REGULARITY PROPERTIES FOR TRIPLE SYSTEMS

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Abstract. Szemerédi's Regularity Lemma proved to be a powerful tool in the area of extremal graph theory [5]. Many of its applications are based on the following technical fact: If G is a k-partite graph with $V(G) = \bigcup_{i=1}^{k} V_i$, $|V_i| = n$ for all $i \in [k]$, and all pairs $\{V_i, V_j\}$, $1 \leq i < j \leq k$, are ϵ-regular of density d, then G contains $d^{\binom{k}{2}} n^k (1 + f(\epsilon))$ cliques $K_k^{(2)}$, where $f(\epsilon) \to 0$ as $\epsilon \to 0$.

The aim of this paper is to establish the analogous statement for 3-uniform hypergraphs. Our result, which we refer to as The Counting Lemma, together with Theorem 3.5 of [2], a Regularity Lemma for Hypergraphs, can be applied in various situations as Szemerédi's Regularity Lemma is for graphs. Some of these applications are discussed in the papers [3], [4] and [7], as well as in upcoming papers of the authors and others.

Contents

1. Introduction

Extremal set theory is a well studied and broad subject within the field of combinatorial mathematics. Problems from this discipline, in the most general sense, are concerned with the determination of thresholds on quantitative properties one set system should have in order to contain another set system as a subsystem.

While containment problems are often difficult in a deterministic setting, one can consider the same problems in a random setting, often with some advantages. Here, we consider a random set system as a hypergraph whose edges were randomly generated from a base set. One nice feature of a random set system is that as a consequence of the edges being randomly generated, one can expect a fairly even distribution of the edges. With an even distribution of the edges of a random set system likely, one is afforded tools for trying to determine if the random set system contains a given subsystem. As a drawback, however, a random set system is not a concrete set system, but rather a probability space. Hence, all conclusions made on the containment of a given subsystem in our random set system can only be in terms of certain probabilities.

In the deterministic setting, one has the possibility of conclusively deciding whether or not a concrete set system contains a given subsystem. However, one is not guaranteed the even distribution of edges that in the random setting played a helpful role in deciding containment. If the set systems under consideration are graphs, one successful approach for the problem of graph containment in the deterministic setting has been the use of a powerful lemma of Szemerédi which in some sense combines the deterministic and random settings.

1.1. **Szemerédi's Regularity Lemma.**

In the course of proving his celebrated Density Theorem, E. Szemerédi established a lemma which decomposes the edge set of any graph into "random like pieces" (cf., [12], [5]). He later established a more applicable version, the well known Regularity Lemma, in [11]. We give a precise account in what follows.

For a graph $G = (V, E)$ and two disjoint sets A, $B \subset V$, we denote by $E(A, B)$ the set of edges $\{a, b\} \in G$ with $a \in A$ and $b \in B$ and put $e(A, B) = |E(A, B)|$.

We also set $d(A,B) = d(G_{AB}) = e(A,B)/|A||B|$ for the *density* of the bipartite graph $G_{AB} = (A \cup B, E(A,B))$.

Let $\epsilon > 0$ be given. We say that a pair A, B is *ϵ-regular* if $|d(A,B) - d(A',B')| < \epsilon$ holds whenever $A' \subset A$, $B' \subset B$, and $|A'| > \epsilon|A|$, $|B'| > \epsilon|B|$. We call a partition $V = V_0 \cup V_1 \cup \cdots \cup V_t$ an *equitable partition* if it satisfies $|V_1| = |V_2| = \cdots = |V_t|$ and $|V_0| < t$; we call an equitable partition *ϵ-regular* if all but $\epsilon\binom{t}{2}$ pairs V_i, V_j are ϵ-regular.

Theorem 1.1.1. Szemerédi's Regularity Lemma. *Let $\epsilon > 0$ be given and let k be a positive integer. There exist positive integers $N = N(\epsilon, k)$ and $K = K(\epsilon, k)$ such that any graph $G = (V, E)$ with $|V| = n \geq N$ vertices admits an ϵ-regular equitable partition $V = V_0 \cup V_1 \cup \cdots \cup V_t$ with t satisfying $k \leq t \leq K$.*

Szemerédi's Lemma is a powerful tool in the area of extremal graph theory. One of its most important consequences is that it can help decide if a graph contains a fixed subgraph. Suppose that a (large) graph G is given along with an ϵ-regular partition $V = V_0 \cup V_1 \ldots \cup V_t$, and let H be a fixed graph. If enough of the ϵ-regular pairs $\{V_i, V_j\}$ are dense enough with respect to ϵ, we may build a copy of H within this collection of bipartite graphs $E(V_i, V_j)$. This observation is due to the following fact which may be appropriately called the Counting Fact for Graphs.

Fact 1.1.2. The Counting Fact for Graphs. *Suppose G is a k-partite graph with $V(G) = \bigcup_{i=1}^{k} V_i$, $|V_i| = n$ for all $i \in [k]$, and all pairs $\{V_i, V_j\}$, $1 \leq i < j \leq k$, are ϵ-regular of density d. Then G contains $d^{\binom{k}{2}} n^k (1 + f(\epsilon))$ cliques $K_k^{(2)}$, where $f(\epsilon) \to 0$ as $\epsilon \to 0$.*

In the following two theorems, we provide examples of when combining Szemerédi's Regularity Lemma with Fact 1.1.2 was useful in containment problems. (in the proceeding two examples, the reader is not expected to see the connection between the examples and Fact 1.1.2)

One of the first applications of Fact 1.1.2 (applied when $k = 3$) was the following result of Ruzsa and Szemerédi [10].

Theorem 1.1.3. Ruzsa, Szemerédi (1976). *For every $\epsilon > 0$, there exists $n_0 = n_0(\epsilon)$ such that if H_n is a 3-uniform hypergraph on $n \geq n_0$ vertices not containing 6 points with 3 or more triples, then $|H_n| < \epsilon n^2$.*

In the next example, for a fixed graph F, we denote by $F_n(F)$ the number of distinct labeled graphs G on n vertices not containing F as a subgraph. We denote by $\mathrm{ex}(n, F)$ the largest number of edges of any graph G on n vertices not containing F as a subgraph.

Theorem 1.1.4. Erdős, Frankl, Rödl (1986). *If F is any graph with $\chi(F) > 2$, then*

$$F_n(F) = 2^{\mathrm{ex}(n,F)(1+o(1))}$$

where $o(1) \to 0$ as $n \to \infty$.

For more on Theorem 1.1.4, see [1].

1.2. **This Paper and its Itinerary.**

The aim of this paper is to establish a similar technique for 3-uniform hypergraphs. More specifically, using the Hypergraph Regularity Lemma from [2] for 3-uniform hypergraphs, one is guaranteed a decomposition of the triples of a hypergraph $\mathcal{H}$ analogous to the decomposition of pairs in Szemerédi's Regularity Lemma for graphs (cf., Theorem 1.1.1). Within the partition obtained from The Hypergraph Regularity Lemma, if there is a sufficient number of "dense and regular" pieces, we show a statement analogous to Fact 1.1.2, that the hypergraph $\mathcal{H}$ contains a specific number of cliques $K_k^{(3)}$ in those pieces. We refer to our main theorem as The Counting Lemma.

In Theorem 3.1.1 of Section 3, we provide a statement for 3-uniform hypergraphs analogous to Fact 1.1.2. We refer to Theorem 3.1.1 as The Counting Lemma. Since its statement is quite complicated, we defer all discussion of it until Section 3, by which time all relevant concepts are defined. In Section 2, we give an exposition of background concepts needed for the remainder of this paper. Over Sections 4-8, we give the proof of Theorem 3.1.1. Note that in Section 6, we needed to develop a regularity lemma, Theorem 6.2.1, which naturally extends Theorem 3.4 of [2].

1.3. **Why the Hypergraph Regularity Lemma of** [2] **is technical.**

One can conclude that the Hypergraph Regularity Lemma must be structurally more complicated than that in Theorem 1.1.1. Indeed, for a hypergraph $\mathcal{H} = (V, E)$ and three disjoint sets $A, B, C \subset V$, we denote by $E(A, B, C)$ the set of triples $\{a, b, c\} \in \mathcal{H}$ with $a \in A$, $b \in B$, and $c \in C$, and put $e(A, B, C) = |E(A, B, C)|$. We also set $d(A, B, C) = d(\mathcal{H}_{ABC}) = e(A, B, C)/|A||B||C|$ for the density of the hypergraph $\mathcal{H}_{ABC} = (A \cup B \cup C, E(A, B, C))$. For a given $\epsilon > 0$, we say the triple A, B, C is *ϵ-regular* if for any A', B', C', $A' \subset A$, $B' \subset B$, $C' \subset C$, $|A'| > \epsilon|A|$, $|B'| > \epsilon|B|$, $|C'| > \epsilon|C|$,

$$|d(A', B', C') - d(A, B, C)| < \epsilon.$$

Following the original proof of Theorem 1.1.1, one can prove a statement for hypergraphs analogous to Theorem 1.1.1. That is, one can prove a statement establishing that every 3-uniform hypergraph (in fact, r-uniform hypergraph) admits an equitable partition $V = V_0 \cup V_1 \cup \ldots \cup V_t$ with all but $\epsilon\binom{t}{3}$ triples V_i, V_j, V_k ϵ-regular, $1 \le i < j < k \le t$. However, no statement like Fact 1.1.2 can be obtained, as the following example illustrates.

Example 1.3.1. For every $\epsilon > 0$, there exists a 3-uniform 4-partite hypergraph $\mathcal{H}$ with partite sets V_1, V_2, V_3, V_4, such that every triple of partite sets V_i, V_j, V_k, $1 \le i < j < k \le 4$, is both dense and ϵ-regular, but still $\mathcal{H}$ contains no copies of the clique $K_4^{(3)}$.

Indeed, let $\epsilon > 0$ be given. Suppose V_1, V_2, V_3, V_4 are four pairwise disjoint sets, each of cardinality n. We construct a random 3-uniform 4-partite hypergraph $\mathcal{H}$ with vertex set $V = V(\mathcal{H}) = V_1 \cup V_2 \cup V_3 \cup V_4$ and edge set given as follows: For all pairs i, j, $1 \le i < j \le 4$, consider the following random coloring of the edges $K(V_i, V_j) = \{\{v_i, v_j\} : v_i \in V_i,\ v_j \in V_j\}$: for each edge $\{v_i, v_j\} \in K(V_i, V_j)$, independently and uniformly at random color the edge $\{v_i, v_j\}$ red with probability $\frac{1}{2}$ or blue with probability $\frac{1}{2}$. Then for $1 \le i < j < k \le 4$, $v_i \in V_i$, $v_j \in V_j$, $v_k \in V_k$, $\{v_i, v_j, v_k\} \in \mathcal{H}$ if and only if edges $\{v_i, v_j\}$ and $\{v_i, v_k\}$ receive different colors. One can show that with probability tending to 1 with n tending to ∞, $\mathcal{H}$ satisfies the following two properties:

1. For any i, j, k, $1 \le i < j < k \le 4$,
$$|d(V_i, V_j, V_k) - \frac{1}{2}| < \epsilon.$$
2. All triples V_i, V_j, V_k are ϵ-regular, $1 \le i < j < k \le 4$.

However, it follows by the construction above that $\mathcal{H}$ contains no copies of $K_4^{(3)}$. Indeed, assume $\{v_1, v_2, v_3, v_4\}$ spans a copy of $K_4^{(3)}$ in $\mathcal{H}$, where for each i, $1 \le i \le 4$, $v_i \in V_i$. It follows by the Pigeon Hole Principle that there exist j, k, $2 \le j < k \le 4$, such that edges $\{v_1, v_j\}$ and $\{v_1, v_k\}$ are colored by the same color. This is, however, a contradiction. $\square$

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2. Definitions, Notation, and Auxiliary Facts

We begin this section by providing some background material. We start by giving some very basic definitions and notation which appear pervasively throughout the paper.

For a natural number n, we use the notation $[n]$ to denote the set $\{1, \dots, n\}$. If X is any finite set and l is any natural number so that $l \le |X|$, we denote by $[X]^l$ the set $\{L \subseteq X : |L| = l\}$. For simplicity, $[n]^l$ denotes $[[n]]^l$. Let X and Y be two sets, $X \cap Y = \emptyset$. We abuse the cartesian product notation $\times$ by defining $X \times Y = \{\{x, y\} : x \in X,\ y \in Y\}$.

2.1. Graphs and Cylinders.

In this subsection, we provide definitions and notation pertaining to graphs. A *graph* G on a finite vertex set $V = \{v_1, \dots, v_n\}$ is defined as a family $G \subseteq [V]^2$. If $G = [V]^2$, we call G a *clique* of size n, and in particular, if $n = 3$, we call G a *triangle*. When we do not specify a vertex set, we denote a clique of size n by $K_n^{(2)}$.

Let k be a natural number, $k \le n$. We often specify that the graph G is *k-partite* with *k-partition* $(V_1, \dots, V_k)$. As usual, this means that there exists a partition of the vertex set V of G, $V = V_1 \cup \dots \cup V_k$, where for any $i \in [k]$, $G \cap [V_i]^2 = \emptyset$. Often, we consider special *bipartite* (that is, 2-partite) subgraphs of a graph G. For nonempty disjoint subsets $A \subseteq V(G)$ and $B \subseteq V(G)$, we denote by $G[A, B]$ that subgraph $\{\{a, b\} \in G : a \in A,\ b \in B\}$.

Definition 2.1.1. We refer to any k-partite graph G with k-partition $(V_1, \dots, V_k)$ as a *k-partite cylinder*, and write G as $G = \bigcup_{1 \le i < j \le k} G^{ij}$ where $G^{ij} = G[V_i, V_j] = \{\{v_i, v_j\} \in G : v_i \in V_i, v_j \in V_j\}$. If $B \subseteq [k]$, $|B| = b$, then the b-partite cylinder $G(B) = \bigcup_{\{i,j\} \in [B]^2} G^{ij}$ is referred to as the *B-cylinder* of G, and if $b = 3$, the B-cylinder $G(B)$ is referred to as a *triad*. Note that if $B = [k], G(B) = G$.

We note that a cylinder is a graph whose vertex set is given with a fixed k-partition.

Definition 2.1.2. Suppose $G \subseteq [V]^2$ is a graph with vertex set $V = V(G)$, and let $X, Y \subseteq V$ be two nonempty disjoint subsets of V. We define the *density* of the pair X, Y with respect to G, denoted $d_G(X, Y)$, as

$$d_G(X,Y) = \frac{|G[X,Y]|}{|X||Y|}.$$

Definition 2.1.3. Suppose $G \subseteq [V]^2$ is a graph with vertex set $V = V(G)$, and let $\epsilon > 0$ be given. Let $X, Y \subseteq V$ be two nonempty disjoint subsets of V. We say that the pair X, Y is *ϵ-regular* if whenever $X' \subseteq X$, $|X'| > \epsilon|X|$ and $Y' \subseteq Y$, $|Y'| > \epsilon|Y|$, then

$$|d_G(X',Y') - d_G(X,Y)| < \epsilon. \tag{1}$$

We use the following slight alteration of the concept of ϵ-regularity. For positive λ and ϵ given, we say that the graph $G[X,Y]$ is a $(\lambda,\epsilon,2)$-*cylinder* provided that whenever $X' \subseteq X$, $|X'| > \epsilon|X|$ and $Y' \subseteq Y$, $|Y'| > \epsilon|Y|$, then

$$\frac{1}{\lambda}(1-\epsilon) < d_G(X',Y') < \frac{1}{\lambda}(1+\epsilon).$$

More generally, we give the following definition.

Definition 2.1.4. Suppose $G = \bigcup_{1\le i<j\le k} G^{ij}$ is a k-partite cylinder with k-partition $(V_1,\dots,V_k)$ and let $\lambda > 0, \epsilon > 0$ be given. We call G a (λ,ϵ,k)-*cylinder* provided all pairs V_i, V_j, $1 \le i < j \le k$, induce G^{ij} satisfying that whenever $V_i' \subseteq V_i$, $|V_i'| > \epsilon|V_i|$, and $V_j' \subseteq V_j$, $|V_j'| > \epsilon|V_j|$ then

$$\frac{1}{\lambda}(1-\epsilon) < d_{G^{ij}}(V_i',V_j') < \frac{1}{\lambda}(1+\epsilon).$$

In other words, each G^{ij} is a $(\lambda,\epsilon,2)$-cylinder. Note that for $B \subseteq [k], |B| = b$, the B-cylinder $G(B) = \bigcup_{\{i,j\}\in[B]^2} G^{ij}$ is a (λ,ϵ,b)-cylinder whenever $G = \bigcup_{1\le i<j\le k} G^{ij}$ is a (λ,ϵ,k)-cylinder. We refer to the B-cylinder $G(B)$ as the (λ,ϵ,B)-*cylinder* of G.

We now compare the two notions of regularity above. A $(\lambda,\epsilon,2)$-cylinder G^{ij} is ϵ-regular with density $d = d_{G^{ij}}(V_i,V_j)$ satisfying

$$|d - \frac{1}{\lambda}| < \frac{\epsilon}{\lambda}.$$

On the other hand, in a graph G, an ϵ-regular pair X, Y with density $d_G(X,Y)$ satisfying $\frac{1}{\lambda} - \epsilon < d_G(X,Y) < \frac{1}{\lambda} + \epsilon$ is a $(\lambda, 2\epsilon\lambda, 2)$-cylinder.

Suppose G is a k-partite cylinder with k-partition $(V_1,\dots,V_k)$. For a vertex $v \in V$, and $j \in [k]$, we denote by $N_j(v)$ the *j-neighborhood* of v, that is, $N_j(v) = \{w \in V_j : \{v,w\} \in G\}$. We now state the following simple fact about (λ,ϵ,k)-cylinders concerning neighborhood sizes.

Fact 2.1.5. *Suppose $G = \bigcup_{1\le i<j\le k} G^{ij}$ is a (λ,ϵ,k)-cylinder with k-partition $(V_1,\dots,V_k)$, $|V_1| = \dots = |V_k| = m$. Fix $i \in [k]$. All but $2k\epsilon m$ vertices $v_i \in V_i$ satisfy that for all $j \in [k]$, $j \ne i$,*

$$\frac{m}{\lambda}(1-\epsilon) < |N_j(v_i)| < \frac{m}{\lambda}(1+\epsilon).$$

For future reference, we also make the following remarks. Sometimes we consider the situation when two k-partite cylinders G and F are simultaneously defined on the same vertex set V with k-partition $(V_1, \dots, V_k)$. In such situations, we still want to denote the j-neighbors in each cylinder. We denote by $N_{G,j}(v)$ the j-neighborhood of the vertex v in the cylinder G, that is, $N_{G,j}(v) = \{w \in V_j : \{v, w\} \in G\}$. Similarly, $N_{F,j}(v) = \{w \in V_j : \{v, w\} \in F\}$.

We now define an auxiliary set system pertaining to a k-partite cylinder G.

Definition 2.1.6. For a k-partite cylinder G, we denote by $\mathcal{K}_j^{(2)}(G)$, $1 \le j \le k$, that j-uniform hypergraph whose edges are precisely those j-element subsets of $V(G)$ which span cliques of order j in G. Note that the quantity $|\mathcal{K}_j^{(2)}(G)|$ counts the total number of cliques in G of order j, that is, $|\mathcal{K}_j^{(2)}(G)| = |\{X \subseteq V(G) : |X| = j, [X]^2 \subseteq G\}|$.

For a (λ, ϵ, k)-cylinder G, the quantity $|\mathcal{K}_j(G)|$ is easy to estimate, as the following fact shows.

Fact 2.1.7. *For any positive integers k, λ, and suitably small positive reals ϵ, there exists a function $\theta_{k,\lambda}(\epsilon)$, $\theta_{k,\lambda}(\epsilon) \to 0$ as $\epsilon \to 0$, such that whenever G is a (λ, ϵ, k)-cylinder with k-partition $(V_1, \dots, V_k), |V_1| = \dots = |V_k| = m$, then*

$$(1 - \theta_{k,\lambda}(\epsilon))\frac{m^k}{\lambda^{\binom{k}{2}}} < |\mathcal{K}_k^{(2)}(G)| < (1 + \theta_{k,\lambda}(\epsilon))\frac{m^k}{\lambda^{\binom{k}{2}}}.$$

The proof of Fact 2.1.7 is an easy exercise.

2.2. **3-Uniform Hypergraphs and 3-cylinders.**

In this subsection, we provide definitions and notation concerning 3-uniform hypergraphs. A *3-uniform hypergraph* $\mathcal{H}$, otherwise called a *triple system*, on a finite vertex set $V = \{v_1, \dots, v_n\}$ is defined as a family $\mathcal{H} \subseteq [V]^3$. If $\mathcal{H} = [V]^3$, we call $\mathcal{H}$ a *clique* of size n. Similar to graphs, when we don't specify a vertex set, we denote a hypergraph clique of size n by $K_n^{(3)}$.

Let k be a natural number, $k \le n$. As in the case of graphs, we often specify that the hypergraph $\mathcal{H}$ is *k-partite* with given *k-partition* $(V_1, \dots, V_k)$. Again, this means that there exists a partition of the vertex set V of $\mathcal{H}$, $V = V_1 \cup \dots \cup V_k$, where for any pair $\{i, j\} \in [k]^2$, $\mathcal{H} \cap [V_i \cup V_j]^3 = \emptyset$.

