust for high approximation accuracies. Similar approaches for constructing the approximation were proposed in the kernel-independent fast multipole methods [16], but they still use additional information from the problem. Different wavelet-based methods [17, 18], that proved to have superior performance over hierarchical matrices, suffer from a similar problems: they require additional information from the problem and require the computation of Galerkin elements with wavelet basis function. Some other approaches consider "hierarchically semiseparable matrices" [19], this format is very close to $\mathcal{H}^{2}$. For this class of matrices (HSS-matrices) randomized algorithm has been proposed to recover the representation from the entries [20]. However, the HSS representation is essentially a one-dimensional $\mathcal{H}^{2}$-matrix, and the randomized algorithm of [20] also requires a fast matrix-by-vector procedure to be available.

We organize our paper as follows. In Section 2 we gather all necessary mathematical tools to work with $\mathcal{H}^{2}$-matrices. In Section 3 we derive the MCBH representation (multicharge Barnes-Hut) for the $\mathcal{H}^{2}$-matrix that is main approximation ansatz, and propose a simple algorithm to compute it. In Section 4 the connection with good submatrices is discussed. In Section 5 the main improvement of the algorithm is proposed. In Section 6 we provide numerical experiments. And, finally, In Section 7 we compare our iterative approach, proposed in Section 5, with methods from [13].

## 2. BASIC NOTATIONS AND DEFINITIONS

A detailed review can be found in the books [7,9]. Here we give a summary of our notations.


Figure 2. Examples of mosaic partitioning in one and two dimensions.

To define the $\mathcal{H}^{2}$-matrix format, we need several standard definitions. The main idea is to split a matrix into large blocks that are approximately of low rank. The matrix element $A_{i j}$ describes interaction between the $i$-th receiver $x_{i}$ and $j$-th source $y_{j}$. Typical examples are function-related matrices $A_{i j}=f\left(x_{i}, y_{j}\right)$ and integral operators, discretized by collocation, Nyström or Galerkin methods with basis functions with local support. The whole set of receivers will be denoted by indices $\mathbf{I}$ and the whole set of sources will be denoted by indices J. For simplicity, we will consider only binary trees, but the extension to an arbitrary number of sons is trivial.

Definition 2.1 (Row and column cluster trees)
Since sources, denoted by J, are presented by columns and receivers, denoted by I, are presented by rows of matrix $A$, let call $\mathcal{T}_{\mathbf{I}}$ as row cluster tree and $\mathcal{T}_{\mathbf{J}}$ as column cluster tree, if

1) I is the root of $\mathcal{T}_{\mathbf{I}}$ and $\mathbf{J}$ is the root of $\mathcal{T}_{\mathbf{J}}$,
2) If $\mathbf{t} \in \mathcal{T}_{\mathbf{I}}$ then $\mathbf{t}$ is a disjoint union of its sons $\mathbf{t}_{\mathbf{1}} \in \mathcal{T}_{\mathbf{I}}$ and $\mathbf{t}_{\mathbf{2}} \in \mathcal{T}_{\mathbf{I}}$, if $\mathbf{t} \in \mathcal{T}_{\mathbf{J}}$ then $\mathbf{t}$ is a disjoint union of its sons $\mathbf{t}_{\mathbf{1}} \in \mathcal{T}_{\mathbf{J}}$ and $\mathbf{t}_{\mathbf{2}} \in \mathcal{T}_{\mathbf{J}}$ :

$$
\operatorname{sons}(\mathbf{t})=\left\{\mathbf{t}_{\mathbf{1}}, \mathbf{t}_{\mathbf{2}}\right\}
$$

The cluster trees are constructed for the rows and the columns of the matrix and are typically based on a certain geometrical construction (i.e., kd-trees, quadtrees and many others). The construction of the partition $\mathbf{P}$ of a given matrix into low-rank blocks is based on the admissibility condition. The standard form of the admissibility condition is also geometrical. A set of receivers $\mathbf{Y}$ and a set of sources $\mathbf{X}$ are said to be admissible, if

$$
\max \{\operatorname{diam} \mathbf{Y}, \operatorname{diam} \mathbf{X}\} \leq \eta \operatorname{dist}(\mathbf{Y}, \mathbf{X})
$$

where $\eta \geq 0$ (it is equal to 0 for Figures 1 and 2). In this case it can be proven that for a class of asymptotically smooth kernels [2] the corresponding block can be well-approximated by a low-rank matrix. This condition is simple to check, and the recursive partition is typically generated by a recursive procedure. It starts from the roots of the cluster trees. If for a given block an admissibility condition is satisfied, the block $(\mathbf{t}, \mathbf{s})$ is added to the partition $\mathbf{P}$. Otherwise, all sons of $\mathbf{t}$ and all sons of $\mathbf{s}$ are processed recursively. This process creates a list of admissible blocks ( $\mathbf{t}, \mathbf{s}$ ).

Definition 2.2 $\left(\chi_{\mathbf{t}}, \chi_{\mathbf{s}}\right)$
Let $\mathbf{t} \in \mathcal{T}_{\mathbf{I}}$, then $\chi_{\mathbf{t}}$ is a diagonal matrix with the following property: $\mathbf{i}$-th diagonal element is 1 if the row $\mathbf{i} \in \mathbf{t}$ and 0 otherwise. Let $\mathbf{s} \in \mathcal{T}_{\mathbf{J}}$, then $\chi_{\mathbf{s}}$ is a diagonal matrix with the following property: $\mathbf{i}$-th diagonal element is 1 if the column $\mathbf{i} \in \mathbf{s}$ and 0 otherwise.

For any pair of admissible nodes $\mathbf{t}$ and $\mathbf{s}$, their corresponding submatrix $A(\mathbf{t}, \mathbf{s})$ can be written as:

$$
A(\mathbf{t}, \mathbf{s})=\chi_{\mathbf{t}} A \chi_{\mathbf{s}}
$$

Definition $2.3\left(\mathbf{S}_{\mathbf{t}}, \mathbf{S}_{\mathbf{s}}\right)$
Assume $\mathbf{t}$ is a node of row cluster tree $\mathcal{T}_{\mathbf{I}}$. Then "set of admissible nodes", $\mathbf{S}_{\mathbf{t}}=\left\{\mathbf{s}_{\mathbf{0}}, \ldots, \mathbf{s}_{\mathbf{k}}\right\}$,
is a set of nodes of a column cluster tree $\mathcal{T}_{\mathbf{J}}$, so that each pair $\left(\mathbf{t}, \mathbf{s}_{\mathbf{i}}\right)$ is admissible if and only if $\mathbf{s}_{\mathbf{i}} \in \mathbf{S}_{\mathbf{t}}$. Similarly, $\mathbf{S}_{\mathbf{s}}$ is a set of admissible nodes for node $\mathbf{s} \in \mathcal{T}_{\mathbf{J}}$.

Definition $2.4\left(\mathbf{R}_{\mathbf{t}}^{*}, \mathbf{C}_{\mathbf{s}}^{*}\right)$
Assume $\mathbf{t}$ is a node of row cluster tree $\mathcal{T}_{\mathbf{I}}$ with a corresponding set of admissible nodes $\mathbf{S}_{\mathbf{t}}$. An admissible block row $\mathbf{R}_{\mathbf{t}}^{*}$ is defined as:

$$
\mathbf{R}_{\mathbf{t}}^{*}=\chi_{\mathbf{t}} A \sum_{\mathbf{s} \in \mathbf{S}_{\mathbf{t}}} \chi_{\mathbf{s}}
$$

Similarly, we define admissible block column $\mathbf{C}_{\mathbf{s}}^{*}$ as:

$$
\mathbf{C}_{\mathbf{s}}^{*}=\left(\sum_{\mathbf{t} \in \mathbf{S}_{\mathbf{s}}} \chi_{\mathrm{t}}\right) A \chi_{\mathbf{s}} .
$$

Definition $2.5(\operatorname{pred}(\mathbf{t}))$
Assume $\mathbf{t}=\mathbf{t}_{\mathbf{0}}$ and $\left\{\mathbf{t}_{\mathbf{1}}, \ldots, \mathbf{t}_{\mathbf{k}}\right\}$ is a set of nodes of cluster tree $\mathcal{T}_{\mathbf{I}}$ or $\mathcal{T}_{\mathbf{J}}$, such that $\mathbf{t}_{\mathbf{k}}$ is the root of tree, and $\mathbf{t}_{\mathbf{k}-\mathbf{i}}$ is the parent of $\mathbf{t}_{\mathbf{k}-\mathbf{i}-\mathbf{1}}$ for each $\mathbf{i}=0, \ldots, \mathbf{k}-\mathbf{1}$. Then, the set $\left\{\mathbf{t}_{\mathbf{1}}, \ldots, \mathbf{t}_{\mathbf{k}}\right\}$ is called "predecessors" of the node $\mathbf{t}$ and can be written as

$$
\operatorname{pred}(\mathbf{t})=\left\{\mathbf{t}_{\mathbf{1}}, \ldots, \mathbf{t}_{\mathbf{k}}\right\} .
$$

