A multivariate triple exponentially weighted moving average control chart

Short running head: MTEWMA monitoring scheme

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Abstract

Statistical process monitoring (SPM) is mostly populated with univariate control charts used to monitor a single variable (or quality characteristic). Nowadays, industries and online environments are filled with processes in which two or more quality characteristics are related. In such situations, univariate control charts are replaced with multivariate control charts for the sake of monitoring several characteristics simultaneously. This paper develops a new multivariate triple exponentially weighted moving average (MTEWMA) chart to serve this purpose. Moreover, the design of the multivariate simple and double exponentially weighted moving average (denoted as MEWMA and MDEWMA) charts are revisited using extensive simulations. It is observed that the MTEWMA chart has very interesting zero-state properties as compared to the steady-state properties. The newly proposed MTEWMA chart is superior over the MEWMA and MDEWMA charts in many situations of the zero-state mode. An illustrative example is provided to demonstrate the sensitivity of the proposed charts.

Keywords: EWMA, MEWMA, MDEWMA, MTEWMA, Multivariate process, Overall performance; Steady-state; Zero-state.

1. Introduction

In nowadays competitive market, a continuous improvement of services and produced items is constantly needed in order to attract customers. This can only be possible if various processes are continuously monitored using appropriate tools and the production or/and manufacturing equipment (or appliance) is also maintained adequately. It is also evident that if the variability of the process is kept at an acceptable level the risk of getting unwanted products can be reduced significantly (Montgomery[1]). The process variation is often caused by natural or/and special causes known as chance and assignable causes of variation. Chance causes of variation do not affect the process negatively. Any significant process variation is attributed to assignable causes of variation and must be reduced or eliminated as soon as possible. Many procedures have been used in statistical process monitoring (SPM) to control and alert the operator about the existence of any assignable causes of variation. Walter A. Shewhart developed a modern monitoring scheme named after his own name (Montgomery[1]) for an efficient monitoring of large sustained shifts in the process parameters. Shewhart chart uses only the latest information to decide on the state of the process; and therefore, it is called memoryless monitoring scheme.

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Page[2] and Roberts[3] introduced the cumulative sum (CUSUM) and exponentially weighted moving average (EWMA) charts, respectively, for monitoring small and large shifts in the process parameters. The CUSUM and EWMA charts use both past and current information to decide on the state of the process, this is the reason why they are called memory-type monitoring schemes. Any old information is assigned smaller weight compared to the weight of the most recent one. Since then, many other researchers have been contributing in the design and improvement of memoryless and memory-type control charts; see for example, Lucas,[4] Klein,[5] Jones et al,[6] Chen and Chen[7] and Sanusi et al.[8] Shamma and Shamma[9] introduced the double EWMA (DEWMA) which is the extension of Roberts[2]'s EWMA chart where the smoothing parameter is used twice (see also Shamma and Shamma[10]). More recently, Alevizakos et al[11] extended the EWMA and DEWMA schemes by developing the triple EWMA control chart where the smoothing parameter is applied three times (see also Alevizakos et al[12]).

The aforementioned control charts are only used to monitor one variable (i.e. one quality characteristic) and are called univariate control charts. When the monitoring of two or more related variables is of interest, the literature recommends the use of multivariate control charts. Hotteling[13] introduced a multivariate control chart based on the χ^2 statistic representing the weighted Mahalanobis distance between a sample observation and the center of the cloud. Crosier[14] and Lowry et al[15] proposed the multivariate CUSUM (MCUSUM) and multivariate EWMA (MEWMA) control charts as alternatives to the CUSUM and EWMA charts, respectively, to efficiently monitor small-to-moderate shifts in the process parameters (see also Pignatiello and Runger[16]). To further improve the MEWMA chart in monitoring small shifts, Alkahtani and Schaffer[17] proposed the multivariate double EWMA (MDEWMA) chart as a multivariate alternative to the DEWMA chart. Thus, this paper proposed the multivariate TEWMA chart as a multivariate alternative of the TEWMA chart to further improve both the MEWMA and MDEWMA charts in monitoring small shifts in the process parameters. Moreover, this paper revisits the design and implementation of the MEWMA and MDEWMA control charts using extensive simulations. For more details on multivariate statistical process control, readers are referred to the review paper by Psarakis and Panaretos.[18] Readers are also referred to the papers by Champ and Jones-Famer,[19] Mahmoud and Maravelakis,[20] Adegoke et al,[21] Haq and Khoo,[22] Sabahno et al,[23] Harris et al,[24] and Katebi and Moghadam[25] for recent developments on multivariate control charts.

The remainder of this paper is organised as follows: in Section 2, we present a short review on the existing MEWMA and MDEWMA control charts. Section 3 introduces the new MTEWMA control chart by laying the mathematical background. Section 4 discusses the in-control (IC) and out-of-control (OOC) performances of the proposed time-varying and asymptotic memory-type control charts. Moreover, the IC robustness of the proposed MTEWMA chart to non-normality is also discussed in terms of the IC characteristics of the run-length distribution. Section 5 provides a numerical example based on real-life data to demonstrate the design and implementation of the proposed control charts. The conclusion and recommendations are given in Section 6. Note that the terms control chart and monitoring scheme as well as chart and scheme can be used interchangeably.

2. Brief review of the multivariate EWMA and double EWMA control charts

In this section, the mathematical background and properties needed for the design of the MEWMA and MDEWMA control charts are provided.

2.1 MEWMA control chart

Let $\boldsymbol{X} = \left(X_1, X_2, \dots, X_p\right)'$ be a p-component random vector representing p jointly distributed random quality characteristics obtained from a process of interest. If the process is in-control (IC), we assume that $\boldsymbol{X}$ follows a p-variate normal distribution with mean vector $\boldsymbol{\mu}_0$ and dispersion matrix $\boldsymbol{\Sigma}_0$, i.e. $\boldsymbol{X} \sim N_p(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$. In course of time, suppose we have observed $\boldsymbol{X}_i$, $(i = 1, 2, \dots)$.

Thus, as natural extension of the univariate EWMA control chart, we define

$$\boldsymbol{Y}_{1i} = \boldsymbol{\Lambda}\boldsymbol{X}_i + (\boldsymbol{I} - \boldsymbol{\Lambda})\boldsymbol{Y}_{1(i-1)}, i = 1, 2, \dots, \tag{1}$$

where $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_p)$, $0 < \lambda_j \leq 1, j = 1, 2, \dots, p$ and $\boldsymbol{Y}_{10} = \boldsymbol{\mu}_0$. Without loss of generality, it can be shown that Equation (1) can be simplified to

$$\boldsymbol{Y}_{1i} = \boldsymbol{\Lambda} \sum_{\boldsymbol{k=1}}^{\boldsymbol{i}} (\boldsymbol{I} - \boldsymbol{\Lambda})^{i-k} \boldsymbol{X}_k + (\boldsymbol{I} - \boldsymbol{\Lambda})^{i} \boldsymbol{\mu}_0. \tag{2}$$

When the process is IC, the process mean vector and dispersion matrix of the charting statistic defined in Equation (1) are defined by

$$E(\boldsymbol{Y}_{1i}|IC) = \boldsymbol{\mu}_0 \text{ and } \boldsymbol{\Sigma}_{\boldsymbol{Y}_{1i}} = \sum_{\boldsymbol{k=1}}^{\boldsymbol{i}} \boldsymbol{G}_{1k}\boldsymbol{\Sigma}_0\boldsymbol{G}'_{1k}, \tag{3}$$

respectively, where $\boldsymbol{G}_{1k} = \boldsymbol{\Lambda}(\boldsymbol{I} - \boldsymbol{\Lambda})^{i-k}$.

Hence, the MEWMA chart gives an OOC signal if

$$u_{1i}^2 = (\boldsymbol{Y}_{1i} - \boldsymbol{\mu}_0)' \boldsymbol{\Sigma}_{\boldsymbol{Y}_{1i}}^{-1} (\boldsymbol{Y}_{1i} - \boldsymbol{\mu}_0) \geq h_{MEWMA}, \tag{4}$$

where $h_{MEWMA} (> 0)$ is chosen to achieve a specified IC ARL (ARL_0) value.

If we have no prior information related to choosing the weights of the observations differently for the p quality characteristics, then we assume $\lambda_1 = \lambda_2 = \ldots = \lambda_p = \lambda$. Under this assumption, the MEWMA charting vector can be written as

$$\boldsymbol{Y}_{1i} = \lambda \boldsymbol{X}_i + (1 - \lambda) \boldsymbol{Y}_{1(i-1)}. \tag{5}$$

Based on Equation (5), the mean vector and time-varying (i.e. zero-state case) dispersion matrix of the charting statistic defined in Equation (5) are given by

$$E(\boldsymbol{Y}_{1i} | IC) = \boldsymbol{\mu}_0 \text{ and } \boldsymbol{\Sigma}_{\boldsymbol{Y}_{1i}} = (c_{11} + c_{12}(i)) \boldsymbol{\Sigma}_0, \tag{6}$$

respectively, where $c_{11} = \frac{\lambda}{2-\lambda}$ and $c_{12}(i) = -\frac{\lambda(1-\lambda)^{2i}}{2-\lambda}$.

When the process has been running for a very long time (i.e. asymptotic case also known as steady-state case), $i \to \infty$ and $c_{12}(i)$ converges toward zero so that $\boldsymbol{\Sigma}_{\boldsymbol{Y}_{1i}} = c_{11} \boldsymbol{\Sigma}_0$.

In the remainder of this paper, we will frequently use the terms time-varying and asymptotic instead of zero-state and steady-steady terms.

2.2 MDEWMA control chart

As a natural extension of the univariate DEWMA control chart, we define the MDEWMA charting vector as follows:

$$\boldsymbol{Y}_{2i} = \boldsymbol{\Lambda} \mathbf{Y}_{1i} + (\boldsymbol{I} - \boldsymbol{\Lambda}) \boldsymbol{Y}_{2(i-1)}, \tag{7}$$

where $\boldsymbol{\Lambda}$ and $\mathbf{Y}_{1i}$ are defined in Equation (1) and $\boldsymbol{Y}_{20} = \boldsymbol{\mu}_0$. Without loss of generality, it can be shown that Equation (7) can be simplified to

$$\boldsymbol{Y}_{2i} = \boldsymbol{\Lambda}^2 \sum_{k=1}^{i} (i - k + 1)(\boldsymbol{I} - \boldsymbol{\Lambda})^{i-k} \boldsymbol{X}_k + (i\boldsymbol{\Lambda} + \boldsymbol{I})(\boldsymbol{I} - \boldsymbol{\Lambda})^i \boldsymbol{\mu}_0. \tag{8}$$

When the process is IC, the process mean vector and dispersion matrix of the charting statistic defined in Equation (7) are defined by

$$E(\boldsymbol{Y}_{2i} | IC) = \boldsymbol{\mu}_0 \text{ and } \boldsymbol{\Sigma}_{\boldsymbol{Y}_{2i}} = \sum_{k=1}^{i} (i - k + 1)^2 \boldsymbol{G}_{2k} \boldsymbol{\Sigma}_0 \boldsymbol{G}_{2k}', \tag{9}$$

where $\boldsymbol{G}_{2k} = \boldsymbol{\Lambda}^2 (\boldsymbol{I} - \boldsymbol{\Lambda})^{i-k}$.

Hence, the MDEWMA chart gives an OOC signal if

$$u_{2i}^2 = (\boldsymbol{Y}_{2i} - \boldsymbol{\mu}_0)' \boldsymbol{\Sigma}_{\boldsymbol{Y}_{2i}}^{-1} (\boldsymbol{Y}_{2i} - \boldsymbol{\mu}_0) \geq h_{MDEWMA}, \tag{10}$$

where $h_{MDEWMA} (> 0)$ is chosen to achieve a specified ARL_0 value.

When $\lambda_1 = \lambda_2 = \ldots = \lambda_p = \lambda$. Under this assumption, the MDEWMA charting vector can be written as

$$\boldsymbol{Y}_{2i} = \lambda\, \boldsymbol{Y}_{1i} + (1-\lambda)\boldsymbol{Y}_{2(i-1)}. \tag{11}$$

Based on Equation (11), the mean vector and time-varying dispersion matrix of the charting statistic defined in Equation (9) become

$$E(\boldsymbol{Y}_{2i}|IC) = \boldsymbol{\mu}_0 \text{ and } \boldsymbol{\Sigma}_{\boldsymbol{Y}_{2i}} = (c_{21} + c_{22}(i))\boldsymbol{\Sigma}_0, \tag{12}$$

respectively, where

$$c_{21} = \lambda^4 \left[\frac{2\theta}{(1-\theta)^3} + \frac{1}{(1-\theta)^2}\right],$$

$$c_{22}(i) = -\lambda^4 \left[\frac{(i+1)^2\theta^i}{1-\theta} + \frac{(2i+3)\theta^{i+1}}{(1-\theta)^2} + \frac{2\theta^{i+2}}{(1-\theta)^3}\right]$$

and $\theta = (1-\lambda)^2$.

When the process has been running for a very long time, $i \to \infty$ and $c_{22}(i)$ converges towards zero so that $\boldsymbol{\Sigma}_{\boldsymbol{Y}_{2i}} = c_{21}\boldsymbol{\Sigma}_0$.

In the next section, we introduce the new time-varying and asymptotic MTEWMA control charts

3. The proposed MTEWMA control chart

More recently, Alevizakos et al[11] introduced the univariate TEWMA scheme for monitoring one quality characteristic for normally distributed observations. As a natural extension of the univariate TEWMA control chart, we define in addition to Equations (1) and (7) the following MTEWMA statistic:

$$\boldsymbol{Y}_{3i} = \boldsymbol{\Lambda}\mathbf{Y}_{2i} + (\boldsymbol{I} - \boldsymbol{\Lambda})\boldsymbol{Y}_{3(i-1)}, \tag{13}$$

where $\boldsymbol{\Lambda}$ and $\mathbf{Y}_{2i}$ are defined in Equation (1) and (7), respectively, and $\boldsymbol{Y}_{30} = \boldsymbol{\mu}_0$. Without loss of generality, it can be shown that Equation (13) can be simplified to

$$\begin{aligned}\boldsymbol{Y}_{3i} = \frac{\boldsymbol{\Lambda}^3}{2}\sum_{k=1}^{i}(i-k+1)(i-k+2)(\boldsymbol{I}-\boldsymbol{\Lambda})^{i-k}\boldsymbol{X}_k \\ + \frac{1}{2}(i(i+1)\boldsymbol{\Lambda}^2 + 2i\boldsymbol{\Lambda} + 2\boldsymbol{I})(\boldsymbol{I}-\boldsymbol{\Lambda})^i\boldsymbol{\mu}_0.\end{aligned} \tag{14}$$

When the process is IC, the process mean vector and dispersion matrix of the charting statistic defined in Equation (13) are defined by

$$E(\boldsymbol{Y}_{3i}|IC) = \boldsymbol{\mu}_0 \text{ and } \boldsymbol{\Sigma}_{\boldsymbol{Y}_{3i}} = \sum_{k=1}^{i}(i-k+1)^2(i-k+2)^2\boldsymbol{G}_{2k}\boldsymbol{\Sigma}_0\boldsymbol{G}'_{2k}, \tag{15}$$

respectively, where $\boldsymbol{G}_{2k} = \frac{\boldsymbol{\Lambda}^3}{2}(\boldsymbol{I}-\boldsymbol{\Lambda})^{i-k}$.

Hence, the MTEWMA chart gives an OOC signal if

$$u_{3i}^2 = (\boldsymbol{Y}_{3i} - \boldsymbol{\mu}_0)'\boldsymbol{\Sigma}_{\boldsymbol{Y}_{3i}}^{-1}(\boldsymbol{Y}_{3i} - \boldsymbol{\mu}_0) \geq h_{MTEWMA}, \quad (16)$$

where $h_{MTEWMA}(> 0)$ is chosen to achieve a specified ARL_0 value.

When $\lambda_1 = \lambda_2 = \ldots = \lambda_p = \lambda$. Under this assumption, the MTEWMA charting vector can be written a

$$\boldsymbol{Y}_{3i} = \lambda \boldsymbol{Y}_{2i} + (1 - \lambda)\boldsymbol{Y}_{3(i-1)}. \quad (17)$$

Based on Equation (17), the mean vector and time-varying dispersion matrix of the charting statistic defined in Equation (15) become

$$E(\boldsymbol{Y}_{3i}|IC) = \boldsymbol{\mu}_0 \text{ and } \boldsymbol{\Sigma}_{\boldsymbol{Y}_{3i}} = (c_{31} + c_{32}(i))\boldsymbol{\Sigma}_0, \quad (12)$$

respectively, where

$$c_{31} = \frac{6(1-\lambda)^6\lambda}{(2-\lambda)^5} + \frac{12(1-\lambda)^4\lambda^2}{(2-\lambda)^4} + \frac{7(1-\lambda)^2\lambda^3}{(2-\lambda)^3} + \frac{\lambda^4}{(2-\lambda)^2},$$

$$\begin{aligned} c_{32}(i) = -\frac{\theta^3\lambda^6}{4}\Bigg[&\frac{i(i^2-1)(i-2)\theta^{i-3}}{1-\theta} + \frac{4i(i^2-1)\theta^{i-2}}{(1-\theta)^2} + \frac{12i(i+1)\theta^{i-1}}{(1-\theta)^3} + \frac{24(i+1)\theta^i}{(1-\theta)^4} \\ &+ \frac{24\theta^{i+1}}{(1-\theta)^5}\Bigg] - 2\theta^2\lambda^6\left[\frac{i(i^2-1)\theta^{i-2}}{1-\theta} + \frac{3i(i+1)\theta^{i-1}}{(1-\theta)^2} + \frac{6(i+1)\theta^i}{(1-\theta)^3} + \frac{6\theta^{i+1}}{(1-\theta)^4}\right] \\ &- \frac{7\theta\lambda^6}{2}\left[\frac{i(i+1)\theta^{i-1}}{1-\theta} + \frac{2(i+1)\theta^i}{(1-\theta)^2} + \frac{2\theta^{i+1}}{(1-\theta)^3}\right] - \lambda^6\left[\frac{\theta^{i+1}}{(1-\theta)^2} + \frac{(i+1)\theta^i}{1-\theta}\right] \end{aligned}$$

and θ is defined in Equation (12).

When the process has been running for a very long time, $i \to \infty$ and $c_{32}(i)$ converges towards zero so that $\boldsymbol{\Sigma}_{\boldsymbol{Y}_{3i}} = c_{31}\boldsymbol{\Sigma}_0$. For more details on how to derive the properties of the MTEWMA statistic, readers are referred to the Appendix.

Note that when $n > 1$, instead of $\boldsymbol{Y}_{1i}$, $\boldsymbol{Y}_{2i}$ and $\boldsymbol{Y}_{3i}$, their respective mean vectors are used and their corresponding dispersion matrix are divided by n.

4. Performance analysis

4.1 Performance measures

The performance of a control chart is evaluated in terms of how fast it detects OOC events. This is quantified in terms of the characteristics of the its run-length distribution. The run-length is defined as the number of rational samples plotted on a chart before the observation of the first OOC state. The average run-length (*ARL*) is the most popular metric used to investigate the performance or sensitivity of a control chart. Several authors reported that the only use of the *ARL* is not enough to provide sufficient information about the sensitivity of a control chart. Therefore, other characteristics of the run-length distribution such as the standard deviation of the run-length (*SDRL*) and percentiles of the run-length (*PRL*) are need to give the missing information. The latter includes the 5th, 25th, 50th, 75th and 95th denoted in this paper as P_5, P_{25}, P_{50}, P_{75} and P_{95}, respectively. The 50th percentile is also known as the median run-length (*MRL*). Note though that the aforementioned metrics evaluate the performance for specific shifts. Thus, the performance of range of shifts or overall performance is evaluated using the expected values of the characteristics of the run-length and the properties of the extra quadratic loss (*EQL*) function. The former includes the expected *ARL* (*EARL*), expected *SDRL* (*ESDRL*) and expected *PRL* (*EPRL*) which are the focus of this paper.

Let RC represents a specific characteristic of the run-length distribution and $RC(\delta)$ its value for a specific shift. Thus, the expected RC (ERC) value is mathematically defined as:

$$ERC = \frac{1}{\Delta} \sum_{\delta=\delta_{min}}^{\delta_{max}} RC(\delta), \tag{13}$$

where $RC(\delta)$ represents the $ARL(\delta)$, $SDRL(\delta)$ and $MRL(\delta)$ according to whether the ERC represents the $EARL$, $ESDRL$ or $EMRL$. The symbol Δ denotes the number of increments between the lower and upper bound shifts (i.e. δ_{min} and δ_{max}). The smaller the ERC, the better the performance of the control chart for a range of shifts between δ_{min} and δ_{max}. Note though that the size of the mean shift in standard deviation is determined by the distance from the IC mean vector $\boldsymbol{\mu}_0$ to the OOC mean vector $\boldsymbol{\mu}_1$, and can be measured by the non-centrality parameter $\delta = \left((\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)'\boldsymbol{\Sigma}_0^{-1}(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)\right)^{1/2}$ (see Harris et al[24]).