Definition 2.2.1. We refer to any k-partite, 3-uniform hypergraph $\mathcal{H}$ with k-partition $(V_1, \dots, V_k)$ as a *k-partite 3-cylinder.* For $B \subseteq [k]$, we define the *B-3-cylinder* of $\mathcal{H}$ as that subhypergraph $\mathcal{H}(B)$ of $\mathcal{H}$ induced on $\bigcup_{i \in B} V_i$. Note that if $B = [k], \mathcal{H}(B) = \mathcal{H}$.

Definition 2.2.2. Suppose that G is a k-partite cylinder given with k-partition $(V_1, \dots, V_k)$, and $\mathcal{H}$ is a k-partite 3-cylinder. We say that G *underlies* the 3-cylinder $\mathcal{H}$ if $\mathcal{H} \subseteq \mathcal{K}_3^{(2)}(G)$.

As in Definition 2.1.6, we define an auxiliary set system pertaining to the 3-cylinder $\mathcal{H}$.

Definition 2.2.3. If $\mathcal{H}$ is a k-partite 3-cylinder, then for $1 \le j \le k$, $\mathcal{K}_j^{(3)}(\mathcal{H})$ denotes that j-uniform hypergraph whose edges are precisely those j-element subsets of $V(\mathcal{H})$ which span a clique of order j in $\mathcal{H}$. Note that the quantity $|\mathcal{K}_j^{(3)}(\mathcal{H})|$ counts the total number of cliques in $\mathcal{H}$ of order j, that is, $|\mathcal{K}_j^{(3)}(\mathcal{H})| = |\{X \subseteq V(\mathcal{H}) : |X| = j, [X]^3 \subseteq \mathcal{H}\}|$.

Definition 2.2.4. Let $\mathcal{H}$ be a k-partite 3-cylinder with underlying k-partite cylinder $G = \bigcup_{1 \le i < j \le k} G^{ij}$, and let $B \subseteq [k]$, $|B| = b$. For the B-cylinder $G(B)$, we define the *density* $d_{\mathcal{H}}(G(B))$ of $\mathcal{H}$ with respect to the B-cylinder $G(B)$ as

$$d_{\mathcal{H}}(G(B)) = \begin{cases} \frac{|\mathcal{K}_b^{(2)}(G(B)) \cap \mathcal{K}_b^{(3)}(\mathcal{H}(B))|}{|\mathcal{K}_b^{(2)}(G(B))|} & \text{if } |\mathcal{K}_b^{(2)}(G(B))| > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

In other words, the density counts the proportion of copies of $K_b^{(2)}$ in $G(B)$ which underlie copies of $K_b^{(3)}$ in $\mathcal{H}(B)$.

More generally, let $Q \subseteq G(B)$, $B \subseteq [k]$, $|B| = b$, where $Q = \bigcup_{\{i,j\} \in [B]^2} Q^{ij}$. One can define the density $d_{\mathcal{H}}(Q)$ of $\mathcal{H}$ with respect to Q as

$$d_{\mathcal{H}}(Q) = \begin{cases} \frac{|\mathcal{K}_b^{(3)}(\mathcal{H}) \cap \mathcal{K}_b^{(2)}(Q)|}{|\mathcal{K}_b^{(2)}(Q)|} & \text{if } |\mathcal{K}_b^{(2)}(Q)| > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

For our purposes, we need an extension of the definition above, and consider a simultaneous density of $\mathcal{H}$ with respect to a fixed r-tuple of b-partite cylinders $(Q(1), \ldots, Q(r))$.

Definition 2.2.5. Let $\mathcal{H}$ be a k-partite 3-cylinder with underlying k-partite cylinder $G = \bigcup_{1 \le i < j \le k} G^{ij}$, and let $B \subset [k], |B| = b$. Let $\overrightarrow{Q} = \overrightarrow{Q}_B = (Q(1), \ldots, Q(r))$ be an r-tuple of B-cylinders $Q(s) = \bigcup_{\{i,j\} \in [B]^2} Q^{ij}(s)$ satisfying that for every $s \in [r]$, $\{i, j\} \in [B]^2$, $Q^{ij}(s) \subseteq G^{ij}$. We define the *density* $d_{\mathcal{H}}(\overrightarrow{Q})$ of $\overrightarrow{Q}$ as

$$d_{\mathcal{H}}(\overrightarrow{Q}) = \begin{cases} \frac{|\mathcal{K}_b^{(3)}(\mathcal{H}) \cap \bigcup_{s=1}^r \mathcal{K}_b^{(2)}(Q(s))|}{|\bigcup_{s=1}^r \mathcal{K}_b^{(2)}(Q(s))|} & \text{if } |\bigcup_{s=1}^r \mathcal{K}_b^{(2)}(Q(s))| > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

We now give a definition which provides a notion of regularity for 3-cylinders.

Definition 2.2.6. Let $\mathcal{H}$ be a k-partite 3-cylinder with underlying cylinder $G = \bigcup_{1 \le i < j \le k} G^{ij}$. Let $B \subseteq [k], |B| = b, r$ and $\delta > 0$ be given. We say that the B-3-cylinder $\mathcal{H}(B)$ is (δ, r)-*regular with respect to* $G(B)$ if the following regularity condition is satisfied: suppose $\overrightarrow{Q} = \overrightarrow{Q}_B = (Q(1), \ldots, Q(r))$ is an r-tuple of B-cylinders $Q(s) = \bigcup_{\{i,j\} \in [B]^2} Q^{ij}(s)$ satisfying that for all $s \in [r]$, and all $\{i, j\} \in [B]^2$, $Q^{ij}(s) \subseteq G^{ij}$. Then $|\bigcup_{s=1}^r \mathcal{K}_b^{(2)}(Q(s))| > \delta |\mathcal{K}_b^{(2)}(G(B))|$ implies

$$d_{\mathcal{H}}(\overrightarrow{Q}) > d_{\mathcal{H}}(G(B)) - \delta. \tag{5}$$

If, moreover, it is specified that $\mathcal{H}(B)$ is (δ, r)-regular with respect to $G(B)$ with density $d_{\mathcal{H}}(G(B)) \ge \alpha - \delta$ for some α, then we say that the B-3-cylinder $\mathcal{H}(B)$ is (α, δ, r)-*regular with respect to* $G(B)$. If the regularity condition fails to be satisfied for any α, we say that $\mathcal{H}(B)$ is (δ, r)-*irregular with respect to* $G(B)$.

Note that this is a weaker definition of regularity since only a lower bound on the density of $\overrightarrow{Q}$ is guaranteed. The more standard notion of ϵ-regularity which replaces (5) with

$$|d_{\mathcal{H}}(\overrightarrow{Q}) - d_{\mathcal{H}}(G(B))| < \delta$$

and corresponds to Szemerédi's definition of an ϵ-regular pair was introduced in [2]. Since most of this paper deals with this weaker concept, we decided to alter the standard vocabulary. However, we also require the stronger notion, and so conclude this section with this definition.

Definition 2.2.7. Let $\mathcal{H}$ be a 3-partite 3-cylinder with underlying cylinder $G = G^{12} \cup G^{23} \cup G^{13}$. Let r and $\delta > 0$ be given. We say that $\mathcal{H}$ is (δ, r)-*fully regular with respect to* G if the following regularity condition is satisfied: suppose $\overrightarrow{Q} = (Q(1), \ldots, Q(r))$ is an r-tuple of triads $Q(s) = \bigcup_{\{i,j\} \in [3]^2} Q^{ij}(s)$, where for all $s \in [r]$, $\{i, j\} \in [3]^2$, $Q^{ij}(s) \subseteq G^{ij}$. Then $|\bigcup_{s=1}^{r} \mathcal{K}_3^{(2)}(Q(s))| > \delta |\mathcal{K}_3^{(2)}(G)|$ implies

$$|d_{\mathcal{H}}(\overrightarrow{Q}) - d_{\mathcal{H}}(G)| < \delta. \tag{6}$$

If, moreover, it is specified that $\mathcal{H}$ is (δ, r)-fully regular with respect to G with density $d_{\mathcal{H}}(G) \in (\alpha - \delta, \alpha + \delta)$ for some α, then we say that $\mathcal{H}$ is (α, δ, r)-*fully regular with respect to* G.

Note that (δ, r)-full regularity implies (δ, r)-regularity, but not conversely.

3. The Counting Lemma

In this section, we present and comment on the main theorem of this work, The Counting Lemma.

3.1. The Statement of the Counting Lemma.

Before we state our theorem, we establish an appropriate context. Since we later alter this context, we refer to the following as Setup 1.

Setup 1:

For a given integer $k \geq 3$, for a given set $\{\alpha_B : B \in [k]^3\}$ of positive reals and for given constants δ, λ, r and ϵ, suppose $\mathcal{H}$ is a triple system and G is an underlying graph satisfying the following:

- (i) $\mathcal{H}$ is a k-partite 3-cylinder with k-partition $(V_1, \ldots, V_k)$, $|V_1| = \ldots = |V_k| = m$.
- (ii) G is an underlying (λ, ϵ, k)-cylinder of $\mathcal{H}$.
- (iii) for all $B \in [k]^3$, $\mathcal{H}(B)$ is (α_B, δ, r)-fully regular with respect to the triad $G(B)$ (cf., Definition 2.2.7).

The aim of this paper is to establish the following theorem.

Theorem 3.1.1. The Counting Lemma. *For all integers $k \geq 4$, sets $\{\alpha_B : B \in [k]^3\}$ of positive reals and $\beta > 0$, there exists a positive constant δ so that for all integers $\lambda \geq \frac{1}{\delta}$, there exist r, ϵ such that the following holds: with parameters k, $\{\alpha_B : B \in [k]^3\}$,*

δ, λ, r and ϵ, suppose $\mathcal{H}$ is a triple system and G is an underlying graph which satisfy the hypothesis of Setup 1. Then

$$\frac{\Pi_{B\in[k]^3}\alpha_B}{\lambda^{\binom{k}{2}}}m^k(1-\beta) \le \left|\mathcal{K}_k^{(3)}(\mathcal{H})\right| \le \frac{\Pi_{B\in[k]^3}\alpha_B}{\lambda^{\binom{k}{2}}}m^k(1+\beta). \tag{7}$$

As mentioned in the Introduction, unlike Fact 1.1.2, the proof of Theorem 3.1.1 is complicated. Sections 4-8 are devoted to proving this theorem.

In proving Theorem 3.1.1, we use the following hierarchy governing the sizes of the constants $\min\{\alpha_B : B \in [k]^3\}$, δ, λ, r, ϵ, m:

$$\min\{\alpha_B : B \in [k]^3\} \gg \delta \ge \frac{1}{\lambda} \gg \frac{1}{r} \gg \epsilon \gg \frac{1}{m}. \tag{8}$$

In Theorem 3.1.1, we make the assumption that $\delta \ge \frac{1}{\lambda}$, thus providing a well arranged (linearly ordered) hierarchy on the constants. However, for the other extreme that $\delta \ll \frac{1}{\lambda}$, Theorem 3.1.1 remains true and its proof becomes considerably simpler. Unfortunately, The Hypergraph Regularity Lemma of [2], the tool which in application provides an environment satisfying the conditions of Setup 1, does not us allow us to make the assumption that $\delta \ll \frac{1}{\lambda}$. The reason for this unfortunate outcome is similar to why in the context of Theorem 1.1.1 one may assume $\epsilon \ge \frac{1}{t}$; one may simply set $t_0 \ge \frac{1}{\epsilon}$.

For the remainder of this paper, the reader may assume the constants with which we work always satisfy the hierarchy given in (8). While we carefully specify the sizes of the promised constants δ, r and ϵ in all of our proofs, we make no effort to estimate the size of the integer m seen in (8). We state here that in this entire paper, we assume the integer m is large enough for any argument we make, and indeed, the value $\frac{1}{m}$ may be assumed to be "infinitely smaller" than any other constant considered.

3.2. **Strategy for Proving the Counting Lemma.**

We found it advantageous to focus on the lower bound of (7) for most of this paper. To that end, we formulate a theorem which exclusively discusses this lower bound. Before doing so, however, we discuss the weaker environment in which we work.

Setup 2:

For a given integer $k \ge 3$, for a given set $\{\alpha_B : B \in [k]^3\}$ of positive reals and for given constants δ, λ, r and ϵ, suppose $\mathcal{H}$ is a triple system and G is an underlying graph satisfying the following:

(i) $\mathcal{H}$ is a k-partite 3-cylinder with k-partition $(V_1, \ldots, V_k)$, $|V_1| = \ldots = |V_k| = m$.
(ii) G is an underlying (λ, ϵ, k)-cylinder of $\mathcal{H}$.
(iii) For all $B \in [k]^3$, $\mathcal{H}(B)$ is (α_B, δ, r)-*regular* with respect to the triad $G(B)$ (cf., Definition 2.2.6).

If moreover, the following also holds:

(iii'). All the constants α_B in the set $\{\alpha_B : B \in [k]^3\}$ are the same, (i.e. $\alpha_B = \alpha$ for all $B \in [k]^3$).

then we refer to this special case of Setup 2 as Setup $2'$.

Theorem 3.2.1. *For all integers $k \geq 4$, sets $\{\alpha_B : B \in [k]^3\}$ of positive reals and $\beta > 0$, there exists a positive constant δ so that for all integers $\lambda \geq \frac{1}{\delta}$, there exist r, ϵ such that the following holds: with parameters k, $\{\alpha_B : B \in [k]^3\}$, δ, λ, r and ϵ, suppose $\mathcal{H}$ is a triple system and G is an underlying graph which satisfy the hypothesis of Setup 2. Then*

$$\left|\mathcal{K}_k^{(3)}(\mathcal{H})\right| \geq \frac{\Pi_{B\in[k]^3}\alpha_B}{\lambda^{\binom{k}{2}}} m^k(1-\beta). \tag{9}$$

We show that (perhaps surprisingly), the upper bound in (7) of Theorem 3.1.1 follows from the lower bound in (7) of Theorem 3.1.1. In other words, Theorem 3.2.1 implies Theorem 3.1.1. We show this implication in Section 8.

To prove Theorem 3.2.1, it suffices to prove the following weaker statement $D(k)$, for all integers $k \geq 4$.

Statement 3.2.2 ($D(k)$)**.** *For all positive α and positive β, there exists a positive constant δ so that for all integers $\lambda \geq \frac{1}{\delta}$, there exist r, ϵ such that the following holds: with constants k, α, δ, λ, r, and ϵ, suppose $\mathcal{H}$ is a k-partite 3-cylinder and G is a k-partite cylinder satisfying the hypothesis of Setup $2'$. Then*

$$\left|\mathcal{K}_k^{(3)}(\mathcal{H})\right| \geq \frac{\alpha^{\binom{k}{3}}}{\lambda^{\binom{k}{2}}} m^k(1-\beta).$$

At the end of this section we prove the following claim.

Claim 3.2.3. *$D(k)$, for all $k \geq 4$, implies Theorem 3.2.1.*

Thus, by Claim 3.2.3, to prove Theorem 3.2.1, it is enough to prove $D(k)$ for all $k \geq 4$. The statement $D(k)$ is proved using induction on k. In our induction scheme, we need to prove the following stronger statement $R(k)$, where $k \geq 3$.

Statement 3.2.4 ($R(k)$)**.** *For all nonnegative α and positive δ_k, there exists a positive constant δ so that for all integers $\lambda \geq \frac{1}{\delta}$, for all integers $r_k \geq 1$, there exist r, ϵ such that the following holds: with constants k, α, δ, λ, r, and ϵ, suppose $\mathcal{H}$ is a k-partite 3-cylinder and G is a k-partite cylinder satisfying the hypothesis of Setup 2'. Then $\mathcal{H} = \mathcal{H}([k])$ is $(\alpha^{\binom{k}{3}}, \delta_k, r_k)$-regular with respect to $G = G([k])$ (*cf., Definition 2.2.6*).*

In other words, the statement $R(k)$ guarantees an "arbitrarily uniform distribution" of the cliques $K_k^{(3)}$ provided all $\binom{k}{3}$ corresponding triads are sufficiently regular.

We proceed according to the following scheme:

$$R(3) \Rightarrow D(4) \Rightarrow R(4) \Rightarrow D(5) \Rightarrow \ldots \Rightarrow R(k-1) \Rightarrow D(k) \Rightarrow R(k) \Rightarrow \ldots \tag{10}$$

The scheme above establishes the validity of the statement $D(k)$, for all $k \geq 4$. We now outline our strategy for proving the induction scheme in (10). Note that the statement $R(3)$ holds by definition. Note also that the first implication, $R(3) \Rightarrow D(4)$, was essentially proved in [2]. We break the inductive step into the following two implications:

(1) $R(k-1) \Rightarrow D(k)$.
(2) $D(k) \Rightarrow R(k)$.

In Section 5, we prove the implication $R(k-1) \Rightarrow D(k)$ from implication (1). We note that our proof of the implication $R(k-1) \Rightarrow D(k)$ in Section 5 follows largely from principles we establish in the upcoming Section 4. We show the implication $D(k) \Rightarrow R(k)$ from implication (2) in Section 7. We note that explicit in our proof of $D(k) \Rightarrow R(k)$ is the use of our Regularity Lemma, which we present in the upcoming Section 6.

3.3. **Proof of Claim 3.2.3.**

We conclude this section now with a proof for Claim 3.2.3.

Proof of Claim 3.2.3.

We begin our proof of Claim 3.2.3 by stating the following fact.

Fact 3.3.1. *Let α, δ, ϵ be given positive reals with $\delta < \frac{\alpha}{4}$ and let λ and r be given positive integers. Let α' be a given positive real satisfying $\alpha' \leq \alpha$, and let 3-partite triple system $\mathcal{H}$ and underlying graph G satisfy the hypothesis of Setup 2 with constants $k = 3$, α, δ, λ, r and ϵ. Then there exists a family*

$$\widetilde{\mathcal{H}} = \{\mathcal{H}_1, \dots, \mathcal{H}_p\}$$

of 3-partite triple systems with the following properties:

(i) $p = \lfloor \frac{\alpha}{\alpha'} \rfloor$,
(ii) $\mathcal{H}_i \cap \mathcal{H}_j = \emptyset$ *for all* $\{i,j\} \in [p]^2$, *and* $\bigcup_{i\in[p]} \mathcal{H}_i \subseteq \mathcal{H}$,
(iii) *for each* $i \in [p]$, $\mathcal{H}_i$ *is* $(\alpha', 2\delta, r)$*-regular with respect to the underlying* $(\lambda, \epsilon, 3)$*-cylinder* G.

The proof of Fact 3.3.1 is found in [6] and [3]. The proof of a similar statement for graphs is given in [2].

The idea of the proof of Claim 3.2.3 is simple. Using Fact 3.3.1, we decompose each of the 3-partite 3-cylinders $\mathcal{H}(B)$, $B \in [k]^3$, into roughly $\frac{\alpha_B}{\alpha}$ 3-partite 3-cylinders, each of density α. For each combination of these sparser "sub-3-cylinders" of $\mathcal{H}(B)$ over all $B \in [k]^3$, we use $D(k)$ to claim each resulting k-partite 3-cylinder has $\frac{\alpha^{\binom{k}{3}}}{\lambda^{\binom{k}{2}}} m^k$ copies of $K_k^{(3)}$. Summing over all $\Pi_{B\in[k]^3} \frac{\alpha_B}{\alpha}$ combinations, we get the required lower bound. We now formally prove Claim 3.2.3.

Let $k \geq 4$ be a given integer, let $\{\alpha_B : B \in [k]^3\}$ be a given set of positive reals, and let $\beta > 0$ be given. We need to define the promised constant $\delta > 0$. Before doing so, we define auxiliary positive constants α and β' to satisfy

$$(1-\beta')\Pi_{B\in[k]^3}\left(1 - \frac{\alpha}{\alpha_B}\right) > 1 - \beta. \tag{11}$$

Now, for the constants $k \geq 4$, α and β' above, let

$$\delta = \frac{\delta_{D(k)}(k, \alpha, \beta')}{2} \tag{12}$$

where $\delta_{D(k)}(k, \alpha, \beta')$ is guaranteed to exist by the statement $D(k)$.

Let $\lambda > \frac{1}{\delta}$ be a given integer. For the constants $k \geq 4$, α, β', δ, λ, let

$$\begin{aligned} r &= r_{D(k)}(k, \alpha, \beta', \delta, \lambda), &(13)\\ \epsilon &= \epsilon_{D(k)}(k, \alpha, \beta', \delta, \lambda) &(14)\end{aligned}$$

be those constants guaranteed to exist by the statement $D(k)$.

With the given parameters k, $\{\alpha_B : B \in [k]^3\}$, δ from (12), the given integer $\lambda > \frac{1}{\delta}$, and r and ϵ from (13) and (14) respectively, suppose $\mathcal{H}$ is a k-partite triple system and G is an underlying graph of $\mathcal{H}$ satisfying the hypothesis of Setup 2. Our goal is to show

$$\left|\mathcal{K}_k^{(3)}(\mathcal{H})\right| \geq \frac{\Pi_{B\in[k]^3}\alpha_B}{\lambda^{\binom{k}{2}}} m^k (1-\beta). \tag{15}$$

We begin by applying Fact 3.3.1 to each 3-partite 3-cylinder $\mathcal{H}(B)$ with underlying $(\lambda, \epsilon, 3)$-cylinder $G(B)$, $B \in [k]^3$. Thus, for each $B \in [k]^3$, we obtain a family $\widetilde{\mathcal{H}(B)} = \{\mathcal{H}_i(B)\}$, $1 \leq i \leq \lfloor \frac{\alpha_B}{\alpha} \rfloor$ of 3-partite 3-cylinders, where:

a. $\mathcal{H}_i(B) \cap \mathcal{H}_j(B) = \emptyset$, for all i, j, $1 \leq i < j \leq \lfloor \frac{\alpha_B}{\alpha} \rfloor$, and $\bigcup_{1\leq i \leq \lfloor \frac{\alpha_B}{\alpha} \rfloor} \mathcal{H}_i(B) \subseteq \mathcal{H}(B)$,
b. for each i, $1 \leq i \leq \lfloor \frac{\alpha_B}{\alpha} \rfloor$, $\mathcal{H}_i(B)$ is $(\alpha, 2\delta, r)$-regular with respect to the $(\lambda, \epsilon, 3)$-cylinder $G(B)$.