Definition $2.6\left(\mathbf{R}_{\mathbf{t}}^{+}, \mathbf{C}_{\mathbf{s}}^{+}, \mathbf{R}_{\mathbf{t}}, \mathbf{C}_{\mathbf{s}}\right)$
Assume $\mathbf{t}=\mathbf{t}_{\mathbf{0}}$ and $\left\{\mathbf{t}_{\mathbf{1}}, \ldots, \mathbf{t}_{\mathbf{k}}\right\}=\boldsymbol{\operatorname { p r e d }}(\mathbf{t})$ are predecessors of the node $\mathbf{t} \in \mathcal{T}_{\mathbf{I}} .\left\{\mathbf{S}_{\mathbf{0}}, \ldots, \mathbf{S}_{\mathbf{k}}\right\}$ are sets of admissible nodes corresponding to $\left\{\mathbf{t}_{\mathbf{0}}, \ldots, \mathbf{t}_{\mathbf{k}}\right\}$. Then $\mathbf{R}_{\mathbf{t}}^{+}$and $\mathbf{R}_{\mathbf{t}}$ are defined as:

$$
\begin{gathered}
\mathbf{R}_{\mathbf{t}}^{+}=\chi_{\mathbf{t}} A \sum_{\mathrm{i}=1}^{\mathbf{k}} \sum_{\mathbf{s} \in \mathbf{S}_{\mathbf{i}}} \chi_{\mathbf{s}}, \\
\mathbf{R}_{\mathbf{t}}=\mathbf{R}_{\mathrm{t}}^{+}+\mathbf{R}_{\mathbf{t}}^{*}=\chi_{\mathbf{t}} A \sum_{\mathbf{i}=0}^{\mathbf{k}} \sum_{\mathbf{s} \in \mathbf{S}_{\mathbf{i}}} \chi_{\mathbf{s}},
\end{gathered}
$$

where $\mathbf{R}_{\mathbf{t}}$ is called "block row". As for $\mathbf{s}_{\mathbf{0}}=\mathbf{s} \in \mathcal{T}_{\mathbf{J}}$, we define the following:

$$
\begin{gathered}
\left\{\mathbf{s}_{1}, \ldots, \mathbf{s}_{\mathbf{k}}\right\}=\operatorname{pred}(\mathbf{s}), \\
\mathbf{S}_{\mathbf{i}}=\mathbf{S}_{\mathbf{s}_{\mathbf{i}}}, \\
\mathbf{C}_{\mathbf{s}}^{+}=\left(\sum_{\mathbf{i}=1}^{\mathbf{k}} \sum_{\mathbf{t} \in \mathbf{S}_{\mathbf{i}}} \chi_{\mathbf{t}}\right) A \chi_{\mathbf{s}}, \\
\mathbf{C}_{\mathbf{s}}=\mathbf{C}_{\mathbf{s}}^{+}+\mathbf{C}_{\mathbf{s}}^{*}=\left(\sum_{\mathbf{i}=0}^{\mathbf{k}} \sum_{\mathbf{t} \in \mathbf{S}_{\mathbf{i}}} \chi_{\mathbf{t}}\right) A \chi_{\mathbf{s}},
\end{gathered}
$$

where $\mathbf{C}_{\mathbf{s}}$ is called "block column".

All kinds of block rows are shown on Figure 3.


Figure 3. Different block rows for node $\mathbf{t}$. $\mathbf{R}_{\mathbf{t}}, \mathbf{R}_{\mathbf{t}}^{+}, \mathbf{R}_{\mathbf{t}}^{*}$.

If $\mathbf{t} \in \mathcal{T}_{\mathbf{I}}$ is a nonleaf node with $\operatorname{sons}(\mathbf{t})=\left\{\mathbf{t}_{\mathbf{1}}, \mathbf{t}_{\mathbf{2}}\right\}$, then block row $\mathbf{R}_{\mathbf{t}}$ can be computed through block rows of its sons:

$$
\mathbf{R}_{\mathbf{t}}=\mathbf{R}_{\mathrm{t}_{1}}^{+}+\mathbf{R}_{\mathrm{t}_{2}}^{+} .
$$

## 3. NESTED CROSS APPROXIMATION

In the following, we will always assume exact low-rank property for simplicity, however all the derivation is true for approximate low-rank case as well. In this case, the equality sign should be replaced by approximate equality sign. The $\mathcal{H}^{2}$-matrix structure has a simple algebraic characterization: all block rows and columns have bounded rank. From this property, we can derive a low-rank representation based on the skeleton decomposition [21]. Recall, that the skeleton decomposition for a low-rank matrix $A$ has the following form:

$$
A=C \hat{A}^{-1} R,
$$

where $C$ consists of basis columns of $A, R$ consists of basis rows of $A$ and $\hat{A}$ is the submatrix of $A$ on the intersection of basis rows and columns. Simple modification of this formula gives us formula for approximating entire matrix with its basis rows or basis columns:

$$
A=\tilde{C} R=C \tilde{R},
$$

with $\tilde{C}=C \hat{A}^{-1}$ and $\tilde{R}=\hat{A}^{-1} R$.


Figure 4. Skeleton decomposition of matrix $A$ by basis rows $R$ and basis columns $C$.

From here, we assume matrix $A$ is a $\mathcal{H}^{2}$-matrix and every block row and block column is a low-rank matrix and all following equations are written as equalities (for convenience). Since any matrix from real problem can only be close to $\mathcal{H}^{2}$-matrix (each block column and block row is $\varepsilon$-low-rank), we refer to error analysis from [13].

Let assume, that $\mathbf{t}$ is a nonleaf node of row cluster tree $\mathcal{T}_{\mathbf{I}}$ with sons $\mathbf{t}_{\mathbf{1}}$ and $\mathbf{t}_{\mathbf{2}}$. From the skeleton decomposition for block rows $\mathbf{R}_{\mathbf{t}}, \mathbf{R}_{\mathbf{t}_{1}}$ and $\mathbf{R}_{\mathbf{t}_{\mathbf{2}}}$ we get the following formulas for $\mathbf{R}_{\mathbf{t}}, \mathbf{R}_{\mathbf{t}_{1}}^{+}$and $\mathbf{R}_{\mathbf{t}_{\mathbf{2}}}^{+}$:

$$
\begin{gathered}
\mathbf{R}_{\mathbf{t}}=U_{\mathbf{t}} \hat{R}_{\mathbf{t}}^{-1} V_{\mathbf{t}} \\
\mathbf{R}_{\mathbf{t}_{\mathbf{1}}}^{+}=U_{\mathbf{t}_{\mathbf{1}}}\left(\hat{R}_{\mathbf{t}_{\mathbf{1}}}^{+}\right)^{-1} V_{\mathbf{t}_{\mathbf{1}}} \\
\mathbf{R}_{\mathbf{t}_{\mathbf{2}}}^{+}=U_{\mathbf{t}_{\mathbf{2}}}\left(\hat{R}_{\mathbf{t}_{\mathbf{2}}}^{+}\right)^{-1} V_{\mathbf{t}_{\mathbf{2}}}
\end{gathered}
$$

where $V_{\mathbf{t}}, V_{\mathbf{t}_{\mathbf{1}}}$ and $V_{\mathbf{t}_{\mathbf{2}}}$ are based on basis rows of block rows $\mathbf{R}_{\mathbf{t}}, \mathbf{R}_{\mathbf{t}_{\mathbf{1}}}$ and $\mathbf{R}_{\mathbf{t}_{\mathbf{2}}}$ correspondingly. Since $\mathbf{R}_{\mathbf{t}}=\mathbf{R}_{\mathbf{t}_{\mathbf{1}}}^{+}+\mathbf{R}_{\mathbf{t}_{\mathbf{2}}}^{+}$, block row $\mathbf{R}_{\mathbf{t}}$ can be rewritten as

$$
\mathbf{R}_{\mathbf{t}}=U_{\mathbf{t}_{\mathbf{1}}} \hat{R}_{\mathbf{t}_{\mathbf{1}}}^{-1} V_{\mathbf{t}_{\mathbf{1}}}+U_{\mathbf{t}_{\mathbf{2}}} \hat{R}_{\mathbf{t}_{\mathbf{2}}}^{-1} V_{\mathbf{t}_{\mathbf{2}}}=\left[\begin{array}{ll}
U_{\mathbf{t}_{\mathbf{1}}}\left(\hat{R}_{\mathbf{t}_{\mathbf{1}}}^{+}\right)^{-1} & U_{\mathbf{t}_{\mathbf{2}}}\left(\hat{R}_{\mathbf{t}_{\mathbf{2}}}^{+}\right)^{-1}
\end{array}\right]\left[\begin{array}{l}
V_{\mathbf{t}_{1}} \\
V_{\mathbf{t}_{\mathbf{2}}}
\end{array}\right]
$$