4.2 IC and OOC performances of the MEWMA, MDEWMA and MTEWMA schemes

The first step in the design of control charts is to determine the control limit coefficients and their corresponding control limits such that the nominal IC *ARL* (ARL_0) is fixed to some high desired values such as 200, 370, 500, etc. The smaller the characteristic of the run-length, the better the performance of the control chart for a specific shift δ. Table 1-3 display the performances of time-varying MEWMA, MDEWMA and MTEWMA control charts in terms of the *ARL*, *SDRL* and *MRL* profiles along with their respective control limits when *n*=1, $\lambda \in${0.05,0.1,0.25,0.5,0.75,0.9} and $p \in${2,3,4,10} for a nominal ARL_0 of 200.

The results in Tables 1-3 can be summarised as follows:

(1) In terms of the *ARL* values:

- The three schemes perform better for small values of *p*. For instance, when p=2 and λ =0.05, for small shifts (say, $\delta = 0.25$), the MEWMA, MDEWMA and

MTEWMA schemes each give a signal on the 59th, 49th and 48th sample, respectively; however, when p=3, λ =0.05 and $\delta = 0.25$, the MEWMA, MDEWMA and MTEWMA schemes give a signal on the 67th, 55th and 53rd sample, respectively. As it can be seen, the performance these three schemes deteriorate if p increases.

- The three schemes perform better for small smoothing parameters regardless of the value of p. The larger the value of λ, the worst the performance of the proposed schemes. For instance, if p=3 and $\delta = 0.25$, the MEWMA scheme gives a signal on the 59th, 74th, 137th, 158th and 169th samples when $\lambda = 0.05$, 0.1, 0.25, 0.5 and 0.9, respectively. The performance deteriorates as λ increases. The same finding is also observed for the MDEWMA and MTEWMA schemes.
- The MDEWMA and MTEWMA schemes outperform the MEWMA scheme regardless of the value of p and the magnitude of the shift in the process mean vector.
- For small values of λ, say $\lambda \in (0,0.25)$, the MTEWMA scheme performs better than the MDEWMA scheme for very small shifts and for moderate, while for large shifts in the mean vector, the two schemes perform similarly. However, when $\lambda \in [0.25,1)$, the MTEWMA scheme outperforms the MDEWMA scheme for small and moderate shifts, while for large shifts they perform similarly.

(2) In terms of the *SDRL* values:

- For all three schemes, regardless of the value of λ, the *SDRL* values increase as p increases, which means that they have higher probabilities of giving false OOC signals as p increases.
- The magnitude of *SDRL* values of the three monitoring schemes are directly proportional to the value of λ. As λ increases, the *SDRL* values increase as well.
- The MDEWMA and MTEWMA schemes have smaller probabilities of giving false OOC signals as compared to the MEWMA scheme. This is reflected by smaller SDRL values of the former two schemes and larger SDRL values for the latter scheme.
- When $\lambda \in (0,0.25)$, for very small shifts in the mean vector, the MTEWMA scheme yields smaller SDRL values when compared to the MDEWMA scheme, while for moderate and large shifts their values are almost similar. When $\lambda \in$

[0.25,1), the MTEWMA scheme yields smaller SDRL value for small and moderate shifts, while for large shifts their values are almost the same.

(3) In terms of the *MRL* values:

- For small and moderate shifts in the mean vector, the proposed schemes yield larger *MRL* values for large value of p. For instance, when $p = 3$, $\lambda = 0.05$ and $\delta = 0.25$, the MEWMA, MDEWMA and MTEWMA schemes yield *MRL* values of 50, 39 and 38, while they yield *MRL* values of 65, 52 and 51, respectively, when $p = 10$, $\lambda = 0.05$ and $\delta = 0.25$. However, for large shifts, there is a slight difference in the MRL values regardless of the value of p. For instance, for both $p = 3$ and 10, $\lambda = 0.05$ and $\delta = 2.50$, MDEWMA scheme yield an *MRL* value of 2, which means that regardless of the p value, there is 50% chance that the MDEWMA scheme gives a signal on the second sample for a shift of 2 standard deviation in the mean vector.
- Regardless of the p value and the magnitude of the shift of the three schemes, the larger the value of λ, the larger the attained *MRL* value. For instance, when $p = 4$, $\delta = 0.5$ and $\lambda = 0.1$, there is 50% chance that the MTEWMA scheme gives a signal on the 22$^{\text{nd}}$ sample, while there 50% chance that it gives a signal on the 79$^{\text{th}}$ sample when $p = 4$, $\delta = 0.5$ and $\lambda = 0.9$.
- The MTEWMA and MDEWMA schemes outperform the MEWMA scheme regardless of the p value and magnitude of the shift in mean vector.

Table 1. *ARL*, *SDRL* and *MRL* profile of the revised time-varying MEWMA control chart along with the control limits when $\lambda \in \{0.05, 0.1, 0.25, 0.5, 0.75, 0.9\}$, n=1 and $p \in \{2,3,4,10\}$ for a nominal $ARL_0 = 200$

	λ	0.05			0.1			0.25			0.5			0.75			0.9		
	Metric	ARL	$SDRL$	MRL	ARL	$SDRL$	MRL	ARL	$SDRL$	MRL	ARL	$SDRL$	MRL	ARL	$SDRL$	MRL	ARL	$SDRL$	MRL
	0.00	199.0	214.1	132.0	200.8	205.0	137.0	199.8	199.7	140.0	200.4	199.3	140.0	199.6	198.8	139.0	200.0	197.7	141.0
	0.25	59.1	56.1	43.0	73.7	70.6	52.0	104.9	104.2	73.0	137.4	136.3	95.0	157.8	159.8	108.0	168.7	169.7	116.0
	0.50	20.7	16.7	17.0	25.2	20.4	20.0	38.8	36.2	28.0	63.6	62.3	45.0	89.7	88.7	63.0	105.6	105.2	74.0
	0.75	10.9	7.9	9.0	12.6	9.0	11.0	17.6	14.6	13.0	29.3	27.9	21.0	47.5	46.5	33.0	60.1	59.8	42.0
	1.00	6.9	4.6	6.0	7.8	5.1	7.0	9.8	7.3	8.0	15.6	13.8	11.0	25.5	24.5	18.0	35.1	34.2	25.0
	1.25	4.8	3.0	4.0	5.4	3.3	5.0	6.4	4.1	5.0	9.1	7.4	7.0	14.6	13.4	11.0	20.0	19.1	14.0
p =2	1.50	3.6	2.2	3.0	4.1	2.4	4.0	4.7	2.8	4.0	6.0	4.5	5.0	9.3	8.1	7.0	12.5	11.9	9.0
	1.75	2.9	1.6	3.0	3.2	1.8	3.0	3.6	2.0	3.0	4.4	3.0	4.0	6.1	5.1	5.0	8.3	7.5	6.0
	2.00	2.4	1.3	2.0	2.6	1.4	2.0	2.9	1.5	3.0	3.4	2.1	3.0	4.3	3.4	3.0	5.7	5.0	4.0
	2.25	2.0	1.0	2.0	2.2	1.1	2.0	2.4	1.2	2.0	2.7	1.5	2.0	3.3	2.3	3.0	4.1	3.3	3.0
	2.50	1.7	0.8	2.0	1.9	0.9	2.0	2.1	1.0	2.0	2.3	1.2	2.0	2.6	1.7	2.0	3.1	2.4	2.0
	2.75	1.6	0.7	1.0	1.7	0.8	2.0	1.8	0.8	2.0	1.9	1.0	2.0	2.2	1.3	2.0	2.4	1.7	2.0
	3.00	1.4	0.6	1.0	1.5	0.7	1.0	1.6	0.7	1.0	1.7	0.8	2.0	1.8	1.0	2.0	2.0	1.3	2.0
	h_{MEWMA}	**7.685**			**8.789**			**9.952**			**10.453**			**10.589**			**10.613**		
	0.00	200.4	217.4	132.0	199.5	207.1	135.0	199.5	202.7	137.0	199.9	199.6	139.0	200.8	198.7	140.0	200.6	199.0	140.0
	0.25	67.3	64.1	50.0	83.9	81.5	60.0	118.2	116.1	81.0	146.9	146.9	101.0	165.9	164.6	115.0	174.5	174.5	120.0
	0.50	23.3	18.6	19.0	28.9	24.1	22.0	45.4	42.1	33.0	74.9	73.1	53.0	102.8	101.3	71.0	118.8	117.1	83.0
	0.75	12.3	8.8	10.0	14.3	10.2	12.0	20.3	17.0	15.0	36.1	34.3	26.0	58.1	57.5	40.0	74.0	73.2	52.0
	1.00	7.6	5.1	7.0	8.7	5.6	8.0	11.3	8.5	9.0	18.7	17.1	14.0	32.0	31.0	23.0	43.8	43.1	31.0
	1.25	5.3	3.3	5.0	6.0	3.6	5.0	7.2	4.8	6.0	10.8	9.1	8.0	18.2	17.1	13.0	25.8	25.3	18.0
p =3	1.50	4.0	2.4	4.0	4.5	2.6	4.0	5.3	3.2	5.0	7.1	5.4	6.0	11.4	10.3	8.0	16.4	15.6	12.0
	1.75	3.2	1.8	3.0	3.5	1.9	3.0	4.0	2.3	4.0	5.0	3.5	4.0	7.5	6.3	6.0	10.3	9.6	7.0
	2.00	2.6	1.4	2.0	2.9	1.5	3.0	3.2	1.7	3.0	3.8	2.4	3.0	5.3	4.2	4.0	7.0	6.3	5.0
	2.25	2.2	1.1	2.0	2.4	1.2	2.0	2.7	1.3	2.0	3.0	1.8	3.0	3.9	2.9	3.0	5.0	4.1	4.0
	2.50	1.9	0.9	2.0	2.1	1.0	2.0	2.3	1.1	2.0	2.5	1.3	2.0	3.0	2.0	2.0	3.7	2.9	3.0
	2.75	1.7	0.8	2.0	1.8	0.8	2.0	2.0	0.9	2.0	2.1	1.1	2.0	2.4	1.5	2.0	2.9	2.1	2.0
	3.00	1.5	0.7	1.0	1.6	0.7	1.0	1.8	0.8	2.0	1.8	0.9	2.0	2.0	1.2	2.0	2.3	1.6	2.0
	h_{MEWMA}	**9.758**			**10.960**			**12.191**			**12.695**			**12.826**			**12.847**		
	0.00	200.1	215.1	132.0	200.5	207.3	137.0	200.1	201.9	138.0	200.1	200.1	139.0	200.1	197.7	140.0	200.2	199.3	139.0
	0.25	71.4	69.4	52.0	90.0	87.7	64.0	124.7	123.8	87.0	153.9	153.4	107.0	169.5	170.2	118.0	177.3	177.9	121.0
	0.50	25.3	20.2	21.0	31.8	26.8	25.0	51.6	48.0	37.0	84.5	83.4	59.0	113.9	111.7	80.0	129.1	126.8	91.0
	0.75	13.3	9.4	11.0	15.5	11.2	13.0	23.1	20.1	17.0	42.5	41.4	30.0	66.5	65.2	46.0	83.1	81.6	58.0
	1.00	8.3	5.5	7.0	9.5	6.1	8.0	12.5	9.5	10.0	21.8	19.9	16.0	37.7	37.1	26.0	51.2	50.8	36.0
	1.25	5.7	3.6	5.0	6.4	3.8	6.0	7.9	5.3	7.0	12.5	10.7	9.0	21.7	20.6	15.0	30.7	29.9	21.0
p =4	1.50	4.4	2.6	4.0	4.8	2.7	4.0	5.6	3.4	5.0	8.0	6.3	6.0	13.5	12.3	10.0	19.3	18.7	13.0
	1.75	3.4	1.9	3.0	3.8	2.1	3.0	4.3	2.4	4.0	5.6	4.0	5.0	8.8	7.7	6.0	12.5	11.7	9.0
	2.00	2.8	1.5	3.0	3.1	1.6	3.0	3.5	1.8	3.0	4.2	2.7	4.0	6.0	4.9	5.0	8.2	7.5	6.0
	2.25	2.3	1.2	2.0	2.6	1.3	2.0	2.9	1.4	3.0	3.3	1.9	3.0	4.4	3.3	3.0	5.8	5.0	4.0
	2.50	2.0	1.0	2.0	2.2	1.1	2.0	2.4	1.2	2.0	2.7	1.5	2.0	3.4	2.4	3.0	4.2	3.5	3.0
	2.75	1.8	0.8	2.0	1.9	0.9	2.0	2.1	1.0	2.0	2.3	1.2	2.0	2.7	1.7	2.0	3.2	2.5	2.0
	3.00	1.6	0.7	1.0	1.7	0.8	2.0	1.9	0.8	2.0	2.0	1.0	2.0	2.2	1.3	2.0	2.5	1.8	2.0
	h_{MEWMA}	**10.627**			**12.904**			**14.199**			**14.718**			**14.829**			**14.866**		
	0.00	200.2	213.7	134.0	200.2	204.8	139.0	199.8	199.3	139.0	200.2	199.1	140.0	200.2	199.3	140.0	200.2	199.8	139.0
	0.25	91.1	88.9	65.0	112.2	111.5	78.0	148.3	147.0	104.0	172.4	174.7	119.0	182.6	186.0	124.0	186.6	187.2	128.0
	0.50	34.2	27.4	28.0	44.0	37.9	33.0	74.7	72.6	52.0	114.9	114.3	79.0	142.5	141.8	99.0	154.0	153.0	106.0
	0.75	17.5	12.2	15.0	21.3	15.6	18.0	35.4	31.8	26.0	65.9	64.2	46.0	97.5	97.3	68.0	114.8	113.8	80.0
	1.00	11.0	7.1	10.0	12.6	8.1	11.0	18.5	15.0	14.0	36.9	35.2	26.0	62.7	61.9	44.0	79.2	78.8	55.0
	1.25	7.7	4.6	7.0	8.7	5.1	8.0	11.4	8.1	9.0	21.1	19.3	15.0	38.7	37.7	27.0	53.5	53.0	37.0
p =10	1.50	5.7	3.3	5.0	6.3	3.5	6.0	7.8	5.0	7.0	13.0	11.0	10.0	24.4	23.4	17.0	35.2	34.5	24.0
	1.75	4.5	2.5	4.0	4.9	2.6	5.0	5.8	3.4	5.0	8.5	6.7	7.0	15.7	14.6	11.0	23.2	22.7	16.0
	2.00	3.6	1.9	3.0	4.0	2.0	4.0	4.5	2.4	4.0	6.0	4.3	5.0	10.4	9.2	8.0	15.6	14.8	11.0
	2.25	3.0	1.5	3.0	3.3	1.6	3.0	3.7	1.8	3.0	4.6	3.0	4.0	7.2	6.0	5.0	10.7	9.8	8.0
	2.50	2.6	1.3	2.0	2.8	1.3	3.0	3.1	1.5	3.0	3.7	2.1	3.0	5.3	4.1	4.0	7.5	6.7	6.0
	2.75	2.2	1.1	2.0	2.4	1.1	2.0	2.6	1.2	2.0	3.0	1.6	3.0	4.0	3.0	3.0	5.4	4.7	4.0
	3.00	2.0	0.9	2.0	2.1	0.9	2.0	2.3	1.0	2.0	2.6	1.3	2.0	3.2	2.1	3.0	4.1	3.3	3.0
	h_{MEWMA}	**21.342**			**22.894**			**24.447**			**25.054**			**25.156**			**25.174**		

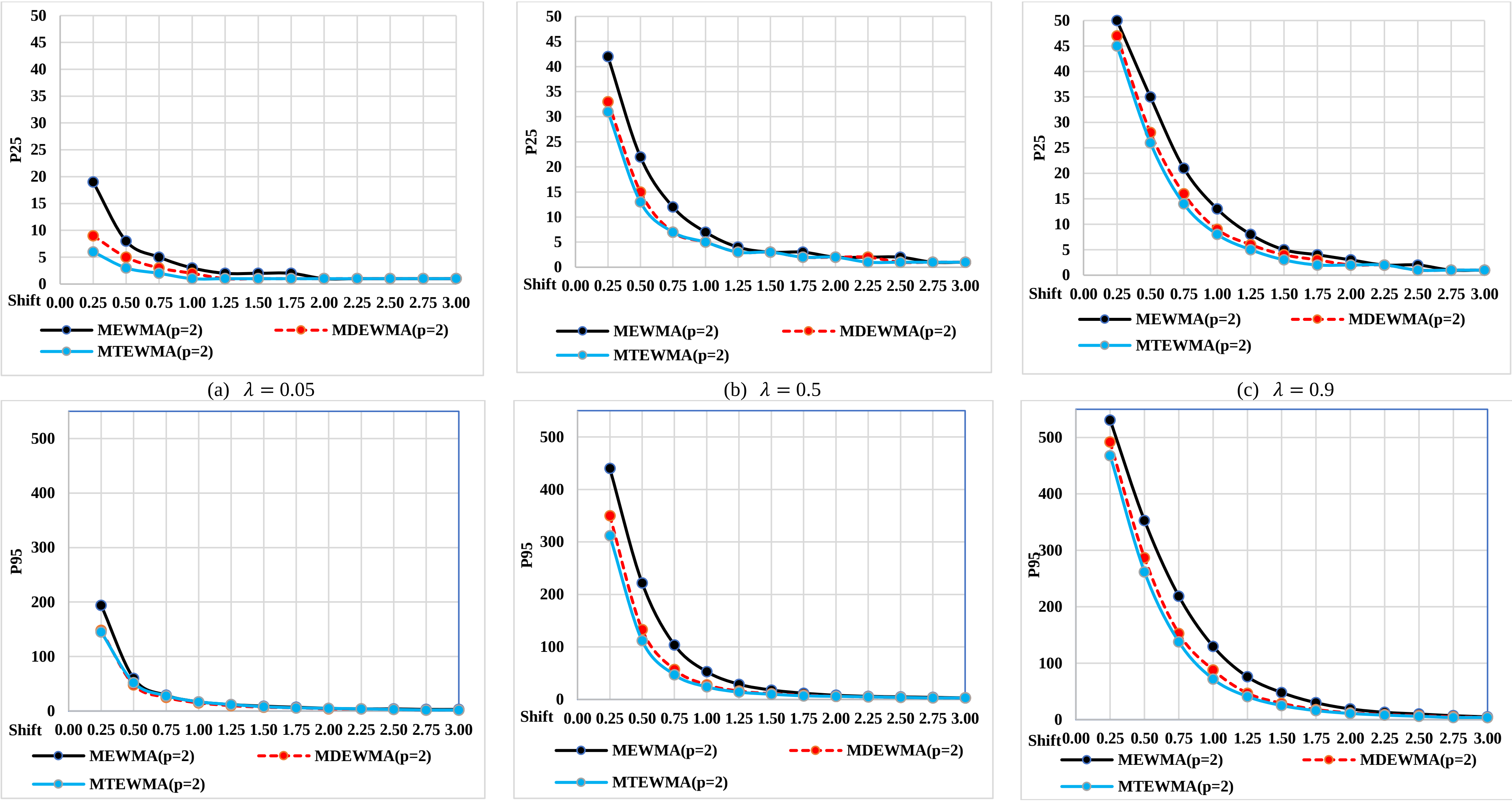
Table 2. *ARL*, *SDRL* and *MRL* profile of the revised time-varying MDEWMA control chart along with the control limits when $\lambda \in \{0.05, 0.1, 0.25, 0.5, 0.75, 0.9\}$, *n*=1 and $p \in \{2,3,4,10\}$ for a nominal $ARL_0 = 200$