For each of the $\Pi_{B\in[k]^3} \lfloor \frac{\alpha_B}{\alpha} \rfloor$ choices of i_B, $1 \leq i_B \leq \lfloor \frac{\alpha_B}{\alpha} \rfloor$, consider a vector $\overrightarrow{i} = (i_B; B \in [k]^3)$. Let I be the set of all of these vectors (note that $|I| = \Pi_{B\in[k]^3} \lfloor \frac{\alpha_B}{\alpha} \rfloor$). Let k-partite 3-cylinder $\mathcal{H}_{\overrightarrow{i}} = \bigcup_{B\in[k]^3} \mathcal{H}_{i_B}(B)$. For all $\overrightarrow{i} \in I$, $\mathcal{H}_{\overrightarrow{i}}$ and G satisfy the conditions of the hypothesis in Setup 2 $'$ with the constants k, α, 2δ, λ, r and ϵ, where δ satisfies (12), and r and ϵ satisfy (13) and (14) respectively. Therefore, we may apply statement $D(k)$ to each k-partite 3-cylinder $\mathcal{H}_{\overrightarrow{i}}$ and G, $\overrightarrow{i} \in I$, to conclude

$$|\mathcal{K}_k^{(3)}(\mathcal{H}_{\overrightarrow{i}})| \geq \frac{\alpha^{\binom{k}{3}}}{\lambda^{\binom{k}{2}}} m^k (1-\beta'). \tag{16}$$

We therefore conclude

$$\begin{aligned} |\mathcal{K}_k^{(3)}(\mathcal{H})| &\geq \left| \bigcup_{\overrightarrow{i}\in I} \mathcal{K}_k^{(3)}(\mathcal{H}_{\overrightarrow{i}}) \right|, \\ &= \sum_{\overrightarrow{i}\in I} |\mathcal{K}_k^{(3)}(\mathcal{H}_{\overrightarrow{i}})|. &(17)\end{aligned}$$

Employing the bound from (16) in (17) yields

$$|\mathcal{K}_k^{(3)}(\mathcal{H})| \geq \frac{\alpha^{\binom{k}{3}}}{\lambda^{\binom{k}{2}}} m^k (1-\beta')|I|. \tag{18}$$

Since $|I| = \Pi_{B\in[k]^3}\lfloor\frac{\alpha_B}{\alpha}\rfloor$, we have from (18) that

$$\begin{aligned} |\mathcal{K}_k^{(3)}(\mathcal{H})| &\geq \frac{\alpha^{\binom{k}{3}}}{\lambda^{\binom{k}{2}}} m^k(1-\beta')\Pi_{B\in[k]^3}\lfloor\frac{\alpha_B}{\alpha}\rfloor, \\ &= \frac{\Pi_{B\in[k]^3}\left(\lfloor\frac{\alpha_B}{\alpha}\rfloor\alpha\right)}{\lambda^{\binom{k}{2}}} m^k(1-\beta'), \\ &\geq \frac{\Pi_{B\in[k]^3}(\alpha_B-\alpha)}{\lambda^{\binom{k}{2}}} m^k(1-\beta'), \\ &= \frac{\Pi_{B\in[k]^3}\alpha_B}{\lambda^{\binom{k}{2}}} m^k(1-\beta')\Pi_{B\in[k]^3}\left(1-\frac{\alpha}{\alpha_B}\right). \end{aligned} \tag{19}$$

By our choice of the auxiliary constants α, β' in (11), we infer from (19) that

$$|\mathcal{K}_k^{(3)}(\mathcal{H})| \geq \frac{\Pi_{B\in[k]^3}\alpha_B}{\lambda^{\binom{k}{2}}} m^k(1-\beta).$$

Hence, (15) is proved, thus proving Claim 3.2.3. □

4. Background Material for the Proof of $R(k-1) \Rightarrow D(k)$

In this section, we present definitions and technical facts, without proof, needed in Section 5.3 for the proof of Lemma 5.1.1. The proofs of these facts are along technical albeit standard lines. The reader interested in the proofs of these facts is encouraged to see [6].

4.1. **Basic Facts.**

We begin with the following fact from [2].

Fact 4.1.1. *Let M, k, λ be given integers, and let $\epsilon > 0$, $\sigma \in (0,1]$ be given real numbers. Suppose $G = \bigcup_{0\leq i<j\leq k} G^{ij}$ is a $(k+1)$-partite cylinder with $(k+1)$-partition $(W_0, W_1, \ldots, W_k)$, $|W_1| = \ldots = |W_k| = M$, and $|W_0| = \sigma M$. Suppose that for all $j \in [k]$, G satisfies these conditions:*

(i) *G^{0j} induced on $W_0 \cup W_j$ satisfies that whenever $W_0' \subseteq W_0$ and $W_j' \subseteq W_j$, $|W_0'| \geq \epsilon M$ and $|W_j'| \geq \epsilon M$, then*

$$d_{G^{0j}}\left(W_0', W_j'\right) \in \left(\frac{1}{\lambda}(1-\epsilon), \frac{1}{\lambda}(1+\epsilon)\right).$$

(ii) *For all $x \in W_0$, $|N_j(x)| \leq \frac{M}{\lambda}(1+\epsilon)$.*

Then the following property holds: there exists an integer $M_0 = M_0(\sigma, k, \epsilon)$ such that for all $M \geq M_0$, there exist $b \geq \sigma/k\epsilon$ vertices $\{x_1, \ldots, x_b\} \subseteq W_0$ satisfying $|N_j(x_u) \cap N_j(x_v)| \leq \frac{M}{\lambda^2}(1+\epsilon)^2$, for all $\binom{b}{2}$ pairs $\{u, v\} \in [b]^2$ and all $j \in [k]$.

We make the following remark about Fact 4.1.1. Suppose that G is a $(\lambda, \epsilon, k+1)$-cylinder with $(k+1)$-partition $(V_0, V_1, \ldots, V_k)$, $|V_0| = |V_1| = \ldots = |V_k| = M$. Let

$$V_0' = \left\{v \in V_0 : |N_i(v)| > \frac{M}{\lambda}(1+\epsilon), \text{for some } i \in [k]\right\}.$$

By Fact 2.1.5, $|V_0'| < k\epsilon M$. Putting $W_0 = V_0 \setminus V_0'$, $W_i = V_i$ for all $i \in [k]$, and $\sigma = 1 - k\epsilon$, we have that the $(k+1)$-partite cylinder G_1, induced from G on the $(k+1)$-partition $(W_0, W_1, \ldots, W_k)$, satisfies the hypothesis of Fact 4.1.1

A consequence of Fact 4.1.1 which is used frequently in Section 5 is stated in the following fact.

Fact 4.1.2. *Let k, r, λ be given integers, and let $\epsilon > 0$, and $\sigma \in (0,1]$ be given real numbers. Suppose $G = \bigcup_{0 \le i < j \le k} G^{ij}$ is a $(k+1)$-partite cylinder with $(k+1)$-partition $(W_0, W_1, \ldots, W_k)$, $|W_1| = \ldots = |W_k| = M$, and $|W_0| = \sigma M$, where M is assumed to be sufficiently large. Suppose that for all $j \in [k]$, G satisfies these conditions:*

(i) G^{0j} induced on $W_0 \cup W_j$ satisfies that whenever $W_0' \subseteq W_0$ and $W_j' \subseteq W_j$, $|W_0'| \ge \epsilon M$ and $|W_j'| \ge \epsilon M$, then

$$d_{G^{0j}}\left(W_0', W_j'\right) \in \left(\frac{1}{\lambda}(1-\epsilon), \frac{1}{\lambda}(1+\epsilon)\right).$$

(ii) For all $x \in W_0$, $|N_j(x)| \le \frac{M}{\lambda}(1+\epsilon)$.

Then there exist pairwise disjoint r-element subsets $S_1, \ldots, S_q$, $q = \lceil \frac{\sigma M}{r}(1 - rk\epsilon/\sigma) \rceil$, satisfying:

(a) for each $i \in [q]$, $S_i = \{x_1^{(i)}, \ldots, x_r^{(i)}\} \subseteq W_0$,

(b) for each $i \in [q]$, for all $\{u, v\} \in [r]^2$, and for all $j \in [k]$,

$$|N_j(x_u^{(i)}) \cap N_j(x_v^{(i)})| \le \frac{M}{\lambda^2}(1+\epsilon)^2.$$

4.2. **Regular Couples.**

We now define a notion of (δ, r)-regularity for cylinders.

Definition 4.2.1. Let γ, δ be positive reals, let r be a positive integer, and let F be a bipartite graph with bipartition (U, V). We say that F is (γ, δ, r)-*regular* if the following property holds: For any r-tuple of pairs of subsets $(\{U_j, V_j\})_{j=1}^r$, $U_j \subseteq U$, $V_j \subseteq V$, $1 \le j \le r$, satisfying

$$|\bigcup_{j=1}^{r}(U_j \times V_j)| > \delta |U||V|,$$

then

$$\frac{|F \cap \bigcup_{j=1}^{r}(U_j \times V_j)|}{|\bigcup_{j=1}^{r}(U_j \times V_j)|} > \gamma. \tag{20}$$

Note that it follows directly from the definition that if F is (γ, δ, r)-regular, then F is also (γ, δ', r')-regular for any $\delta' \ge \delta$ and positive integer $r' \le r$. We use this fact repeatedly in later sections.

We proceed with the following easy fact.

Fact 4.2.2. *Let γ, δ, and r be given, and suppose F is a bipartite graph with bipartition (U, V). If F is (γ, δ, r)-regular, then all but less than $\delta|U|$ vertices $u \in U$ satisfy*

$$|N(u)| > \gamma |V|.$$

When we encounter (γ, δ, r)-regular bipartite graphs, they actually are *subgraphs* of other highly regular bipartite graphs. This situation prompts the following definition.

Definition 4.2.3. Let γ, δ, ϵ be given positive reals, let r, λ be given positive integers, and let F, G be two given bipartite graphs, each with bipartition (U, V). We call the ordered pair of graphs (F, G) a $(\gamma, \delta, r, \lambda, \epsilon)$*-regular couple* provided

(i) $F \subseteq G$,
(ii) F is (γ, δ, r)-regular,
(iii) G is a $(\lambda, \epsilon, 2)$-cylinder.

Note that in the definition above, F being (γ, δ, r)-regular only ensures a lower bound of γ on the density described in (20). With G being a (λ, ϵ, k)-cylinder, Definition 2.1.4 ensures that the density of bipartite subgraphs of G on subsets $U' \subseteq U$, $|U'| > \epsilon|U|$, $V' \subseteq V$, $|V'| > \epsilon|V|$, is roughly $\frac{1}{\lambda}$.

The following important fact relates to Fact 2.1.7.

Fact 4.2.4. *For all integers $k \geq 3$ and all positive α, β, there exists $\delta > 0$ so that for all integers $\lambda \geq \frac{1}{\delta}$, there exist r, ϵ so that the following property holds: Suppose*

(i) *$G = \bigcup_{1 \leq i < j \leq k} G^{ij}$ is a (λ, ϵ, k)-cylinder with k-partition $(V_1, \ldots, V_k)$, $|V_1| = \ldots = |V_k| = m$.*
(ii) *$F = \bigcup_{1 \leq i < j \leq k} F^{ij}$ is a k-partite cylinder defined on the same k-partition $(V_1, \ldots, V_k)$.*
(iii) *For all i, j, $1 \leq i < j \leq k$, (F^{ij}, G^{ij}) is a $(\frac{\alpha - 2\delta}{\lambda}, \delta, r, \lambda, \epsilon)$-regular couple.*

Then F satisfies

$$\left|\mathcal{K}_k^{(2)}(F)\right| \geq (\frac{\alpha}{\lambda})^{\binom{k}{2}} m^k (1 - \beta).$$

4.3. **Link Graphs.**

For our final fact, we need the following definition.

Definition 4.3.1. Let $\mathcal{H}$ be a k-partite 3-cylinder with k-partition $(V_1, \ldots, V_k)$, and let $G = \bigcup_{1 \leq i < j \leq k} G^{ij}$ be an underlying k-partite cylinder. Let i, j, $1 < i < j \leq k$, be integers and let $x \in V_1$. We define the $\{i, j\}$*-link graph of* x, L_x^{ij}, as

$$L_x^{ij} \quad = \quad \Big\{ \{y, z\} \in G^{ij} : y \in N_{G,i}(x), z \in N_{G,j}(x), \{x, y, z\} \in \mathcal{H} \Big\}. \tag{21}$$

We further define the *link graph of* x, L_x, as

$$L_x = \bigcup_{1 < i < j \leq k} L_x^{ij}.$$

Before continuing with our next fact, we describe some notation we need. Suppose $G = \bigcup_{1 \leq i < j \leq k} G^{ij}$ is a k-partite cylinder with k-partition $(V_1, \ldots, V_k)$, and let $x \in V_1$. For fixed i, j, $1 < i < j \leq k$, define

$$G^{ij}(x, G) = \Big\{ \{x, y\} \in G^{ij} : x \in N_{G,i}(x), y \in N_{G,j}(x) \Big\} = G^{ij}[N_{G,i}(v), N_{G,j}(v)].$$

We conclude this section by stating that any (α, δ, r)-regular hypergraph $\mathcal{H}$ with underlying (λ, ϵ, k)-cylinder $G = \bigcup_{1 \leq i < j \leq k} G^{ij}$ (on partite sets $(V_1, \ldots, V_k)$) satisfies that for all i, j, $1 < i < j \leq k$, and most vertices $x \in V_1$, the $\{i, j\}$-link graph L_x^{ij} is $(\frac{\alpha - 2\sqrt{\delta}}{\lambda}, \sqrt{\delta}, r)$-regular. We now present this crucial fact.

Fact 4.3.2. *For all positive reals α, δ, $\alpha > 2\delta$, and for all positive integers k, λ, and r, there exists $\epsilon > 0$ so that the following property holds: suppose*

(i) *$\mathcal{H}$ is a k-partite 3-cylinder with k-partition $(V_1, \ldots, V_k)$, $|V_1| = \ldots = |V_k| = m$.*
(ii) *$G = \bigcup_{1 \le i < j \le k} G^{ij}$ is an underlying (λ, ϵ, k)-cylinder of $\mathcal{H}$.*
(iii) *For all i, j, $1 < i < j \le k$, $\mathcal{H}(\{1, i, j\})$ is (α, δ, r)-regular with respect to $G(\{1, i, j\})$.*

Then

(a) *all but $4\epsilon(k-1)m$ vertices $v \in V_1$ satisfy that for each $i \in \{2, \ldots, k\}$,*
$$\frac{m}{\lambda}(1-\epsilon) \le |N_{G,i}(v)| \le \frac{m}{\lambda}(1+\epsilon),$$
(b) *all but $2\binom{k-1}{2}\sqrt{\delta}m$ vertices $v \in V_1$ satisfy that for each i, j, $1 < i < j \le k$, $(L_x^{ij}, G^{ij}(x, G))$ is an $(\frac{\alpha - 2\sqrt{\delta}}{\lambda}, \sqrt{\delta}, r, \lambda, 2\lambda\epsilon)$-regular couple.*

5. $R(k-1) \Rightarrow D(k)$

The objective of this section is to prove that $R(k-1) \Rightarrow D(k)$. To that end, we give an auxiliary statement in Lemma 5.1.1 that implies $R(k-1) \Rightarrow D(k)$. In Claim 5.1.2, we prove the sufficiency of Lemma 5.1.1 for the implication $R(k-1) \Rightarrow D(k)$. Afterwards, we verify Lemma 5.1.1. Note that our proof of Lemma 5.1.1 largely follows from principles given in the previous section.

5.1. Lemma 5.1.1 and $R(k-1) \Rightarrow D(k)$.

We now state Lemma 5.1.1.

Lemma 5.1.1. *For all integers $k \ge 4$, for all $\alpha > 0$ and $\beta > 0$, there exists a positive constant δ so that for all integers $\lambda \ge \frac{1}{\delta}$, there exist r, ϵ such that the following holds: suppose*

(i) *$\mathcal{H}$ is a k-partite 3-cylinder with k-partition $(V_1, \ldots, V_k)$, $|V_1| = \ldots = |V_k| = m$.*
(ii) *G is an underlying (λ, ϵ, k)-cylinder of $\mathcal{H}$.*
(iii) *For all $B \in ([k]^3 \cup [k]^{k-1})$, $|B| = b \in \{3, k-1\}$, $\mathcal{H}(B)$ is $(\alpha^{\binom{b}{3}}, \delta, r)$-regular with respect to $G(B)$.*

Then
$$\left|\mathcal{K}_k^{(3)}(\mathcal{H})\right| \ge \frac{\alpha^{\binom{k}{3}}}{\lambda^{\binom{k}{2}}} m^k (1-\beta).$$

We now prove in Claim 5.1.2 that in order to show the implication $R(k-1) \Rightarrow D(k)$, it is enough to show Lemma 5.1.1.

Claim 5.1.2. *For $k \ge 4$, Lemma 5.1.1 implies the implication $R(k-1) \Rightarrow D(k)$.*

Before proceeding to the proof of Claim 5.1.2, we first discuss the main ideas behind the proof of Claim 5.1.2. Fix $k \ge 4$, and assume both Lemma 5.1.1 and the statement $R(k-1)$ are true. Our goal is to show that the statement $D(k)$ is true for the fixed integer $k \geq 4$. Before formally disclosing the constants δ, r, and ϵ guaranteed to exist by statement $D(k)$, we reveal our strategy for proving $D(k)$.

Note that in the hypothesis of $D(k)$, we assume that all $\mathcal{H}(B)$, $B \in [k]^3$, are sufficiently (α, δ, r)-regular. The statement $R(k-1)$ being true ensures us that having all $\mathcal{H}(B)$ sufficiently regular guarantees that all $(k-1)$-partite 3-cylinders $\mathcal{H}(C)$, $C \in [k]^{k-1}$, are arbitrarily regular. Lemma 5.1.1 being true ensures us that with both the triads $\mathcal{H}(B)$ and the $(k-1)$-partite 3-cylinders $\mathcal{H}(C)$ sufficiently regular, $\mathcal{H}$ must contain at least $\frac{\alpha^{\binom{k}{3}}}{\lambda^{\binom{k}{2}}} m^k(1-\beta)$ copies of $K_k^{(3)}$, thus establishing $D(k)$. With this strategy revealed, we formally disclose the constants.

5.2. **Proof of Claim 5.1.2.**

With $k \geq 4$ already given, let $\alpha, \beta > 0$ be given. Our first goal is to define the constants guaranteed by the statement $D(k)$, starting with the promised constant δ.

Definitions of the Constants:

We first define the constant δ promised by the statement $D(k)$. Recall that we assume that Lemma 5.1.1 is true for the fixed integer $k \geq 4$. Choose the constants α and β as they are given above. Lemma 5.1.1 guarantees a constant $\delta_L = \delta_L(\alpha, \beta)$. Recall we also assume that the statement $R(k-1)$ is true for the fixed integer $k \geq 4$. Choose α as it was given above, and let $\delta_{k-1} = \delta_L$. The statement $R(k-1)$ guarantees a constant $\delta_R = \delta_R(\alpha, \delta_L)$. Note that $\delta_R \leq \delta_L$. Set $\delta = \delta_R$. Thus, we have produced the constant δ promised by the statement $D(k)$.

Let $\lambda \geq \frac{1}{\delta}$ be given. Our next goal is to produce the constants r and ϵ promised by the statement $D(k)$. Again, recall that we assume Lemma 5.1.1 and the statement $R(k-1)$ are true for the fixed integer $k \geq 4$.

For the constants α and β given above, $\delta = \delta_R$, and $\lambda \geq \frac{1}{\delta}$ chosen above, Lemma 5.1.1 guarantees constants $r_L = r_L(\alpha, \beta, \delta, \lambda)$ and $\epsilon_L = \epsilon_L(\alpha, \beta, \delta, \lambda)$.

Set $r_{k-1} = r_L$. With constants α, $\delta_{k-1} = \delta_L$, δ, λ, $r_{k-1} = r_L$ determined above, the statement $R(k-1)$ guarantees constants $r_R = r_R(\alpha, \delta_L, \delta, \lambda, r_L)$ and $\epsilon_R = \epsilon_R(\alpha, \delta_L, \delta, \lambda, r_L)$. Set

$$r = \max\{r_L, r_R\} \tag{22}$$

and

$$\epsilon = \min\{\epsilon_L, \epsilon_R\}. \tag{23}$$

Thus, we have determined the constants guaranteed by the statement $D(k)$. We now prove that $D(k)$ is true with the constants δ, r and ϵ defined above.

Proof of Claim 5.1.2.