Since $V_{\mathbf{t}_{\mathbf{1}}}$ and $V_{\mathbf{t}_{\mathbf{2}}}$ are rows of matrix $\mathbf{R}_{\mathbf{t}}$, they can be obtained via skeleton decomposition of $\mathbf{R}_{\mathbf{t}}$ :

$$
\left[\begin{array}{l}
V_{\mathbf{t}_{1}} \\
V_{\mathbf{t}_{2}}
\end{array}\right]=\hat{U}_{\mathbf{t}} \hat{R}_{\mathbf{t}}^{-1} V_{\mathbf{t}}
$$

Matrix $\hat{U}_{\mathbf{t}} \hat{R}_{\mathbf{t}}^{-1}$ is a local transfer matrix from basis rows of block row $\mathbf{R}_{\mathbf{t}}$ to basis rows of block rows $\mathbf{R}_{\mathbf{t}_{\mathbf{1}}}$ and $\mathbf{R}_{\mathbf{t}_{\mathbf{2}}}$. Here and later we mark it as $M_{\mathbf{t}}$. Matrices $U_{\mathbf{t}_{\mathbf{1}}} \hat{R}_{\mathbf{t}_{1}}^{-1}$ and $U_{\mathbf{t}_{\mathbf{2}}} \hat{R}_{\mathbf{t}_{\mathbf{2}}}^{-1}$ are global transfer matrices from basises of block rows $\mathbf{R}_{\mathbf{t}_{1}}$ and $\mathbf{R}_{\mathbf{t}_{\mathbf{2}}}$ to all corresponding rows. They are marked as $P_{\mathbf{t}_{1}}$ and $P_{\mathbf{t}_{\mathbf{2}}}$. Within new notations, we can rewrite low-rank decomposition of block row $\mathbf{R}_{\mathbf{t}}$ :

$$
\mathbf{R}_{\mathbf{t}}=\left[\begin{array}{ll}
P_{\mathbf{t}_{1}} & P_{\mathbf{t}_{2}}
\end{array}\right] M_{\mathbf{t}} V_{\mathbf{t}}
$$

For the following argumentation, we will need additional definition:
Definition $3.1\left(\hat{\chi}_{\mathbf{t}}, \hat{\chi}_{\mathbf{s}}\right)$
Let $\hat{\chi}_{\mathbf{t}}$ for $\mathbf{t} \in \mathcal{T}_{\mathbf{I}}$ be a diagonal matrix with the following property: $\mathbf{i}$-th diagonal element is 1 if the row $\mathbf{i}$ is in the set of basis rows for the block row $\mathbf{R}_{\mathbf{t}}$ and 0 otherwise. Let $\hat{\chi}_{\mathbf{s}}$ for
$\mathbf{s} \in \mathcal{T}_{\mathbf{J}}$ be a diagonal matrix with the following property: $\mathbf{i}$-th diagonal element is 1 if the column $\mathbf{i}$ is in the set of basis columns for the block column $\mathbf{C}_{\mathbf{s}}$ and 0 otherwise.

For any admissible pair of nodes $(\mathbf{t}, \mathbf{s})$, corresponding submatrix $A(\mathbf{t}, \mathbf{s})$ can be written as follows:

$$
A(\mathbf{t}, \mathbf{s})=\mathbf{R}_{\mathbf{t}} \chi_{\mathbf{s}}=P_{\mathbf{t}} V_{\mathbf{t}} \chi_{\mathbf{s}}=P_{\mathbf{t}} \hat{\chi}_{\mathbf{t}} A \chi_{\mathbf{s}}=P_{\mathbf{t}} \hat{\chi}_{\mathbf{t}} A(\mathbf{t}, \mathbf{s})
$$

where $P_{\mathrm{t}}$ is a global transfer matrix and $V_{\mathrm{t}}$ is a matrix, based on basis rows of $\mathbf{R}_{\mathbf{t}}$. Since $\mathbf{C}_{\mathbf{s}}=U_{\mathbf{s}} P_{\mathbf{s}}$, where $U_{\mathbf{s}}$ is a matrix, based on basis columns of $\mathbf{C}_{\mathbf{s}}$, and $P_{\mathbf{s}}$ is a global transfer matrix, we also get

$$
A(\mathbf{t}, \mathbf{s})=A(\mathbf{t}, \mathbf{s}) \hat{\chi}_{\mathbf{s}} P_{\mathbf{s}} .
$$

Transforming two equations into one:

$$
A(\mathbf{t}, \mathbf{s})=P_{\mathbf{t}} \hat{\chi}_{\mathbf{t}} A(\mathbf{t}, \mathbf{s}) \hat{\chi}_{\mathbf{s}} P_{\mathbf{s}} .
$$

Finally, we get:

$$
A(\mathbf{t}, \mathbf{s})=P_{\mathbf{t}} \hat{A}(\mathbf{t}, \mathbf{s}) P_{\mathbf{s}},
$$

where $\hat{A}(\mathbf{t}, \mathbf{s})$ is a submatrix, based on basis rows of $\mathbf{R}_{\mathbf{t}}$ and basis columns of $\mathbf{C}_{\mathbf{s}}$. Taking into account that $P_{\mathbf{t}}$ and $P_{\mathbf{s}}$ can be computed recursively, we get a $\mathcal{H}^{2}$-type factorization.

First and most simple idea, proposed in the article, is to choose basis rows for each block row $\mathbf{R}_{\mathbf{t}}$ hierarchically: use $\hat{\chi}_{\mathbf{t}_{1}}$ and $\hat{\chi}_{\mathbf{t}_{\mathbf{2}}}$ to obtain $\hat{\chi}$, where $\mathbf{t}_{\mathbf{1}}$ and $\mathbf{t}_{\mathbf{2}}$ are sons of the node $\mathbf{t}$. Technically, we can write it as follows:

$$
\hat{\chi}_{\mathbf{t}_{1}} \mathbf{R}_{\mathbf{t}}+\hat{\chi}_{\mathbf{t}_{2}} \mathbf{R}_{\mathbf{t}}=M_{\mathrm{t}} \hat{\chi}_{\mathrm{t}} \mathbf{R}_{\mathrm{t}}
$$

Obviously, matrices $M_{\mathbf{t}}$ and $\hat{\chi}_{\mathbf{t}}$ can be computed with help of skeleton decomposition. The same holds true for basis columns of each block column $\mathbf{C}_{\mathbf{s}}$ :

$$
\mathbf{C}_{\mathbf{s}} \hat{\chi}_{\mathbf{s}_{1}}+\mathbf{C}_{\mathbf{s}} \hat{\chi}_{\mathbf{s}_{2}}=\mathbf{C}_{\mathbf{s}} \hat{\chi}_{\mathbf{s}} M_{\mathbf{s}} .
$$