	λ	**0.05**			**0.1**			**0.25**			**0.5**			**0.75**			**0.9**		
	Metric	***ARL***	***SDRL***	***MRL***	***ARL***	***SDRL***	***MRL***	***ARL***	***SDRL***	***MRL***	***ARL***	***SDRL***	***MRL***	***ARL***	***SDRL***	***MRL***	***ARL***	***SDRL***	***MRL***
***p* =2**	0.00	200.5	245.9	116.0	200.5	221.4	130.0	199.7	202.8	137.0	200.1	199.1	139.0	199.9	199.9	139.0	200.0	198.4	139.0
	0.25	48.6	50.2	35.0	60.0	59.3	43.0	84.6	83.2	59.0	116.1	116.1	81.0	145.0	144.1	101.0	162.7	163.5	112.0
	0.50	17.5	15.7	14.0	20.7	17.0	17.0	28.7	25.3	22.0	45.9	43.8	32.0	71.7	70.3	51.0	96.6	95.8	68.0
	0.75	9.4	8.1	7.0	10.9	8.2	9.0	13.6	10.4	11.0	20.3	18.3	15.0	34.8	33.6	25.0	52.1	51.2	36.0
	1.00	5.8	4.8	4.0	6.8	4.9	6.0	8.1	5.5	7.0	10.9	8.9	8.0	18.0	16.8	13.0	29.5	28.8	21.0
	1.25	3.9	3.1	3.0	4.7	3.3	4.0	5.5	3.5	5.0	6.8	4.8	6.0	10.4	9.0	8.0	16.5	15.5	12.0
	1.50	3.0	2.3	2.0	3.5	2.4	3.0	4.1	2.5	4.0	4.8	3.1	4.0	6.7	5.5	5.0	10.4	9.6	8.0
	1.75	2.3	1.6	2.0	2.7	1.8	2.0	3.2	1.9	3.0	3.6	2.1	3.0	4.7	3.5	4.0	6.8	5.8	5.0
	2.00	1.9	1.2	1.0	2.2	1.4	2.0	2.6	1.5	2.0	2.9	1.6	3.0	3.5	2.4	3.0	4.8	3.9	4.0
	2.25	1.6	1.0	1.0	1.9	1.1	2.0	2.2	1.2	2.0	2.4	1.2	2.0	2.8	1.7	2.0	3.5	2.7	3.0
	2.50	1.4	0.7	1.0	1.6	0.9	1.0	1.9	0.9	2.0	2.0	1.0	2.0	2.3	1.3	2.0	2.8	1.9	2.0
	2.75	1.3	0.6	1.0	1.4	0.7	1.0	1.6	0.8	1.0	1.8	0.8	2.0	2.0	1.0	2.0	2.2	1.4	2.0
	3.00	1.2	0.5	1.0	1.3	0.6	1.0	1.5	0.7	1.0	1.6	0.7	1.0	1.7	0.8	2.0	1.9	1.1	2.0
	h_{MDEWMA}		**4.924**			**6.229**			**8.278**			**9.879**			**10.492**			**10.603**	
***p* =3**	0.00	199.9	247.1	115.0	199.1	219.2	129.5	199.8	207.0	137.0	199.7	199.9	138.0	200.2	198.2	140.0	200.3	198.5	140.0
	0.25	55.4	57.5	39.0	68.3	67.6	49.0	95.9	94.3	67.0	128.7	127.3	89.0	154.0	154.2	106.0	168.7	168.6	117.0
	0.50	20.0	17.7	16.0	23.1	19.2	19.0	33.2	29.8	25.0	54.3	52.0	39.0	84.5	83.1	59.0	109.1	107.9	76.0
	0.75	10.4	8.9	8.0	12.2	9.0	11.0	15.3	11.9	12.0	24.2	22.0	18.0	42.4	41.0	30.0	64.0	63.1	44.0
	1.00	6.6	5.3	5.0	7.6	5.4	7.0	9.1	6.2	8.0	12.9	10.6	10.0	22.5	21.0	16.0	36.3	35.3	25.0
	1.25	4.5	3.5	4.0	5.2	3.6	5.0	6.1	3.8	5.0	7.7	5.7	6.0	12.7	11.3	9.0	20.9	20.1	15.0
	1.50	3.3	2.5	3.0	3.9	2.6	3.0	4.6	2.7	4.0	5.4	3.5	5.0	8.1	6.7	6.0	13.0	12.3	9.0
	1.75	2.6	1.8	2.0	3.0	1.9	3.0	3.5	2.0	3.0	4.1	2.4	4.0	5.5	4.2	4.0	8.4	7.5	6.0
	2.00	2.1	1.4	2.0	2.5	1.5	2.0	2.9	1.6	3.0	3.2	1.7	3.0	4.0	2.7	3.0	5.8	4.8	4.0
	2.25	1.8	1.1	1.0	2.0	1.2	2.0	2.4	1.3	2.0	2.6	1.3	2.0	3.1	1.9	3.0	4.2	3.4	3.0
	2.50	1.5	0.8	1.0	1.8	1.0	1.0	2.1	1.0	2.0	2.3	1.1	2.0	2.6	1.5	2.0	3.2	2.3	3.0
	2.75	1.4	0.7	1.0	1.5	0.8	1.0	1.8	0.9	2.0	2.0	0.9	2.0	2.2	1.1	2.0	2.6	1.7	2.0
	3.00	1.3	0.5	1.0	1.4	0.6	1.0	1.6	0.7	1.0	1.7	0.8	2.0	1.9	0.9	2.0	2.1	1.3	2.0
	h_{MDEWMA}		**6.677**			**8.111**			**10.375**			**12.092**			**12.729**			**12.825**	
***p* =4**	0.00	200.5	242.1	116.0	200.2	219.4	132.0	200.3	205.3	136.0	200.3	201.9	138.0	200.0	199.0	139.0	200.8	200.4	139.0
	0.25	58.1	60.6	41.0	73.1	73.6	52.0	104.0	103.4	72.0	136.0	135.9	94.0	159.9	161.1	110.0	173.3	173.8	119.0
	0.50	21.2	18.8	17.0	25.2	20.7	21.0	36.8	32.9	28.0	61.3	58.8	43.0	95.7	94.9	67.0	120.6	118.7	84.0
	0.75	11.4	9.5	9.0	13.2	9.7	12.0	16.8	13.2	14.0	27.6	25.5	20.0	49.4	48.2	34.0	73.5	71.7	51.0
	1.00	7.1	5.7	6.0	8.3	5.8	7.0	10.0	6.8	9.0	14.5	12.1	11.0	26.6	25.4	19.0	42.7	41.5	30.0
	1.25	4.8	3.7	4.0	5.6	3.9	5.0	6.6	4.1	6.0	8.5	6.3	7.0	14.6	12.9	11.0	25.3	24.5	18.0
	1.50	3.6	2.7	3.0	4.3	2.8	4.0	4.9	2.9	4.0	6.0	4.0	5.0	9.2	7.7	7.0	15.8	14.7	11.0
	1.75	2.8	2.0	2.0	3.3	2.1	3.0	3.9	2.2	4.0	4.4	2.6	4.0	6.2	4.8	5.0	10.0	9.1	7.0
	2.00	2.3	1.5	2.0	2.6	1.6	2.0	3.1	1.7	3.0	3.5	1.9	3.0	4.5	3.2	4.0	6.8	5.8	5.0
	2.25	1.9	1.2	2.0	2.2	1.3	2.0	2.6	1.4	2.0	2.9	1.5	3.0	3.4	2.2	3.0	4.8	3.9	4.0
	2.50	1.6	0.9	1.0	1.9	1.0	2.0	2.2	1.1	2.0	2.4	1.2	2.0	2.8	1.6	2.0	3.6	2.7	3.0
	2.75	1.5	0.8	1.0	1.6	0.9	1.0	1.9	0.9	2.0	2.1	1.0	2.0	2.3	1.3	2.0	2.9	2.0	2.0
	3.00	1.3	0.6	1.0	1.5	0.7	1.0	1.7	0.8	2.0	1.8	0.8	2.0	2.0	1.0	2.0	2.3	1.5	2.0
	h_{MDEWMA}		**8.253**			**9.853**			**12.295**			**14.099**			**14.751**			**14.847**	
***p* =10**	0.00	200.2	244.7	116.0	200.7	218.8	133.0	199.7	203.6	138.0	200.2	200.8	138.0	200.1	199.9	139.0	200.8	200.3	139.0
	0.25	73.9	78.6	52.0	93.1	94.5	65.0	126.9	127.2	88.0	159.2	158.4	111.0	177.2	179.5	122.0	185.9	188.1	127.0
	0.50	28.2	24.5	23.0	34.4	29.3	28.0	52.9	49.3	38.0	87.3	86.5	60.0	125.0	124.8	86.0	149.0	148.5	103.0
	0.75	15.1	12.2	13.0	17.5	12.5	16.0	24.4	20.0	19.0	43.8	41.7	31.0	76.6	76.0	53.0	104.8	104.0	73.0
	1.00	9.5	7.3	8.0	10.9	7.4	10.0	13.7	9.7	12.0	23.0	20.3	17.0	45.0	43.3	32.0	69.6	69.6	49.0
	1.25	6.5	4.9	5.0	7.6	4.9	7.0	8.9	5.5	8.0	13.1	10.7	10.0	25.9	24.5	18.0	44.8	44.7	31.0
	1.50	4.8	3.5	4.0	5.6	3.6	5.0	6.4	3.7	6.0	8.5	6.0	7.0	15.8	14.3	11.0	28.4	27.5	20.0
	1.75	3.8	2.6	3.0	4.3	2.7	4.0	5.0	2.7	5.0	6.0	3.8	5.0	10.1	8.7	8.0	18.4	17.6	13.0
	2.00	3.0	2.0	2.0	3.5	2.1	3.0	4.0	2.1	4.0	4.6	2.6	4.0	6.9	5.4	5.0	12.1	11.0	9.0
	2.25	2.5	1.6	2.0	2.9	1.7	3.0	3.3	1.7	3.0	3.7	1.9	3.0	5.1	3.7	4.0	8.3	7.3	6.0
	2.50	2.1	1.3	2.0	2.4	1.4	2.0	2.8	1.4	3.0	3.1	1.5	3.0	3.9	2.5	3.0	6.0	5.0	4.0
	2.75	1.8	1.0	2.0	2.1	1.1	2.0	2.4	1.2	2.0	2.6	1.2	2.0	3.1	1.9	3.0	4.5	3.5	3.0
	3.00	1.6	0.8	1.0	1.8	1.0	2.0	2.1	1.0	2.0	2.3	1.0	2.0	2.6	1.4	2.0	3.5	2.5	3.0
	h_{MDEWMA}		**16.797**			**18.962**			**22.073**			**24.347**			**25.085**			**25.179**	

Table 3. *ARL*, *SDRL* and *MRL* profile of the time-varying MTEWMA control chart along with the control limits when $\lambda \in \{0.05,0.1,0.25,0.5,0.75,0.9\}$, n=1 and $p \in \{2,3,4,10\}$ for a nominal $ARL_0 = 200$

	λ	0.05			0.1			0.25			0.5			0.75			0.9		
	Metric	ARL	$SDRL$	MRL	ARL	$SDRL$	MRL	ARL	$SDRL$	MRL	ARL	$SDRL$	MRL	ARL	$SDRL$	MRL	ARL	$SDRL$	MRL
p =2	0.00	200.0	258.4	106.0	200.7	228.2	128.0	200.1	206.2	136.0	200.2	200.9	138.0	200.1	199.5	139.0	200.4	199.3	139.0
	0.25	47.5	50.0	34.0	56.6	55.5	42.0	77.1	74.6	55.0	105.8	105.0	74.0	135.6	135.4	94.0	156.9	157.4	109.0
	0.50	18.0	17.2	13.0	20.8	17.1	18.0	26.1	22.1	20.0	39.1	37.0	28.0	61.7	60.5	44.0	88.4	87.0	62.0
	0.75	9.9	9.3	7.0	11.3	8.9	10.0	13.0	9.4	11.0	17.4	15.0	13.0	28.2	26.9	20.0	46.5	45.4	32.0
	1.00	6.0	5.6	4.0	7.2	5.6	6.0	8.1	5.3	7.0	9.6	7.3	8.0	14.7	13.1	11.0	24.7	23.7	18.0
	1.25	4.1	3.7	3.0	4.8	3.8	4.0	5.6	3.5	5.0	6.1	4.1	5.0	8.6	7.1	6.0	14.2	13.0	10.0
	1.50	3.0	2.6	2.0	3.6	2.8	3.0	4.2	2.6	4.0	4.5	2.7	4.0	5.7	4.3	5.0	8.9	7.9	7.0
	1.75	2.3	1.9	2.0	2.8	2.1	2.0	3.3	2.1	3.0	3.5	2.0	3.0	4.1	2.8	3.0	5.9	5.0	4.0
	2.00	1.9	1.4	1.0	2.2	1.5	2.0	2.6	1.6	2.0	2.8	1.5	3.0	3.2	1.9	3.0	4.2	3.3	3.0
	2.25	1.6	1.0	1.0	1.8	1.2	1.0	2.2	1.3	2.0	2.4	1.2	2.0	2.6	1.4	2.0	3.2	2.2	3.0
	2.50	1.4	0.8	1.0	1.6	0.9	1.0	1.8	1.0	1.0	2.0	1.0	2.0	2.2	1.1	2.0	2.5	1.6	2.0
	2.75	1.2	0.6	1.0	1.4	0.7	1.0	1.6	0.8	1.0	1.8	0.8	2.0	1.9	0.9	2.0	2.1	1.2	2.0
	3.00	1.2	0.4	1.0	1.2	0.6	1.0	1.4	0.7	1.0	1.6	0.7	1.0	1.7	0.7	2.0	1.7	0.7	2.0
	h_{MTEWMA}	**4.111**			**5.297**			**7.307**			**9.177**			**10.317**			**10.584**		
p =3	0.00	200.0	259.4	105.5	200.5	227.1	127.0	200.3	209.0	135.0	199.7	201.2	138.0	200.5	200.9	140.0	200.2	197.6	140.0
	0.25	53.2	55.9	38.0	63.6	63.3	47.0	88.1	87.0	62.0	117.4	116.0	82.0	145.6	145.6	100.0	164.1	163.7	114.0
	0.50	20.5	19.2	16.0	23.2	18.9	20.0	30.2	25.9	23.0	45.9	43.2	33.0	73.0	71.2	51.0	100.6	99.3	70.0
	0.75	11.1	10.2	8.0	12.6	9.6	11.0	14.4	10.6	12.0	20.3	17.7	15.0	34.5	32.8	24.0	56.6	56.0	39.0
	1.00	6.9	6.2	5.0	8.0	6.1	7.0	9.0	5.9	8.0	11.0	8.5	9.0	17.9	16.3	13.0	31.1	30.1	22.0
	1.25	4.6	4.1	3.0	5.5	4.2	5.0	6.2	3.8	6.0	6.9	4.7	6.0	10.2	8.6	8.0	17.3	16.2	12.0
	1.50	3.4	2.9	2.0	4.1	3.1	3.0	4.7	2.9	4.0	5.1	3.1	4.0	6.7	5.2	5.0	11.1	9.8	8.0
	1.75	2.6	2.1	2.0	3.1	2.3	2.0	3.6	2.2	3.0	3.9	2.2	4.0	4.7	3.3	4.0	7.2	6.1	5.0
	2.00	2.1	1.6	1.0	2.4	1.7	2.0	2.9	1.8	3.0	3.1	1.7	3.0	3.6	2.2	3.0	5.0	4.0	4.0
	2.25	1.7	1.2	1.0	2.0	1.4	1.0	2.4	1.4	2.0	2.6	1.3	2.0	2.9	1.6	2.0	3.7	2.7	3.0
	2.50	1.5	0.9	1.0	1.7	1.1	1.0	2.0	1.1	2.0	2.2	1.1	2.0	2.4	1.2	2.0	2.9	1.9	2.0
	2.75	1.3	0.7	1.0	1.5	0.8	1.0	1.7	0.9	1.0	1.9	0.9	2.0	2.0	1.0	2.0	2.4	1.5	2.0
	3.00	1.2	0.5	1.0	1.3	0.7	1.0	1.5	0.8	1.0	1.7	0.8	2.0	1.8	0.8	2.0	2.0	1.1	2.0
	h_{MTEWMA}	**5.711**			**7.079**			**9.307**			**11.350**			**12.561**			**12.807**		
p =4	0.00	200.1	258.6	105.0	200.1	224.6	129.0	200.2	206.7	136.0	200.5	199.1	140.0	200.4	201.3	138.0	200.5	198.3	140.0
	0.25	56.3	58.9	40.0	68.2	67.9	50.0	94.0	93.6	66.0	126.2	124.3	88.0	152.1	151.0	106.0	168.5	170.1	117.0
	0.50	21.9	20.4	17.0	25.1	20.1	22.0	33.2	29.1	25.0	51.7	49.5	37.0	81.7	80.5	57.0	112.0	109.6	79.0
	0.75	12.0	10.9	9.0	13.7	10.3	12.0	15.8	11.6	13.0	23.0	20.5	17.0	40.5	39.2	28.0	64.6	63.1	45.0
	1.00	7.5	6.7	6.0	8.8	6.6	8.0	9.7	6.3	9.0	12.4	9.6	10.0	20.9	19.2	15.0	36.6	36.0	25.0
	1.25	5.0	4.4	4.0	6.0	4.5	5.0	6.6	4.1	6.0	7.6	5.3	6.0	11.7	10.1	9.0	21.0	20.0	15.0
	1.50	3.8	3.2	3.0	4.4	3.3	4.0	5.0	3.0	5.0	5.4	3.3	5.0	7.6	6.0	6.0	12.9	11.7	9.0
	1.75	2.9	2.3	2.0	3.4	2.5	3.0	3.9	2.3	4.0	4.2	2.3	4.0	5.2	3.7	4.0	8.5	7.4	6.0
	2.00	2.3	1.7	2.0	2.7	1.9	2.0	3.1	1.9	3.0	3.3	1.8	3.0	3.9	2.5	3.0	5.8	4.8	4.0
	2.25	1.9	1.3	1.0	2.2	1.4	2.0	2.6	1.5	2.0	2.8	1.4	3.0	3.1	1.8	3.0	4.2	3.2	3.0
	2.50	1.6	1.0	1.0	1.8	1.2	1.0	2.2	1.3	2.0	2.3	1.1	2.0	2.6	1.4	2.0	3.2	2.3	3.0
	2.75	1.4	0.8	1.0	1.6	0.9	1.0	1.9	1.0	2.0	2.0	1.0	2.0	2.2	1.1	2.0	2.6	1.6	2.0
	3.00	1.3	0.6	1.0	1.4	0.7	1.0	1.6	0.9	1.0	1.8	0.8	2.0	1.9	0.9	2.0	2.2	1.3	2.0
	h_{MTEWMA}	**7.202**			**8.712**			**11.135**			**13.324**			**14.567**			**14.812**		
p =10	0.00	200.5	257.9	108.0	200.5	224.9	129.0	200.6	206.7	138.0	200.3	202.7	138.0	200.1	199.6	138.0	200.2	198.8	139.0
	0.25	70.8	76.4	51.0	86.4	88.0	61.0	118.2	118.4	81.0	149.3	149.3	104.0	172.1	172.4	119.0	182.3	184.2	125.0
	0.50	28.5	25.6	23.0	32.9	27.0	28.0	46.4	42.0	34.0	75.7	73.4	53.0	112.4	112.6	78.0	141.2	141.1	98.0
	0.75	15.9	13.8	13.0	17.7	12.8	16.0	22.0	17.1	18.0	35.8	33.4	26.0	64.0	62.8	45.0	95.5	94.9	66.0
	1.00	10.2	8.6	8.0	11.5	8.1	10.0	13.0	8.5	12.0	18.4	15.6	14.0	35.0	33.8	24.0	61.1	60.6	42.0
	1.25	6.9	5.8	5.0	8.0	5.7	7.0	8.9	5.3	8.0	11.2	8.3	9.0	20.0	18.3	14.0	37.6	36.3	26.0
	1.50	5.1	4.2	4.0	6.0	4.2	5.0	6.6	3.7	6.0	7.5	4.9	6.0	12.1	10.3	9.0	23.5	22.7	17.0
	1.75	3.9	3.1	3.0	4.6	3.2	4.0	5.1	2.9	5.0	5.6	3.2	5.0	8.0	6.3	6.0	15.1	14.0	11.0
	2.00	3.0	2.4	2.0	3.6	2.5	3.0	4.1	2.3	4.0	4.3	2.3	4.0	5.7	4.0	5.0	9.9	8.7	7.0
	2.25	2.5	1.8	2.0	2.9	2.0	2.0	3.4	1.9	3.0	3.5	1.8	3.0	4.3	2.7	4.0	6.9	5.8	5.0
	2.50	2.1	1.4	1.0	2.4	1.6	2.0	2.8	1.6	3.0	3.0	1.4	3.0	3.4	2.0	3.0	5.1	3.9	4.0
	2.75	1.8	1.2	1.0	2.1	1.3	2.0	2.4	1.3	2.0	2.6	1.2	2.0	2.8	1.5	3.0	3.9	2.7	3.0
	3.00	1.5	0.9	1.0	1.8	1.1	1.0	2.1	1.1	2.0	2.3	1.0	2.0	2.4	1.2	2.0	3.1	2.1	3.0
	h_{MTEWMA}	**15.304**			**17.395**			**20.594**			**23.389**			**24.906**			**25.148**		

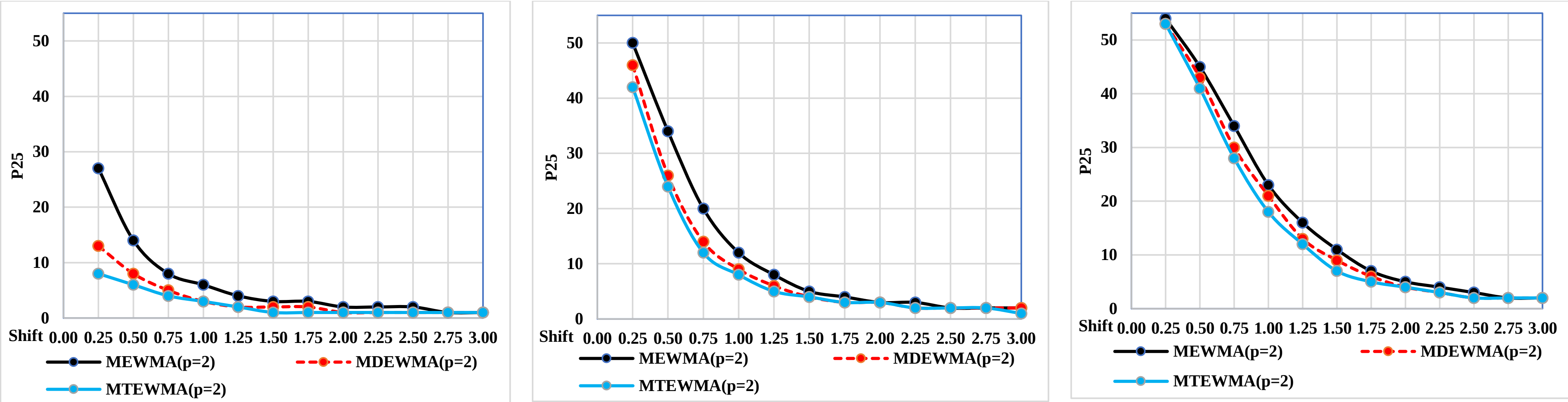
Figures 1 and 2 compare the three time-varying schemes (i.e. the time-varying MEWMA, MDEWMA and MTEWMA schemes) in terms of the P_{25} and P_{95} profiles when when *n*=1, $\lambda \in \{0.05,0.5,0.9\}$ for p =2 and 10, respectively, with a nominal $ARL_0 = 200$. To picture the difference between different cases, we use the same scale to construct the graphs. From Figures 1 and 2, it can be seen that the MTEWMA scheme has a better PRL profile compared to the MEWMA and MDEWMA schemes from small to moderate shifts in the process mean vector. However, for large shifts, the three schemes are almost similar in terms of the *PRL* profile. The MDEWMA scheme outperforms the MEWMA scheme for small and moderate shifts. As λ increases, the values of the P_{25} and P_{95} profiles increase as well (see Figures 1 (a)-(c), Figures 1 (d)-(f), Figures 2 (a)-(c) and Figures 2 (d)-(f)). The larger the p value, the larger the *PRL* values.