As in the hypothesis of statement $D(k)$, suppose that $k \geq 4$, α and $\beta > 0$ are given constants. Let the constant δ be given in the Definitions of the Constants. Let $\lambda > \frac{1}{\delta}$ be a given integer, and suppose constants r and ϵ are given in the Definitions of the Constants. Suppose triple system $\mathcal{H}$ and underlying graph G satisfy the following setup with constants k, α, δ, λ, r and ϵ:

(i) $\mathcal{H}$ is a k-partite 3-cylinder with k-partition $(V_1, \ldots, V_k)$, $|V_1| = \ldots = |V_k| = m$.
(ii) Let G be an underlying (λ, ϵ, k)-cylinder of $\mathcal{H}$.

(iii) For all $B \in [k]^3$, $\mathcal{H}(B)$ is (α, δ, r)-regular with respect to the triad $G(B)$.

In order to prove the statement $D(k)$ is true, we must show

$$\left|\mathcal{K}_k^{(3)}(\mathcal{H})\right| \geq \frac{\alpha^{\binom{k}{3}}}{\lambda^{\binom{k}{2}}} m^k (1-\beta).$$

Let $C \in [k]^{k-1}$, and consider the restricted $(k-1)$-partite 3-cylinder $\mathcal{H}(C)$ with underlying restricted $(\lambda, \epsilon, k-1)$-cylinder $G(C)$. We first prepare the $(k-1)$-partite 3-cylinder $\mathcal{H}(C)$ with underlying $(\lambda, \epsilon, k-1)$-cylinder $G(C)$ for the statement $R(k-1)$. Specifically, note that all triads $\mathcal{H}(B)$ of $\mathcal{H}(C)$, $B \in [C]^3$, are (α, δ, r)-regular, where δ, r, and ϵ satisfy the following: with $\delta_{k-1} = \delta_L$, we have $\delta = \delta_R(\alpha, \delta_L)$. With $\lambda \geq \frac{1}{\delta}$, we have $r \geq r_R$ and $\epsilon \leq \epsilon_R$. Thus, all triads $\mathcal{H}(B)$ of $\mathcal{H}(C)$ are (α, δ_R, r_R)-regular with respect to the $(\lambda, \epsilon_R, 3)$-cylinder $G(B)$, where δ_R, r_R, and ϵ_R, are parameters which verify the applicability of statement $R(k-1)$ for the choices α, $\delta_{k-1} = \delta_L$, $\lambda \geq \frac{1}{\delta}$, and $r_{k-1} = r_L$. The statement $R(k-1)$ guarantees that the $(k-1)$-partite 3-cylinder $\mathcal{H}(C)$ is $(\alpha^{\binom{k-1}{3}}, \delta_L, r_L)$-regular with respect to the $(\lambda, \epsilon, k-1)$-cylinder $G(C)$. This property holds for all $C \in [k]^{k-1}$.

Now we prepare the k-partite 3-cylinder $\mathcal{H}$ with underlying (λ, ϵ, k)-cylinder G for Lemma 5.1.1. By assumption, all triads $\mathcal{H}(B)$, $B \in [k]^3$, are (α, δ, r)-regular with respect to $G(B)$, so in particular, with $\delta = \delta_R \leq \delta_L$ and $r \geq r_L, r_R$, all triads $\mathcal{H}(B)$ are both (α, δ_R, r_R)-regular and (α, δ_L, r_L)-regular with respect to $G(B)$. By the application of $R(k-1)$ above, since all triads $\mathcal{H}(B)$ are (α, δ_R, r_R)-regular, we also conclude that for all $C \in [k]^{k-1}$, the $(k-1)$-partite 3-cylinder $\mathcal{H}(C)$ is $(\alpha^{\binom{k-1}{3}}, \delta_L, r_L)$-regular with respect to the $(\lambda, \epsilon_L, k-1)$-cylinder $G(C)$. Since all triads $\mathcal{H}(B)$ are (α, δ_L, r_L)-regular, as well as all $(k-1)$-partite 3-cylinders $\mathcal{H}(C)$ are $(\alpha^{\binom{k-1}{3}}, \delta_L, r_L)$-regular, where δ_L, r_L, and ϵ_L verify the applicability of Lemma 5.1.1 for the choices α, β, and λ, we have the hypothesis of Lemma 5.1.1 satisfied. By that lemma,

$$\left|\mathcal{K}_k^{(3)}(\mathcal{H})\right| \geq \frac{\alpha^{\binom{k}{3}}}{\lambda^{\binom{k}{2}}} m^k (1-\beta).$$

Thus the statement $D(k)$ is true for the fixed integer $k \geq 4$. □

5.3. **Proof of Lemma 5.1.1.**

We conclude this section by proving Lemma 5.1.1. We begin by defining the constants involved.

Definitions of the Constants:

Let $k \geq 3$ be an arbitrary integer, and α, β be arbitrary positive constants. Before precisely defining the constants δ, r and ϵ guaranteed to exist by Lemma 5.1.1, we first informally introduce these and other constants, and emphasize the hierarchy which governs their relative sizes. With α, β given, we first define constants γ, δ_1, δ_2, and δ so that

$$\alpha \gg \gamma \gg \min\{\delta_1, \delta_2^2\} = \delta.$$

For $\lambda \geq \frac{1}{\delta}$, we next define constants r_1, r_2, r, ϵ_1, ϵ_2, ϵ_3, and ϵ so that

$$\frac{1}{\lambda} \gg \min\{\frac{1}{r_1}, \frac{1}{r_2}\} = \frac{1}{r} \gg \min\{\frac{\epsilon_1}{2\lambda}, \epsilon_2, \epsilon_3\} = \epsilon.$$

We now define these constants formally, however, for quick reference, one may think of the following hierarchy,

$$\alpha \gg \gamma \gg \sqrt{\delta} \gg \delta > \frac{1}{\lambda} \gg \frac{1}{r} \gg \epsilon,$$

as it is this hierarchy which justifies all of our calculations in the proof of Lemma 5.1.1.

With α, β given positive constants, we first define an auxiliary positive constant $\overline{\beta}$ to satisfy

$$1 - \overline{\beta} > (1-\beta)^{1/6}. \tag{24}$$

Further define auxiliary positive constant γ to satisfy both

$$\sqrt{\gamma} \leq \frac{\alpha^{\binom{k-1}{2}}(1-\overline{\beta})}{4 \cdot 8^{k-1}}, \tag{25}$$

$$1 - \sqrt{\gamma} > (1-\beta)^{1/6}. \tag{26}$$

We now produce the constant δ promised in Lemma 5.1.1. Let $\delta_1 > 0$ satisfy

$$1 - \frac{2\delta_1}{\alpha^{\binom{k-1}{3}}} > (1-\beta)^{1/6}, \tag{27}$$

$$1 - 5\sqrt{\delta_1}\binom{k-1}{2} > (1-\beta)^{1/6}, \tag{28}$$

$$2\delta_1 < \frac{\gamma(1-\sqrt{\gamma})(1-\overline{\beta})\alpha^{\binom{k-1}{2}}}{2^k}. \tag{29}$$

We want to ensure that our choice of δ is sufficient for an application of Fact 4.2.4. For the constants α, $\overline{\beta}$, and $k-1$ from above, let

$$\delta_2 = \delta_2(\alpha, \overline{\beta}, k-1) \tag{30}$$

be the constant guaranteed by Fact 4.2.4. Set

$$\delta = \min\{\delta_1, \delta_2^2\}. \tag{31}$$

Let $\lambda \geq \frac{1}{\delta} = \max\{\frac{1}{\delta_1}, \frac{1}{\delta_2^2}\}$ be a given integer. Fact 4.2.4 guarantees the existence of constants

$$\epsilon_1 = \epsilon_1(\alpha, \overline{\beta}, k-1, \delta, \lambda), \tag{32}$$

$$r_1 = r_1(\alpha, \overline{\beta}, k-1, \delta, \lambda). \tag{33}$$

Let

$$r_2 = \lceil \gamma \lambda^{k-1} \rceil \tag{34}$$

and

$$r = \max\{r_1, r_2\}. \tag{35}$$

Finally, we define the constant ϵ guaranteed by Lemma 5.1.1. In order to define ϵ, we need to first define additional constants ϵ_2 and ϵ_3. We want to ensure that our choice of ϵ will be sufficient for an application of Fact 4.3.2. With constants k, α, δ, λ, and r set above, let

$$\epsilon_2 = \epsilon_2(\alpha, \delta, k, \lambda, r) \tag{36}$$

be that constant guaranteed by Fact 4.3.2.

Recall the function $\theta_{k,\lambda}(\epsilon)$ defined in Fact 2.1.7. This function has the property that for λ fixed, $\theta_{k,\lambda}(\epsilon) \to 0$ as $\epsilon \to 0$. Let ϵ_3 be a positive constant satisfying each of the inequalities below

$$\begin{aligned} \epsilon_3 &< \frac{\sqrt{\delta}}{2}, && (37)\\ \theta_{k-1,\lambda}(2\lambda^2\epsilon_3) &< 1, && (38)\\ (1-\epsilon_3)^{k-1} &> (1-\beta)^{1/6}, && (39)\\ 1 - \frac{(k-1)r\epsilon_3}{1-4\sqrt{\delta}\binom{k-1}{2}} &> (1-\beta)^{1/6}. && (40) \end{aligned}$$

Having defined the constants ϵ_1, ϵ_2, and ϵ_3, set

$$\epsilon = \min\{\frac{\epsilon_1}{2\lambda}, \epsilon_2, \epsilon_3\}. \tag{41}$$

Note that the constants α, γ, λ, r, and ϵ defined above do indeed satisfy the following hierarchy

$$\alpha \gg \gamma \gg \sqrt{\delta} \gg \delta > \frac{1}{\lambda} \gg \frac{1}{r} \gg \epsilon.$$

We see that the first 2 inequalities follow from (25) and (29), (31). We see in (35) that $r \gg \lambda$, and the last inequality follows from (40) and (41).

Having defined all the promised constants, we proceed with the proof of Lemma 5.1.1.

Proof of Lemma 5.1.1.

Let $\alpha, \beta > 0$ and integer $k \geq 4$ be given. Let δ be given in (31). Let integer $\lambda > \frac{1}{\delta}$ be given. Let r, ϵ be given in (35) and (41) respectively. Let $\mathcal{H}$ be a k-partite 3-cylinder with k-partition $(V_1, \ldots, V_k)$, $|V_1| = \ldots = |V_k| = m$, and let $G = \bigcup_{1\leq i<j\leq k} G^{ij}$ be an underlying (λ, ϵ, k)-cylinder of $\mathcal{H}$. Suppose further that for all $B \in ([k]^3 \cup [k]^{k-1})$, $|B| = b \in \{3, k-1\}$, $\mathcal{H}(B)$ is $(\alpha^{\binom{b}{3}}, \delta, r)$-regular with respect to $G(B)$. Our goal is to show that $\mathcal{H}$ satisfies

$$\left|\mathcal{K}_k^{(3)}(\mathcal{H})\right| \geq \frac{\alpha^{\binom{k}{3}}}{\lambda^{\binom{k}{2}}} m^k (1-\beta).$$

We first state a definition.

Definition 5.3.1. Let $v \in V_1$ be a vertex from the vertex set V_1. We say that v is a *good vertex* provided that v satisfies the following conditions:

(i) For all i, $1 < i \leq k$,

$$\frac{m}{\lambda}(1-\epsilon) < |N_{G,i}(v)| < \frac{m}{\lambda}(1+\epsilon). \tag{42}$$

(ii) For all i, j, $1 < i < j \leq k$, the pair of graphs $(L_v^{ij}, G^{ij}(v, G))$, the $\{i, j\}$-link graph of v and the subgraph of G^{ij} induced on the sets $N_{G,i}(v)$ and $N_{G,j}(v)$, form an $(\frac{\alpha - 2\sqrt{\delta}}{\lambda}, \sqrt{\delta}, r, \lambda, 2\lambda\epsilon)$-regular couple.

Set
$$V_1' = \{v \in V_1 : v \text{ is a good vertex}\}.$$
We prove the following claim.

Claim 5.3.2.
$$|V_1'| \geq \left(1 - 4\sqrt{\delta}\binom{k-1}{2}\right) m.$$

Proof of Claim 5.3.2.

We would like to apply Fact 4.3.2 to G to infer that the size of V_1' is approximately the size of V_1. First, we note that the hypothesis of Fact 4.3.2 is satisfied by $\mathcal{H}$ and G. Indeed, by assumption, $\mathcal{H}$ and G satisfy that all $\binom{k}{3}$ triads are (α, δ, r)-regular, and that G is a (λ, ϵ, k)-cylinder with $\epsilon \leq \epsilon_2$. Since ϵ_2 is set in (36) to verify the applicability of Fact 4.3.2 for the parameters k, α, δ, λ, and r, and since $\epsilon \leq \epsilon_2$, then ϵ also verifies the applicability of Fact 4.3.2 for the same parameters. Statement (a) of Fact 4.3.2 tells us that all but $4(k-1)\epsilon m$ vertices satisfy (42) for all i, $1 \leq i \leq k$. Statement (b) of Fact 4.3.2 tells us that for fixed i, j, $1 < i < j \leq k$, all but $2\binom{k-1}{2}\sqrt{\delta}m$ vertices $x \in V_1$ satisfy that $(L_x^{ij}, G^{ij}(x, G))$ is an $(\frac{\alpha-2\sqrt{\delta}}{\lambda}, \sqrt{\delta}, r, \lambda, 2\lambda\epsilon)$-regular couple. Thus, we have
$$\begin{aligned} |V_1'| &> \left(1 - \binom{k-1}{2} 2\sqrt{\delta} - 4(k-1)\epsilon\right) m, \\ &> \left(1 - 4\sqrt{\delta}\binom{k-1}{2}\right) m, \end{aligned}$$
where the last inequality follows from the fact that we chose ϵ in (37) and (41) to satisfy $\epsilon < \frac{\sqrt{\delta}}{2}$. □

For convenience, set
$$\sigma = 1 - 4\sqrt{\delta}\binom{k-1}{2} \tag{43}$$
so that we may simply say $|V_1'| > \sigma m$.

We now pause to reveal our strategy for the proof of Lemma 5.1.1. Our main technique in this proof is counting copies of $K_{k-1}^{(2)}$ in the link graphs L_v of good vertices $v \in V_1'$. It follows from the definition of the link graph L_v that any copy of $K_{k-1}^{(2)} \subset L_v$ corresponds to a copy of $K_k^{(3)} \subset \mathcal{H}$. Hence, obtaining a lower bound on the total number of copies of $K_{k-1}^{(2)}$ contained within the link graphs L_v, $v \in V_1'$, implies a lower bound on the number of copies of $K_k^{(3)}$ contained within $\mathcal{H}$. This lower bound will match the one promised by Lemma 5.1.1.

We consider good vertices in our argument outlined above for the following reason. A good vertex v satisfies that for all i, j, $1 < i < j \leq k$, $(L_v^{ij}, G^{ij}(v, G))$ is a

$(\frac{\alpha-2\sqrt{\delta}}{\lambda}, \sqrt{\delta}, r, \lambda, 2\lambda\epsilon)$-regular couple, and therefore (as we prove in Claim 5.3.4) satisfies the hypothesis of Fact 4.2.4. Fact 4.2.4 then provides a lower bound on the number of copies of $K_{k-1}^{(2)}$ contained within L_v.

We proceed by proving the following claim which will be useful in the strategy above.

Claim 5.3.3. *There exists a family $\mathcal{S} = \{S_1, \dots, S_q\}$ of pairwise disjoint, r_2-element subsets of V_1' which satisfies:*

(i) $q \geq \frac{\sigma m}{r_2}(1 - \frac{(k-1)r_2\epsilon}{\sigma})$, where σ is given in (43) ,

(ii) for each $i \in [q]$, $S_i = \{v_1^{(i)}, v_2^{(i)}, \dots, v_{r_2}^{(i)}\} \subset V_1'$,

(iii) for each $i \in [q]$, for each $\{x, y\} \in [r_2]^2$, for all j, $1 < j \leq k$,

$$|N_{G,j}(v_x^{(i)}) \cap N_{G,j}(v_y^{(i)})| \leq \frac{m}{\lambda^2}(1+\epsilon)^2. \tag{44}$$

Proof of Claim 5.3.3.

We apply Fact 4.1.2 to the subgraph $\widetilde{G}$ of G induced on the sets V_1', $V_2, \dots, V_k$. Fact 4.1.2 would guarantee the existence of a family $\mathcal{S}$ described precisely in Claim 5.3.3, so we need only check that the hypothesis of Fact 4.1.2 is met. To that effect, given the notation used in Fact 4.1.2, set $V_1' = W_0$, $V_2 = W_1, \dots, V_k = W_{k-1}$, $m = M$, and let σ be given as in (43). Under this translation of notation, we write $\widetilde{G} = \bigcup_{0 \leq i < j \leq k-1} \widetilde{G^{ij}}$, where for all i, j, $0 < i < j \leq k-1$, $\widetilde{G^{ij}} = G[W_i, W_j] = G[V_{i+1}, V_{j+1}]$, and for $0 = i < j \leq k-1$, $\widetilde{G^{0j}} = G[W_0, W_j] = G[V_1', V_{j+1}]$. The condition of Fact 4.1.2 in (i) requires that for every $j \in [k-1]$, the graph $\widetilde{G^{0j}}$ satisfies that for every $W_0' \subseteq W_0$, $|W_0'| \geq \epsilon M$ and for every $W_j' \subseteq W_j$, $|W_j'| \geq \epsilon M$, then

$$d_{\widetilde{G^{0j}}}\left(W_0', W_j'\right) \in \left(\frac{1}{\lambda}(1-\epsilon), \frac{1}{\lambda}(1+\epsilon)\right).$$

However, this condition is easily satisfied by the fact that G is a (λ, ϵ, k) cylinder. The condition of Fact 4.1.2 (ii) requires that all $v \in W_0 = V_1'$ satisfy that for all $j \in [k-1]$,

$$|N_{\widetilde{G},j}(v)| \leq \frac{M}{\lambda}(1+\epsilon).$$

However, this is satisfied by the fact that all $v \in V_1'$ satisfy the property in (42) of Definition 5.3.1. Thus, we may apply Fact 4.1.2, and hence obtain a family $\mathcal{S}$ described precisely in Claim 5.3.3 above. □

Let $\mathcal{S} = \{S_1, \dots, S_q\}$ be that collection guaranteed by Claim 5.3.3 above satisfying (i), (ii), and (iii) in the statement of Claim 5.3.3. From the collection $\mathcal{S}$, consider the set $S_1 = \{v_1^{(1)}, v_2^{(1)}, \dots, v_{r_2}^{(1)}\} \in \mathcal{S}$. For convenience of notation, we drop the superscripts to obtain $S_1 = \{v_1, v_2, \dots, v_{r_2}\}$. For $\mu \in [r_2]$, fix vertex $v = v_\mu \in S_1$, and consider the link graph $L_v = \bigcup_{1<i<j\leq k} L_v^{ij}$. We conclude a lower bound on $|\mathcal{K}_{k-1}^{(2)}(L_v)|$ in the following claim.

Claim 5.3.4. *For the fixed vertex $v \in S_1$,*

$$|\mathcal{K}_{k-1}^{(2)}(L_v)| \geq \left(\frac{\alpha}{\lambda}\right)^{\binom{k-1}{2}} \left(\frac{m}{\lambda}(1-\epsilon)\right)^{k-1} (1-\overline{\beta}). \tag{45}$$

Proof of Claim 5.3.4.

We apply Fact 4.2.4 to the pair of graphs $L_v = \bigcup_{1<i<j\le k} L_v^{ij}$ and $\bigcup_{1<i<j\le k} G^{ij}(v,G)$. Note that the hypothesis of Fact 4.2.4 is met with this pair of graphs. Indeed, the graph $\bigcup_{1<i<j\le k} G^{ij}(v,G)$ is a $(\lambda, 2\lambda\epsilon, k-1)$-cylinder with $(k-1)$-partition $(N_{G,2}(v), \dots, N_{G,3}(v))$, where by virtue of $v \in V_1'$ being a good vertex, the inequality $|N_{G,i}(v)| > \frac{m}{\lambda}(1-\epsilon)$ from (42) is satisfied for all i, $1 < i \le k$. The graph $L_v = \bigcup_{1<i<j\le k} L_v^{ij}$ is a $(k-1)$-partite cylinder defined on the same $(k-1)$-partition, where for all i, j, $1 < i < j \le k$, $L_v^{ij} \subseteq G^{ij}(v,G)$. Since by virtue of $v \in V_1'$ being a good vertex, the pair $(L_v^{ij}, G^{ij}(v,G))$ is a $(\frac{\alpha - 2\sqrt{\delta}}{\lambda}, \sqrt{\delta}, r, \lambda, 2\lambda\epsilon)$-regular couple for all i, j, $1 < i < j \le k$ (cf., Definition 4.2.3). The only condition left to check before applying Fact 4.2.4 is that the constants δ, r, and ϵ are appropriate to invoke Fact 4.2.4.

Recall $\delta_2 = \delta_2(\alpha, \overline{\beta}, k-1)$ was that constant set in (30) to verify the applicability of Fact 4.2.4 for the parameters α, $\overline{\beta}$, and $k-1$. Similarly, $r_1 = r_1(\alpha, \overline{\beta}, k-1, \delta, \lambda)$ and $\epsilon_1 = \epsilon_1(\alpha, \overline{\beta}, k-1, \delta, \lambda)$ were those constants set in (33) and (32) respectively to verify the applicability of Fact 4.2.4 for the parameters α, $\overline{\beta}$, $k-1$, δ, and λ, $\lambda \ge \frac{1}{\delta}$. Since $\sqrt{\delta} \le \delta_2$ in (31), $r \ge r_1$ in (35), and $\epsilon \le \frac{\epsilon_1}{2\lambda}$ in (41), we have that $\sqrt{\delta}$, r, and $2\lambda\epsilon$ verifies the applicability of Fact 4.2.4 for the same parameters.