Since all basises are chosen hierarchically, this kind of $\mathcal{H}^{2}$-decomposition can be named as nested cross approximation. In assumption, we generalize proposed proto-method for row cluster tree $\mathcal{T}_{\mathbf{I}}$ in Algorithm 1, which can be easily adapted for column cluster tree $\mathcal{T}_{\mathbf{J}}$.

```
Algorithm 1 Computation of basis rows and transfer matrices for each node of row cluster
tree
Require: Row cluster tree \(\mathcal{T}_{\mathbf{I}}\), block row \(\mathbf{R}_{\mathbf{t}}\) for each node \(\mathbf{t} \in \mathcal{T}_{\mathbf{I}}\).
Ensure: nested basises \(\hat{\chi}_{\mathbf{t}}\) and transfer matrices \(M_{\mathbf{t}}\) for each node of \(\mathcal{T}_{\mathbf{I}}\).
    for each \(\mathbf{t} \in \mathcal{T}_{\mathbf{I}}\) \{from bottom of the row cluster tree to top\} do
        \{define auxiliary matrix \(\hat{R}\) to compute basis rows and transfer matrix\}
        if \(\operatorname{sons}(\mathbf{t})\) is empty then
            \(\hat{R}=\mathbf{R}_{\mathbf{t}}\)
        else \(\left\{\boldsymbol{\operatorname { s o n s }}(\mathbf{t})\right.\) is not empty, \(\left.\left\{\mathbf{t}_{\mathbf{1}}, \mathbf{t}_{\mathbf{2}}\right\}=\boldsymbol{\operatorname { s o n s }}(\mathbf{t})\right\}\)
            \(\hat{R}=\left(\hat{\chi}_{\mathbf{t}_{1}}+\hat{\chi}_{\mathbf{t}_{2}}\right) \mathbf{R}_{\mathbf{t}}\)
        end if
        \(\left\{\right.\) compute basis rows \(\hat{\chi}_{\mathbf{t}}\) and transfer matrix \(M_{\mathbf{t}}\) \}
        \(\hat{R}=M_{\mathbf{t}} \hat{\chi}_{\mathbf{t}} \hat{R}\) \{obviously, \(\left.\hat{\chi}_{\mathbf{t}} \hat{R}=\hat{\chi}_{\mathbf{t}} \mathbf{R}_{\mathbf{t}}\right\}\)
    end for
```


## 4. MAXIMUM VOLUME PRINCIPLE

Main idea of the nested cross approximation is to select basis rows and basis columns for special auxiliary matrices ( $\hat{R}$ from the Algorithm 1). If each block row has precise rank it does not matter what rows we choose for basises. Problem arises when block rows are close to low rank matrices. Skeleton approximation

$$
A \approx C \hat{A}^{-1} R
$$

is about choosing submatrix $\hat{A}$. If it has very small volume (modulus of determinant), it is very close to singular and approximation error is close to infinity. Guaranteed approximation error can be obtained with help of maximum volume principle [10]: if $\hat{A}$ is a submatrix of maximum volume (modulus of determinant) amongst all $r \times r$ submatrices, following estimation holds true:

$$
\left\|A-C \hat{A}^{-1} R\right\|_{C} \leq(r+1) \sigma_{r+1},
$$

where $\sigma_{r+1}$ is a $r+1$ singular value of matrix $A$ and $\|\cdot\|_{C}$ is an infinite or Chebyshev norm (maximum in modulus element in matrix).

Finding maximum volume submatrix has exponential complexity, but suboptimal submatrices can be found in a fast way by using greedy algorithms. First of all, we can find a good submatrix in a low rank matrix, close to $A$ :

$$
A \approx U V^{T}+E, A \in \mathbb{C}^{n \times m}, E \in \mathbb{C}^{n \times m}, U \in \mathbb{C}^{n \times r}, V \in \mathbb{C}^{m \times r}
$$

Since

$$
\operatorname{det}(X Y)=\operatorname{det}(X) \operatorname{det}(Y)
$$

for any given square matrices $X$ and $Y$, computation of good submatrix in $A$ requires computation of $r$ good rows in matrix $U$ and $r$ good rows in matrix $V$. Recall, that our
initial problem was to define basis rows for block rows and basis columns for block columns, so we need either left factor or right factor. So without prejudice to the generality, we need to find good $r \times r$ submatrix in a $n \times r$ matrix. Following Algorithm 2, maximum-volume algorithm [22] (which we call maxvol), solves this problem.

Suppose we need to find a good $r \times r$ submatrix in a matrix $A \in \mathbb{C}^{n \times r}$, with $n>r$. First step of maxvol algorithm is to find a nonsingular submatrix of matrix $A$, so we use the LU decomposition with row pivoting, which requires $O\left(n r^{2}\right)$ operations. In other words, initialization gives us a submatrix $\hat{A} \in \mathbb{C}^{r \times r}$ and coefficients $C \in \mathbb{C}^{n \times r}$ such that $A=C \hat{A}$. Each iteration of the algorithm swaps two rows in order to increase the volume of the submatrix. As shown in [22] each iteration requires only $O(n r)$ operations since it finds absolute maximum element in $C$ and applies rank-1 update to it. In practice the algorithm converges very fast, thus So, we can estimate the complexity of maxvol algorithm as $O\left(n r^{2}\right)$ operations.

```
Algorithm 2 maxvol [22] algorithm
Require: Nonsingular matrix \(A \in \mathbb{C}^{n \times r}, n>r\)
Ensure: Good submatrix \(\hat{A} \in \mathbb{C}^{r \times r}\) and coefficients \(C \in \mathbb{C}^{n \times r}\) such, that \(A=C \hat{A}\)
    Find nonsingular \(r \times r\) submatrix \(\hat{A}\) in matrix \(A\)
    \{i.e. by pivots from LU factorization \(\}\)
    \(C \leftarrow A \hat{A}^{-1},\{i, j\} \leftarrow \operatorname{argmax}(|C|)\)
    \(\{|C|\) means element-wise modulus, \(i\) and \(j\) are row and column numbers of maximum
    in modulus element in \(C\}\)
    while \(\left|C_{i j}\right|>1\) do
        \(\left\{C_{i j}\right.\) is element on intersection of \(i\)-th row and \(j\)-th column of \(\left.C\right\}\)
        \(\hat{A}_{j} \leftarrow A_{i}\)
        \(\left\{A_{i}\right.\) is \(i\)-th row of \(A, \hat{A}_{j}\) is \(j\)-th row of \(\left.\hat{A}\right\}\)
        \(C \leftarrow C-C_{j}\left(C_{i}-e_{j}\right) / C_{i j}\)
        \(\left\{C_{i}\right.\) is \(i\)-th row and \(C_{j}\) is \(j\)-th column of \(C, e_{j}\) is \(j\)-th row of identity matrix of size
        \(r \times r\}\)
        \(\{i, j\} \leftarrow \operatorname{argmax}(|C|)\)
    end while
    return \(C, \hat{A}\)
```


## 5. PRACTICAL ALGORITHM FOR NESTED CROSS APPROXIMATION

Main disadvantage of the Algorithm 1 is computational cost. Auxiliary matrix $\hat{R}$ of each leaf node $\mathbf{t} \in \mathcal{T}_{\mathbf{I}}$ contains $n_{\mathbf{t}}$ (number of receivers, corresponding to node $\mathbf{t}$ ) nonzero rows and $N_{\mathbf{t}}$ (number of sources in far zone of $\mathbf{t}$ or $\operatorname{pred}(\mathbf{t})$ ) nonzero columns. Since $\mathbf{t}$ is a nonleaf node, $N_{\mathbf{t}}$ is close to number of sources. Obviously, minimum number of operations to compute basis rows for node $\mathbf{t}$ is a multiplication of $n_{\mathbf{t}}$ and $N_{\mathbf{t}}$. Summing it for all leaf nodes we get a $O(N M)$ complexity, where $N$ is number of sources and $M$ is number of receivers. So we need to reduce the number of nonzero columns of $\hat{R}$, i.e. to select a certain representing set.

Let assume that we already have representing sets for each node of $\mathcal{T}_{\mathbf{I}}$ and $\mathcal{T}_{\mathbf{J}}$. It is again convenient to use diagonal matrices to work with representing sets.

Definition $5.1\left(\psi_{\mathbf{t}}, \psi_{\mathbf{s}}\right)$
Let $\psi_{\mathbf{t}}$ be a diagonal matrix with the following property: $\mathbf{i}$-th diagonal element is 1 if the column $\mathbf{i}$ is in the representing set for the block row $\mathbf{R}_{\mathbf{t}}$ and 0 otherwise. Let $\psi_{\mathbf{s}}$ be a diagonal matrix with the following property: $\mathbf{i}$-th diagonal element is 1 if the row $\mathbf{i}$ is in the representing set for the block column $\mathbf{C}_{\mathbf{s}}$ and 0 otherwise.