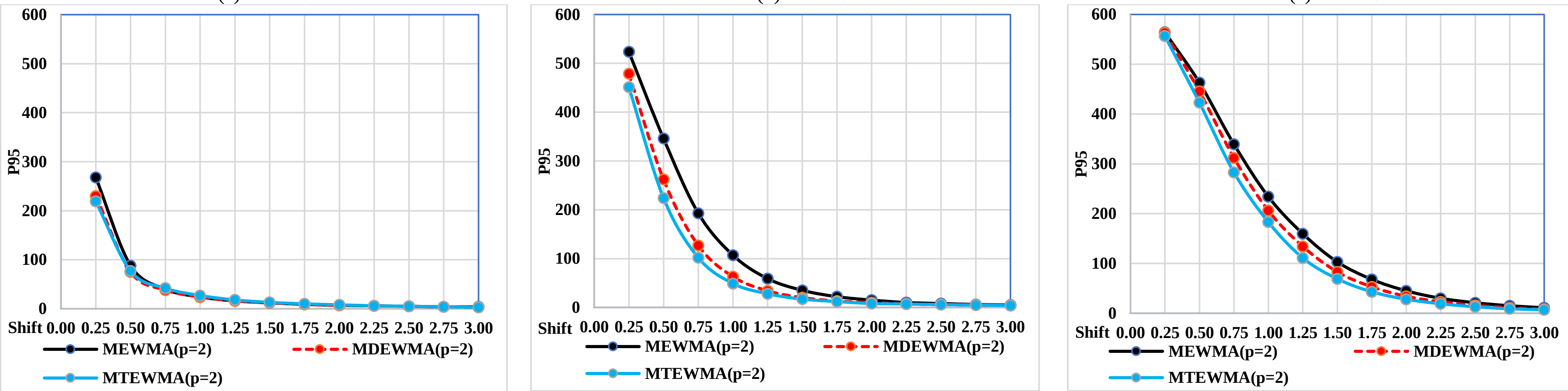
(a) $\lambda = 0.05$ (b) $\lambda = 0.5$ (c) $\lambda = 0.9$

(d) $\lambda = 0.05$ (e) $\lambda = 0.5$ (f) $\lambda = 0.9$

Figure 1. The comparison of the performance of the time-varying MEWMA, MDEWMA and TEWMA scheme in terms of the OOC *PRL* when $\lambda = 0.05$, 0.5 and 0.9 for p=2



(a) $\lambda = 0.05$ (b) $\lambda = 0.5$ (c) $\lambda = 0.9$

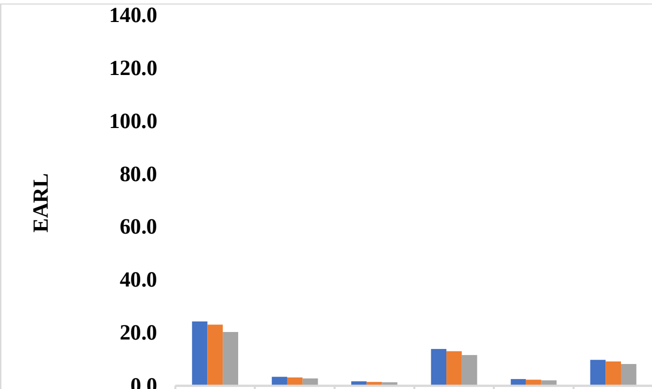


(d) $\lambda = 0.05$ (e) $\lambda = 0.5$ (f) $\lambda = 0.9$

Figure 2. The comparison of the performance of the time-varying MEWMA, MDEWMA and TEWMA scheme in terms of the OOC *PRL* when $\lambda = 0.05$, 0.5 and 0.9 for p =10

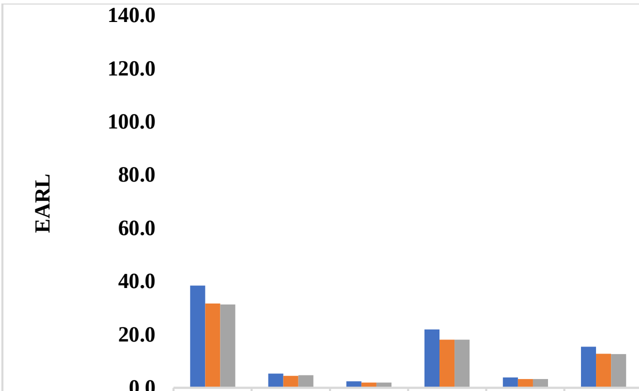
Figure 3 compares the performances of the time-varying MEWMA, MDEWMA and MTEWMA schemes for different ranges of shifts in terms of the *EARL* profile. In Figure 3, these schemes are compared for small, moderate, large, small-to-moderate, moderate-to-large and small-to-large shifts denoted by S, M, L, S-to-M, M-to-L and S-to-L where $\delta \in (0,1]$, $\delta \in (1,2]$, $\delta \in (2,3]$, $\delta \in (0,2]$, $\delta \in (1,3]$ and $\delta \in (0,3]$, respectively. In this paper, we only display the results of $p \in \{2,10\}$ and $p \in \{0.05,0.5,0.9\}$ to preserve space.

From Figure 3, it can be observed that the time-varying MTEWMA scheme outperforms both the MEWMA and MDEWMA schemes regardless of the values of p, λ and range of shifts under investigation except for large values of p combined with smaller values of λ where the MDEWMA scheme performs slightly better than the MTEWMA scheme for moderate shifts; however, for these two schemes are similar in performance for other ranges of shifts. In terms of the EARL profile, the MDEWMA scheme outperforms the MEWMA scheme for all ranges of shifts. Moreover, for all these three schemes, the smaller the value of p, the better the overall performance. The larger the value of λ, the worst the overall performance of the three schemes.



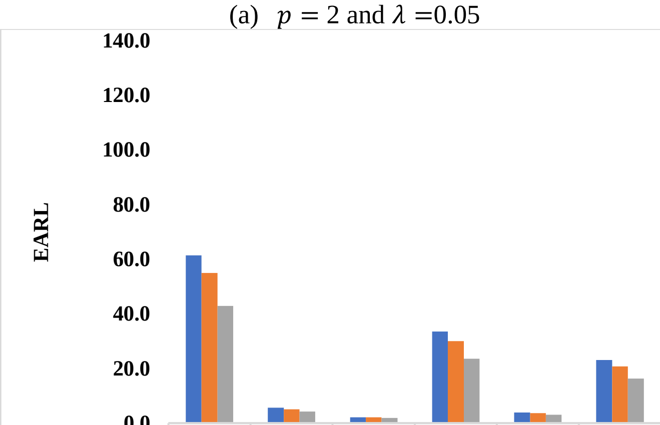
	S	M	L	S-to-M	M-to-L	S-to-L
MEWMA	24.4	3.4	1.7	13.9	2.5	9.8
MDEWMA	23.1	3.1	1.5	13.1	2.3	9.2
MTEWMA	20.4	2.8	1.3	11.6	2.1	8.2

(a) $p = 2$ and $\lambda = 0.05$

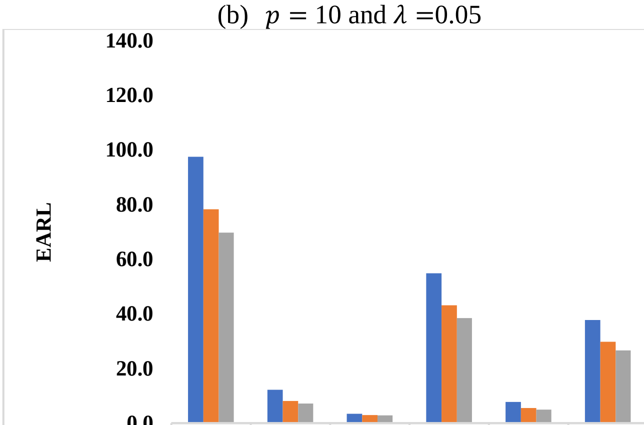


	S	M	L	S-to-M	M-to-L	S-to-L
MEWMA	38.5	5.3	2.4	21.9	3.9	15.4
MDEWMA	31.7	4.5	2.0	18.1	3.3	12.7
MTEWMA	31.4	4.7	2.0	18.0	3.3	12.7

(b) $p = 10$ and $\lambda = 0.05$

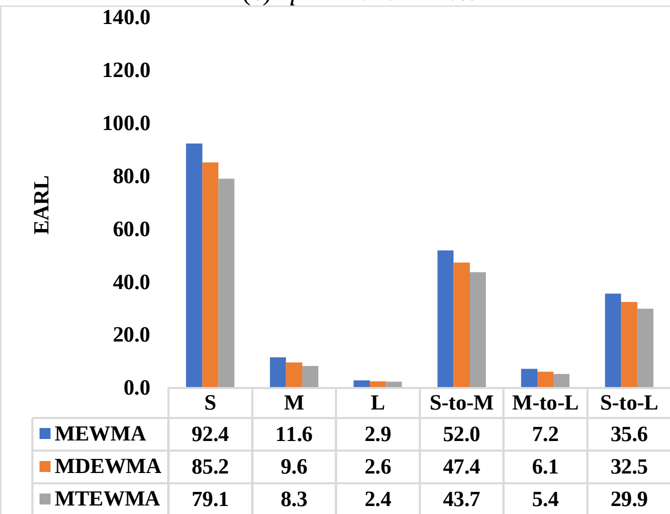


	S	M	L	S-to-M	M-to-L	S-to-L
MEWMA	61.5	5.7	2.1	33.6	3.9	23.1
MDEWMA	55.0	5.1	2.1	30.1	3.6	20.8
MTEWMA	43.0	4.2	1.9	23.6	3.1	16.4



	S	M	L	S-to-M	M-to-L	S-to-L
MEWMA	97.5	12.2	3.4	54.9	7.8	37.7
MDEWMA	78.3	8.1	2.9	43.2	5.5	29.8
MTEWMA	69.8	7.1	2.8	38.5	5.0	26.6

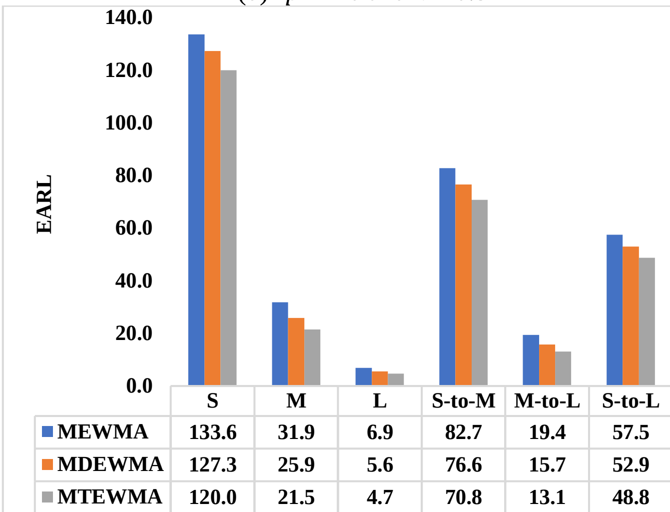
(c) $p\,=2$ and λ =0.5



	S	M	L	S-to-M	M-to-L	S-to-L
MEWMA	92.4	11.6	2.9	52.0	7.2	35.6
MDEWMA	85.2	9.6	2.6	47.4	6.1	32.5
MTEWMA	79.1	8.3	2.4	43.7	5.4	29.9

(e) $p\,=2$ and λ =0.9

(d) $p\,=10$ and λ =0.5



	S	M	L	S-to-M	M-to-L	S-to-L
MEWMA	133.6	31.9	6.9	82.7	19.4	57.5
MDEWMA	127.3	25.9	5.6	76.6	15.7	52.9
MTEWMA	120.0	21.5	4.7	70.8	13.1	48.8

(f) $p\,=10$ and λ =0.9

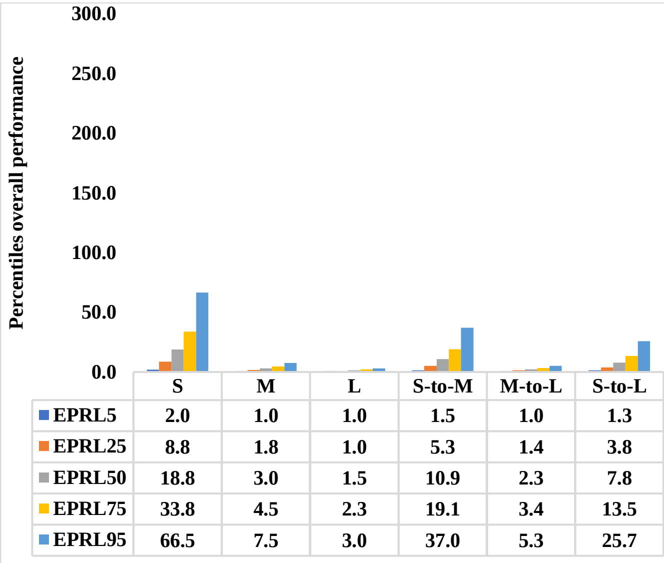
Figure 3. Overall performance comparison of the time-varying MEWMA, MDEWMA and MTEWMA schemes in terms of the *EARL* profile when *n*=1, *p*=2 and 10 and λ =0.05, 0.5 and 0.9

In Figure 4, the performances proposed time-varying MEWMA, MDEWMA and MTEWMA schemes are investigated in terms of the *EPRL* values for all the ranges considered in this paper when $n\,=1$, $p\,=2$ and $\lambda\in\{0.05,0.9\}$. The results in Figure 4 can be summarised as followed:

- When $\lambda=0.05$ and 0.9, for small shifts in the process mean vector, on average, there is 5% chance that the time-varying MEWMA scheme gives a signal on the 2nd and 6th samples, respectively. In the same situation, the MDEWMA and MTEWMA schemes are expected to give a signal on the 1st and 5th samples on average. Moreover, when $\lambda=$ 0.05 and 0.9, for small shifts in the process mean vector, on average, there is 95% chance that the time-varying MEWMA scheme gives a signal on the 67th and 277th samples, respectively. In the same situation, the MDEWMA scheme is expected to give a signal on the 59th and 255th samples on average, while the MTEWMA scheme is expected to give a signal on the 61st and 235th sample on average (see Figures 4 (a)-(b)). These findings show that larger values of λ deteriorate the performance of the proposed schemes in monitoring small shifts and the MEWMA scheme is outperformed by the MDEWMA and MTEWMA schemes for small shifts (see Figures (d)-(f)). Other percentiles can be interpreted in a similar way.
- For large shifts, when $\lambda=0.05$ and 0.9, on average, there is 5% chance that the three schemes give a signal on the 1st sample in both cases (see Figures 4 (a)-(b)). Moreover, when $\lambda=0.05$ and 0.9, for small shifts in the process mean vector, on average, there is 95% chance that the time-varying MEWMA scheme gives a signal on the 3rd and 8th samples, respectively. In the same situation, the MDEWMA and MTEWMA schemes

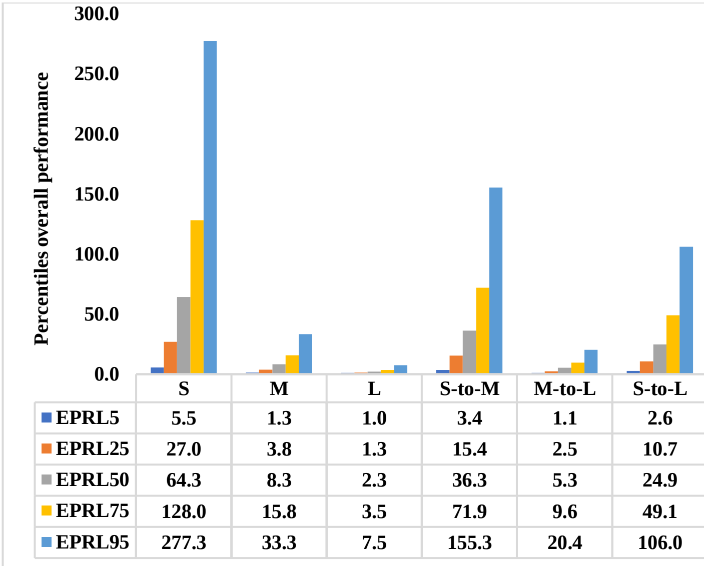
are expected to give a signal on the 3rd and 6th samples on average (see Figures 4 (c)-(f)).

- The results in Figure 4 also show that there is more chance to get a signal earlier when using a small value of λ than using a larger one. For instance, for small-to-moderate shifts in the process mean vector, there is 95% chance that the time-varying MTEWMA scheme gives a signal on the 37th sample when $\lambda = 0.05$; whereas, there 95% that it gives a signal on 155th sample when $\lambda = 0.9$.
- Also note that there is 95% that on average the time-varying MTEWMA scheme will give a signal on sample number 61, 8, 3, 34, 5 and 24 for S, M, L, S-to-M, M-to-L and S-to-L shifts when $\lambda = 0.05$, respectively. However, when $\lambda = 0.9$, it will give a signal on sample number 235, 23, 6, 129, 14 and 80, respectively.