Therefore, we apply Fact 4.2.4 to the graphs L_v and $\bigcup_{1<i<j\le k} G^{ij}(v,G)$. Since the size of each partite set $N_{G,j}(v)$, $1 < j \le k$, of these graphs L_v, $\bigcup_{1<i<j\le k} G^{ij}(v,G)$ satisfies

$$|N_{G,j}(v)| > \frac{m}{\lambda}(1-\epsilon),$$

we apply Fact 4.2.4 to the graphs L_v and $\bigcup_{1<i<j\le k} G^{ij}(v,G)$ to conclude

$$|\mathcal{K}_{k-1}^{(2)}(L_v)| \ge \left(\frac{\alpha}{\lambda}\right)^{\binom{k-1}{2}} \left(\frac{m}{\lambda}(1-\epsilon)\right)^{k-1} (1-\overline{\beta}).$$

□

We now use (45) to prove the following claim. Recall the collection $\mathcal{S} = \{S_1, \dots, S_q\}$ guaranteed by Claim 5.3.3, where for convenience, we put $S_1 = \{v_1, \dots, v_{r_2}\}$. The following claim asserts that the number of $(k-1)$-cliques with all edges belonging to a link graph of some vertex v_μ in S_1, $1 \le \mu \le r_2$, is a large portion of all the $(k-1)$-cliques in the $(k-1)$-partite subgraph of G spanned by the sets $V_2, \dots, V_k$.

Claim 5.3.5. *Let $B = \{2, \dots, k\}$ so that $G(B) = \bigcup_{1<i<j\le k} G^{ij}$. Then*

$$|\bigcup_{\mu=1}^{r_2} \mathcal{K}_{k-1}^{(2)}(L_{v_\mu})| > \delta |\mathcal{K}_{k-1}^{(2)}(G(B))|. \tag{46}$$

Proof of Claim 5.3.5.

We apply inclusion-exclusion to $|\bigcup_{\mu=1}^{r_2} \mathcal{K}_{k-1}^{(2)}(L_{v_\mu})|$ to obtain

$$|\bigcup_{\mu=1}^{r_2} \mathcal{K}_{k-1}^{(2)}(L_{v_\mu})| \ge \sum_{\mu=1}^{r_2} |\mathcal{K}_{k-1}^{(2)}(L_{v_\mu})| - \sum_{\{\mu,\nu\}\in[r_2]^2} |\mathcal{K}_{k-1}^{(2)}(L_{v_\mu}) \cap \mathcal{K}_{k-1}^{(2)}(L_{v_\nu})|. \tag{47}$$

To provide an upper bound for the second order term above, we first recall that for all $\{\mu, \nu\} \in [r_2]^2$, the vertices $v_\mu, v_\nu \in S_1$ satisfy the property in (44). That property tells us that for all j, $1 < j \leq k$,

$$|N_{G,j}(v_\mu) \cap N_{G,j}(v_\nu)| \leq \frac{m}{\lambda^2}(1+\epsilon)^2.$$

It follows then from (44) and Fact 2.1.7 that for each $\{\mu, \nu\} \in [r_2]^2$,

$$\begin{aligned} |\mathcal{K}^{(2)}_{k-1}(L_{v_\mu}) \cap \mathcal{K}^{(2)}_{k-1}(L_{v_\nu})| &< (1+\theta_{k-1,\lambda}(2\lambda^2\epsilon))\frac{1}{\lambda^{\binom{k-1}{2}}}\left(\frac{m}{\lambda^2}(1+\epsilon)^2\right)^{k-1}, \\ &< \frac{2}{\lambda^{\binom{k-1}{2}}}\left(\frac{m}{\lambda^2}(1+\epsilon)^2\right)^{k-1}, \qquad (48) \end{aligned}$$

where the last inequality follows from the fact that we chose ϵ in (38) and (41) to satisfy $\theta_{k-1,\lambda}(2\lambda^2\epsilon) < 1$. Applying (45) and (48) to the right hand side of (47), we obtain the following further lower bound on $|\bigcup_{\mu=1}^{r_2} \mathcal{K}^{(2)}_{k-1}(L_{v_\mu})|$:

$$\begin{aligned} |\bigcup_{\mu=1}^{r_2} \mathcal{K}^{(2)}_{k-1}(L_{v_\mu})| &\geq r_2\left(\frac{\alpha}{\lambda}\right)^{\binom{k-1}{2}}\left(\frac{m}{\lambda}(1-\epsilon)\right)^{k-1}(1-\overline{\beta}) - \qquad (49) \\ &\quad \binom{r_2}{2}\frac{2}{\lambda^{\binom{k-1}{2}}}\left(\frac{m}{\lambda^2}(1+\epsilon)^2\right)^{k-1}, \qquad (50) \\ &\geq \frac{m^{k-1}}{\lambda^{\binom{k-1}{2}}}P(r_2), \qquad (51) \end{aligned}$$

where

$$P(r_2) = \left[r_2\frac{\alpha^{\binom{k-1}{2}}}{\lambda^{k-1}}(1-\epsilon)^{k-1}(1-\overline{\beta}) - r_2^2\left(\frac{1+\epsilon}{\lambda}\right)^{2(k-1)}\right].$$

We now use our value for r_2 given in (34) and begin by making the following estimations of r_2.

Note that it trivially follows that $r_2 \geq \gamma\lambda^{k-1}$. It also trivially follows that

$$r_2 \leq \gamma\lambda^{k-1} + 1 = \gamma\lambda^{k-1}(1 + \frac{1}{\gamma\lambda^{k-1}}). \qquad (52)$$

From our hypothesis that $\lambda \geq \frac{1}{\delta}$, we easily infer that $\delta^{k-1} \geq \frac{1}{\lambda^{k-1}}$. Using this lower bound on δ^{k-1} in the right hand side of (52), we see

$$r_2 \leq \gamma\lambda^{k-1}(1 + \frac{\delta^{k-1}}{\gamma}).$$

However, it follows from (29) and (31) that $\delta \leq \gamma$, thus we have

$$\gamma\lambda^{k-1}(1 + \frac{\delta^{k-1}}{\gamma}) \leq \gamma\lambda^{k-1}(1 + \frac{\delta^{k-2}\gamma}{\gamma}) \leq \gamma\lambda^{k-1}(1+\delta),$$

where the last inequality follows from the fact that $k \geq 3$. In sum,

$$\gamma\lambda^{k-1} \leq r_2 = \lceil \gamma\lambda^{k-1} \rceil \leq \gamma\lambda^{k-1}(1+\delta). \qquad (53)$$

With γ defined in (25), (26), we substitute the bounds on r_2 given in (53) into the expression in (51) to obtain a further lower bound on $|\bigcup_{\mu=1}^{r_2} \mathcal{K}_{k-1}^{(2)}(L_{v_\mu})|$. This substitution yields

$$|\bigcup_{\mu=1}^{r_2} \mathcal{K}_{k-1}^{(2)}(L_{v_\mu})| \geq \frac{m^{k-1}}{\lambda^{\binom{k-1}{2}}} \left[\gamma \alpha^{\binom{k-1}{2}} (1-\epsilon)^{k-1}(1-\overline{\beta}) - \gamma^2 (1+\delta)^2 (1+\epsilon)^{2(k-1)} \right],$$

the right hand side of which we equivalently write as

$$\frac{m^{k-1}}{\lambda^{\binom{k-1}{2}}} \gamma \left[\alpha^{\binom{k-1}{2}} (1-\epsilon)^{k-1}(1-\overline{\beta}) - \sqrt{\gamma}\sqrt{\gamma}(1+\delta)^2 (1+\epsilon)^{2(k-1)} \right]. \tag{54}$$

Recall from (25) that γ satisfies

$$\sqrt{\gamma} \leq \frac{\alpha^{\binom{k-1}{2}}(1-\overline{\beta})}{4 \cdot 8^{k-1}}.$$

Since $\delta < 1$ and $\epsilon < 1/2$, we easily have that

$$\sqrt{\gamma} \leq \frac{\alpha^{\binom{k-1}{2}}(1-\overline{\beta})(1-\epsilon)^{k-1}}{(1+\delta)^2(1+\epsilon)^{2(k-1)}}. \tag{55}$$

Substituting (55) into one of the factors of $\sqrt{\gamma}$ in (54) yields

$$|\bigcup_{\mu=1}^{r_2} \mathcal{K}_{k-1}^{(2)}(L_{v_\mu})| \geq \frac{m^{k-1}}{\lambda^{\binom{k-1}{2}}} \gamma (1-\sqrt{\gamma}) \alpha^{\binom{k-1}{2}} (1-\overline{\beta})(1-\epsilon)^{k-1}. \tag{56}$$

Recall that by our choice of δ in (29) and (31), we have

$$2\delta < \frac{\gamma(1-\sqrt{\gamma})\alpha^{\binom{k-1}{2}}(1-\overline{\beta})}{2^k}.$$

Since $\epsilon < 1/2$, we easily have that

$$2\delta < \gamma(1-\sqrt{\gamma})\alpha^{\binom{k-1}{2}}(1-\overline{\beta})(1-\epsilon)^{k-1}. \tag{57}$$

Hence, we have from (56) and (57) that

$$|\bigcup_{\mu=1}^{r_2} \mathcal{K}_{k-1}^{(2)}(L_{v_\mu})| \geq 2\delta \frac{m^{k-1}}{\lambda^{\binom{k-1}{2}}}. \tag{58}$$

On the other hand, it follows from Fact 2.1.7 that

$$|\mathcal{K}_{k-1}^{(2)}(G(B))| < (1+\theta_{k-1,\lambda}(\epsilon)) \frac{m^{k-1}}{\lambda^{\binom{k-1}{2}}}. \tag{59}$$

Since $\theta_{k-1,\lambda}(\epsilon)$ decreases as ϵ decreases, we infer by (38) and (41) that

$$\theta_{k-1,\lambda}(\epsilon) < \theta_{k-1,\lambda}(2\lambda^2\epsilon) < 1. \tag{60}$$

Combining (58), (59), and (60), we infer that

$$|\bigcup_{\mu=1}^{r_2} \mathcal{K}_{k-1}^{(2)}(L_{v_\mu})| > \delta |\mathcal{K}_{k-1}^{(2)}(G(B))|.$$

Thus Claim 5.3.5 is proved. □

We use Claim 5.3.5 to invoke the $(\alpha^{\binom{k-1}{3}}, \delta, r)$-regularity of the $(k-1)$-partite 3-cylinder $\mathcal{H}(B)$ (recall $B = \{2, \dots, k\}$). Viewing $(L_{v_1}, \dots, L_{v_{r_2}})$ as an r_2-tuple of $(k-1)$-partite cylinders satisfying (46) and recalling that $r_2 \leq r$, the $(\alpha^{\binom{k-1}{3}}, \delta, r)$-regularity of $\mathcal{H}(B)$ implies

$$|\mathcal{K}_{k-1}^{(3)}(\mathcal{H}(B)) \cap \bigcup_{\mu=1}^{r_2} \mathcal{K}_{k-1}^{(2)}(L_{v_\mu})| > (\alpha^{\binom{k-1}{3}} - 2\delta)|\bigcup_{\mu=1}^{r_2} \mathcal{K}_{k-1}^{(2)}(L_{v_\mu})|.$$

Using the bound in (56) in the right hand side of the above inequality yields

$$|\mathcal{K}_{k-1}^{(3)}(\mathcal{H}(B)) \cap \bigcup_{\mu=1}^{r_2} \mathcal{K}_{k-1}^{(2)}(L_{v_\mu})| >$$

$$(\alpha^{\binom{k-1}{3}} - 2\delta)\gamma(1-\sqrt{\gamma})\alpha^{\binom{k-1}{2}}(1-\overline{\beta})(1-\epsilon)^{k-1}\frac{m^{k-1}}{\lambda^{\binom{k-1}{2}}}. \tag{61}$$

In view of (24), (26), (39), and (41), we have

$$(1-\sqrt{\gamma})(1-\overline{\beta})(1-\epsilon)^{k-1} > (1-\beta)^{\frac{1}{2}}, \tag{62}$$

and also, in view of (27) and (31),

$$(\alpha^{\binom{k-1}{3}} - 2\delta)\alpha^{\binom{k-1}{2}} = \alpha^{\binom{k}{3}}\left(1 - \frac{2\delta}{\alpha^{\binom{k-1}{3}}}\right) > \alpha^{\binom{k}{3}}(1-\beta)^{\frac{1}{6}}. \tag{63}$$

Thus it follows from (61), (62), and (63) that

$$|\mathcal{K}_{k-1}^{(3)}(\mathcal{H}(B)) \cap \bigcup_{\mu=1}^{r_2} \mathcal{K}_{k-1}^{(2)}(L_{v_\mu})| > \alpha^{\binom{k}{3}}\gamma(1-\beta)^{\frac{2}{3}}\frac{m^{k-1}}{\lambda^{\binom{k-1}{2}}}. \tag{64}$$

We now conclude our argument for the proof of Lemma 5.1.1. It follows from the definition of link graph L_{v_μ}, $1 \leq \mu \leq r_2$, that for all

$$\{w_2, \dots, w_k\} \in \left(\mathcal{K}_{k-1}^{(3)}(\mathcal{H}(B)) \cap \bigcup_{\mu=1}^{r_2} \mathcal{K}_{k-1}^{(2)}(L_{v_\mu})\right),$$

the set $\{v_\mu, w_2, \dots, w_k\} \in \mathcal{K}_k^{(3)}(\mathcal{H})$. Using this fact and repeating the argument for (64) above for all sets $S_i \in \mathcal{S}$, $1 \leq i \leq q$, we obtain

$$\begin{aligned} \left|\mathcal{K}_k^{(3)}(\mathcal{H})\right| &\geq \sum_{i=1}^{q} |\mathcal{K}_{k-1}^{(3)}(\mathcal{H}(B)) \cap \bigcup_{\mu=1}^{r_2} \mathcal{K}_{k-1}^{(2)}(L_{v_\mu})|, \\ &> q\alpha^{\binom{k}{3}}\gamma(1-\beta)^{\frac{2}{3}}\frac{m^{k-1}}{\lambda^{\binom{k-1}{2}}}. \end{aligned} \tag{65}$$

Recall $q = (1 - \frac{(k-1)r_2\epsilon}{\sigma})\frac{\sigma m}{r_2}$. In light of the fact that $r_2 \leq \gamma\lambda^{\binom{k-1}{2}}(1+\delta)$, and due to (40) and (43), we have

$$q = (1 - \frac{(k-1)r_2\epsilon}{\sigma})\frac{\sigma m}{r_2} \geq (1-\beta)^{\frac{1}{6}}\frac{\sigma m}{\gamma\lambda^{k-1}(1+\delta)}. \tag{66}$$

Combining the lower bound for q in (66) with the inequality in (65) implies

$$\left|\mathcal{K}_k^{(3)}(\mathcal{H})\right| > \frac{\sigma}{1+\delta}\alpha^{\binom{k}{3}}(1-\beta)^{\frac{5}{6}}\frac{m^k}{\lambda^{\binom{k}{2}}}. \tag{67}$$

Recall from (43) that $\sigma = 1 - 4\sqrt{\delta}\binom{k-1}{2}$, and thus

$$\frac{\sigma}{1+\delta} \geq 1 - 5\sqrt{\delta}\binom{k-1}{2}.$$

Recall from our choice of δ in (28) and (31) that

$$1 - 5\sqrt{\delta}\binom{k-1}{2} > (1-\beta)^{\frac{1}{6}}. \tag{68}$$

Using the inequality in (68), we may bound the quantity in the right hand side of (67) further from below to imply

$$\left|\mathcal{K}_k^{(3)}(\mathcal{H})\right| > \frac{\alpha^{\binom{k}{3}}}{\lambda^{\binom{k}{2}}}m^k(1-\beta),$$

thus Lemma 5.1.1 is proved. $\square$

6. The Regularity Lemma

Our next goal is to prove the implication $D(k) \Rightarrow R(k)$. However, much of our proof of this implication involves the use of a regularity lemma. In this section, we state a regularity lemma for 3-uniform hypergraphs which slightly extends Theorem 3.5 in [2]. The proof of this lemma follows the same lines as that of Theorem 3.5, and we do not include it here. First, we state a number of supporting definitions which are analogous to those found in [2].

6.1. Definitions for the Regularity Lemma.

Definition 6.1.1. Let t be an integer and let $V = \bigcup_{i=1}^k V_i$, $|V_1| = \ldots = |V_k| = N$, be a partition of a kN element set V. We define an *equitable refinement* of $V = \bigcup_{i=1}^k V_i$ as a partition $V = W = W_0 \cup \bigcup_{1\leq i\leq k}\bigcup_{1\leq x_i\leq t} W_{x_i}$, where

(i) for any $i \in [k]$, $\bigcup_{1\leq x_i\leq t} W_{x_i} \subseteq V_i$,
(ii) for each i, $1 \leq i \leq k$, for each x_i, $1 \leq x_i \leq t$, $|W_{x_i}| = \lfloor \frac{N}{t} \rfloor = m$,
(iii) $|W_0| < kt$.

Note that $W_0 = \emptyset$ if t divides N. We use

$$\bigcup_{1\leq i\leq k}\ \bigcup_{1\leq x_i\leq t} x_i = \{1_1, \ldots, t_1, 2_1, \ldots, t_2, \ldots, t_1, \ldots, t_k\}$$

as double-indices.

Definition 6.1.2. Let k, λ, l, t be positive integers, ϵ, ϵ_1, ϵ_2 be positive reals. Suppose $G = \bigcup_{1\leq i<j\leq k} G^{ij}$ is a (λ, ϵ, k)-cylinder with k-partition $(V_1, \ldots, V_k)$, $|V_1| = \ldots = |V_k| = N$. We define a $(\lambda l, kt, \epsilon_1, \epsilon_2)$-*partition* $\mathcal{P}$ of $V(G) = V_1 \cup \ldots \cup V_k$ as a partition $V =$

$W = W_0 \cup \bigcup_{1\le i\le k} \bigcup_{1\le x_i\le t} W_{x_i}$ of $V(G)$, together with a system of bipartite graphs $P^{x_i y_j}_{\alpha_{x_i y_j}}$, $1 \le i < j \le k$, $1 \le x_i, y_j \le t$, $1 \le \alpha_{x_i y_j} \le l_{x_i y_j} \le l$, such that:

(0) $W = W_0 \cup \bigcup_{1\le i\le k} \bigcup_{1\le x_i\le t} W_{x_i}$ is an equitable refinement of $V = V_1 \cup \ldots \cup V_k$.

(i) For all i, j, $1 \le i < j \le k$, for all x_i, y_j, $1 \le x_i, y_j \le t$,

$$\begin{aligned} \bigcup_{1\le \alpha_{x_i y_j} \le l_{x_i y_j}} P^{x_i y_j}_{\alpha_{x_i y_j}} &\subseteq G[W_{x_i}, W_{y_j}], \\ &= \Big\{\{v_{x_i}, v_{y_j}\} \in G : v_{x_i} \in W_{x_i}, v_{y_j} \in W_{y_j}\Big\}. \end{aligned}$$

(ii) All but $\epsilon_1 \binom{kt}{2} m^2/\lambda$ pairs $\{v_{x_i}, v_{y_j}\} \in G$, $v_{x_i} \in W_{x_i}$, $v_{y_j} \in W_{y_j}$, are edges of $\frac{\epsilon_2}{2\lambda l}$-regular bipartite graphs $P^{x_i y_j}_{\alpha_{x_i y_j}}$, $1 \le i < j \le k$, $1 \le x_i, y_j \le t$, $1 \le \alpha_{xy} \le l_{xy} \le l$, satisfying

$$\frac{1}{\lambda l} - \frac{\epsilon_2}{2\lambda l} < d_{P^{x_i y_j}_{\alpha_{x_i y_j}}}(W_{x_i}, W_{y_j}) < \frac{1}{\lambda l} + \frac{\epsilon_2}{2\lambda l}. \tag{69}$$

Note the following about Definition 6.1.2.

- In statement (0) of Definition 6.1.2, if it is further assumed that t divides N, then we have that $W_0 = \emptyset$, where W_0 is that residual class of vertices described in Definition 6.1.1.
- Note that since $|G| \sim \binom{k}{2}\frac{N^2}{\lambda} \le \binom{kt}{2}\frac{m^2}{\lambda}$, statement ($ii$) of Definition 6.1.2 says that all but an ϵ_1-portion of the edges of G belong to $\frac{\epsilon_2}{2\lambda l}$-regular graphs $P^{xy}_{\alpha_{xy}}$ of density satisfying (69).
- Note that the $\frac{\epsilon_2}{2\lambda l}$-regular bipartite graphs $P^{x_i y_j}_{\alpha_{x_i y_j}}$ each with density satisfying (69) are $(\lambda l, \epsilon_2, 2)$-cylinders. For the remainder of this paper, we refer to such bipartite graphs $P^{x_i y_j}_{\alpha_{x_i y_j}}$ specifically as being $(\lambda l, \epsilon_2, 2)$-cylinders.

We now state a definition relating to Definition 2.2.7.