If a good representing set for each block row $\mathbf{R}_{\mathbf{t}}$ is already known, then basis rows and transfer matrix for each block row can be computed with help of a small matrix $\hat{R} \psi_{\mathbf{t}}$ instead of $\hat{R}$. Maxvol algorithm 2 can be used to find basis rows or columns together with transfer matrices in an efficient way. Finally, we get a prototype Algorithm 3.

```
Algorithm 3 Computation of basis rows and transfer matrices for row cluster tree with given
representing sets.
Require: Row cluster tree \(\mathcal{T}_{\mathbf{I}}\), block row \(\mathbf{R}_{\mathbf{t}}\) and representing set \(\psi_{\mathbf{t}}\) for each node \(\mathbf{t} \in \mathcal{T}_{\mathbf{I}}\),
    accuracy parameter \(\varepsilon\).
Ensure: basis rows \(\hat{\chi}_{\mathbf{t}}\) and transfer matrix \(M_{\mathbf{t}}\) for each block row \(\mathbf{R}_{\mathbf{t}}\).
    for \(\mathbf{t} \in \mathcal{T}_{\mathbf{I}}\{\) from bottom of the row cluster tree to top\} do
        \{define auxiliary matrix \(\hat{R}\) to compute basis rows and transfer matrix\}
        if \(\operatorname{sons}(\mathbf{t})\) is empty then
            \(\hat{R}=\mathbf{R}_{\mathbf{t}} \psi_{\mathbf{t}}\)
        else \(\left\{\boldsymbol{\operatorname { s o n s }}(\mathbf{t})\right.\) is not empty, \(\left.\left\{\mathbf{t}_{\mathbf{1}}, \mathbf{t}_{\mathbf{2}}\right\}=\boldsymbol{\operatorname { s o n s }}(\mathbf{t})\right\}\)
            \(\hat{R}=\left(\hat{\chi}_{\mathbf{t}_{\mathbf{1}}}+\hat{\chi}_{\mathbf{t}_{\mathbf{2}}}\right) \mathbf{R}_{\mathbf{t}} \psi_{\mathbf{t}}\)
        end if
        \(\{\) truncated svd + maxvol \(\}\)
        \(U, S, V=\operatorname{svd}(\hat{R}\), tol \(=\varepsilon) ; M_{\mathbf{t}}, \hat{\chi}_{\mathbf{t}}=\operatorname{maxvol}(U)\)
    end for
```


### 5.1. Partially fixed representing sets

The main problem is how to select these representing sets adaptively. Previous approaches $[13,16,23]$ used geometrical constructions. In this paper we propose a purely algebraical method: each representing set is divided into self and predecessors parts, each of which is calculated separately. First of all, assume that predecessors part of each representing set is predefined, then to find rather good self part of representing set for a node $\mathbf{t}$, we can use basis rows or columns of all nodes in $\mathbf{S}_{\mathbf{t}}$ or $\boldsymbol{\operatorname { s o n s }}\left(\mathbf{S}_{\mathbf{t}}\right)$ if these basises are already defined. This leads us to a level-by-level algorithm: start on the bottom of trees $\mathcal{T}_{\mathbf{I}}$ and $\mathcal{T}_{\mathbf{J}}$, calculate basis columns for each node of the $\mathcal{T}_{\mathbf{J}}$ on current level of the tree, calculate basis rows for each node of the $\mathcal{T}_{\mathbf{I}}$ on the same level and then go up one level. Under assumption of fixed predecessors part $\psi_{\mathbf{t}}^{p}$ of each representing set, we summarize level-by-level idea in Algorithm 4.

```
Algorithm 4 Computation of basis rows/columns and transfer matrices for each node of
rows/columns cluster tree with given predecessors part of each representing set.
Require: Cluster trees \(\mathcal{T}_{\mathbf{I}}\) and \(\mathcal{T}_{\mathbf{J}}\), block row/column \(\mathbf{R}_{\mathbf{t}}\left(\mathbf{C}_{\mathbf{t}}\right)\) and predecessors part \(\psi_{\mathbf{t}}^{p}\) of
    representing set for each node \(\mathbf{t}\), accuracy \(\varepsilon\)
Ensure: basis rows/columns \(\hat{\chi}_{\mathbf{t}}\) and transfer matrix \(M_{\mathbf{t}}\) for each node \(t \in \mathcal{T}_{\mathbf{I}}\) or \(\mathbf{t} \in \mathcal{T}_{\mathbf{J}}\)
    \{Assume, that both \(\mathcal{T}_{\mathbf{I}}\) and \(\mathcal{T}_{\mathbf{J}}\) have level_count levels
    for current_level = level_count to 1 \{from bottom of trees to top\} do
        for \(\mathbf{s} \in \mathcal{T}_{\mathbf{J}}\{\) on the current_level \(\}\) do
            \(\hat{\psi}=\psi_{\mathbf{s}}^{p}\{\) initialize \(\hat{\psi}\) with predecessors part \(\}\)
            for \(\mathbf{t} \in \mathbf{S}_{\mathbf{s}}\) do
                if \(\operatorname{sons}(\mathbf{t})\) is empty then
                \(\hat{\psi}+=\hat{\chi}_{\mathbf{t}}\) or \(\hat{\psi}+=\chi_{\mathbf{t}}\) (whether \(\hat{\chi}_{\mathbf{t}}\) is defined or not)
            else \(\left\{\boldsymbol{\operatorname { s o n s }}(\mathbf{t})\right.\) is not empty, \(\left.\left\{\mathbf{t}_{\mathbf{1}}, \mathbf{t}_{\mathbf{2}}\right\}=\boldsymbol{\operatorname { s o n s }}(\mathbf{t})\right\}\)
                \(\hat{\psi}+=\hat{\chi}_{\mathbf{t}_{\mathbf{1}}}+\hat{\chi}_{\mathbf{t}_{\mathbf{2}}}\)
                    end if
            end for
            if \(\operatorname{sons}(\mathbf{s})\) is empty then
                \(\hat{R}=\hat{\psi} \mathbf{C}_{\mathbf{s}}\), in other terms \(\hat{R}=\hat{\psi} A \chi_{\mathbf{s}}\)
            else \(\left\{\operatorname{sons}(\mathbf{s})\right.\) is not empty, \(\left.\left\{\mathbf{s}_{\mathbf{1}}, \mathbf{s}_{\mathbf{2}}\right\}=\boldsymbol{\operatorname { s o n s }}(\mathbf{s})\right\}\)
                    \(\hat{R}=\hat{\psi} \mathbf{C}_{\mathbf{s}}\left(\hat{\chi}_{\mathbf{s}_{1}}+\hat{\chi}_{\mathbf{s}_{\mathbf{2}}}\right)\), in other terms \(\hat{R}=\hat{\psi} A\left(\hat{\chi}_{\mathbf{s}_{1}}+\hat{\chi}_{\mathbf{s}_{\mathbf{2}}}\right)\)
            end if
            \(U, S, V=\operatorname{svd}(\hat{R}\), tol \(=\varepsilon) ; M_{\mathbf{s}}, \hat{\chi}_{\mathbf{s}}=\operatorname{maxvol}(V)\{\) truncated \(\operatorname{svd}+\operatorname{maxvol}\}\)
        end for
        for \(\mathbf{t} \in \mathcal{T}_{\mathbf{I}}\{\) on the current_level \(\}\) do
            \(\hat{\psi}=\psi_{\mathbf{t}}^{p}\) \{initialize \(\hat{\psi}\) with predecessors part \(\}\)
            \(\hat{\psi}+=\sum_{\mathbf{s} \in \mathbf{S}_{\mathbf{t}}} \hat{\chi}_{\mathbf{s}}\)
            if \(\operatorname{sons}(\mathbf{t})\) is empty then
                \(\hat{R}=\mathbf{R}_{\mathbf{t}} \hat{\psi}\), in other terms \(\hat{R}=\chi_{\mathbf{t}} A \hat{\psi}\)
            else \(\left\{\boldsymbol{\operatorname { s o n s }}(\mathbf{t})\right.\) is not empty, \(\left.\left\{\mathbf{t}_{\mathbf{1}}, \mathbf{t}_{\mathbf{2}}\right\}=\boldsymbol{\operatorname { s o n s }}(\mathbf{t})\right\}\)
                \(\hat{R}=\left(\hat{\chi}_{\mathbf{t}_{\mathbf{1}}}+\hat{\chi}_{\mathbf{t}_{\mathbf{2}}}\right) \mathbf{R}_{\mathbf{t}} \hat{\psi}\), in other terms \(\hat{R}=\left(\hat{\chi}_{\mathbf{t}_{\mathbf{1}}}+\hat{\chi}_{\mathbf{t}_{\mathbf{2}}}\right) A \hat{\psi}\)
            end if
            \(U, S, V=\operatorname{svd}(\hat{R}\), tol \(=\varepsilon) ; M_{\mathbf{t}}, \hat{\chi}_{\mathbf{t}}=\operatorname{maxvol}(U)\{\) truncated \(\operatorname{svd}+\operatorname{maxvol}\}\)
        end for
    end for
```


### 5.2. Representing sets and iterations

How to get predecessors part $\psi_{\mathbf{t}}^{p}$ of each representing set? Let assume that all basis rows and columns are given. Obviously, union of basis columns of nodes from $\mathbf{S}_{\mathbf{t}} \cup \mathbf{S}_{\mathbf{p r e d}(\mathbf{t})}$ is a good representing set for block row $\mathbf{R}_{\mathbf{t}}$. If nonleaf node $\mathbf{t}$ has sons $\left\{\mathbf{t}_{\mathbf{1}}, \mathbf{t}_{\mathbf{2}}\right\}=\operatorname{sons}(\mathbf{t})$, then predecessors part of representing set for node $\mathbf{t}_{\mathbf{1}}$ can be set equal to just full representing set of $\mathbf{t}$. This simple idea works perfectly with our previous approach:

1. with a given predecessors part of representing sets, compute basis rows and columns,
2. with given basises, update representing sets,
3. shift newly calculated representing sets to produce predecessors part of representing sets
4. go to step 1 .