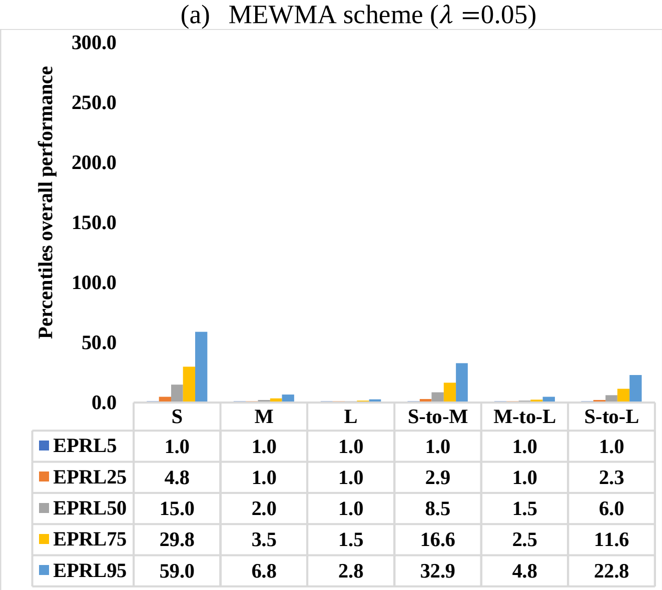
	S	M	L	S-to-M	M-to-L	S-to-L
EPRL5	2.0	1.0	1.0	1.5	1.0	1.3
EPRL25	8.8	1.8	1.0	5.3	1.4	3.8
EPRL50	18.8	3.0	1.5	10.9	2.3	7.8
EPRL75	33.8	4.5	2.3	19.1	3.4	13.5
EPRL95	66.5	7.5	3.0	37.0	5.3	25.7

(a) MEWMA scheme (λ =0.05)



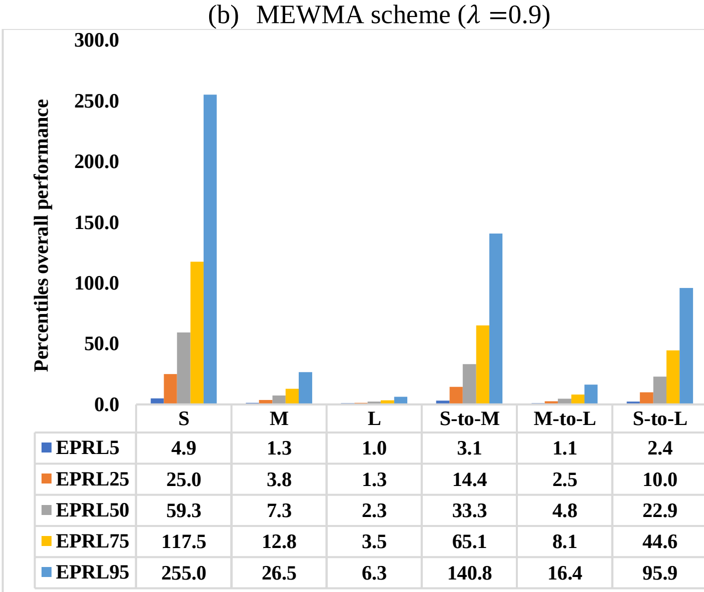
	S	M	L	S-to-M	M-to-L	S-to-L
EPRL5	5.5	1.3	1.0	3.4	1.1	2.6
EPRL25	27.0	3.8	1.3	15.4	2.5	10.7
EPRL50	64.3	8.3	2.3	36.3	5.3	24.9
EPRL75	128.0	15.8	3.5	71.9	9.6	49.1
EPRL95	277.3	33.3	7.5	155.3	20.4	106.0

(b) MEWMA scheme (λ =0.9)



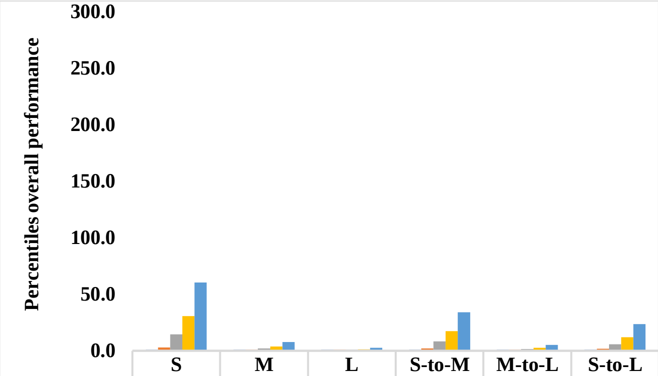
	S	M	L	S-to-M	M-to-L	S-to-L
EPRL5	1.0	1.0	1.0	1.0	1.0	1.0
EPRL25	4.8	1.0	1.0	2.9	1.0	2.3
EPRL50	15.0	2.0	1.0	8.5	1.5	6.0
EPRL75	29.8	3.5	1.5	16.6	2.5	11.6
EPRL95	59.0	6.8	2.8	32.9	4.8	22.8

(c) MDEWMA scheme (λ =0.05)



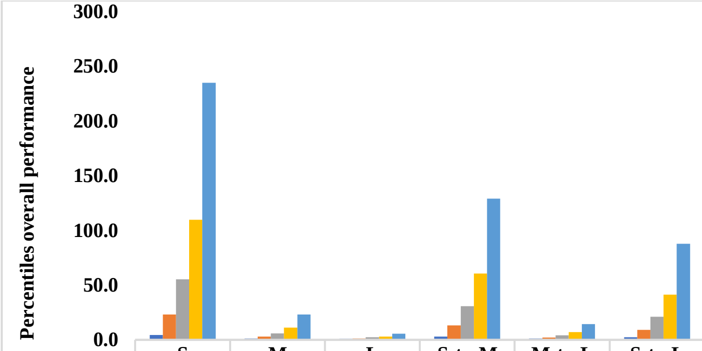
	S	M	L	S-to-M	M-to-L	S-to-L
EPRL5	4.9	1.3	1.0	3.1	1.1	2.4
EPRL25	25.0	3.8	1.3	14.4	2.5	10.0
EPRL50	59.3	7.3	2.3	33.3	4.8	22.9
EPRL75	117.5	12.8	3.5	65.1	8.1	44.6
EPRL95	255.0	26.5	6.3	140.8	16.4	95.9

(d) MDEWMA scheme (λ =0.9)



	S	M	L	S-to-M	M-to-L	S-to-L
EPRL5	1.0	1.0	1.0	1.0	1.0	1.0
EPRL25	3.0	1.0	1.0	2.0	1.0	1.7
EPRL50	14.5	2.0	1.0	8.3	1.5	5.8
EPRL75	30.8	3.8	1.3	17.3	2.5	11.9
EPRL95	60.5	7.8	2.8	34.1	5.3	23.7

(e) MTEWMA scheme (λ =0.05)



	S	M	L	S-to-M	M-to-L	S-to-L
EPRL5	4.5	1.3	1.0	2.9	1.1	2.3
EPRL25	23.3	3.0	1.3	13.1	2.1	9.2
EPRL50	55.3	6.0	2.3	30.6	4.1	21.2
EPRL75	109.8	11.3	3.0	60.5	7.1	41.3
EPRL95	235.0	23.3	5.5	129.1	14.4	87.9

(f) MTEWMA scheme (λ =0.9)

Figure 4. Overall performance of the proposed schemes in terms of the *EPRL* when n=1, p=2 and $\lambda \in\{0.05,0.9\}$

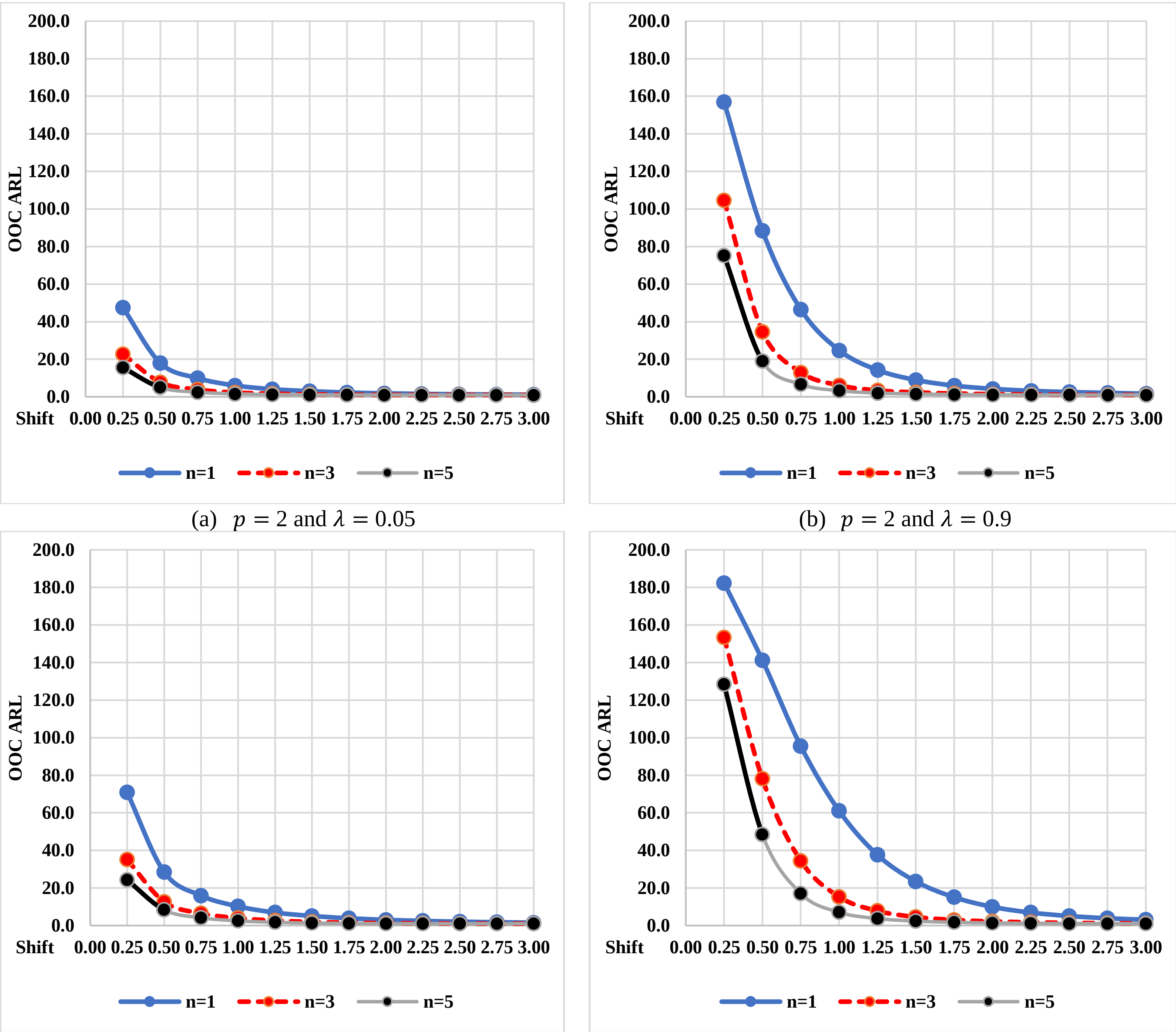
Next, Since the MTEWMA scheme outperforms the MEWMA and MDEWMA schemes in many cases, in the remainder of this paper, we will focus on the MTEWMA scheme. Thus, let us first investigate the effect of the sample size on the performance of the proposed time-varying MTEWMA scheme before investigating further its properties.

Table 4 displays the performance of the MTEWMA control charts in terms of the *ARL*, *SDRL* and *MRL* profiles along with the control limits when n=3, $\lambda \in\{0.05,0.1,0.25,0.5,0.75,0.9\}$ and $p \in\{2,3,4,10\}$ for nominal ARL_0 of 200 to investigate the effect of the sample size. From Table 4, it can be seen that when the sample size increases, the control limit constants remain the same and the sensitivity of the proposed MTEWMA scheme increases as well. In other words, the MTEWMA scheme presents better OOC *ARL*, *SDRL* and *MRL* properties at the expense of higher costs related to the use of large sample sizes. The patterns in the performance results are similar to that of the MTEWMA scheme with n=1.

Table 4. *ARL*, *SDRL* and *MRL* profile of the time-varying MTEWMA control chart along with the control limits when $\lambda \in \{0.05,0.1,0.25,0.5,0.75,0.9\}$, *n*=3 and $p \in \{2,3,4,10\}$ for a nominal $ARL_0 = 200$

	λ	0.05			0.1			0.25			0.5			0.75			0.9		
	Metric	ARL	$SDRL$	MRL	ARL	$SDRL$	MRL	ARL	$SDRL$	MRL	ARL	$SDRL$	MRL	ARL	$SDRL$	MRL	ARL	$SDRL$	MRL
p =2	0.00	203.2	261.6	108.0	202.7	228.8	129.0	203.1	209.8	137.0	200.7	200.0	140.0	199.5	198.4	139.0	200.8	198.8	140.0
	0.25	22.8	21.6	17.0	25.9	21.6	22.0	33.3	29.2	25.0	50.1	48.1	35.0	76.1	73.7	54.0	104.7	103.7	73.0
	0.50	7.8	7.2	6.0	9.0	7.0	8.0	10.1	7.0	9.0	12.8	10.4	10.0	20.5	19.3	15.0	34.6	33.3	24.0
	0.75	3.9	3.5	3.0	4.6	3.6	4.0	5.3	3.3	5.0	5.8	3.7	5.0	8.0	6.4	6.0	13.0	11.9	9.0
	1.00	2.3	1.9	1.0	2.8	2.1	2.0	3.3	2.1	3.0	3.6	2.0	3.0	4.2	2.8	3.0	6.1	5.1	5.0
	1.25	1.7	1.1	1.0	1.9	1.3	1.0	2.3	1.4	2.0	2.5	1.3	2.0	2.7	1.5	2.0	3.4	2.5	3.0
	1.50	1.3	0.7	1.0	1.5	0.8	1.0	1.7	1.0	1.0	1.9	0.9	2.0	2.0	1.0	2.0	2.4	1.5	2.0
	1.75	1.1	0.4	1.0	1.2	0.5	1.0	1.4	0.7	1.0	1.5	0.7	1.0	1.6	0.7	2.0	1.8	1.0	2.0
	2.00	1.1	0.3	1.0	1.1	0.3	1.0	1.2	0.5	1.0	1.3	0.5	1.0	1.4	0.6	1.0	1.4	0.7	1.0
	2.25	1.0	0.2	1.0	1.0	0.2	1.0	1.1	0.3	1.0	1.2	0.4	1.0	1.2	0.4	1.0	1.2	0.5	1.0
	2.50	1.0	0.1	1.0	1.0	0.1	1.0	1.0	0.2	1.0	1.1	0.3	1.0	1.1	0.3	1.0	1.1	0.3	1.0
	2.75	1.0	0.1	1.0	1.0	0.1	1.0	1.0	0.1	1.0	1.0	0.2	1.0	1.1	0.2	1.0	1.1	0.2	1.0
	3.00	1.0	0.0	1.0	1.0	0.0	1.0	1.0	0.1	1.0	1.0	0.1	1.0	1.0	0.1	1.0	1.0	0.1	1.0
	h_{MTEWMA}	**4.111**			**5.297**			**7.307**			**9.177**			**10.317**			**10.584**		
p =3	0.00	201.0	264.1	104.0	203.0	227.4	131.0	200.5	207.9	135.0	200.0	201.4	136.0	202.3	202.3	139.0	200.6	199.2	139.0
	0.25	25.2	23.9	19.0	28.6	24.3	24.0	38.3	34.3	28.0	58.7	56.5	42.0	89.8	87.6	63.0	116.7	116.8	81.0
	0.50	8.7	8.0	6.0	10.1	7.7	9.0	11.3	7.7	10.0	14.9	12.3	11.0	25.0	23.5	18.0	42.7	41.7	30.0
	0.75	4.4	3.9	3.0	5.2	3.9	4.0	5.9	3.7	5.0	6.6	4.4	6.0	9.5	8.0	7.0	16.2	15.1	12.0
	1.00	2.6	2.2	2.0	3.1	2.3	2.0	3.7	2.2	3.0	3.9	2.2	4.0	4.8	3.3	4.0	7.4	6.3	5.0
	1.25	1.8	1.3	1.0	2.1	1.5	2.0	2.5	1.5	2.0	2.7	1.4	3.0	3.0	1.8	3.0	4.0	3.0	3.0
	1.50	1.4	0.8	1.0	1.6	1.0	1.0	1.9	1.1	2.0	2.1	1.0	2.0	2.2	1.1	2.0	2.7	1.7	2.0
	1.75	1.2	0.5	1.0	1.3	0.7	1.0	1.5	0.8	1.0	1.7	0.8	2.0	1.8	0.8	2.0	2.0	1.1	2.0
	2.00	1.1	0.3	1.0	1.2	0.4	1.0	1.3	0.5	1.0	1.4	0.6	1.0	1.5	0.6	1.0	1.6	0.7	1.0
	2.25	1.0	0.2	1.0	1.1	0.3	1.0	1.1	0.4	1.0	1.2	0.4	1.0	1.3	0.5	1.0	1.3	0.5	1.0
	2.50	1.0	0.1	1.0	1.0	0.2	1.0	1.1	0.2	1.0	1.1	0.3	1.0	1.2	0.4	1.0	1.2	0.4	1.0
	2.75	1.0	0.1	1.0	1.0	0.1	1.0	1.0	0.2	1.0	1.1	0.2	1.0	1.1	0.3	1.0	1.1	0.3	1.0
	3.00	1.0	0.0	1.0	1.0	0.1	1.0	1.0	0.1	1.0	1.0	0.1	1.0	1.0	0.2	1.0	1.0	0.2	1.0
	h_{MTEWMA}	**5.711**			**7.079**			**9.307**			**11.350**			**12.561**			**12.807**		
p =4	0.00	199.8	261.0	105.0	200.8	223.0	130.0	199.5	204.6	136.0	200.4	200.0	137.0	198.8	199.0	138.0	198.1	197.5	137.0
	0.25	27.1	25.3	21.0	31.1	26.3	26.0	42.5	38.7	31.0	66.1	63.9	46.0	97.9	97.4	68.0	126.9	126.7	87.0
	0.50	9.5	8.6	7.0	11.0	8.2	10.0	12.4	8.5	11.0	16.8	14.2	13.0	29.1	27.8	21.0	49.5	48.7	34.0
	0.75	4.8	4.2	3.0	5.7	4.2	5.0	6.4	3.9	6.0	7.2	4.8	6.0	10.8	9.1	8.0	19.2	18.3	14.0
	1.00	2.9	2.4	2.0	3.5	2.6	3.0	4.0	2.4	4.0	4.2	2.4	4.0	5.4	3.9	4.0	8.6	7.5	6.0
	1.25	2.0	1.4	1.0	2.3	1.6	2.0	2.7	1.6	2.0	2.9	1.5	3.0	3.3	2.0	3.0	4.6	3.5	4.0
	1.50	1.5	0.9	1.0	1.7	1.1	1.0	2.0	1.2	2.0	2.2	1.1	2.0	2.4	1.2	2.0	3.0	2.0	2.0
	1.75	1.3	0.6	1.0	1.4	0.7	1.0	1.6	0.8	1.0	1.8	0.8	2.0	1.9	0.9	2.0	2.1	1.2	2.0
	2.00	1.1	0.4	1.0	1.2	0.5	1.0	1.3	0.6	1.0	1.5	0.6	1.0	1.6	0.7	1.0	1.7	0.8	1.0
	2.25	1.1	0.2	1.0	1.1	0.3	1.0	1.2	0.4	1.0	1.3	0.5	1.0	1.3	0.5	1.0	1.4	0.6	1.0
	2.50	1.0	0.1	1.0	1.0	0.2	1.0	1.1	0.3	1.0	1.2	0.4	1.0	1.2	0.4	1.0	1.2	0.5	1.0
	2.75	1.0	0.1	1.0	1.0	0.1	1.0	1.0	0.2	1.0	1.1	0.3	1.0	1.1	0.3	1.0	1.1	0.3	1.0
	3.00	1.0	0.0	1.0	1.0	0.1	1.0	1.0	0.1	1.0	1.0	0.2	1.0	1.0	0.2	1.0	1.1	0.2	1.0
	h_{MTEWMA}	**7.202**			**8.712**			**11.135**			**13.324**			**14.567**			**14.812**		
p =10	0.00	199.7	253.3	107.0	197.8	224.9	126.0	196.6	203.6	134.0	196.6	195.2	137.0	201.4	202.4	140.0	198.3	200.1	136.0
	0.25	35.1	32.4	28.0	41.1	35.4	34.0	60.0	56.5	43.0	92.6	91.0	64.0	128.0	124.9	90.0	153.4	151.6	107.0
	0.50	12.6	10.8	10.0	14.3	10.1	13.0	16.7	12.0	14.0	25.8	23.0	19.0	48.6	46.9	35.0	78.1	77.1	54.0
	0.75	6.5	5.5	5.0	7.6	5.3	7.0	8.3	4.9	8.0	10.2	7.2	8.0	17.9	16.1	13.0	34.5	32.9	24.0
	1.00	3.9	3.2	3.0	4.6	3.3	4.0	5.2	2.9	5.0	5.6	3.3	5.0	8.2	6.5	6.0	15.3	14.3	11.0
	1.25	2.7	2.0	2.0	3.1	2.1	3.0	3.6	2.0	3.0	3.8	1.9	4.0	4.7	3.1	4.0	7.8	6.6	6.0
	1.50	1.9	1.3	1.0	2.3	1.5	2.0	2.7	1.5	2.0	2.8	1.3	3.0	3.2	1.8	3.0	4.5	3.4	4.0
	1.75	1.5	0.9	1.0	1.8	1.0	1.0	2.1	1.1	2.0	2.2	1.0	2.0	2.4	1.1	2.0	3.0	2.0	2.0
	2.00	1.3	0.6	1.0	1.4	0.7	1.0	1.7	0.8	1.0	1.8	0.8	2.0	1.9	0.8	2.0	2.2	1.3	2.0
	2.25	1.2	0.4	1.0	1.2	0.5	1.0	1.4	0.6	1.0	1.5	0.6	1.0	1.6	0.7	2.0	1.7	0.9	2.0
	2.50	1.1	0.3	1.0	1.1	0.4	1.0	1.2	0.5	1.0	1.3	0.5	1.0	1.4	0.5	1.0	1.5	0.6	1.0
	2.75	1.0	0.2	1.0	1.1	0.2	1.0	1.1	0.3	1.0	1.2	0.4	1.0	1.3	0.4	1.0	1.3	0.5	1.0
	3.00	1.0	0.1	1.0	1.0	0.2	1.0	1.1	0.2	1.0	1.1	0.3	1.0	1.1	0.4	1.0	1.2	0.4	1.0
	h_{MTEWMA}	**15.304**			**17.395**			**20.594**			**23.389**			**24.906**			**25.148**		

In Figure 5 compares the performance of the MTEWMA scheme for different sample sizes when $p \in \{2,10\}$ and $\lambda \in \{0.05,0.9\}$. In this figure, it is very clear that the MTEWMA scheme performs better for large sample sizes regardless of the values of p and λ. Moreover, regardless of the sample size, the higher the value of λ or/and p, the worst the performance of the proposed MTEWMA scheme. For instance, for small p and λ values (say, $p = 2$ and $\lambda = 0.05$), when n = 1, 3 and 5, the proposed MTEWMA scheme give an OOC signal before samples number 50, 25, 20, respectively, while for large λ value (say, $p = 2$ and $\lambda = 0.9$), when n=1, 3 and 5, the MTEWMA scheme gives an OOC signal on after samples number 155, 110 and 70, respectively. A similar increasing pattern is also observed for large p values.