Definition 6.1.3. Let positive integers k, λ, l, t, r and positive reals δ, ϵ , ϵ_1, ϵ_2 be given. Suppose

(i) $G = \bigcup_{1\le i<j\le k} G^{ij}$ is a (λ, ϵ, k)-cylinder with k-partition $(V_1, \ldots, V_k)$, $|V_1| = \ldots = |V_k| = N$.

(ii) $\mathcal{H}$ is a k-partite 3-cylinder with underlying k-partite cylinder $G = \bigcup_{1\le i<j\le k} G^{ij}$.

(iii) $\mathcal{P}$ is a $(\lambda l, kt, \epsilon_1, \epsilon_2)$-partition of G with equitable refinement $W_0 \cup \bigcup_{1\le i\le k} \bigcup_{1\le x_i\le t} W_{x_i}$.

We say that the $(\lambda l, kt, \epsilon_1, \epsilon_2)$- partition $\mathcal{P}$ of G is (δ, r)-*regular* with respect to $\mathcal{H}$ if

$$\sum\{|\mathcal{K}^{(2)}_3(P_3)| : P_3 \text{ is not a } (\delta, r)\text{-fully regular triad of } \mathcal{P}\} < \delta \left(\frac{kN}{\lambda}\right)^3. \tag{70}$$

Suppose $P_3 = P^{x_h y_i}_{\alpha} \cup P^{x_h z_j}_{\beta} \cup P^{y_i z_j}_{\gamma}$, $1 \le h < i < j \le k$, $\alpha \in [l_{x_h y_i}]$, $\beta \in [l_{x_h z_j}], \gamma \in [l_{y_i z_j}]$, is a triad of the partition $\mathcal{P}$ defined on $W_{x_h}, W_{y_i}, W_{z_j}$. For the triad P_3, set

$$\mu_{P_3} = \frac{|\mathcal{K}_3^{(2)}(P_3)|}{m^3}.$$

We may equivalently write the inequality in (70) with the notation above as

$$\sum\{\mu_{P_3} : P_3 \text{ is not a } (\delta, r)\text{-fully regular triad of } \mathcal{P}\} < \delta \left(\frac{kN}{m\lambda}\right)^3. \tag{71}$$

6.2. **Statement of the Regularity Lemma.**

We now state our regularity lemma.

Theorem 6.2.1. *For all constants ϵ_1, δ, positive integers k, λ, l_0, t_0, $k \geq 3$, integer valued functions $r(t, l)$, and functions $\epsilon_2(l)$, there exist $\epsilon > 0$ and integers T_0, L_0, N_0 so that if*

(i) G is a (λ, ϵ, k)-cylinder with k partition $(V_1, \ldots, V_k)$, $|V_1| = \ldots = |V_k| = N \geq N_0$,

(ii) $\mathcal{H}$ is a k-partite 3-cylinder with G underlying $\mathcal{H}$,

then $\mathcal{H}$ admits a $(\delta, r(t, l))$-regular $(\lambda l, kt, \epsilon_1, \epsilon_2(l))$ partition $\mathcal{P}$, where $l_0 \leq l \leq L_0$, $t_0 \leq t \leq T_0$.

Note that in Theorem 6.2.1, the numbers ϵ, T_0, L_0, N_0 do not only depend on ϵ, δ, k, λ, l_0, and t_0, but also functions $r(t, l)$, and $\epsilon_2(l)$.

We mention that there is only one slight difference between Theorem 6.2.1 and the original Theorem 3.5 of Frankl and Rödl in [2]. Indeed, recall in Definition 6.1.2 and Theorem 6.2.1 the k-partite graph $G = \bigcup_{1 \leq i < j \leq k} G^{ij}$ had each G^{ij}, $1 \leq i < j \leq k$, a $(\lambda, \epsilon, 2)$-cylinder. In Theorem 3.5 of [2], essentially speaking, the graph G was $G = K_{kN}$. To be precisely compatible with Definition 6.1.2, Theorem 3.5 of [2] could be formulated with $G = \bigcup_{1 \leq i < j \leq k} K[V_i, V_j]$, where $K[V_i, V_j]$ denotes the complete bipartite graph on $V_i \cup V_j$, $1 \leq i < j \leq k$. In this way, we see each $K[V_i, V_j]$, $1 \leq i < j \leq k$, is a $(1, \epsilon, 2)$-cylinder and hence the easy generalization from Theorem 3.5 in [2] to Theorem 6.2.1.

It is also possible to prove a slight generalization of Theorem 6.2.1 above. The following statement is analogous to that of Theorem 3.11 in [2], and its proof is also along the same lines as Theorem 3.5 in [2].

Theorem 6.2.2. *For all constants ϵ_1, δ, positive integers s, k, λ, l_0, t_0, $k \geq 3$, integer valued functions $r(t, l)$, and functions $\epsilon_2(l)$, there exists $\epsilon > 0$ and integers T_0, L_0, N_0 so that if*

(i) G is a (λ, ϵ, k)-cylinder with k-partition $(V_1, \ldots, V_k)$, $|V_1| = \ldots = |V_k| = N \geq N_0$,

(ii) $\{\mathcal{H}_1, \ldots, \mathcal{H}_s\}$ is any family of k-partite 3-cylinders satisfying that for each $i \in [s]$, G underlies $\mathcal{H}_i$,

then there exists a $(\lambda l, kt, \epsilon_1, \epsilon_2(l))$ partition $\mathcal{P}$ which is $(\delta, r(t, l))$-regular with respect to each $\mathcal{H}_i$, where $l_0 \leq l \leq L_0$, $t_0 \leq t \leq T_0$.

7. $D(k) \Longrightarrow R(k)$

In this section, we prove the implication $D(k) \Rightarrow R(k)$. Our proof is handled through the upcoming Lemma 7.1.1.

7.1. **Lemma 7.1.1.**

We recall the familiar Setup 2', and for convenience, we restate this setup. For an integer $k \geq 3$, $\mathcal{H}$ is as usual a k-partite 3-cylinder, and $G = \bigcup_{1 \leq i < j \leq k} G^{ij}$ is an underlying k-partite cylinder of $\mathcal{H}$. In particular, for specified constants k, α, δ, λ, r, ϵ and N, $\mathcal{H}$ and G satisfy the following Setup:

Setup:

(i) $\mathcal{H}$ is a k-partite 3-cylinder with k-partition $(V_1, \dots, V_k)$, $|V_1| = \dots = |V_k| = N$,
(ii) $G = \bigcup_{1 \leq i < j \leq k} G^{ij}$ is a (λ, ϵ, k)-cylinder underlying $\mathcal{H}$,
(iii) for all $B \in [k]^3$, $\mathcal{H}(B)$ is (α, δ, r)-regular with respect to $G(B)$.

We prove the following lemma.

Lemma 7.1.1. *For every integer $k \geq 4$, the statement $D(k)$ implies the statement $R(k)$.*

Before giving any formal proofs, we first discuss our strategy for verifying Lemma 7.1.1. For the moment, let us suppose that given constants $k \geq 4$, α, δ_k, a constant $\delta > 0$ has been disclosed, and given an integer $\lambda \geq \frac{1}{\delta}$ and an integer $r_k \geq 1$, constants r, ϵ, and N_0 have been disclosed. Suppose 3-cylinder $\mathcal{H}$ and underlying cylinder G satisfy the conditions of the Setup with the constants k, α, δ, λ, r, ϵ, and N, $N \geq N_0$. To say that $\mathcal{H}$ is $(\alpha^{\binom{k}{3}}, \delta_k, r_k)$-regular with respect to G means the following (cf., Definition 2.2.6): let $\overrightarrow{Q} = (Q(s))$, $1 \leq s \leq r_k$, be an r_k-tuple, where for all $s \in [r_k]$, $Q(s) = \bigcup_{1 \leq i < j \leq k} Q^{ij}(s)$, and for all $\{i, j\} \in [k]^2$, $s \in [r_k]$, $Q^{ij}(s) \subseteq G^{ij}$. If $\overrightarrow{Q}$ satisfies

$$|\bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))| > \delta_k |\mathcal{K}_k^{(2)}(G)|,$$

then $\overrightarrow{Q}$ also satisfies $d_{\mathcal{H}}(\overrightarrow{Q}) > \alpha^{\binom{k}{3}} - 2\delta_k$, or equivalently,

$$|\mathcal{K}_k^{(3)}(\mathcal{H}) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))| > (\alpha^{\binom{k}{3}} - 2\delta_k) |\bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))|. \tag{72}$$

Therefore, to prove Lemma 7.1.1, we need only show that for any such $\overrightarrow{Q}$, the inequality in (72) is indeed satisfied.

While the proof of Lemma 7.1.1 is complicated in its technical details, the idea behind it is simple. We now give an outline for the proof of Lemma 7.1.1 for the case that $r_k = 1$. Suppose that 3-cylinder $\mathcal{H}$ and underlying cylinder G satisfy the conditions of the Setup with constants k, α, δ, λ, r, ϵ and N. Suppose $Q = \bigcup_{1 \leq i < j \leq k} Q^{ij}$ is given so that for all $\{i, j\} \in [k]^2$, $Q^{ij} \subseteq G^{ij}$, and $|\mathcal{K}_k^{(2)}(Q))| > \delta_k |\mathcal{K}_k^{(2)}(G)|$. We show

$$|\mathcal{K}_k^{(3)}(\mathcal{H}) \cap \mathcal{K}_k^{(2)}(Q)| > (\alpha^{\binom{k}{3}} - 2\delta_k) |\bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))|. \tag{73}$$

We begin by restricting the hypergraph $\mathcal{H}$ to $\mathcal{H}^Q = \mathcal{H} \cap \mathcal{K}_3^{(2)}(Q)$. We use our Regularity Lemma, Theorem 6.2.1, to obtain a $(\lambda l, kt, \epsilon_1, \epsilon_2(l))$ partition $\mathcal{P}_0$ of the hypergraph $\mathcal{H}^Q$ which is $(\delta, r(t, l))$-regular with respect to underlying cylinder G, where $W = W_0 \cup \bigcup_{1 \leq i \leq k} \bigcup_{1 \leq x_i \leq t} W_{x_i}$ is the equitable refinement of $V = V_1 \cup \ldots \cup V_k$ associated with $\mathcal{P}_0$, and $P^{x_i y_j}_{\alpha_{x_i y_j}}$, $1 \leq i < j \leq k$, $1 \leq x_i, y_j \leq t$, $1 \leq \alpha_{x_i y_j} \leq l_{x_i y_j} \leq l$ is the system of bipartite graphs associated with $\mathcal{P}_0$. We consider *k-partite small cylinders* $C_k = \bigcup_{\{x_i,y_j\}\in[B]^2} P^{x_iy_j}_{\alpha_{x_iy_j}}$ within the partition $\mathcal{P}_0$, where the sets B satisfy $B \subseteq \bigcup_{1\le i\le k}\bigcup_{1\le x_i\le t} x_i$, $|B| = k$, and for each pair $\{x_i, y_j\} \in [B]^2$, $i \neq j$. Our goal is to apply the statement $D(k)$ to "appropriate" small k-partite cylinders C_k. To that end, we characterize a class of k-partite small cylinders C_k, the class Π_k of *perfect cylinders*. Perfect cylinders are those small k-partite cylinders C_k for which the statement $D(k)$ applies (i.e. all triads of C_k are "dense and regular", and all bipartite graphs $P^{x_iy_j}_{\alpha_{x_iy_j}}$ of C_k are "dense and regular"). We show that very few copies of $K^{(2)}_k$ from $\mathcal{K}^{(2)}_k(Q)$ belong to non perfect cylinders (we call these cylinders *defective cylinders*). Since nearly all copies of $K^{(2)}_k$ from $\mathcal{K}^{(2)}_k(Q)$ belong to perfect cylinders $C_k \in \Pi_k$, we conclude a lower bound on the number of perfect cylinders $|\Pi_k|$. We apply the statement $D(k)$ to each perfect cylinder $C_k \in \Pi_k$ to conclude a lower bound on $|\mathcal{K}^{(3)}_k(\mathcal{H}^Q) \cap \mathcal{K}^{(2)}_k(C_k)|$. We then sum this number over all such perfect C_k (we can estimate $|\Pi_k|$) to obtain the inequality in (73).

However, in order to make our plan precise, we need to disclose the constants promised by Lemma 7.1.1. In what follows, we define these constants and then return to prove (72) in the upcoming Proposition 7.2.4.

7.2. **Proving Lemma 7.1.1.** .

We now begin with the definitions of the constants involved in Lemma 7.1.1. However, the Reader may feel free to skip through the tedious definitions and observe that, similarly to Lemma 5.1.1, we commit ourselves to the following hierarchy

$$\alpha, \delta_k \gg \delta > \frac{1}{\lambda} \gg \frac{1}{r} \gg \epsilon \gg \frac{1}{N_0}.$$

We mention that it is necessary for

$$r \ge r_k \tag{74}$$

We explain the reason for this (subtle) requirement later in context.

Definitions of the Constants in Lemma 7.1.1

In order to start with the formal definitions, observe that the statement $R(k)$ compactly stated says "$\forall\alpha, \delta_k$, $\exists\delta : \forall\lambda \ge \frac{1}{\delta}, \forall r_k \ge 1$, $\exists r, \epsilon, N_0$ so that ...". Let $k \ge 4$, and let α, δ_k be two given positive reals. Note that we may assume that $2\delta_k < \alpha^{\binom{k}{3}}$, since otherwise Lemma 7.1.1 would be trivial. With α, and δ_k given, we first define the promised constant δ.

Definition of δ

In the definition of δ, we invoke the statement $D(k)$, which recall compactly stated says "$\forall\alpha, \beta$, $\exists\delta_{D(k)} = \delta_{D(k)}(\alpha, \beta) : \forall\lambda \ge \frac{1}{\delta}$, $\exists r_{D(k)} = r_{D(k)}(\alpha, \beta, \delta, \lambda), \epsilon_{D(k)} = \epsilon_{D(k)}(\alpha, \beta, \delta, \lambda)$ so that ...". Apply the parameters $\frac{\alpha}{2}$, and $\beta = \delta_k$ to the statement $D(k)$, where α, δ_k, k are fixed above. Let

$$\delta_{D(k)} = \delta_{D(k)}(\frac{\alpha}{2}, \delta_k) \tag{75}$$

be that value guaranteed by statement $D(k)$. Set $\delta > 0$ to be small enough so that each of the following inequalities are satisfied:

$$\begin{aligned} \delta &< \frac{\alpha}{4}, & (76)\\ \delta &\leq \delta_{D(K)}, & (77) \end{aligned}$$

$$\frac{\alpha^{\binom{k}{3}}\left(1-\frac{12\sqrt{\delta}k^3}{\delta_k}\right)(1-\frac{2\delta}{\alpha})^{\binom{k}{3}}(1-\delta_k)}{1+\delta} > \alpha^{\binom{k}{3}} - 2\delta_k. \tag{78}$$

Thus, we have defined the promised constant δ. Note that with δ fixed, it follows that

$$\delta \leq \delta_{D(k)}(\frac{\alpha}{2}, \delta_k) \leq \delta_{D(k)}(\alpha - 2\delta, \delta_k). \tag{79}$$

Let $\lambda \geq \frac{1}{\delta}$, $r_k \geq 1$ be given integers. Our next goal is to produce the promised constants r, ϵ, and N_0.

Definitions of r, ϵ, and $\mathbf{N_0}$

The definitions of the constants r, ϵ, and N_0 depend on the output of two previous statements, the statement $D(k)$ and our Regularity Theorem, Theorem 6.2.1. As a result, we are only momentarily able to define these constants (cf., (97) for the definition of r, (90) for the definition of ϵ and (93) for the definition of N_0).

First, we consider the statement $D(k)$. Recall that with α and δ_k given, we have defined δ and subsequently have $\lambda \geq \frac{1}{\delta}$ and $r_k \geq 1$ arbitrarily given. Let l be an integer variable. For the parameters $\alpha - 2\delta$, $\beta = \delta_k$, δ, λl let

$$\begin{aligned} r_{D(k)}(l) &= r_{D(k)}(\alpha - 2\delta, \delta_k, \delta, \lambda l), & (80)\\ \epsilon_{D(k)}(l) &= \epsilon_{D(k)}(\alpha - 2\delta, \delta_k, \delta, \lambda l) & (81) \end{aligned}$$

be guaranteed by the statement $D(k)$. We specify the value l seen in (80) and (81) only after an application of the Regularity Theorem, Theorem 6.2.1.

Recall the Regularity Theorem, as it is stated in Section 6. We may more compactly write Theorem 6.2.1 as the following "$\forall \epsilon_1, \delta, \forall k, \lambda, l_0, t_0 : k \geq 3, \forall r(t,l), \epsilon_2(l), \exists \epsilon, T_0, L_0, N_0$, so that …". We now describe the input for Theorem 6.2.1. Let δ be given in (76)-(78), let $k \geq 4$ be given in the beginning of the Definitions of the Constants in Lemma 7.1.1, and let $\lambda \geq \frac{1}{\delta}$ be given as above. Since δ, k, and λ were already chosen, we now define the remaining constants:

$$\begin{aligned} \epsilon_1 &= \frac{\sqrt{\delta}}{\lambda^{\binom{k}{2}}}, & (82)\\ l_0 &= \lceil \frac{1}{\delta} \rceil, & (83)\\ t_0 &= 1. & (84) \end{aligned}$$

Now we must produce the input functions $r(t,l)$ and $\epsilon_2(l)$. Recall the definition of $r_{D(k)}(l)$ in (80). For arbitrary positive integers l and t, set

$$r(t,l) = \max\{2r_k t^3 l^3, r_{D(k)}(l)\}. \tag{85}$$

Recall Fact 2.1.7. For the definition of the function $\epsilon_2(l)$, we need this fact, as well as the following two consequences of it. We state these easy consequences of Fact 2.1.7 without proof.

Fact 7.2.1. *For any positive integers k, λ, and suitably small positive reals ϵ, there exists a function $\theta'_{k,\lambda}(\epsilon)$, $\theta'_{k,\lambda}(\epsilon) \to 0$ as $\epsilon \to 0$, such that whenever G is a (λ, ϵ, k)-cylinder with k-partition $(V_1, \ldots, V_k), |V_1| = \ldots = |V_k| = m$, the following holds.*

For each i, j, $1 \le i < j \le k$, all but $2k\epsilon m^2$ edges $e = \{v_i, v_j\} \in G$, $v_i \in V_i$, $v_j \in V_j$, satisfy

$$(1 - \theta'_{k,\lambda}(\epsilon)) \frac{m^{k-2}}{\lambda^{\binom{k}{2}-1}} < |\{Y \in \mathcal{K}_k^{(2)}(G) : v_i, v_j \in Y\}| < (1 + \theta'_{k,\lambda}(\epsilon)) \frac{m^{k-2}}{\lambda^{\binom{k}{2}-1}}.$$

Fact 7.2.2. *For any positive integers k, λ, and suitably small positive reals ϵ, there exists a function $\theta''_{k,\lambda}(\epsilon)$, $\theta''_{k,\lambda}(\epsilon) \to 0$ as $\epsilon \to 0$, such that whenever G is a (λ, ϵ, k)-cylinder with k-partition $(V_1, \ldots, V_k), |V_1| = \ldots = |V_k| = m$, the following holds.*

For each i_1, i_2, i_3, $1 \le i_1 < i_2 < i_3 \le k$, all but $6k\epsilon m^3$ triangles $t = \{v_{i_1}, v_{i_2}, v_{i_3}\} \in \mathcal{K}_3^{(2)}(G)$, $v_{i_1} \in V_{i_1}$, $v_{i_2} \in V_{i_2}$, $v_{i_3} \in V_{i_3}$ satisfy

$$\begin{aligned}(1 - \theta''_{k,\lambda}(\epsilon)) \frac{m^{k-3}}{\lambda^{\binom{k}{2}-3}} \quad &< \quad |\{Y \in \mathcal{K}_k^{(2)}(G) : v_{i_1}, v_{i_2}, v_{i_3} \in Y\}|, \\ &< \quad (1 + \theta''_{k,\lambda}(\epsilon)) \frac{m^{k-3}}{\lambda^{\binom{k}{2}-3}}.\end{aligned}$$

The definition of the function $\epsilon_2(l)$ depends on functions $\theta_{k,\lambda l}(\epsilon)$, $\theta'_{k,\lambda l}(\epsilon)$, $\theta''_{k,\lambda l}(\epsilon)$ defined in Facts 2.1.7, 7.2.1, 7.2.2. Note that these functions all tend to zero as their argument ϵ tends to zero. For an arbitrary positive integer l, let $\epsilon_2(l)$ be a positive quantity satisfying each of the following inequalities

$$\epsilon_2(l) \le \epsilon_{D(k)}(l), \tag{86}$$

$$\theta_{3,\lambda l}(\epsilon_2(l)) < \frac{1}{10}, \tag{87}$$

$$\theta_{k,\lambda l}(\epsilon_2(l)), \theta'_{k,\lambda l}(\epsilon_2(l)), \theta''_{k,\lambda l}(\epsilon_2(l)) < 1/2, \tag{88}$$

$$\theta_{k,\lambda l}(\epsilon_2(l)) < \delta. \tag{89}$$

For the input values ϵ_1, δ, λ, l_0, t_0, $r(t, l)$, and $\epsilon_2(l)$ given above, let

$$\epsilon = \epsilon(\epsilon_1, \delta, \lambda, l_0, t_0, r(t, l), \epsilon_2(l)), \tag{90}$$

$$T_0 = T_0(\epsilon_1, \delta, \lambda, l_0, t_0, r(t, l), \epsilon_2(l)), \tag{91}$$

$$L_0 = L_0(\epsilon_1, \delta, \lambda, l_0, t_0, r(t, l), \epsilon_2(l)), \tag{92}$$

$$N_0 = N_0(\epsilon_1, \delta, \lambda, l_0, t_0, r(t, l), \epsilon_2(l)) \tag{93}$$

be the constants guaranteed by Theorem 6.2.1. Note that we may assume without loss of generality that ϵ from (90) satisfies

$$\epsilon < \frac{\sqrt{\delta}}{6k\lambda^{\binom{k}{2}}}, \tag{94}$$

$$\theta_{k,\lambda}(\epsilon), \theta'_{k,\lambda}(\epsilon), \theta''_{k,\lambda}(\epsilon) < \frac{1}{2}, \tag{95}$$

$$\theta_{3,\lambda}(\epsilon) < \frac{1}{2}. \tag{96}$$

We stress here that the values ϵ, T_0, L_0, and N_0 are now fixed constants. We define the constant r as

$$r = r(T_0, L_0) \tag{97}$$

where the function $r(t,l)$ is given in (85). Having defined the constants above, we now proceed to the proof of Lemma 7.1.1.