Obviously, we get an alternating iterative method of computing basises and representings. If we recalculate representings just as union of corresponding basises, size of each representing set will depend on number of nodes in $\mathbf{S}_{\mathbf{t}} \cup \mathbf{S}_{\mathbf{p r e d}(\mathbf{t})}$. This problem can be easily solved by additional resampling in top-to-bottom fashion, proposed in Algorithm 5.

```
Algorithm 5 Computation of full representing sets with given basises
Require: Cluster trees \(\mathcal{T}_{\mathbf{I}}\) and \(\mathcal{T}_{\mathbf{J}}\), basis rows/columns \(\hat{\chi}_{\mathbf{t}}\) for each node \(\mathbf{t}\)
Ensure: Representing set \(\psi_{\mathbf{t}}\) for each node \(\mathbf{t}\)
    \{Assume, that both \(\mathcal{T}_{\mathbf{I}}\) and \(\mathcal{T}_{\mathbf{J}}\) have level_count levels\}
    for current_level \(=1\) to level__count \(\{\) from bottom of trees to top \(\}\) do
        for \(\mathbf{t} \in \mathcal{T}_{\mathbf{I}}\{\) on the current_level \(\}\) do
            \(\left\{\right.\) Initialize \(\hat{\psi}\) as a resampled union of basis rows/columns from \(\left.\mathbf{S}_{\operatorname{pred}(\mathbf{t})}\right\}\)
            if current_level is 1 \{check if node \(\mathbf{t}\) is the root node\} then
                \(\hat{\psi}=0\)
            else \(\{\mathbf{p}\) is the parent node of \(\mathbf{t}\}\)
                \(\hat{\psi}=\psi_{\mathbf{p}}\)
            end if
            \(\hat{\psi}+=\sum_{\mathbf{s} \in \mathbf{S}_{\mathbf{t}}} \hat{\chi}_{\mathbf{s}}\left\{\right.\) Add basis rows/columns from \(\left.\mathbf{S}_{\mathbf{t}}\right\}\)
            \(\psi_{\mathbf{t}}=\operatorname{maxvol}\left(\hat{\chi}_{\mathbf{t}} A \hat{\psi}\right)\left\{\right.\) resample \(\hat{\psi}\) into \(\left.\psi_{\mathbf{t}}\right\}\)
        end for
    end for
```


### 5.3. Iterative Multicharge method

Finally, the proposed MCBH algorithm is an iterative method and is summarized in Algorithm 6. Initialization is very easy: basis rows and columns are empty. Each iteration consists of 3 steps:

1. Obtaining representing sets with Algorithm 5 for each cluster tree,
2. Shifting representing sets into predecessors part of representing sets,
3. Computing basis rows/columns and transfer matrices with Algorithm 4.

Each iteration uses all the information of the previous iteration and increases accuracy. Numerical examples showed, that 1-2 iterations are usually enough to get the desired accuracy.

```
Algorithm 6 Iterative MCBH algorithm.
    Initialize basis rows/columns as empty
    while did not get required accuracy do
        run Algorithm 5 for each cluster tree to get representing sets
        reevaluate representing sets into predecessors part of representing sets
        run Algorithm 4 to obtain new improved basis rows/columns and transfer matrices
    end while
```


### 5.4. Complexity estimates

Since the form of $\mathcal{H}^{2}$-factorization presented in 3 is the same, as in [13], storage and error analysis holds the same as in [13]. Only thing to be analised is complexity. Main approach is formulated as step-by-step algorithm 6 , so we analyse each step separately.

Let assume we have a matrix $A \in \mathbb{C}^{N \times M}$, that is already partitioned into blocks with row cluster tree $\mathcal{T}_{\mathbf{I}}$ and column cluster tree $\mathcal{T}_{\mathbf{J}}$. For simplicity, assume that all the ranks are equal to $r$. Then we have to do the following steps.
Step 1: Initialize basis rows/columns as empty. Obviously, it requires $O\left(\left|\mathcal{T}_{\mathbf{I}}\right|+\left|\mathcal{T}_{\mathbf{J}}\right|\right)$ operations $(|\mathcal{T}|$ is a number of nodes in cluster tree $\mathcal{T})$.
Step 2: run Algorithm 5 for each cluster tree to get representing sets. Algorithm 5 works in a top-to-bottom manner with each node of clusters $\mathcal{T}_{\mathbf{I}}$ and $\mathcal{T}_{\mathbf{J}}$. On the zero iteration (we enumerate iterations from 0), basises are empty, so this step does not require any operations. Assume, that for any other iteration basis size for each node of cluster trees $\mathcal{T}_{\mathbf{I}}$ and $\mathcal{T}_{\mathbf{J}}$ is equal to constant $r$. Also, assume any node $t \in \mathcal{T}_{\mathbf{I}} \cup \mathcal{T}_{\mathbf{J}}$ has $C_{F}$ admissibly far nodes and any leaf node $t \in \mathcal{T}_{\mathbf{I}} \cup \mathcal{T}_{\mathbf{J}}$ has $C_{C}$ admissibly close nodes. Then, computation of representing sets with given basises for all nodes will require $O\left(\operatorname{maxvol}\left(\left(C_{F}+1\right) r \times r\right)\right) *\left(\left|\mathcal{T}_{\mathbf{I}}\right|+\left|\mathcal{T}_{\mathbf{J}}\right|\right)=$ $O\left(C_{F} r^{3}\left(\left|\mathcal{T}_{\mathbf{I}}\right|+\left|\mathcal{T}_{\mathbf{J}}\right|\right)\right)$ operations.
Step 3: reevaluate representing sets into the predecessors part of representing sets. This step is a simple move of representing sets from parent to child for each node (except leaves) of cluster trees $\mathcal{T}_{\mathbf{I}}$ and $\mathcal{T}_{\mathbf{J}}$. Obviously, it requires $O\left(\left|\mathcal{T}_{\mathbf{I}}\right|+\left|\mathcal{T}_{\mathbf{J}}\right|\right)$ operations.
Step 4: run Algorithm 4 to obtain new improved basis rows/columns and transfer matrices. Assume each non-leaf node has $K$ children, each leaf node corresponds to $K r$ rows or columns. Then, SVD reduction of Algorithm 4 works with $K r \times\left(C_{F} K r+r\right)$ matrices for each node $t \in \mathcal{T}_{\mathbf{J}}$ and with $K r \times\left(C_{F} r+r\right)$ for each node $t \in \mathcal{T}_{\mathbf{I}}$. Since $K \geq 2$, we can estimate the complexity of all SVD reductions in Algorithm 4 as $O\left(\left|\mathcal{T}_{\mathbf{J}}\right| C_{F} K^{3} r^{3}\right)$ operations. Each maxvol procedure works with a $K r \times r$ matrix, so, totally for all nodes, it requires $O\left(\left(\left|\mathcal{T}_{\mathbf{I}}\right|+\left|\mathcal{T}_{\mathbf{J}}\right|\right) K r^{3}\right)$ operations.
With further simplifications ( $N=M$, complexity of SVD is of the same order as the complexity of maxvol), total complexity is about $O\left(N_{\text {iters }} C_{F} K^{3} r^{3}\left|\mathcal{T}_{\mathbf{J}}\right|\right)$ operations. As we already assumed each leaf node has exactly $K r$ elements in it, so we have $\left|\mathcal{T}_{\mathbf{J}}\right|=$ $O\left(M K^{-1} r^{-1}\right)=O\left(N K^{-1} r^{-1}\right)$. So, the complexity of the approximation of far interactions in the matrix $A$ requires $O\left(C_{F} N K^{2} r^{2}\right)$ operations. Since number of leaf nodes is $O\left(N K^{-1} r^{-1}\right)$ and each leaf node requires $O\left(C_{C} K^{2} r^{2}\right)$ to save close interactions, we get following complexity on close interactions: $O\left(C_{C} N K r\right)$ operations. Totally, complexity of proposed MCBH algorithm 6 is $O\left(C_{F} N K^{2} r^{2}+C_{C} N K r\right)$ operations. where $C_{F}$ is a mean number of admissibly far nodes for all nodes, $C_{C}$ is a mean number of admissibly close nodes for all leaf nodes, $N$ is a number of rows and columns, $K$ is a number of children nodes for each node of cluster trees and $r$ is a mean basis size (rank) of each node.