(a) $p = 2$ and $\lambda = 0.05$ (b) $p = 2$ and $\lambda = 0.9$

(c) $p = 10$ and $\lambda = 0.05$ (d) $p = 10$ and $\lambda = 0.9$

Figure 5. Effect of the sample size on the performance of the time varying MTEWMA scheme when $p \in \{2,10\}$ and $\lambda \in \{0.05,0.9\}$ for a nominal $ARL_0 = 200$

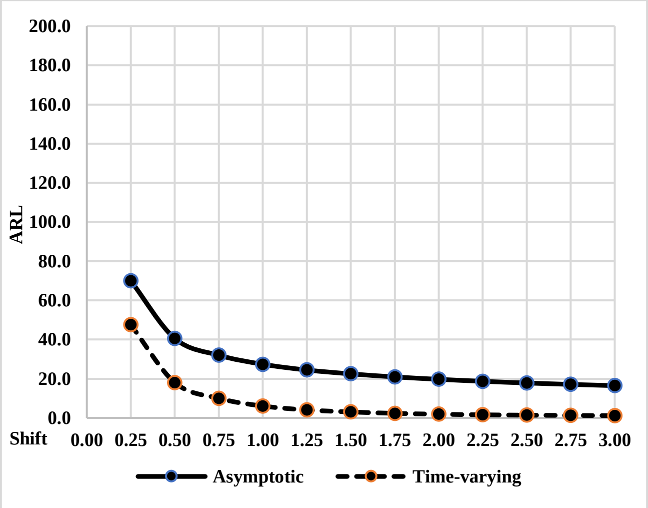
4.3 Time varying versus asymptotic performance

Table 5 investigates the performance of the asymptotic MTEWMA scheme when n=1, $p \in \{2,3,4,10\}$ and $\lambda \in \{0.05,0.2,0.25,0.75,0.9\}$ for a nominal $ARL_0 = 200$. From Table 5, it can be observed that the performance of the proposed MTEWMA scheme degrades considerably when the process has been running for a very long time (i.e. in steady-state) when $\lambda \in (0,0.75)$. In other words, for the asymptotic case, the MTEWMA scheme performs worst for small and moderate values of λ regardless of the value of p. For instance, for shift of 0.25 standard deviation (i.e. δ =0.25), when $\lambda = 0.05$, p =2 and n=1, the time-varying MTEWMA scheme gives an OOC signal on the 48th sample, while in the same situation the asymptotic one gives an OOC signal on the 70th sample. In addition, for a shift of 3 standard deviation (i.e. δ =3), when $\lambda = 0.25$, p =2 and n=1, the time-varying MTEWMA scheme gives an OOC signal on the 1st sample, while in the same situation the asymptotic one gives an OOC signal on the 5th sample. These findings explain the slowness of the MTEWMA scheme in steady-state and can be extended to large values of p. For instance, for a shift of 3 standard deviation, when $\lambda = 0.05$, p =10 and n=1, the time-varying MTEWMA scheme gives an OOC signal on the 2nd sample, while in the same situation the asymptotic one gives an OOC signal on the 22nd sample. However, for large values of λ, the proposed MTEWMA scheme performs almost similarly in both zero-state and time-varying cases. For instance, for shift of 0.25 standard deviation, when $\lambda = 0.9$, p =2 and n=1, both the time-varying and asymptotic MTEWMA schemes give an OOC signal on the 157th.

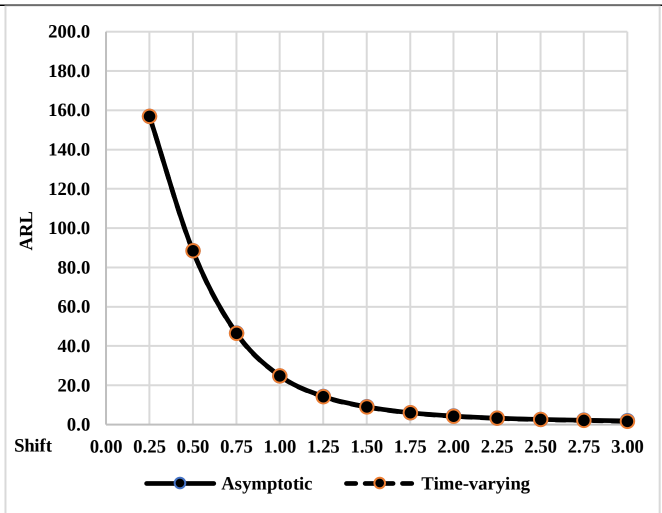
Table 5. *ARL*, *SDRL* and *MRL* profile of the asymptotic MTEWMA control chart along with the control limits when $\lambda \in\{0.05,0.1,0.25,0.5,0.75,0.9\}$, n=1 and $p \in\{2,3,4,10\}$ for a nominal $ARL_0 = 200$

	λ	**0.05**			**0.1**			**0.25**			**0.5**			**0.75**			**0.9**		
	Metric	ARL	$SDRL$	MRL	ARL	$SDRL$	MRL	ARL	$SDRL$	MRL	ARL	$SDRL$	MRL	ARL	$SDRL$	MRL	ARL	$SDRL$	MRL
p =2	0.00	200.0	164.8	149.0	200.2	180.3	145.0	200.1	191.3	140.0	200.2	196.7	140.0	200.2	197.8	139.0	200.3	198.8	139.5
	0.25	70.0	35.3	59.0	67.0	46.7	52.0	80.4	70.2	59.0	106.8	102.7	75.0	135.8	134.5	94.0	157.0	157.0	109.0
	0.50	40.6	9.8	38.0	30.9	12.4	28.0	29.1	20.8	23.0	40.2	36.3	29.0	62.4	60.3	44.0	88.5	86.8	62.0
	0.75	32.0	5.0	31.0	21.8	5.2	21.0	16.0	8.3	14.0	18.4	14.7	14.0	28.5	26.7	20.0	46.7	45.4	32.0
	1.00	27.4	3.2	27.0	18.1	3.1	18.0	11.2	4.1	10.0	10.6	7.0	9.0	15.1	13.0	11.0	24.8	23.6	18.0
	1.25	24.4	2.3	24.0	15.8	2.1	16.0	9.0	2.3	8.0	7.1	3.8	6.0	9.0	7.1	7.0	14.4	13.0	10.0
	1.50	22.5	1.8	22.0	14.3	1.6	14.0	7.8	1.6	8.0	5.5	2.4	5.0	6.1	4.3	5.0	9.1	7.9	7.0
	1.75	20.9	1.5	21.0	13.2	1.3	13.0	7.0	1.2	7.0	4.5	1.6	4.0	4.4	2.7	4.0	6.0	4.9	5.0
	2.00	19.7	1.3	20.0	12.3	1.1	12.0	6.4	1.0	6.0	4.0	1.1	4.0	3.5	1.8	3.0	4.3	3.2	3.0
	2.25	18.7	1.1	19.0	11.6	0.9	12.0	6.0	0.8	6.0	3.5	0.9	3.0	2.9	1.3	3.0	3.3	2.2	3.0
	2.50	17.8	1.0	18.0	11.0	0.8	11.0	5.6	0.7	6.0	3.2	0.7	3.0	2.5	1.0	2.0	2.6	1.6	2.0
	2.75	17.1	0.9	17.0	10.5	0.7	10.0	5.3	0.6	5.0	3.0	0.6	3.0	2.3	0.8	2.0	2.2	1.2	2.0
	3.00	16.5	0.8	16.0	10.1	0.7	10.0	5.0	0.6	5.0	2.8	0.6	3.0	2.1	0.6	2.0	1.9	1.0	2.0
	h_{MTEWMA}	**3.116**			**4.777**			**7.141**			**9.121**			**10.301**			**10.580**		
p =3	0.00	200.3	161.2	150.0	200.2	179.5	143.0	200.3	192.9	141.0	200.3	196.4	140.0	200.0	198.6	140.0	200.0	197.0	140.0
	0.25	77.2	38.9	65.0	75.2	52.8	59.0	91.3	81.4	67.0	118.7	114.3	83.0	145.6	144.0	101.0	164.1	163.2	114.0
	0.50	44.3	10.4	42.0	33.9	13.9	30.0	33.4	24.1	26.0	47.1	42.4	34.0	73.4	70.6	52.0	100.6	98.9	70.0
	0.75	34.7	5.2	34.0	23.7	5.7	23.0	17.6	9.5	15.0	21.5	17.3	16.0	34.8	32.5	25.0	56.7	55.8	40.0
	1.00	29.7	3.3	29.0	19.4	3.3	19.0	12.1	4.5	11.0	12.1	8.3	10.0	18.3	16.2	13.0	31.2	30.1	22.0
	1.25	26.5	2.4	26.0	16.9	2.2	17.0	9.6	2.5	9.0	8.0	4.4	7.0	10.6	8.6	8.0	17.5	16.2	13.0
	1.50	24.4	1.9	24.0	15.4	1.7	15.0	8.3	1.7	8.0	6.1	2.8	5.0	7.1	5.1	5.0	11.2	9.8	8.0
	1.75	22.7	1.6	23.0	14.2	1.4	14.0	7.5	1.3	7.0	4.9	1.8	4.0	5.1	3.2	4.0	7.3	6.1	5.0
	2.00	21.3	1.3	21.0	13.2	1.1	13.0	6.8	1.0	7.0	4.2	1.3	4.0	3.9	2.2	3.0	5.2	4.0	4.0
	2.25	20.2	1.1	20.0	12.5	1.0	12.0	6.3	0.8	6.0	3.8	0.9	4.0	3.2	1.5	3.0	3.8	2.7	3.0
	2.50	19.3	1.0	19.0	11.8	0.9	12.0	5.9	0.7	6.0	3.5	0.7	3.0	2.7	1.1	2.0	3.0	1.9	2.0
	2.75	18.5	0.9	18.0	11.3	0.8	11.0	5.6	0.6	6.0	3.2	0.6	3.0	2.4	0.9	2.0	2.5	1.4	2.0
	3.00	17.8	0.8	18.0	10.8	0.7	11.0	5.4	0.6	5.0	3.0	0.5	3.0	2.2	0.7	2.0	2.1	1.1	2.0
	h_{MTEWMA}	**4.503**			**6.483**			**9.114**			**11.293**			**12.539**			**12.799**		
p=4	0.00	200.5	158.1	152.0	200.4	174.6	146.0	200.3	190.2	142.0	200.3	194.1	141.0	200.2	199.4	139.0	200.1	197.4	140.0
	0.25	81.3	41.1	69.0	79.4	56.8	62.0	97.4	87.2	71.0	126.5	121.3	89.0	152.3	149.7	106.0	168.6	169.5	117.0
	0.50	46.9	11.1	45.0	36.1	15.1	32.0	36.7	27.0	29.0	53.0	48.5	38.0	82.2	79.9	57.0	112.0	109.2	79.0
	0.75	36.7	5.4	36.0	25.0	6.0	24.0	19.0	10.3	16.0	24.2	20.0	18.0	41.0	39.1	29.0	64.7	62.9	45.0
	1.00	31.5	3.5	31.0	20.5	3.4	20.0	13.0	5.0	12.0	13.4	9.4	11.0	21.4	19.2	16.0	36.7	35.8	26.0
	1.25	28.0	2.5	28.0	17.8	2.3	18.0	10.2	2.7	10.0	8.6	4.9	7.0	12.1	10.0	9.0	21.1	19.9	15.0
	1.50	25.8	2.0	26.0	16.2	1.8	16.0	8.8	1.8	8.0	6.5	3.0	6.0	7.9	5.8	6.0	13.1	11.7	9.0
	1.75	24.0	1.6	24.0	14.9	1.4	15.0	7.8	1.3	8.0	5.2	1.9	5.0	5.6	3.7	4.0	8.6	7.4	6.0
	2.00	22.6	1.4	22.0	13.9	1.2	14.0	7.1	1.0	7.0	4.5	1.4	4.0	4.2	2.4	4.0	5.9	4.7	5.0
	2.25	21.4	1.2	21.0	13.1	1.0	13.0	6.6	0.9	7.0	3.9	1.0	4.0	3.5	1.7	3.0	4.3	3.2	3.0
	2.50	20.4	1.0	20.0	12.4	0.9	12.0	6.2	0.7	6.0	3.6	0.8	3.0	2.9	1.2	3.0	3.4	2.2	3.0
	2.75	19.5	0.9	19.0	11.9	0.8	12.0	5.9	0.7	6.0	3.3	0.6	3.0	2.6	0.9	2.0	2.7	1.6	2.0
	3.00	18.8	0.8	19.0	11.4	0.7	11.0	5.6	0.6	6.0	3.1	0.5	3.0	2.3	0.7	2.0	2.3	1.2	2.0
	h_{MTEWMA}	**5.804**			**8.029**			**10.929**			**13.258**			**14.547**			**14.803**		
p =10	0.00	200.3	151.7	153.0	200.1	172.5	148.0	200.2	190.2	142.0	200.4	197.6	139.0	200.9	198.6	139.0	200.3	198.3	140.0
	0.25	97.7	50.2	82.0	97.8	70.7	76.0	120.6	108.8	87.0	150.2	145.9	106.0	172.8	171.3	120.0	182.6	183.9	126.0
	0.50	56.5	13.5	54.0	45.2	20.4	39.0	50.4	39.3	38.0	76.8	71.5	55.0	113.2	112.3	78.0	141.5	141.0	99.0
	0.75	44.1	6.1	43.0	30.1	7.7	28.0	25.4	15.5	21.0	37.1	32.5	27.0	64.7	62.3	45.0	95.7	94.6	67.0
	1.00	37.9	3.9	37.0	24.4	4.0	24.0	16.4	7.1	14.0	19.7	15.1	15.0	35.7	33.6	25.0	61.4	60.5	43.0
	1.25	33.8	2.9	34.0	21.3	2.7	21.0	12.5	3.7	12.0	12.3	7.9	10.0	20.4	18.1	15.0	37.8	36.3	27.0
	1.50	31.0	2.2	31.0	19.1	2.0	19.0	10.4	2.2	10.0	8.6	4.5	7.0	12.7	10.3	9.0	23.9	22.8	17.0
	1.75	28.8	1.8	29.0	17.6	1.6	17.0	9.2	1.6	9.0	6.6	2.8	6.0	8.4	6.2	7.0	15.2	13.8	11.0
	2.00	27.0	1.5	27.0	16.4	1.3	16.0	8.4	1.2	8.0	5.5	1.9	5.0	6.0	3.9	5.0	10.1	8.8	7.0
	2.25	25.6	1.3	25.0	15.5	1.1	15.0	7.8	1.0	8.0	4.8	1.3	4.0	4.7	2.7	4.0	7.1	5.8	5.0
	2.50	24.3	1.2	24.0	14.7	1.0	15.0	7.3	0.8	7.0	4.3	1.0	4.0	3.8	1.9	3.0	5.2	3.9	4.0
	2.75	23.3	1.0	23.0	14.0	0.9	14.0	6.9	0.7	7.0	3.9	0.8	4.0	3.2	1.3	3.0	4.0	2.8	3.0
	3.00	22.4	0.9	22.0	13.4	0.8	13.0	6.5	0.7	6.0	3.7	0.7	4.0	2.8	1.0	3.0	3.2	2.0	3.0
	h_{MTEWMA}	**13.083**			**16.393**			**20.292**			**23.303**			**24.892**			**25.143**		

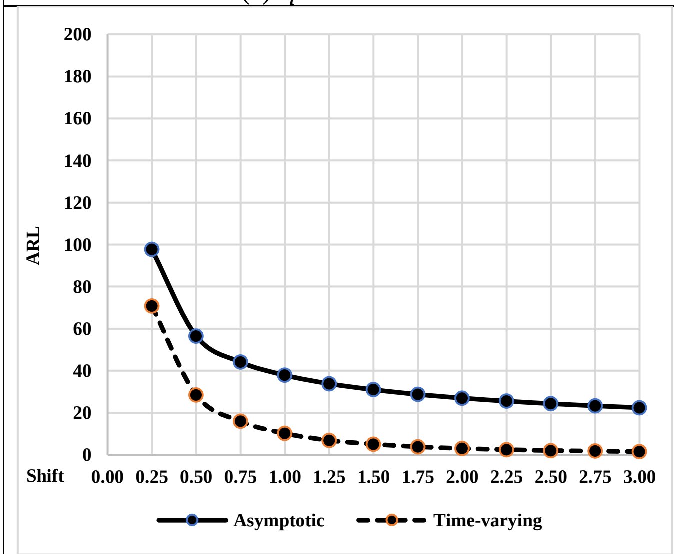
The above findings are also displayed in Figure 6. Figures 6, 7 and 8 compare the performance of the time-varying and asymptotic MTEWMA schemes when $\lambda \in \{0.05, 0.9\}$ and $p \in \{2,10\}$ in terms of the *ARL*, *SDRL* and *MRL* profiles, respectively. From Figure 6, it can be observed that for small λ the proposed MTEWMA scheme performs better under the zero-state case compared to the asymptotic one in terms of the *ARL* profile. This confirm the findings of Knoth et al[26] reporting the worst performance of the TEWMA scheme and other memory-type schemes in steady-state. For large λ values, the MTEWMA scheme performs similarly for both cases. However, in terms of the *SDRL* profile, Figure 7 shows that when $\lambda \in (0,0.75)$, the asymptotic MTEWMA scheme has a better *SDRL* properties than the time-varying MTEWMA scheme for small and moderate shifts and they are slightly similar for large shifts. For large values of λ, their *SDRL* profiles are almost similar regardless of the size of the shift. In terms of the *MRL* profile, Figure 8 shows the pattern of the findings is similar the one in terms of the *ARL* profile where for small values of λ the MTEWMA scheme performs better under the zero-state case compared to the asymptotic case and for large values of λ there is a similar performance in both cases.



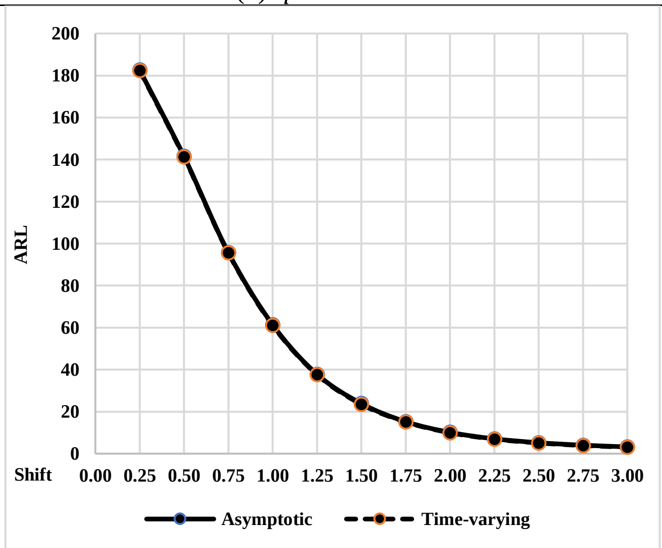
(a) $p = 2$ and $\lambda = 0.05$



(b) $p = 2$ and $\lambda = 0.9$

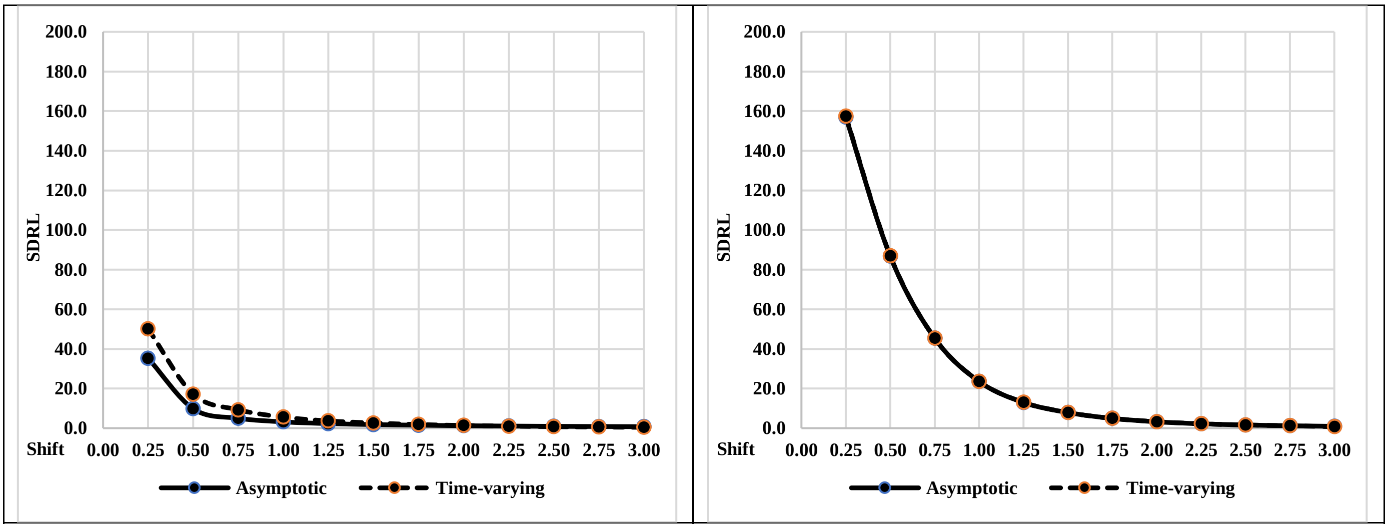


(c) $p = 10$ and $\lambda = 0.05$

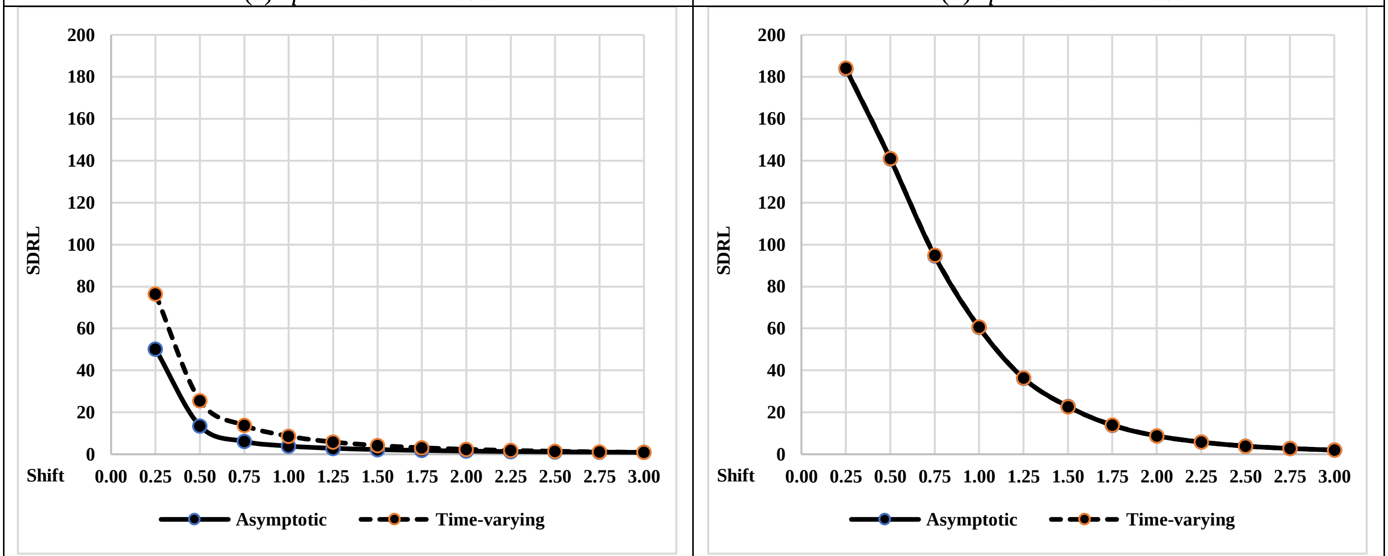


(d) $p = 10$ and $\lambda = 0.9$

Figure 6. Asymptotic versus time-varying MTEWMA schemes in terms of the OOC *ARL* profile when n=1, p=2 and 10 and λ =0.05 and 0.9