Proof of Lemma 7.1.1.

Given α and δ_k, let δ be given in (76), (77) and (78). Given integers $\lambda \geq \frac{1}{\delta}$ and $r_k \geq 1$ let r, ϵ, and N_0 be given in (97), (90), and (93) respectively, and let $N \geq N_0$. We may think of the constants satisfying the following hierarchy:

$$\alpha \gg \delta_k \gg \delta > \frac{1}{\lambda} \gg \frac{1}{r} \gg \epsilon. \tag{98}$$

We make the following remark describing a small simplifying assumption we make in our proof.

Remark 7.2.3. To simplify some of the details in this proof, we assume that $(T_0)!$ divides N, where T_0 is given in (91). This assumption is not essential for the proof of Lemma 7.1.1, but it is more convenient to assume it. We make further notes on this assumption later in context. This concludes our remark. $\square$

Let $\mathcal{H}$ be a k-partite 3-cylinder and G a k-partite underlying cylinder of $\mathcal{H}$ satisfying the conditions of the Setup in the beginning of the section with the constants k, α, δ, λ, r, ϵ and N. That is, suppose

(*i*) $\mathcal{H}$ is a k-partite 3-cylinder with k-partition $(V_1, \ldots, V_k)$, $|V_1| = \ldots = |V_k| = N$,
(*ii*) $G = \bigcup_{1 \leq i < j \leq k} G^{ij}$ is a (λ, ϵ, k)-cylinder underlying $\mathcal{H}$,
(*iii*) for all $B \in [k]^3$, $\mathcal{H}(B)$ is (α, δ, r)-regular with respect to $G(B)$.

Our goal is to show that $\mathcal{H}$ is $(\alpha^{\binom{k}{3}}, \delta_k, r_k)$-regular with respect to G.

Recall that directly after the statement of Lemma 7.1.1, we discussed what it meant for $\mathcal{H}$ to be $(\alpha^{\binom{k}{3}}, \delta_k, r_k)$-regular with respect to G; we said we need only show the inequality in (72). More precisely, suppose $\overrightarrow{Q} = (Q(s))$, $1 \leq s \leq r_k$, is an r_k-tuple, where for all $s \in [r_k]$, $Q(s) = \bigcup_{1 \leq i < j \leq k} Q^{ij}(s)$, and for all $\{i,j\} \in [k]^2$, $s \in [r_k]$, $Q^{ij}(s) \subseteq G^{ij}$. We show that if

$$|\bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))| > \delta_k |\mathcal{K}_k^{(2)}(G)|, \tag{99}$$

then

$$d_{\mathcal{H}}(\overrightarrow{Q}) > \alpha^{\binom{k}{3}} - 2\delta_k,$$

or equivalently,

$$|\mathcal{K}_k^{(3)}(\mathcal{H}) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))| > (\alpha^{\binom{k}{3}} - 2\delta_k)|\bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))|. \tag{100}$$

To this effect, assume $\overrightarrow{Q} = (Q(s))$, $1 \leq s \leq r_k$, where for all $s \in [r_k]$, $Q(s) = \bigcup_{1 \leq i < j \leq k} Q^{ij}(s)$, and for all $\{i,j\} \in [k]^2$, $s \in [r_k]$, $Q^{ij}(s) \subseteq G^{ij}$. Assume further that $\overrightarrow{Q}$ satisfies (99). Thus, in view of the discussion above, we will have proved Lemma 7.1.1 once we have established (100). This is the subject of the following Proposition 7.2.4.

Proposition 7.2.4.

$$|\mathcal{K}_k^{(3)}(\mathcal{H}) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))| > (\alpha^{\binom{k}{3}} - 2\delta_k)|\bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))|.$$

7.3. **Itinerary for the Proof of Proposition 7.2.4.**

The proof of this proposition depends largely on one single fact which we prove in the upcoming Lemma 7.3.14. However, we will not be able to state Lemma 7.3.14 until we establish some definitions and notation. Therefore, the following material is used to describe the situation discussed in Lemma 7.3.14. We break this material into the following parts.

- **A. Applying the Regularity Lemma:** Here we apply Theorem 6.2.1 to the hypergraph $\mathcal{H}^Q = \mathcal{H} \cap \bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s))$ with underlying cylinder G to obtain a $(\lambda l, kt, \epsilon_1, \epsilon_2)$-partition $\mathcal{P}_0$. Note that as a consequence of this application, we disclose specific values of l and t, $l_0 \leq l \leq L_0$, $t_0 \leq t \leq T_0$, where l_0, L_0, t_0 and T_0 are given in (83), (92), (84) and (91) respectively.
- **B. Big and Small Cylinders:** Here we define structures from the partition $\mathcal{P}_0$ over which we work.
- **C. Defective and Perfect Cylinders:** Here we define special classes of small cylinders.

When we fill in the above outline with details, we then state Lemma 7.3.14. After stating Lemma 7.3.14, we proceed directly to the proof of Proposition 7.2.4 in Subsection 7.4. After concluding the proof of Proposition 7.2.4, we begin the task of proving Lemma 7.3.14 in Subsection 7.5.

To begin, restrict $\mathcal{H}$ to the k-partite 3-cylinder $\mathcal{H}^Q$ defined by

$$\mathcal{H}^Q = \mathcal{H} \cap \bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s)) \subset \mathcal{H}. \tag{101}$$

Clearly, these are the triples of $\mathcal{H}$ with which (100) is concerned.

We make the following remark about the hypergraph $\mathcal{H}^Q$ with respect to the requirement we mentioned in (74).

Remark 7.3.1. We note that here we use the fact that $r \geq r_k$ specified in (74). It is this condition which guarantees that the 3-cylinder $\mathcal{H}^Q$ is nonempty. Indeed, we are given an r_k-tuple $\overrightarrow{Q} = (Q(s))$, satisfying (99). It is not hard to show that for any $B \in [k]^3$, $|(\bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s))) \cap \mathcal{K}_3^{(2)}(G(B))| > \delta |\mathcal{K}_3^{(2)}(G(B))|$. Therefore, with $r \geq r_k$, it follows from the (α, δ, r)-regularity of the triad $G(B)$ that $|\mathcal{H}^Q(B)| = |\mathcal{H}(B) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s))| > (\alpha - 2\delta)|(\bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s))) \cap \mathcal{K}_3^{(2)}(G(B))| > (\alpha - 2\delta)\delta|\mathcal{K}_3^{(2)}(G(B))| > \frac{(\alpha-2\delta)\delta}{2}\frac{N^3}{\lambda^3}$ where the last inequality follows from the fact that $G(B)$ is a $(\lambda, \epsilon, 3)$-cylinder and since in Fact 2.1.7, we have that $\theta_{3,\lambda}(\epsilon) < \frac{1}{2}$ from (96). Provided that N is sufficiently large, we thus conclude that $|\mathcal{H}^Q(B)|$ is positive for all $B \in [k]^3$. This concludes our remark. $\Box$ $\Box$

A. Applying the Regularity Theorem:

We now appeal to the Regularity Lemma, Theorem 6.2.1. We apply Theorem 6.2.1 to the k-partite 3-cylinder $\mathcal{H}^Q$ given above with the underlying (λ, ϵ, k)-cylinder $\bigcup_{1 \leq i < j \leq k} G^{ij}$. Using parameters

$$\epsilon_1, \delta, \lambda, l_0, t_0, r(t,l), \epsilon_2(l) \tag{102}$$

given by (82)-(89) and $\lambda \geq \frac{1}{\delta}$ given at the beginning of the Definitions of the Constants, Theorem 6.2.1 guarantees constants

$$\epsilon, L_0, T_0, N_0 \tag{103}$$

which were disclosed in (90)-(93) in the Definitions of the Constants. For $\mathcal{H}^Q$ and G defined above, the application of the Regularity Lemma also gives a $(\lambda l, kt, \epsilon_1, \epsilon_2(l))$-partition $\mathcal{P}_0$ which is $(\delta, r(t,l))$-fully regular with respect to $\mathcal{H}^Q$, where $l_0 \leq l \leq L_0$, $t_0 \leq t \leq T_0$. We emphasize to the Reader the following subtle point. All the parameters seen in (102) and (103) were already defined in the Definitions of the Constants. Moreover, they did not depend on the choice of the hypergraph $\mathcal{H}^Q$, nor did they depend on the choice of graph G, so they could be disclosed without reference to specific hypergraphs or graphs. Now that we have a specific hypergraph $\mathcal{H}^Q$ and underlying graph G, we *also* obtain a concrete partition $\mathcal{P}_0$ of the specific hypergraph $\mathcal{H}^Q$ and G. Therefore, the only thing new here is the partition $\mathcal{P}_0$, and specific *constant* values l and t satisfying $l_0 \leq l \leq L_0$ and $t_0 \leq t \leq T_0$. Since after the application of Theorem 6.2.1 above, the values l, t become fixed constants, consequently, $r(t,l)$, $\epsilon_2(l)$, are fixed constants. We set $r_1 = r(t,l)$, recall from (97) that we set $r = r(T_0, L_0)$, and observe that

$$r_k \leq r_1 \leq r \tag{104}$$

holds. We also set $\epsilon_2 = \epsilon_2(l)$.

Now we are able to summarize *all* the constants used in Section 7. Together with previously defined constants, we note that our set of fixed constants now consists of

$$k, \alpha, \delta_k, \delta, \lambda, r_k$$

given at the beginning of the Definitions of the Constants (specifically, δ given in (77)-(78)),

$$\epsilon_1, l_0, t_0$$

given as input for the Regularity Lemma in (82)-(84),

$$\epsilon, L_0, T_0, N_0, l, t$$

given as output of the Regularity Lemma in (90)-(93) (l and t just obtained in the most recent application of the Regularity Lemma), and consequently

$$r, r_1, \epsilon_2$$

given by (97), $r_1 = r(t,l)$, and $\epsilon_2 = \epsilon_2(l)$. We think of these constants satisfying the following hierarchy

$$\alpha \gg \delta_k \gg \delta > \frac{1}{\lambda} \gg \epsilon_1, \frac{1}{(l\lambda)^k} \gg \epsilon_2 \gg \frac{1}{T_0} \gg \epsilon \gg \frac{1}{N}. \tag{105}$$

Note that the hierarchy above updates the one given in (98).

Throughout the remainder of Section 7, we denote by $\mathcal{P}_0$ that $(\lambda l, kt, \epsilon_1, \epsilon_2)$-partition which is (δ, r_1)-fully regular with respect to $\mathcal{H}^Q$. We use the following notation to characterize $\mathcal{P}_0$.

Let W_{x_i}, $1 \leq i \leq k$, $1 \leq x_i \leq t$ denote the vertex classes of $\mathcal{P}_0$ so that $W = \bigcup_{1\leq i\leq k} \bigcup_{1\leq x_i \leq t} W_{x_i}$ is an equitable refinement of $V = V_1 \cup \ldots \cup V_k$, that is, so that:

(i) For all i, $1 \leq i \leq k$, and all x_i, $1 \leq x_i \leq t$, $|W_{x_i}| = m_{\mathcal{P}_0} = m$.

(ii) For all $i \in [k]$, $\bigcup_{1\leq x_i\leq t} W_{x_i} = V_i$.

Note that it follows from our assumption that $(T_0)!$ divides N that $mt = N$ holds. Hence, we now see that this auxiliary divisibility assumption made above was so that the garbage class of vertices W_0 (cf., Definition 6.1.1) would satisfy $W_0 = \emptyset$.

Further, we let $P^{x_iy_j}_{\alpha_{x_iy_j}}$, $1 \leq i \leq k$, $1 \leq x_i, y_j \leq t$, $1 \leq \alpha_{x_iy_j} \leq l_{x_iy_j} \leq l$, denote the system of bipartite graphs of $\mathcal{P}_0$ such that:

(iii) For all i, j, $1 \leq i < j \leq k$, for all x_i, y_j, $1 \leq x_i, y_j \leq t$,

$$\begin{aligned} \bigcup_{1\leq \alpha_{x_iy_j} \leq l_{x_iy_j} \leq l} P^{x_iy_j}_{\alpha_{x_iy_j}} &\subseteq G[W_{x_i}, W_{y_j}] \\ &= \{\{v_{x_i}, v_{y_j}\} \in G : v_{x_i} \in W_{x_i}, v_{y_j} \in W_{y_j}\}. \end{aligned}$$

(iv) All but $\epsilon_1 t^2 \binom{k}{2} m^2/\lambda$ pairs $\{v_{x_i}, v_{y_j}\} \in G$, $v_{x_i} \in W_{x_i}, v_{y_j} \in W_{y_j}$, are edges of $(\lambda l, \epsilon_2, 2)$-cylinders $P^{x_iy_j}_{\alpha_{x_iy_j}}$, $1 \leq i < j \leq k$, $1 \leq x_i, y_j \leq t$, $1 \leq \alpha_{x_iy_j} \leq l_{x_iy_j} \leq l$.

We now proceed to Part B. Before doing so, we make the following brief comment. Note that Lemma 7.1.1 is used to show the implication $D(k) \Longrightarrow R(k)$, which is a statement for $k \geq 4$, and we fixed $k \geq 4$ in the Definitions of the Constants. It is this reason why in what follows, we distinguish between triads and k-partite cylinders.

B. Big and Small Cylinders

Definition 7.3.2. For $B \subseteq [k], |B| = b$, we refer to the B-cylinder $G(B) = \bigcup_{\{i,j\}\in[B]^2} G^{ij}$ as the *big B-cylinder*, or more generally, as a b-partite *big cylinder*. In the frequently occurring case that $b = 3$, we refer to this cylinder as a *big triad*.

Analogous to Definition 7.3.2, we define "small" cylinders with partite sets W_{x_i}, $1 \leq i \leq k$, $1 \leq x_i \leq t$, where W_{x_i} is given as a class of the equitable refinement described above. More formally, let $(V_1, \ldots, V_k)$ be the k-partition of $\mathcal{H}$, and let W_{x_i}, $1 \leq i \leq k$, $1 \leq x_i \leq t$ denote the vertex classes of the equitable refinement of $V = V_1 \cup \ldots \cup V_k$ described above associated with the fixed partition $\mathcal{P}_0$. For convenience, we use the following definition.

Definition 7.3.3. We call any set $B \subseteq \bigcup_{1\leq i\leq k} \bigcup_{1\leq x_i\leq t} x_i$ *transversal* if it satisfies the following conditions:

(i) $2 \leq |B| \leq k$.

(ii) For all $\{x_i, y_j\} \in [B]^2$, $i \neq j$.

We are now ready for the following definition. Note that in what follows, with the notation ${}^sB$, ${}^sb$ used, we think of the "s" as standing for "small".

Definition 7.3.4. Let ${}^sB$ be transversal, where $|{}^sB| = {}^sb \leq k$. We call the ${}^sb$-partite cylinder $\bigcup_{\{x_i,y_j\}\in[{}^sB]^2} P^{x_iy_j}_{\alpha_{x_iy_j}}$ of $\mathcal{P}_0$, $1 \leq \alpha_{x_iy_j} \leq l_{x_iy_j} \leq l$, the *small ${}^sB$-cylinder*, or more generally, a *small cylinder*. In the frequently occurring case that ${}^sb = 3$, we refer to this cylinder as a *small triad*.

Note that a small cylinder is uniquely determined by the choice of the transversal set ${}^sB$, and choice of integers $\alpha_{x_iy_j}$, $1 \leq \alpha_{x_iy_j} \leq l_{x_iy_j}$ for all $\{x_i, y_j\} \in [{}^sB]^2$. Thus, if a small cylinder is k-partite, we may write it as $C_k = C_k({}^sB, (\alpha_{x_iy_j})_{\{x_i,y_j\}\in[{}^sB]^2}) = \bigcup_{\{x_i,y_j\}\in[{}^sB]^2} P^{x_iy_j}_{\alpha_{x_iy_j}}$, for the choice of the transversal set ${}^sB$, $|{}^sB| =^s b = k$, and choice of integers $\alpha_{x_iy_j}$, $1 \leq \alpha_{x_iy_j} \leq l_{x_iy_j}$, for all $\{x_i, y_j\} \in [{}^sB]^2$. If the small cylinder is 3-partite, that is, a small triad, then we write the cylinder as $C_3 = C_3(\{x_h, y_i, z_j\}, (\alpha_{x_hy_i}, \alpha_{x_hz_j}, \alpha_{y_iz_j})) = P^{x_hy_i}_{\alpha_{x_hy_i}} \cup P^{x_hz_j}_{\alpha_{x_hz_j}} \cup P^{y_iz_j}_{\alpha_{y_iz_j}}$ where $\{x_h, y_i, z_j\}$ is transversal, $1 \leq \alpha_{x_hy_i} \leq l_{x_hy_i}$, $1 \leq \alpha_{x_hz_j} \leq l_{x_hz_j}$, $1 \leq \alpha_{y_iz_j} \leq l_{y_iz_j}$.

In what follows, any k-partite small cylinder C_k of $\mathcal{P}_0$ is of the form

$$C_k = C_k({}^sB, (\alpha_{x_iy_j})_{\{x_i,y_j\}\in[{}^sB]^2}) = \bigcup_{\{x_i,y_j\}\in[{}^sB]^2} P^{x_iy_j}_{\alpha_{x_iy_j}},$$

where ${}^sB$ is a transversal set, $|{}^sB| =^s b = k$, and $\alpha_{x_iy_j}$ are integers such that $1 \leq \alpha_{x_iy_j} \leq l_{x_iy_j}$ for all $\{x_i, y_j\} \in [{}^sB]^2$. Similarly, any small triad C_3 of $\mathcal{P}_0$ is of the form

$$C_3 = C_3(\{x_h, y_i, z_j\}, (\alpha_{x_hy_i}, \alpha_{x_hz_j}, \alpha_{y_iz_j})) = P^{x_hy_i}_{\alpha_{x_hy_i}} \cup P^{x_hz_j}_{\alpha_{x_hz_j}} \cup P^{y_iz_j}_{\alpha_{y_iz_j}},$$

where $\{x_h, y_i, z_j\}$ is transversal, $1 \leq \alpha_{x_hy_i} \leq l_{x_hy_i}$, $1 \leq \alpha_{x_hz_j} \leq l_{x_hz_j}$, $1 \leq \alpha_{y_iz_j} \leq l_{y_iz_j}$.

C. Defective and Perfect Cylinders

Here we consider 2-defective cylinders and several kinds of 3-defective cylinders. We note here that defective cylinders are *always* small cylinders. We begin with the following definition.

Definition 7.3.5. We say that the k-partite small cylinder $C_k = \bigcup_{\{x_i,y_j\}\in[^sB]^2} P^{x_iy_j}_{\alpha_{x_iy_j}}$ is a *2-defective cylinder* if for some $\{x_i, y_j\} \in [^sB]^2$, the bipartite graph $P^{x_iy_j}_{\alpha_{x_iy_j}}$ is not a $(\lambda l, \epsilon_2, 2)$-cylinder.

For some of the definitions which follow, it is necessary to have the following notion.

Definition 7.3.6. Let C_3 be a small triad. We define the 3-partite 3-cylinder $\mathcal{H}^Q_{C_3}$ to be that subhypergraph of $\mathcal{H}^Q$ induced on the underlying triangles of C_3, that is,

$$\mathcal{H}^Q_{C_3} = \mathcal{H}^Q \cap \mathcal{K}^{(2)}_3(C_3).$$

We now describe specific classes of small triads C_3 which are in some way "defective".

Definition 7.3.7. Let $C_3 = P^{x_hy_i}_{\alpha_{x_hy_i}} \cup P^{x_hz_j}_{\alpha_{x_hz_j}} \cup P^{y_iz_j}_{\alpha_{y_iz_j}}$ be a small $(\lambda l, \epsilon_2, 3)$-cylinder of $\mathcal{P}_0$. We say that C_3 is a *regular-defective triad* provided it satisfies that $\mathcal{H}^Q_{C_3}$ is not (δ, r_1)-fully regular with respect to C_3.

We easily extend the notion of regular-defective from small triads C_3 to k-partite small cylinders C_k.