Another interesting question is the number of matrix entries, required to compute approximation with Algorithm 6. Often, each matrix element is computed via complex function, which require many operations just for a single matrix element. Matrix elements are only used in close interactions and steps 2 and 4 of Algorithm 6.

Close interactions: storage cost is the same, as number of operations $\left(O\left(C_{C} N K r\right)\right)$. Step 2: run Algorithm 5 for each cluster tree to get representing sets. This operation requires computation of maxvol for $\left(C_{F}+1\right) r \times r$ submatrices of $A$ for each node.
Step 4: run Algorithm 4 to obtain new improved basis rows/columns and transfer matrices. Each iteration uses $K r \times\left(C_{F} K r+r\right)$ submatrix of $A$ for each node $t \in \mathcal{T}_{\mathbf{J}}$ and $K r \times\left(C_{F} r+\right.$ $r)$ submatrix of $A$ for each node $t \in T I$.
So, each iteration uses $O\left(C_{F} K^{2} r^{2}\left|\mathcal{T}_{\mathbf{J}}\right|\right)$ matrix entries. Assuming $\left|\mathcal{T}_{\mathbf{J}}\right|=O\left(N K^{-1} r^{-1}\right)$, we summarize total number of matrix elements used: $N_{\text {iter }}$ iterations of MCBH require $O\left(N_{i t e r} C_{F} N K r+C_{C} N K r\right)$ matrix values.

## 6. NUMERICAL EXPERIMENTS

This section contains two numerical examples. The first is the following: compute electric field potential, created by $n$ given particles $X_{1}, \ldots, X_{n}$ with charges $q_{1}, \ldots, q_{n}$, in coordinates of particles $X_{1}, \ldots, X_{n}$. Particles are distributed randomly (uniform distribution) over a cube $[0,1]^{3}$. This problem can be reformulated as computation of a matrix-vector product:

$$
\begin{equation*}
A q=f \tag{1}
\end{equation*}
$$

where $q$ is a vector of charges of particles, $f$ is desired electric field potential and matrix $A$ is defined as follows:

$$
A_{i j}= \begin{cases}\frac{1}{X X_{i}-X_{j} \mid}, & i \neq j  \tag{2}\\ 0, & i=j\end{cases}
$$

The second example is a solvation problem in the framework of polarized continuum model [24, 25, 26]. It arises in computer modeling of drugs: find the surface charge density $\sigma$ on a given solvent excluded surface $\Omega$, such that

$$
\begin{equation*}
\sigma(\mathbf{r})=\frac{1-\varepsilon}{2 \pi(1+\varepsilon)}\left(\sum_{i} \frac{Q_{i}\left(\left(\mathbf{r}-\mathbf{R}_{\mathbf{i}}\right) \cdot \mathbf{n}\right)}{\left|\mathbf{r}-\mathbf{R}_{\mathbf{i}}\right|^{3}}+\int_{\Omega} \frac{\sigma\left(\mathbf{r}^{\prime}\right)\left(\left(\mathbf{r}-\mathbf{r}^{\prime}\right) \cdot \mathbf{n}\right)}{\left|\mathbf{r}-\mathbf{r}^{\prime}\right|^{3}} d S^{\prime}\right), \tag{3}
\end{equation*}
$$

where $Q_{i}$ is a charge of $i$-th atom in molecule, $\mathbf{R}_{\mathbf{i}}$ is a position vector of $i$-th atom in molecule, $\mathbf{r}$ is a radius vector to surface, $\mathbf{n}$ is a perpendicular from surface to solvent and $\varepsilon$ is a relative permittivity. The surface is approximated by discrete elements with the Nyström method for the off-diagonal elements and the diagonal elements are computed from the identities:

$$
\begin{align*}
& \int_{\Omega} \frac{\left(\left(\mathbf{r}-\mathbf{r}^{\prime}\right) \cdot \mathbf{n}^{\prime}\right)}{\left|\mathbf{r}-\mathbf{r}^{\prime}\right|^{3}} d S^{\prime}=2 \pi, \\
& \int_{\Omega_{\varepsilon}} \frac{\sigma\left(\mathbf{r}^{\prime}\right)\left(\left(\mathbf{r}-\mathbf{r}^{\prime}\right) \cdot \mathbf{n}\right)}{\left|\mathbf{r}-\mathbf{r}^{\prime}\right|^{3}} d S^{\prime} \approx \sigma(\mathbf{r})\left(2 \pi-\int_{\Omega \backslash \Omega_{\varepsilon}} \frac{\left(\left(\mathbf{r}-\mathbf{r}^{\prime}\right) \cdot \mathbf{n}^{\prime}\right)}{\left|\mathbf{r}-\mathbf{r}^{\prime}\right|^{3}} d S^{\prime}\right) . \tag{4}
\end{align*}
$$

So, after discretizing and integrating over each discrete element, we get following system:

$$
A q=B Q,
$$

where $q$ is a vector of charges of discrete elements and $Q$ is a vector of charges of atoms in molecule. Matrices $A$ and $B$ are defined as follows:

$$
\begin{gather*}
A_{i j}=\left\{\begin{array}{ll}
\frac{(\varepsilon-1)}{4 \pi(1+\varepsilon)} \frac{\left(\left(\mathbf{r}_{\mathbf{i}}-\mathbf{r}_{\mathbf{j}}\right) \cdot \mathbf{n}_{\mathbf{i}}\right) S_{i}}{\left|\mathbf{r}_{\mathbf{i}}-\mathbf{r}_{\mathbf{j}}\right|^{3}}, & i \neq j \\
\frac{\varepsilon}{1+\varepsilon}-\sum_{k \neq j} A_{k j}, & i=j
\end{array},\right.  \tag{5}\\
B_{i j}=\frac{1-\varepsilon}{4 \pi(1+\varepsilon)} \frac{\left(\left(\mathbf{r}_{\mathbf{i}}-\mathbf{R}_{\mathbf{j}}\right) \cdot \mathbf{n}_{\mathbf{i}}\right) S_{i}}{\left|\mathbf{r}_{\mathbf{i}}-\mathbf{R}_{\mathbf{j}}\right|^{3}},
\end{gather*}
$$

where $\mathbf{r}_{\mathbf{i}}$ is a radius vector to center of $i$-th discrete element, $\mathbf{n}_{\mathbf{i}}$ is a perpendecular to $i$-th discrete element and $S_{i}$ is an area of $i$-th discrete element.

In the following numerical examples we study only the approximation of the matrix (5); we are not considering the problem of solution of linear systems with such matrices.

### 6.1. Implementation remarks

MCBH algorithm 6 is implemented in Python with the most computationally intensive parts reimplemented in Cython [http://cython.org]. For the basic linear algebra tasks the MKL library is used. The code can be obtained at http://bitbucket.org/muxas/h2tools. To build the hierarchical tree, we used recursive inertial bisection with admissibility parameter $\eta=0$ and block_size $=50$, where block_size is a constant, such that we divide cluster into subclusters only if it has more than block_size elements. For Figure 5, Figure 6, Figure 13 and Figure 14 each SVD reduction of Algorithm 6 is computed with the relative tolerance $\frac{\tau}{\text { level_count }}$. We used it instead of $\tau$ to align relative error, reached in solvation problem on 1 iteration of our algorithm (see Figure 13). All the tests, except tests on number of iterations, were performed with 1 iteration of MCBH. In addition, we tested the optimization of the ranks by the SVD recompression of the $\mathcal{H}^{2}$ matrices [9] with different tolerances. As a starting point for the SVD recompression we used the MCBH approximation with the accuracy parameter $\tau=\frac{10^{-8}}{\text { level_count }}$ and 1 additional iteration. Memory requirement in the figures below is the memory to store both transfer matrices and interaction matrices. The error in figures is relative error of only far field approximation in the spectral norm. It is computed with the help of Propack package [http://sun.stanford.edu/~rmunk/PROPACK/].