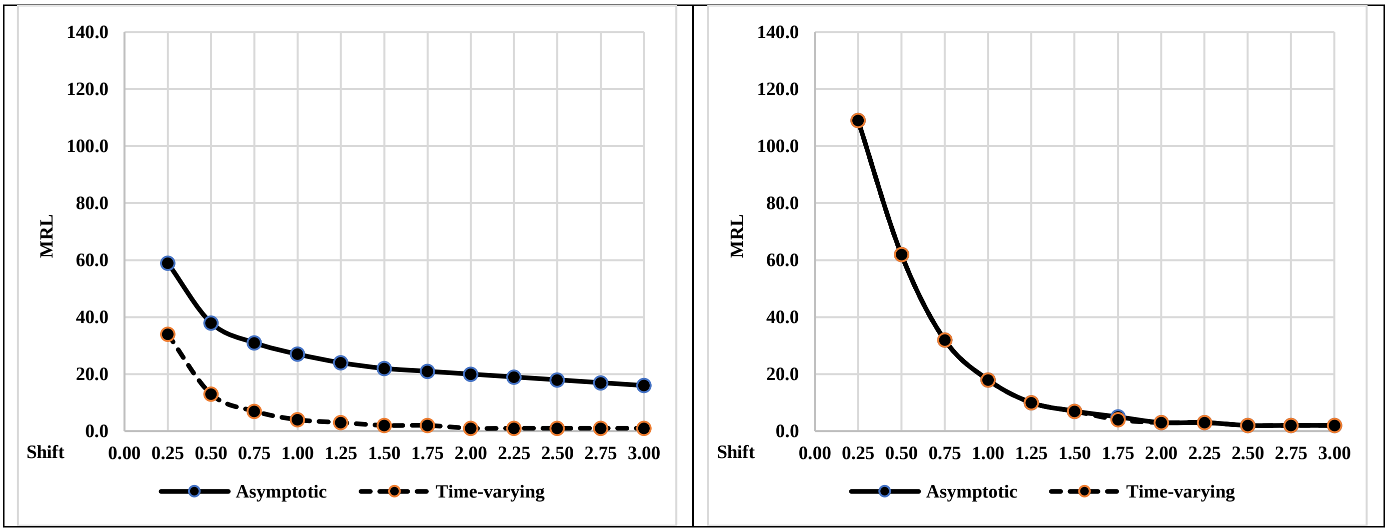


(a) $p\,=2$ and $\lambda=0.05$ | (b) $p\,=2$ and $\lambda=0.9$



(c) $p\,=10$ and $\lambda=0.05$ | (d) $p\,=10$ and $\lambda=0.9$

Figure 7. Asymptotic versus time-varying MTEWMA schemes in terms of the OOC *SDRL* profile when n=1, p=2 and 10 and λ =0.05 and 0.9



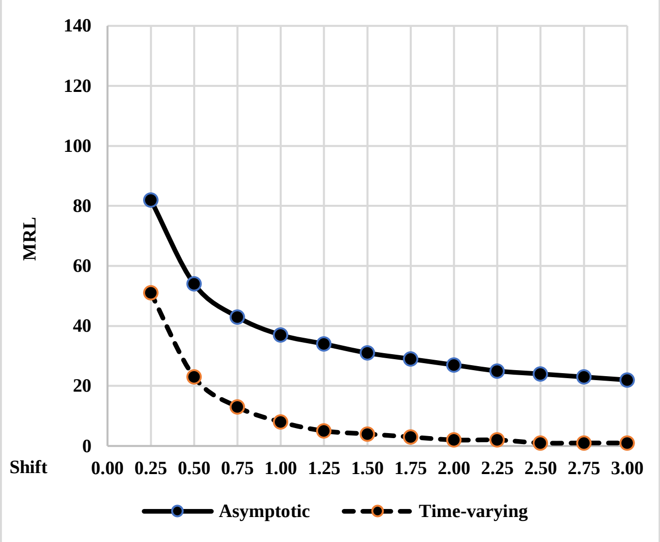
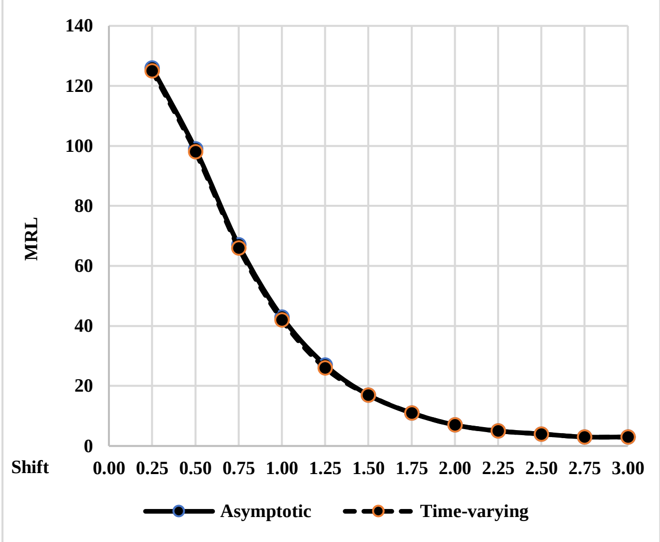
(e) $p = 2$ and $\lambda = 0.05$	(f) $p = 2$ and $\lambda = 0.9$
	
(g) $p = 10$ and $\lambda = 0.05$	(h) $p = 10$ and $\lambda = 0.9$

Figure 8. Asymptotic versus time-varying MTEWMA schemes in terms of the OOC *MRL* profile when n=1, p=2 and 10 and λ =0.05 and 0.9

To investigate the reason why the proposed MTEWMA scheme behaves differently for different values of λ, we analysed the pattern of the control limits for different λ values. Thus, Figure 9 compares the time-varying and asymptotic control limits when $\lambda \in$ {0.05, 0.1,0.25,0.5,0.9} and $p \in \{2,3,4,10\}$ for a nominal $ARL_0 = 200$. From Figure 9, it can be observed that the control limits of the asymptotic MTEWMA scheme are narrower than those of the time-varying ones when $\lambda \in (0,0.75)$. However, for large values of λ, the control limits are almost the same in both cases.

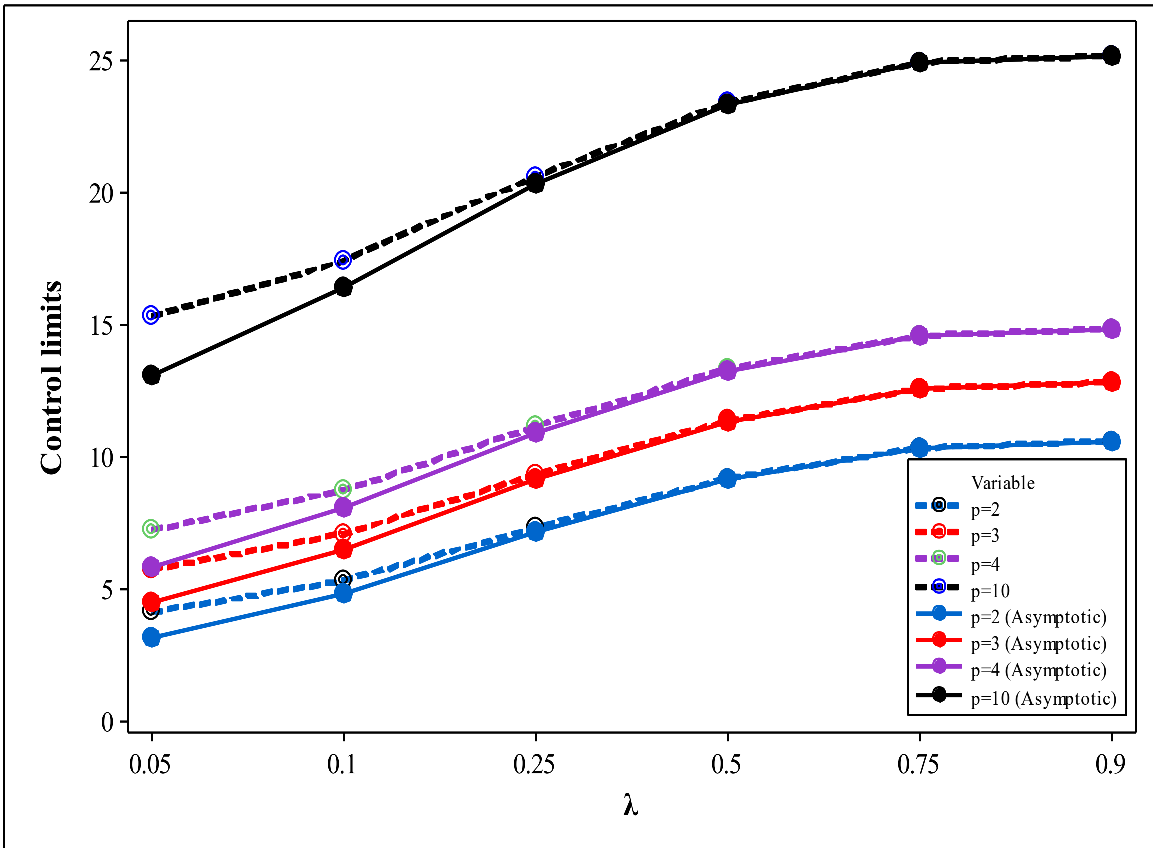


Figure 9. Comparison of the asymptotic and time-varying control limits for a nominal ARL_0 = 200

4.4 IC Robustness to non-normality of the MEWMA, MDEWMA and MTEWMA schemes

To study the IC robustness of the proposed control charts, we consider eight distributions for different values of p; these are: (i) the $N_p(\mathbf{0},\mathbf{I}_p)$ distribution which represents the bivariate or multivariate normal distribution with vector mean $\mathbf{0}$ and covariance matrix $\mathbf{I}_p$ where $\mathbf{0}$ is a $1 \times p$ vector and $\mathbf{I}_p$ is a $p \times p$ identity matrix, (ii) the $t_p(\nu)$ representing the multivariate Student's t distribution with v degree of freedom where ν = 3, 10, 30, 100 and 1000, and (iii) the Dirichlet distribution denoted as $D(\alpha_i)$ with $i = 1, 2, \ldots, p+1$. In this study, the proposed control chart is declared to be IC robust if the IC characteristics are significantly much closer to the nominal value across all continuous distribution. Table 6 presents a summary of results on the IC characteristics of the run-length distribution of the proposed schemes when $p \in \{2,4,10\}$ and $\lambda = 0.05$ for a nominal $ARL_0 = 200$ regardless of the value of n. The symbol ">" indicates that the value of the IC run-length characteristic is extremely high.

From Table 6, it can be observed that as p increases the proposed control charts lose their IC robustness. This can be noticed by smaller values of the IC run-length characteristics for large

p values. Under the $t_p(v)$ distribution, the larger the value of v the more robust the proposed charts are. The converse is true for small v values. For instance, the attained ARL_0 values of the MEWMA, MDEWMA and MTEWMA charts are (20.1,29.5,34.6) and (156.2,164.9,170.4) under the $t_2(3)$ and $t_2(30)$ distributions, respectively. From these results it can also be noticed that the larger the degrees of freedom, the more robust the charts are. In addition, these results show that the MTEWMA chart is more robust than the MEWMA and MDEWMA charts. Under the Dirichlet distribution, for small values of p the proposed charts yield very small IC run-length characteristics and as p increases the IC run-length characteristics increase dramatically. From these results, it is clear that the proposed charts are not IC robust under the Dirichlet distribution regardless of the magnitude of parameters.

Table 6. Robustness to non-normality of the MEWMA, MDEWMA and MTEWMA control charts when $p \in \{2,4\ ,10\}$ and $\lambda = 0.05$ for a nominal $ARL_0 = 200$ regardless of the value of n

		MEWMA							MDEWMA							MTEWMA						
		ARL	***SDRL***	**P5**	**P25**	**P50**	**P75**	**P95**	***ARL***	***SDRL***	**P5**	**P25**	**P50**	**P75**	**P95**	***ARL***	***SDRL***	**P5**	**P25**	**P50**	**P75**	**P95**
p=2	$N_2(0,I_2)$	199.0	214.1	3.0	43.0	132.0	282.0	635.0	200.5	245.9	1.0	18.0	116.0	290.0	696.5	200.0	258.4	1.0	10.0	106.0	291.0	729.0
	$t_2(3)$	20.1	26.4	1.0	3.0	9.0	28.0	73.0	29.5	46.3	1.0	2.0	9.0	38.0	125.0	34.6	57.0	1.0	1.0	9.0	44.0	152.5
	$t_2(10)$	92.6	107.2	1.0	14.0	56.0	133.0	307.0	112.6	150.1	1.0	6.0	54.0	163.0	416.0	119.8	167.1	1.0	4.0	50.0	175.0	467.0
	$t_2(30)$	156.2	171.2	2.0	32.0	102.0	223.0	494.0	164.9	208.7	1.0	11.0	88.0	241.0	590.0	170.4	225.9	1.0	7.0	84.0	249.5	632.0
	$t_2(100)$	187.6	203.0	3.0	42.0	123.0	265.0	594.0	188.0	231.0	1.0	16.0	107.0	272.0	659.0	189.2	245.9	1.0	8.0	96.0	275.0	692.0
	$t_2(1000)$	199.5	215.0	4.0	45.0	131.0	281.0	635.0	199.9	245.8	1.0	19.0	115.0	287.0	691.0	199.1	259.6	1.0	9.0	103.0	289.0	719.0
	$D(2,1,1)$	29.0	4.6	22.0	26.0	29.0	32.0	37.0	18.2	2.4	14.0	17.0	18.0	20.0	22.0	19.4	2.8	15.0	17.0	19.0	21.0	24.0
	$D(1,1,1)$	50.1	12.2	34.0	42.0	48.0	56.0	73.0	24.7	3.4	19.0	22.0	24.0	27.0	31.0	25.9	3.8	20.0	23.0	26.0	28.0	32.0
p=4	$N_4(0,I_4)$	200.1	215.1	4.0	45.0	132.0	283.0	622.0	200.5	242.1	1.0	18.0	116.0	294.0	691.0	200.1	258.6	1.0	10.0	105.0	293.0	720.0
	$t_4(3)$	10.8	14.9	1.0	2.0	5.0	14.0	41.0	19.0	31.1	1.0	1.0	5.0	23.0	84.0	22.9	39.4	1.0	1.0	5.0	26.0	105.0
	$t_4(10)$	54.1	68.7	1.0	6.0	28.0	77.0	191.0	94.4	130.3	1.0	4.0	41.0	136.0	360.0	99.9	144.9	1.0	3.0	35.0	143.0	399.0
	$t_4(30)$	99.7	115.9	1.0	15.0	60.0	143.0	330.0	154.8	198.1	1.0	10.0	83.0	224.0	554.0	159.5	213.5	1.0	6.0	77.0	231.0	593.0
	$t_4(100)$	124.3	138.8	2.0	23.0	78.0	178.0	402.0	185.8	230.4	1.0	16.0	105.0	270.0	651.0	187.0	245.5	1.0	8.0	95.0	274.0	681.0
	$t_4(1000)$	135.9	149.1	2.0	27.0	88.0	193.0	436.0	199.2	245.5	1.0	18.0	114.0	289.0	682.0	198.0	257.0	1.0	10.0	103.0	290.0	711.0
	$D(2,1,1,1,1))$	>	>	>	>	>	>	>	45.3	3.1	40.0	43.0	45.0	47.0	51.0	46.2	2.8	42.0	44.0	46.0	48.0	51.0
	$D(1,1,1,1,1)$	>	>	>	>	>	>	>	55.7	3.8	50.0	53.0	55.0	58.0	62.0	54.0	2.8	50.0	52.0	54.0	56.0	59.0
p=10	$N_{10}(0,I_{10})$	200.2	213.7	4.0	46.0	134.0	283.0	626.0	200.2	244.7	1.0	19.0	116.0	291.0	695.5	200.5	257.9	1.0	11.0	108.0	292.0	717.0
	$t_{10}(3)$	7.1	9.7	1.0	1.0	3.0	9.0	27.0	10.3	17.0	1.0	1.0	3.0	12.0	44.0	12.5	21.5	1.0	1.0	3.0	14.0	58.0
	$t_{10}(10)$	48.4	63.2	1.0	4.0	23.0	68.0	179.0	66.4	96.4	1.0	2.0	23.0	95.0	264.0	71.8	110.2	1.0	1.0	19.0	101.0	301.0
	$t_{10}(30)$	122.1	137.7	1.0	20.0	78.0	175.0	398.0	136.5	177.9	1.0	7.0	71.0	199.0	497.0	140.7	194.6	1.0	4.0	62.0	206.0	533.0
	$t_{10}(100)$	172.0	185.0	3.0	36.0	113.0	242.0	553.0	180.5	224.9	1.0	14.0	103.0	260.0	631.0	180.2	236.7	1.0	8.0	92.0	262.0	664.0
	$t_{10}(1000)$	197.1	211.5	4.0	45.0	131.0	279.0	615.0	198.8	242.7	1.0	19.0	116.0	287.0	690.0	197.3	255.5	1.0	10.0	103.0	287.0	719.0
	$D(2,\alpha_i)$, $\alpha_i = 1$, $i = 2, 3, \ldots, 10$	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>
	$D(\alpha_i)$, $\alpha_i = 1$, $i = 1, 2, 3, \ldots, 11$	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>	>

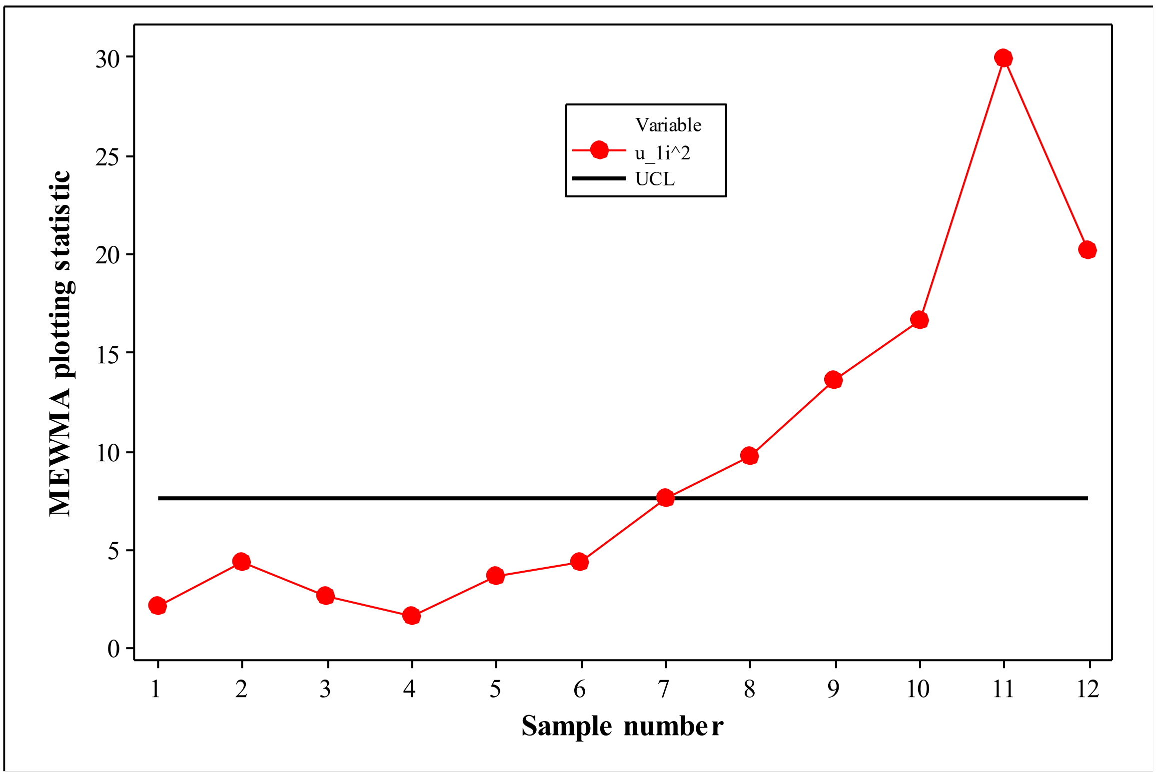
Note: The symbol “>” indicates that the values are extremely high

5. Illustrative example

In this section, we use real-life data from a spring manufacturing process to demonstrate the implementation and application of the proposed MEWMA, MDEWMA and MTEWMA schemes. These data are retrieved from Chen et al[27] and contain two variables X_1 representing the spring inner diameter and X_2 representing the spring elasticity. During the process monitoring, twelve samples each of size 5 (i.e. n=5) are collected for each variable. From the historical data, $\boldsymbol{X}$ follows a bivariate normal distribution (i.e. $\boldsymbol{X}\sim N_2(\boldsymbol{\mu}_0, \boldsymbol{\Sigma})$) with IC process mean vector and dispersion matrix given by

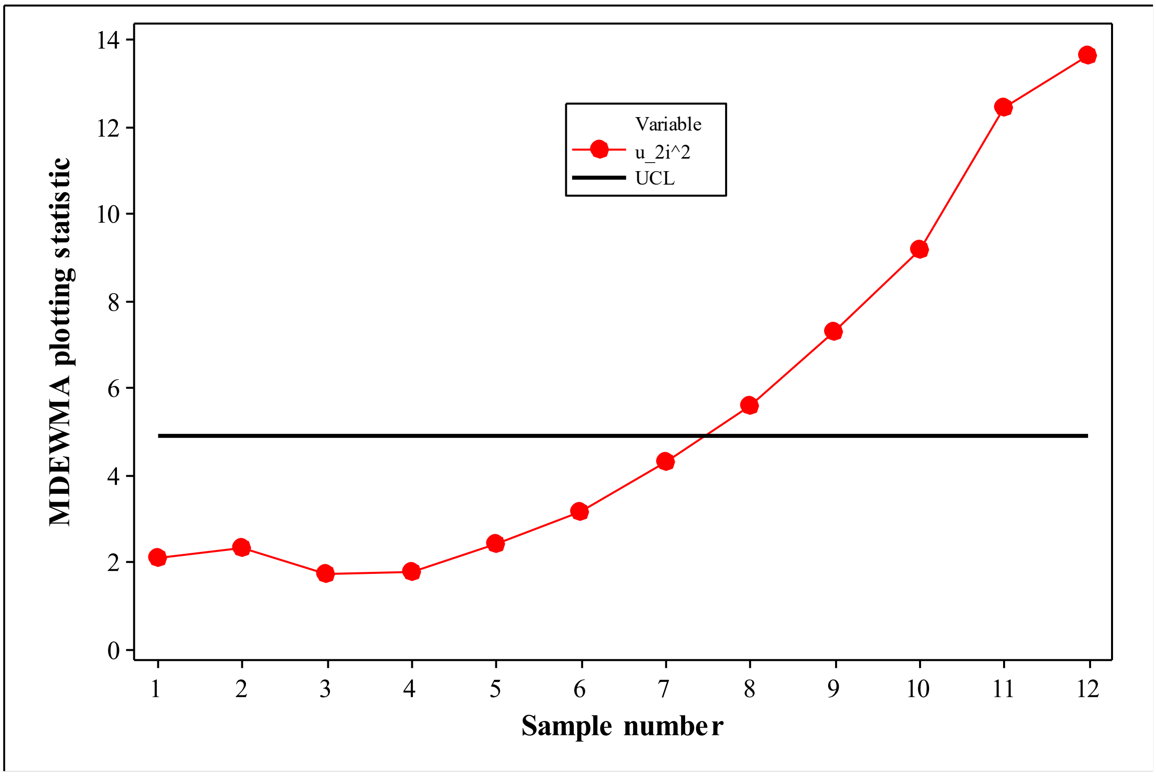
$$\boldsymbol{\mu}_0 = (28.29\ 48.85)' \text{ and } \boldsymbol{\Sigma} = \begin{pmatrix} 0.0035 & -0.0046 \\ -0.0046 & 0.0026 \end{pmatrix},$$

respectively. The proposed schemes are implemented using a nominal $ARL_0 = 200$. Thus, when $\lambda = 0.05$, n=5 and $p = 2$, the control limit of the MEWMA, MDEWMA and MTEWMA schemes are found to be equal to 7.685, 4.924 and 4.111 so that the attained ARL_0 are 199, 200.5 and 200, respective. The plots of the three schemes are shown in Figure 10 and Table 7. It can be seen that the MEWMA and MDEWMA schemes give a signal on the 8th sample, while the MTEWMA scheme give an OOC signal on the 9th sample. Therefore, in this particular application, the MEWMA and MDEWMA schemes outperform the MTEWMA scheme.

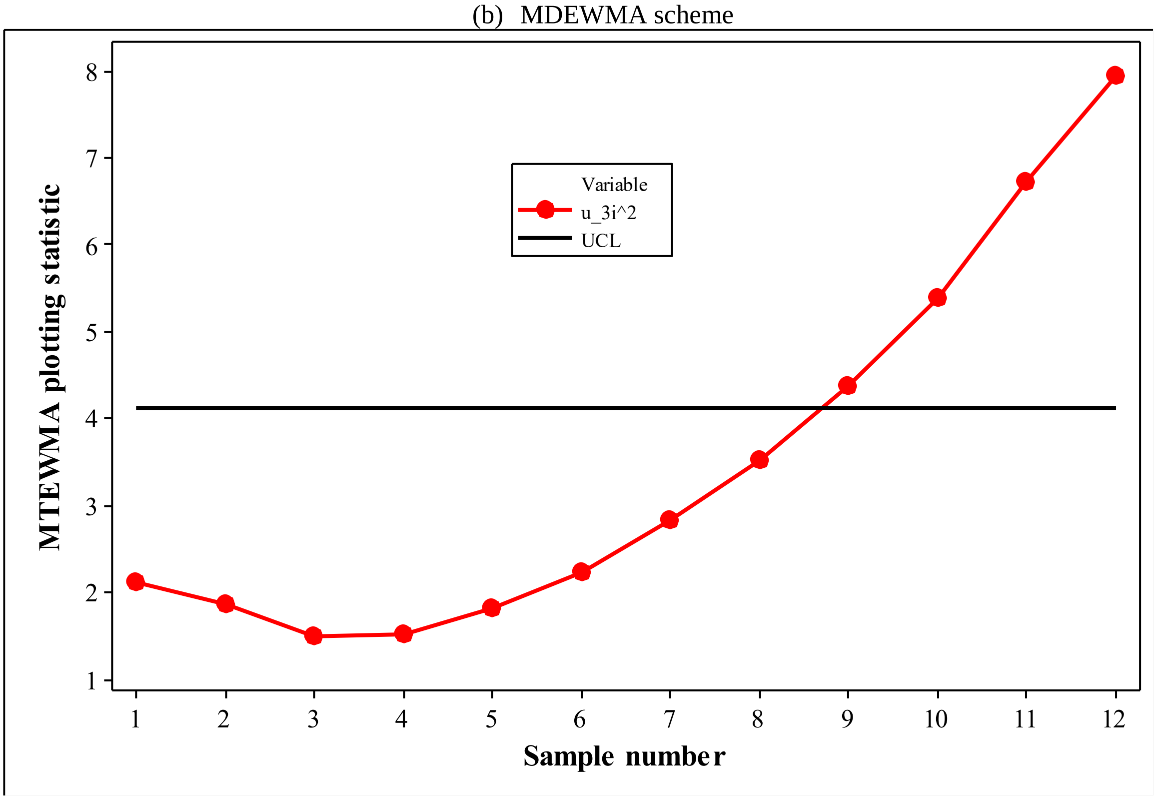


(a) MEWMA scheme

MTEWMA monitoring scheme



(b) MDEWMA scheme



(c) MTEWMA scheme

Figure 10. MEWMA, MDEWMA and MTEWMA schemes for the spring manufacturing process

Table 7. MEWMA, MDEWMA and MTEWMA schemes for the spring manufacturing process data along with their corresponding plotting statistics and control limits when n=5, $\lambda = 0.05$ and $p = 2$

Sample number	**Spring manufacturing process data**										**MEWMA scheme**			**MDEWMA scheme**			**MTEWMA scheme**		
	$\boldsymbol{X_1}$					$\boldsymbol{X_2}$					$\boldsymbol{u_{1i}^2}$	$\boldsymbol{h_{MEWMA}}$	**OOC**	$\boldsymbol{u_{2i}^2}$	$\boldsymbol{h_{MDEWMA}}$	**OOC**	$\boldsymbol{u_{3i}^2}$	$\boldsymbol{h_{MTEWMA}}$	**OOC**
1	28.1	28.3	28.2	28.2	28.2	46.3	45.7	45.8	45.8	45.8	2.12	7.685	No	2.12	4.924	No	2.1	4.111	No
2	28.5	28.3	28.3	28.3	28.2	45.8	45.9	45.8	45.9	45.9	4.40	7.685	No	2.34	4.924	No	1.8	4.111	No
3	28.2	28.3	28.2	28.3	28.2	45.8	45.7	45.7	45.5	45.5	2.70	7.685	No	1.71	4.924	No	1.4	4.111	No
4	28.2	28.2	28.2	28.2	28.2	45.8	45.9	45.7	46.0	45.8	1.62	7.685	No	1.76	4.924	No	1.5	4.111	No
5	28.3	28.3	28.2	28.3	28.3	45.7	45.9	46.0	45.7	45.6	3.72	7.685	No	2.44	4.924	No	1.8	4.111	No
6	28.3	28.2	28.3	28.2	28.1	45.7	45.9	45.7	45.9	46.0	4.40	7.685	No	3.16	4.924	No	2.2	4.111	No
7	28.2	28.3	28.3	28.3	28.4	45.9	45.8	45.6	45.7	45.7	7.67	7.685	No	4.32	4.924	No	2.8	4.111	No
8	28.2	28.3	28.3	28.3	28.3	45.7	45.8	45.6	45.8	45.7	9.80	7.685	**Yes**	5.59	4.924	**Yes**	3.5	4.111	No
9	28.2	28.3	28.3	28.3	28.3	45.5	46.1	45.8	45.5	45.8	13.6	7.685	No	7.28	4.924	No	4.3	4.111	**Yes**
10	28.3	28.2	28.3	28.3	28.3	45.7	45.7	45.7	45.8	45.9	16.6	7.685	No	9.17	4.924	No	5.3	4.111	No
11	28.3	28.3	28.3	28.4	28.3	45.8	45.3	45.7	45.8	45.8	29.9	7.685	No	12.4	4.924	No	6.7	4.111	No
12	28.1	28.2	28.2	28.1	28.3	45.3	45.2	45.7	45.8	45.8	20.2	7.685	No	13.6	4.924	No	7.9	4.111	No

6. Conclusion and recommendations

In this paper, we proposed a new MTEWMA scheme based on the newly proposed TEWMA scheme by Alevizakos et al[11]. In addition, we revisited the design of the MEWMA and MDEWMA schemes. It observed that the MTEWMA scheme outperforms the MEWMA and MTEWMA schemes in many situations in zero-state. The performances of the proposed schemes decrease for large values of p and λ. It is also observed that the performance of the MTEWMA scheme deteriorate considerably in steady-state for small and moderate smoothing parameters (i.e. λ values). Therefore, engineers and operators in the industries are advised to maintain memory-type monitoring tools (i.e. control chart) by re-starting the monitoring process in a regular basis when the process runs in steady-state. It was also observed that the proposed schemes perform better for large sample size. This will generally incur additional costs. To avoid huge additional costs, a rational balance must be strike between the sensitivity of the proposed schemes and the resulting costs of implementation.

Researchers who are interested in this topic may consider the design of MTEWMA schemes under the assumption of estimated process parameters taking into account the presence of autocorrelation.

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Appendix: Properties of the MEWMA, MDEWMA and MTEWMA statistics

This appendix shows how to derive the mean vector and covariance matrix of the MEWMA, MDEWMA and MTEWMA statistics.

The properties of the MEWMA statistic can be derived as follows:

From Equation (1), we have:

$$\begin{aligned}\boldsymbol{Y}_{1i} &= \boldsymbol{\Lambda}\boldsymbol{X}_i + (\boldsymbol{I}-\boldsymbol{\Lambda})\boldsymbol{Y}_{1(i-1)} \\ &= \boldsymbol{\Lambda}\boldsymbol{X}_i + (\boldsymbol{I}-\boldsymbol{\Lambda})\left[\boldsymbol{\Lambda}\boldsymbol{X}_{i-1} + (\boldsymbol{I}-\boldsymbol{\Lambda})\boldsymbol{Y}_{1(i-2)}\right] \\ &= \boldsymbol{\Lambda}\boldsymbol{X}_i + \boldsymbol{\Lambda}(\boldsymbol{I}-\boldsymbol{\Lambda})\boldsymbol{X}_{i-1} + (\boldsymbol{I}-\boldsymbol{\Lambda})^2\boldsymbol{Y}_{1(i-2)}\end{aligned}$$

Continuing with the process of recursive substitution, for $\boldsymbol{Y}_{1(i-k)}$, we get

$$\boldsymbol{Y}_{1i} = \boldsymbol{\Lambda}\sum_{\boldsymbol{k}=\boldsymbol{1}}^{\boldsymbol{i}}(\boldsymbol{I}-\boldsymbol{\Lambda})^{i-k}\,\boldsymbol{X}_k + (\boldsymbol{I}-\boldsymbol{\Lambda})^i\boldsymbol{Y}_{10}. \tag{A.1}$$

Thus, the mean vector of $\boldsymbol{Y}_{1i}$ given that the process is IC is given by

$$E(\boldsymbol{Y}_{1i}|IC) = \boldsymbol{\Lambda}\sum_{\boldsymbol{k}=\boldsymbol{1}}^{\boldsymbol{i}}(\boldsymbol{I}-\boldsymbol{\Lambda})^{i-k}\,\boldsymbol{\mu}_0 + (\boldsymbol{I}-\boldsymbol{\Lambda})^i\boldsymbol{\mu}_0 = \boldsymbol{\mu}_0. \tag{A.2}$$

The covariance matrix of the MEWMA statistic, $\boldsymbol{Y}_{1i}$, given that the process is IC is then derived as follows:

Let $\boldsymbol{G}_{1k} = \boldsymbol{\Lambda}(\boldsymbol{I}-\boldsymbol{\Lambda})^{i-k}$. Then,

$$\boldsymbol{\Sigma}_{\boldsymbol{Y}_{1i|IC}} = \sum_{k=1}^{i}\boldsymbol{G}_{1k}\boldsymbol{\Sigma}_{\boldsymbol{X}_{k|IC}}\boldsymbol{G}_{1k}' = \sum_{k=1}^{i}\boldsymbol{G}_{1k}\boldsymbol{\Sigma}_0\boldsymbol{G}_{1k}'. \tag{A.3}$$

Equation (5) follows immediately from Equation (1) noting that $\boldsymbol{\Lambda} = \lambda\mathbf{I}$ and $\boldsymbol{I}-\boldsymbol{\Lambda} = (1-\lambda)\boldsymbol{I}$.

Properties of the MDEWMA statistics can be derived as follows:

Following Equation (A.1), the expression of $\boldsymbol{Y}_{2i}$ can be written as

$$\boldsymbol{Y}_{2i} = \boldsymbol{\Lambda} \sum_{k=1}^{i} (\boldsymbol{I} - \boldsymbol{\Lambda})^{i-k} \boldsymbol{Y}_{1k} + (\boldsymbol{I} - \boldsymbol{\Lambda})^{i} \boldsymbol{Y}_{20}. \tag{A.4}$$

From Equations (A.4) and (A.1),

$$\boldsymbol{Y}_{2i} = \boldsymbol{\Lambda} \sum_{k=1}^{i} (\boldsymbol{I} - \boldsymbol{\Lambda})^{i-k} \left[\boldsymbol{\Lambda} \sum_{l=1}^{k} (\boldsymbol{I} - \boldsymbol{\Lambda})^{k-l} \boldsymbol{X}_{l} + (\boldsymbol{I} - \boldsymbol{\Lambda})^{k} \boldsymbol{Y}_{10}\right] + (\boldsymbol{I} - \boldsymbol{\Lambda})^{i} \boldsymbol{Y}_{20}$$

Then,

$$\boldsymbol{Y}_{2i} = \boldsymbol{\Lambda}^{2} \sum_{k=1}^{i} (i - k + 1)(\boldsymbol{I} - \boldsymbol{\Lambda})^{i-k} \boldsymbol{X}_{k} + (i\boldsymbol{\Lambda} + \boldsymbol{I})(\boldsymbol{I} - \boldsymbol{\Lambda})^{i} \boldsymbol{\mu}_{0}. \tag{A.5}$$

Thus, the mean vector of $\boldsymbol{Y}_{2i}$ given that the process is IC is then derived as follows:

$$E(\boldsymbol{Y}_{2i}|IC) = \boldsymbol{\Lambda}^{2} \sum_{k=1}^{i} (i - k + 1)(\boldsymbol{I} - \boldsymbol{\Lambda})^{i-k} \boldsymbol{\mu}_{0} + (i\boldsymbol{\Lambda} + \boldsymbol{I})(\boldsymbol{I} - \boldsymbol{\Lambda})^{i} \boldsymbol{\mu}_{0} = \boldsymbol{\mu}_{0}. \tag{A.6}$$

The covariance matrix of the MDEWMA statistic, $\boldsymbol{Y}_{2i}$, given that the process is IC is then derived as follows:

Let $\boldsymbol{G}_{2k} = \boldsymbol{\Lambda}^{2}(\boldsymbol{I} - \boldsymbol{\Lambda})^{i-k}$. Then,

$$\boldsymbol{\Sigma}_{\boldsymbol{Y}_{2i|IC}} = \sum_{k=1}^{i} (i - k + 1)^{2} \boldsymbol{G}_{2k} \boldsymbol{\Sigma}_{0} \boldsymbol{G}_{2k}'. \tag{A.7}$$

Properties of the MTEWMA statistics can be derived as follows:

From Equations (A.1) and (A.4), the expression of $\boldsymbol{Y}_{3i}$ can be written as

$$\boldsymbol{Y}_{3i} = \boldsymbol{\Lambda} \sum_{k=1}^{i} (\boldsymbol{I} - \boldsymbol{\Lambda})^{i-k} \boldsymbol{Y}_{2k} + (\boldsymbol{I} - \boldsymbol{\Lambda})^{i} \boldsymbol{Y}_{30}. \tag{A.8}$$

Following Equations (A.4) and (A.1), Equation (A.8) can be simplified to

$$\boldsymbol{Y}_{3i} = \frac{\boldsymbol{\Lambda}^{3}}{2} \sum_{k=1}^{i} (i - k + 1)(i - k + 2)(\boldsymbol{I} - \boldsymbol{\Lambda})^{i-k} \boldsymbol{X}_{k} + \frac{1}{2}(i(i+1)\boldsymbol{\Lambda}^{2} + 2i\boldsymbol{\Lambda} + 2\mathbf{I})(\boldsymbol{I} - \boldsymbol{\Lambda})^{i} \boldsymbol{\mu}_{0}. \tag{A.9}$$

Thus, the mean vector of $\boldsymbol{Y}_{3i}$ given that the process is IC is then derived as follows:

$$\begin{aligned} E(\boldsymbol{Y}_{3i}|IC) = {} & \frac{\boldsymbol{\Lambda}^{3}}{2} \sum_{k=1}^{i} (i - k + 1)(i - k + 2)(\boldsymbol{I} - \boldsymbol{\Lambda})^{i-k} \boldsymbol{\mu}_{0} \\ & + \frac{1}{2}(i(i+1)\boldsymbol{\Lambda}^{2} + 2i\boldsymbol{\Lambda} + 2\mathbf{I})(\boldsymbol{I} - \boldsymbol{\Lambda})^{i} \boldsymbol{\mu}_{0} = \boldsymbol{\mu}_{0}. \end{aligned} \tag{A.10}$$

MTEWMA monitoring scheme

The covariance matrix of the MTEWMA statistic, $\boldsymbol{Y}_{3i}$, given that the process is IC is then derived as follows:

Let $\boldsymbol{G}_{3k} = \frac{\boldsymbol{\Lambda}^3}{2}(\boldsymbol{I} - \boldsymbol{\Lambda})^{i-k}$. Then,

$$\boldsymbol{\Sigma}_{\boldsymbol{Y}_{3i|IC}} = \sum_{k=1}^{i} (i-k+1)^2(i-k+2)^2 \boldsymbol{G}_{3k} \boldsymbol{\Sigma}_0 \boldsymbol{G}'_{3k}. \quad \text{(A.11)}$$