Tests were performed on a server with $2 \operatorname{Intel}(\mathrm{R}) \mathrm{Xeon}(\mathrm{R}) \mathrm{CPU}$ E5504@ 2.00GHz processors with 72 GB of RAM. However, only 2 threads were used (this is default number of threads for MKL). Python, Cython and MKL are from the Enthought Python Distribution (EPD 7.3-1 ,64-bit) [https://www.enthought.com/]. Python version is "2.7.3", Cython version is " 0.16 ", MKL version is " $10.3-1$ ".

### 6.2. Experiment with particles and interaction matrix (2)

Figure 5 shows dependence of relative error on number of iterations and accuracy parameter. Figure 6 shows dependence of approximation time and memory on accuracy parameter. To show dependence of approximation time and memory on number of particles and accuracy parameter, we measured maximum and mean values for 100 different random particle configurations. Figures $7,8,9,10,11,12$ tests used $\tau$ accuracy parameter instead of
mentioned $\frac{\tau}{\text { level_count }}$. Figure 7 and Figure 8 show dependence of maximum and mean approximation time on accuracy parameter and number of particles. Figure 9 and Figure 10 show dependence of maximum and mean approximation memory on accuracy parameter and number of particles. Figures 11 and 12 show compress ratio (approximation size vs matrix size).


Figure 5. Dependence of approximation error on number of iterations (left) and accuracy parameter $\tau$ (right) for the electrostatics problem (2), $N=100000$.


Figure 6. Dependence of approximation time (right) and memory (left) on accuracy parameter $\tau$ for the electrostatics problem, $N=100000$.


Figure 7. Dependence of maximum (100 tests) approximation time on accuracy parameter (left) and number of particles (right) for the electrostatics problem (2).


Figure 8. Dependence of mean (100 tests) approximation time on accuracy parameter (left) and number of particles (right) for the electrostatics problem (2).


Figure 9. Dependence of maximum (100 tests) approximation memory on accuracy parameter (left) and number of particles (right) for the electrostatics problem (2).


Figure 10. Dependence of mean (100 tests) approximation memory on accuracy parameter (left) and number of particles (right) for the electrostatics problem (2).


Figure 11. Dependence of worst (100 tests) compress ratio on accuracy parameter (left) and number of particles (right) for the electrostatics problem (2).


Figure 12. Dependence of mean (100 tests) compress ratio on accuracy parameter (left) and number of particles (right) for the electrostatics problem (2).

### 6.3. Boundary integral equation experiment

In this experiment the test surface consists of 222762 triangles. Figure 13 shows dependence of relative error on number of iterations and accuracy parameter. Figure 14 shows dependence of approximation time and memory on accuracy parameter.


Figure 13. Dependence of approximation error on number of iterations (left) and accuracy parameter $\tau$ (right) for the boundary integral equation (3), surface consists of 222762 discrete elements.


Figure 14. Dependence of approximation time (right) and memory (left) on accuracy parameter $\tau$ for the boundary integral operator (3), surface consists of 222762 discrete elements.

## 7. COMPARISON WITH OTHER METHODS

For the comparison we have chosen the methods describe in the paper [13], namely ACAMERGE and ACAGEO.

We have found that ACAMERGE method is equivalent to the zero iteration of proposed multicharge method (MCBH). As it can be seen from the numerical experiments, this iteration can have limited accuracy. Impact of the number of iterations on relative accuracy for different problems is shown in Figure 5 and Figure 13. So, ACAMERGE cannot get
relative error of far field approximation lower than $10^{-3}$ at least for the double-layer-type problem considered above.

ACAGEO method is based on the Chebyshev grid introduced in the bounding box of a given cluster node. Since the original code from [13] is not available, for a fair comparison we have reimplemented the ACAGEO method.

Table I and Table II correspond to application of MCBH and ACAGEO to approximation of matrices for two problems from Section 6. Notation used is the following:
$\tau$ - accuracy parameter,
block_size - maximum size of leaf node (number of elements in it),
iters - number of iterations for MCBH method,
time - factorization time in seconds,
error - relative spectral error of approximation, measured only by far field,
points - number of Chebyshev points for ACAGEO method (number of potential basis elements for each node of cluster trees).

Table I. Comparison of time and error of MCBH and ACAGEO for electrostatics problem (2).

|  |  | MCBH |  |  | ACAGEO |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\tau$ | block_size | iters | time | error | points | time | error |
| $10^{-2}$ | 25 | 0 | 9.7 | $2.2 * 10^{-2}$ | 8 <br> 27 | 10.7 <br> 29.5 | $1.8 * 10^{-2}$ <br> $2.2 * 10^{-2}$ |
|  | 25 | 0 | 15.5 | $3.2 * 10^{-3}$ | 27 | 30.5 | $3.8 * 10^{-3}$ |
| $10^{-4}$ | 25 | 0 | 26.8 | $4.3 * 10^{-4}$ | 27 | 32.3 | $6.3 * 10^{-4}$ |
| $10^{-5}$ | 25 | 0 | 46.3 | $1.2 * 10^{-4}$ | 27 | 35 | $2.4 * 10^{-4}$ |
|  | 1 | 148 | $3 * 10^{-5}$ | 64 | 217.4 | $1.8 * 10^{-4}$ |  |
| $10^{-6}$ | 25 | 1 | 262 | $3.3 * 10^{-6}$ | 27 <br> 64 | 33.4 <br> 221 | $1.8 * 10^{-4}$ <br> $4 * 10^{-5}$ |

Table II. Comparison of time and error of MCBH and ACAGEO for solvation problem.

|  |  | MCBH |  |  | ACAGEO |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\tau$ | block_size | iters | time | error | points | time | error |
| $10^{-2}$ | 25 | 0 | 11.5 | $8.4 * 10^{-2}$ | 8 | 13.2 | $2.6 * 10^{-1}$ |
|  | 25 | 1 | 29 | $6.4 * 10^{-2}$ | 27 | 35.7 | $9.6 * 10^{-2}$ |
| $10^{-3}$ | 25 | 0 | 18.6 | $2.4 * 10^{-2}$ | 8 | 13.3 | $2.6 * 10^{-1}$ |
|  | 25 | 1 | 50 | $1.1 * 10^{-2}$ | 27 | 37 | $3.3 * 10^{-2}$ |
| $10^{-4}$ | 25 | 0 | 30.8 | $1.2 * 10^{-2}$ | 8 | 13.2 | $2.6 * 10^{-1}$ |
|  | 25 | 1 | 101.9 | $1.6 * 10^{-3}$ | 27 | 38.2 | $1.7 * 10^{-2}$ |
|  | 25 | 2 | 174.9 | $1.6 * 10^{-3}$ | 64 | 163 | $1.5 * 10^{-2}$ |
| $10^{-5}$ | 25 | 0 | 45.9 | $9 * 10^{-3}$ | 27 | 38.6 | $1.4 * 10^{-2}$ |
|  | 25 | 1 | 181.6 | $2.1 * 10^{-4}$ | 64 | 166.3 | $7.6 * 10^{-3}$ |
|  | 25 | 2 | 321.8 | $2 * 10^{-4}$ | 125 | 609 | $8.6 * 10^{-3}$ |

## 8. CONCLUSIONS AND FUTURE WORK

The proposed algorithm is robust in the sense that high approximation accuracies are achievable using at most 2 additional iterations. It also does not require the construction of apriori representing sets. An important question that needs to be solved is the existence of a good MCBH-type approximation, provided that a good $\mathcal{H}^{2}$-approximation exists. A preliminary study shows that it is possible to derive the existence of a quasi-optimal approximation in the spirit of the recent paper [27].

Computational speed and memory requirements of the algorithm can be improved in several ways. The algorithm can be readily parallelized since it has the usual FMM structure. Also, in the case when the elements of a matrix can be computed in a cheap way, the interaction matrices can be computed online, and the memory consumption becomes much smaller. For the case when the computation of a single element of a matrix is expensive (for example, in the Galerkin method for the solution of a boundary integral equation), the application of the $\mathcal{H}^{2}$-recompression algorithm may give a noticeable reduction in the storage.

The code is available as a part of the h2tools package http://bitbucket.org/muxas/h2tools, and we plan to extended it with different compression algorithms and solvers. It will be interesting to apply the new method to different problems, for example to the problem of vortex ring dynamics [28].

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    Contract/grant sponsor: This work was supported by Russian Science Foundation Grant 14-11-00659