Definition 7.3.8. Suppose that $C_k = \bigcup_{\{x_i,y_j\}\in[^sB]^2} P^{x_iy_j}_{\alpha_{x_iy_j}}$ is a $(\lambda l, \epsilon_2, k)$-cylinder of $\mathcal{P}_0$. We say that C_k is a *regular-defective cylinder* if one of the small triads C_3 of C_k is a regular-defective triad, that is, for some $\{x_h, y_i, z_j\} \in [^sB]^3$, $C_3 = P^{x_hy_i}_{\alpha_{x_hy_i}} \cup P^{x_hz_j}_{\alpha_{x_hz_j}} \cup P^{y_iz_j}_{\alpha_{y_iz_j}}$ is a regular-defective triad.

In addition to the class of regular-defective triads C_3, we are also interested in the following class of small triads, also "defective" in some way, and disjoint from the class of regular-defective triads described above.

Definition 7.3.9. Let $C_3 = P^{x_hy_i}_{\alpha_{x_hy_i}} \cup P^{x_hz_j}_{\alpha_{x_hz_j}} \cup P^{y_iz_j}_{\alpha_{y_iz_j}}$ be a small $(\lambda l, \epsilon_2, 3)$-cylinder of $\mathcal{P}_0$. We say that C_3 is a *dense-defective triad* provided

- (i) C_3 is not regular-defective, but
- (ii) C_3 satisfies that $d_{\mathcal{H}^Q}(C_3) \le \alpha - 3\delta$.

We now easily extend the notion of dense-defective from small triads C_3 to k-partite small cylinders C_k.

Definition 7.3.10. Suppose that $C_k = \bigcup_{\{x_i,y_j\}\in[^sB]^2} P^{x_iy_j}_{\alpha_{x_iy_j}}$ is a $(\lambda l, \epsilon_2, k)$-cylinder of $\mathcal{P}_0$. We say that C_k is a *dense-defective cylinder* if

a. C_k is not a regular-defective cylinder, but
b. one of the small triads C_3 of C_k is dense-defective, that is, for some $\{x_h, y_i, z_j\} \in [^sB]^3$ $C_3 = P^{x_hy_i}_{\alpha_{x_hy_i}} \cup P^{x_hz_j}_{\alpha_{x_hz_j}} \cup P^{y_iz_j}_{\alpha_{y_iz_j}}$ is a dense-defective triad.

Putting together the definitions in Definition 7.3.8 and Definition 7.3.10, we define 3-defective cylinders.

Definition 7.3.11. Suppose $C_k = \bigcup_{\{x_i,y_j\}\in[^sB]^2} P^{x_iy_j}_{\alpha_{x_iy_j}}$ is a k-partite small cylinder of $\mathcal{P}_0$. We call the k-partite small cylinder C_k a *3-defective cylinder* if it is either a regular-defective cylinder or a dense-defective cylinder.

We now come to two classes of cylinders with which the remainder of the paper is concerned.

Definition 7.3.12. Suppose $C_k = \bigcup_{\{x_i,y_j\}\in[^sB]^2} P^{x_iy_j}_{\alpha_{x_iy_j}}$ is a k-partite small cylinder of $\mathcal{P}_0$. We call the k-partite small cylinder C_k a *defective-cylinder* if it is either a 2-defective cylinder or a 3-defective cylinder.

Now that we have described many different ways in which a k-partite small cylinder or a small triad can be defective, we identify a class of k-partite small cylinders which are not defective in the ways we saw above.

Definition 7.3.13. Suppose $C_k = \bigcup_{\{x_i,y_j\}\in[^sB]^2} P^{x_iy_j}_{\alpha_{x_iy_j}}$ is a k-partite small cylinder of $\mathcal{P}_0$. We call the k-partite small cylinder C_k a *perfect cylinder* if it is not a defective cylinder.

Note that a perfect cylinder $C_k = \bigcup_{\{x_i,y_j\}\in[^sB]^2} P^{x_iy_j}_{\alpha_{x_iy_j}}$ satisfies all of the following:

(i) C_k is a $(\lambda l, \epsilon_2, k)$-cylinder.

(ii) All $\binom{k}{3}$ small triads $C_3 = P^{x_hy_i}_{\alpha_{x_hy_i}} \cup P^{x_hz_j}_{\alpha_{x_hz_j}} \cup P^{y_iz_j}_{\alpha_{y_iz_j}}$, $\{x_h, y_i, z_j\} \in [^sB]^3$, satisfy that $\mathcal{H}^Q_{C_3} = \mathcal{H}^Q \cap \mathcal{K}^{(2)}_3(C_3)$ is $(\alpha - 2\delta, \delta, r_1)$- regular with respect to C_3.

In what remains, we heavily use the following notation to denote the classes of k-partite small cylinders and small triads that were described above. To denote the k-partite small cylinders C_k described above, let

$$\begin{aligned}
2\mathcal{D}_k &= \{C_k : C_k \text{ is a 2-defective cylinder of } \mathcal{P}_0\}, && (106)\\
\mathcal{RD}_k &= \{C_k : C_k \text{ is a regular-defective cylinder of } \mathcal{P}_0\}, && (107)\\
\mathcal{DD}_k &= \{C_k : C_k \text{ is a dense-defective cylinder of } \mathcal{P}_0\}, && (108)\\
3\mathcal{D}_k &= \{C_k : C_k \text{ is a 3-defective cylinder of } \mathcal{P}_0\}, && (109)\\
\mathcal{D}_k &= \{C_k : C_k \text{ is a defective cylinder of } \mathcal{P}_0\}, && (110)\\
\Pi_k &= \{C_k : C_k \text{ is a perfect cylinder of } \mathcal{P}_0\}. && (111)
\end{aligned}$$

To denote the small triads C_3 described above, let

$$\begin{aligned}
\mathcal{RD}_3 &= \{C_3 : C_3 \text{ is a regular-defective triad of } \mathcal{P}_0\}, && (112)\\
\mathcal{DD}_3 &= \{C_3 : C_3 \text{ is a dense-defective triad of } \mathcal{P}_0\}. && (113)
\end{aligned}$$

Note that in the notation above, the following trivial identities hold

$$\begin{aligned}
\mathcal{RD}_k \cup \mathcal{DD}_k &= 3\mathcal{D}_k, && (114)\\
2\mathcal{D}_k \cup 3\mathcal{D}_k &= \mathcal{D}_k. && (115)
\end{aligned}$$

For convenience, for any family $\mathcal{A}$ given by (106)-(111), we set

$$\mathcal{K}^{(2)}_k(\mathcal{A}) = \bigcup_{C_k\in\mathcal{A}} \mathcal{K}^{(2)}_k(C_k).$$

Similarly, for any family A given by (112) or (113), we set

$$\mathcal{K}^{(2)}_3(\mathcal{A}) = \bigcup_{C_3\in\mathcal{A}} \mathcal{K}^{(2)}_3(C_3).$$

Our goal is to provide an upper bound on the quantity

$$|\mathcal{K}_k^{(2)}(\mathcal{D}_k) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))|. \tag{116}$$

This upper bound is crucial for the proof of Proposition 7.2.4. We state this upper bound as a lemma itself. This is Lemma 7.3.14 advertised earlier in this section.

Lemma 7.3.14.

$$|\mathcal{K}_k^{(2)}(\mathcal{D}_k) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))| < \ 6\sqrt{\delta}k^3 \ N^k/\lambda^{\binom{k}{2}}.$$

As mentioned earlier, we defer the proof of Lemma 7.3.14 until after the end of the proof of Proposition 7.2.4. Therefore, in that vein, we proceed with the proof of Proposition 7.2.4. However, a fact which we need there is the following corollary to Lemma 7.3.14. In this corollary, we are able to estimate the total number of perfect cylinders C_k.

Corollary 7.3.15.

$$|\Pi_k| > (|\bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))| - 6\sqrt{\delta}k^3 \frac{N^k}{\lambda^{\binom{k}{2}}}) \ \frac{(\lambda l)^{\binom{k}{2}}}{m^k(1+\theta_{\lambda l,k}(\epsilon_2))}.$$

Proof of Corollary 7.3.15.
We use the following 2 obvious facts

$$|\mathcal{K}_k^{(2)}(\Pi_k)| \geq |\mathcal{K}_k^{(2)}(\Pi_k) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))| \tag{117}$$

and

$$|\mathcal{K}_k^{(2)}(\Pi_k) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))| = |\bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))| - |\mathcal{K}_k^{(2)}(\mathcal{D}_k) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))|. \tag{118}$$

Combining (117) and (118) yields

$$|\mathcal{K}_k^{(2)}(\Pi_k)| \geq |\bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))| - |\mathcal{K}_k^{(2)}(\mathcal{D}_k) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))|,$$

which in tandem with Lemma 7.3.14 implies

$$\sum\nolimits_{C_k \in \Pi_k} |\mathcal{K}_k^{(2)}(C_k)| > |\bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))| - 6\sqrt{\delta}k^3 \frac{N^k}{\lambda^{\binom{k}{2}}}. \tag{119}$$

By Fact 2.1.7, each perfect cylinder $C_k \in \Pi_k$ satisfies

$$|\mathcal{K}_k^{(2)}(C_k)| < (1+\theta_{\lambda l,k}(\epsilon_2)) \frac{m^k}{(\lambda l)^{\binom{k}{2}}}. \tag{120}$$

We thus infer from (119) and (120) that

$$|\Pi_k| > (|\bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))| - 6\sqrt{\delta}k^3 \frac{N^k}{\lambda^{\binom{k}{2}}}) \ (\lambda l)^{\binom{k}{2}}/m^k(1+\theta_{\lambda l,k}(\epsilon_2)),$$

and so Corollary 7.3.15 is proved. $\square$

7.4. **The Proof of Proposition 7.2.4.**

With Lemma 7.3.14 stated, we return to our original goal; proving Proposition 7.2.4. Recall that we were given $\overrightarrow{Q} = (Q(s))$, $1 \leq s \leq r_k$, where for all $s \in [r_k]$, $Q(s) = \bigcup_{1 \leq i < j \leq k} Q^{ij}(s)$, and for all $\{i, j\} \in [k]^2$, for all $s \in [r_k]$, $Q^{ij}(s) \subseteq G^{ij}$. Moreover, $\overrightarrow{Q}$ satisfied

$$|\bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))| > \delta_k |\mathcal{K}_k^{(2)}(G)|. \tag{121}$$

Proposition 7.2.4 states that

$$|\mathcal{K}_k^{(3)}(\mathcal{H}) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))| > (\alpha^{\binom{k}{3}} - 2\delta_k)| \bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))|.$$

Recall Lemma 7.3.14 states

$$|\mathcal{K}_k^{(2)}(\mathcal{D}_k) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))| < 6\sqrt{\delta} k^3 \frac{N^k}{\lambda^{\binom{k}{2}}},$$

where we recall $\mathcal{D}_k$ is the set of all k-partite defective cylinders. Recall that a cylinder C_k is a defective cylinder if it is either

- (i) 2-defective (i.e. not a $(\lambda l, \epsilon_2, k)$-cylinder),
- (ii) regular-defective but not 2-defective (i.e. one of its $\binom{k}{3}$ triads C_3 gives rise to $\mathcal{H}^Q_{C_3}$ which is not (δ, r_1)-fully regular with respect to C_3),
- (iii) dense-defective but not regular-defective or 2-defective (i.e. one of its $\binom{k}{3}$ triads C_3 has density $d_{\mathcal{H}^Q}(C_3)$ no more than $\alpha - 3\delta$).

We now turn our attention to $\mathcal{K}_k^{(2)}(\Pi_k) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))$, where we recall that Π_k is the set of all k-partite perfect cylinders. Recall that a cylinder C_k is a perfect cylinder if it is a $(\lambda l, \epsilon_2, k)$-cylinder, all of whose $\binom{k}{3}$ triads C_3 give rise to $\mathcal{H}^Q_{C_3}$ which is $(\alpha - 2\delta, \delta, r_1)$-regular with respect to C_3.

We first note that as a consequence of Lemma 7.3.14, nearly all of the copies of $\bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))$ belong to perfect cylinders C_k. Said differently, the quantity $|\mathcal{K}_k^{(2)}(\Pi_k) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))|$ is nearly identical to $|\bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))|$. To see this, recall that the r_k-tuple $\overrightarrow{Q}$ satisfied

$$|\bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))| > \delta_k \left|\mathcal{K}_k^{(2)}(G)\right|. \tag{122}$$

Since G is a (λ, ϵ, k)-cylinder, Fact 2.1.7 says that

$$\left|\mathcal{K}_k^{(2)}(G)\right| > (1 - \theta_{k,\lambda}(\epsilon)) \frac{N^k}{\lambda^{\binom{k}{2}}} > \frac{1}{2} \frac{N^k}{\lambda^{\binom{k}{2}}}, \tag{123}$$

where the last inequality follows from the fact that ϵ satisfies (96). Thus, we may apply the bound in (123) to further bound the inequality in (122) to conclude

$$|\bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))| > \frac{\delta_k}{2} \frac{N^k}{\lambda^{\binom{k}{2}}}. \tag{124}$$

Now, it is easy to bound the fraction

$$\frac{|\mathcal{K}_k^{(2)}(\Pi_k) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))|}{|\bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))|}.$$

It follows from Lemma 7.3.14 and (124) that

$$\begin{aligned}
\frac{|\mathcal{K}_k^{(2)}(\Pi_k) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))|}{|\bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))|} &= \frac{|\bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))| - |\mathcal{K}_k^{(2)}(\mathcal{D}_k) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))|}{|\bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))|}, \\
&> 1 - \frac{6\sqrt{\delta}k^3 N^k}{\lambda^{\binom{k}{2}} |\bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))|}, \qquad (125) \\
&> 1 - \frac{12\sqrt{\delta}k^3}{\delta_k}. \qquad (126)
\end{aligned}$$

Note that in (126), δ satisfies $\delta \ll \delta_k$. In terms of the proportion, nearly all of the elements of $\bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))$ are elements of $\mathcal{K}_k^{(2)}(\Pi_k)$.

We now complete the proof of Proposition 7.2.4. We know that almost all of the copies of $K_k^{(2)}$ from $\bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))$ belong to perfect cylinders C_k. For each perfect cylinder C_k, we want to apply the statement $D(k)$ to 3-cylinder $\mathcal{H}^Q \cap \mathcal{K}_3^{(2)}(C_k)$ with underlying cylinder C_k to conclude that there are a sufficient number of copies of $K_k^{(3)}$ contained in $\mathcal{H}^Q \cap \mathcal{K}_3^{(2)}(C_k)$. We then sum the number of such copies of $K_k^{(3)}$ over all $C_k \in \Pi_k$ (the number of which we know from Corollary 7.3.15).

Let us first recall the formulation of the statement $D(k)$ (with a change in notation).

For all positive $\tilde{\alpha}, \tilde{\beta}$, there exists $\tilde{\delta} = \delta_{D(k)}(\tilde{\alpha}, \tilde{\beta}) > 0$ so that for all $\tilde{\lambda} \geq \frac{1}{\tilde{\delta}}$, there exist positive constants $\tilde{r} = r_{D(k)}(\tilde{\alpha}, \tilde{\beta}, \tilde{\delta}, \tilde{\lambda})$, $\tilde{\epsilon} = \epsilon_{D(k)}(\tilde{\alpha}, \tilde{\beta}, \tilde{\delta}, \tilde{\lambda})$ so that the following holds: suppose

(i) *$\tilde{\mathcal{H}}$ is a k-partite 3-cylinder with k-partition $(\tilde{V}_1, \dots, \tilde{V}_k), |\tilde{V}_1| = \dots = |\tilde{V}_k| = \tilde{N}$.*

(ii) *$\tilde{G} = \bigcup_{1 \leq i < j \leq k} \tilde{G}^{ij}$ is a $(\tilde{\lambda}, \tilde{\epsilon}, \tilde{k})$ cylinder underlying $\tilde{\mathcal{H}}$.*

(iii) *For all $B \in [k]^3, \tilde{\mathcal{H}}(B)$ is $(\tilde{\alpha}, \tilde{\delta}, \tilde{r})$-regular with respect to $\tilde{G}(B)$.*

Then $\tilde{\mathcal{H}}$ satisfies

$$|\mathcal{K}_k^{(3)}(\tilde{\mathcal{H}})| \geq (1 - \tilde{\beta}) \frac{(\tilde{\alpha})^{\binom{k}{3}}}{(\tilde{\lambda})^{\binom{k}{2}}} (\tilde{N})^k.$$

Fix $C_k \in \Pi_k$ with k-partition $(W_{x_1}, \dots, W_{x_k})$. By definition of Π_k, C_k is a $(\lambda l, \epsilon_2, k)$-cylinder which has all $\binom{k}{3}$ triads C_3 giving rise to $\mathcal{H}_{C_3}^Q$ which is $(\alpha - 2\delta, \delta, r_1)$-regular with respect to C_3. Since we apply the statement $D(k)$ to the fixed cylinder C_k and the 3-cylinder $\mathcal{H}^Q \cap \mathcal{K}_3^{(2)}(C_k)$, we set

$$(\tilde{V}_1, \dots, \tilde{V}_k) = (W_{x_1}, \dots, W_{x_k}), \qquad (127)$$

$$\tilde{N} = m, \qquad (128)$$

$$\tilde{\mathcal{H}} = \mathcal{H}^Q \cap \mathcal{K}_3^{(2)}(C_k) \qquad (129)$$

$$\tilde{G} = C_k. \tag{130}$$

Moreover, we need to verify that all the relevant constants α, δ, λl, r_1, and ϵ_2 that have already been fixed satisfy $D(k)$. More precisely, set

$$\tilde{\alpha} = \alpha - 2\delta, \tag{131}$$

$$\tilde{\beta} = \delta_k. \tag{132}$$

The statement $D(k)$ guarantees constant $\tilde{\delta}$ given by $\tilde{\delta} = \delta_{D(k)}(\alpha - 2\delta, \delta_k)$. We see from (79) that

$$\delta \le \delta_{D(k)}(\alpha/2, \delta_k) \le \delta_{D(k)}(\alpha - 2\delta, \delta_k) = \tilde{\delta}.$$

Therefore, our choice of δ verifies the statement $D(k)$ for the choices $\tilde{\alpha} = \alpha - 2\delta$ and $\tilde{\beta} = \delta_k$. Since $\tilde{G}$ is a $(\lambda l, \epsilon_2, k)$-cylinder, set

$$\tilde{\lambda} = \lambda l. \tag{133}$$

It follows from (77) and (83) that

$$\frac{1}{\tilde{\delta}} = \frac{1}{\delta_{D(k)}(\alpha - 2\delta, \delta_k)} \le \frac{1}{\delta_{D(k)}(\alpha/2, \delta_k)} \le \lceil \frac{1}{\delta} \rceil \le \lambda l_0 \le \lambda l = \tilde{\lambda}.$$

Thus, $\tilde{\lambda} \ge \frac{1}{\tilde{\delta}}$. The statement $D(k)$ guarantees constants

$$\tilde{r} = r_{D(k)}(\alpha - 2\delta, \delta_k, \delta, \lambda l) \tag{134}$$

and

$$\tilde{\epsilon} = \epsilon_{D(k)}(\alpha - 2\delta, \delta_k, \delta, \lambda l). \tag{135}$$

By (85), we have

$$r_1 = r(t, l) = \max\{2r_k t^3 l^3, r_{D(k)}(\alpha - 2\delta, \delta_k, \delta, \lambda l)\}. \tag{136}$$

Thus $r_1 \ge r_{D(k)}(\alpha - 2\delta, \delta_k, \delta, \lambda l)\} = \tilde{r}$ is an appropriate choice for $D(k)$ given the parameters in (131), (132), (133). By our choice of ϵ_2 in (86),

$$\epsilon_2(l) \le \epsilon_{D(k)}(\alpha - 2\delta, \delta_k, \delta, \lambda l) = \tilde{\epsilon}. \tag{137}$$

Thus we have that ϵ_2 is an appropriate choice for $D(k)$ given the parameters in (131), (132), (133).

With the parameters given in (131)-(137), apply $D(k)$ to the cylinder $\tilde{G}$ and 3-cylinder $\tilde{\mathcal{H}}$. By that application,

$$|\mathcal{K}_k^{(3)}(\tilde{\mathcal{H}})| \ge (1 - \delta_k) \frac{(\alpha - 2\delta)^{\binom{k}{3}}}{(\lambda l)^{\binom{k}{2}}} m^k. \tag{138}$$

Since $\mathcal{K}_k^{(3)}(\tilde{\mathcal{H}}) = \mathcal{K}_k^{(3)}(\mathcal{H}^Q) \cap \mathcal{K}_k^{(2)}(C_k)$, (138) may be reformulated as

$$|\mathcal{K}_k^{(3)}(\mathcal{H}^Q) \cap \mathcal{K}_k^{(2)}(C_k)| \ge (1 - \delta_k) \frac{(\alpha - 2\delta)^{\binom{k}{3}}}{(\lambda l)^{\binom{k}{2}}} m^k. \tag{139}$$

We repeat the argument above for all $C_k \in \Pi_k$ using (139) and the lower bound in Corollary 7.3.15 that

$$|\Pi_k| > (|\bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))| - 6\sqrt{\delta}k^3 \frac{N^k}{\lambda^{\binom{k}{2}}}) \frac{(\lambda l)^{\binom{k}{2}}}{m^k(1+\theta_{\lambda l,k}(\epsilon_2))}.$$

In particular, we obtain

$$\begin{aligned} |\mathcal{K}_k^{(3)}(\mathcal{H}^Q) \cap \mathcal{K}_k^{(2)}(\Pi_k)| &= |\mathcal{K}_k^{(3)}(\mathcal{H}^Q) \cap \bigcup_{C_k \in \Pi_k} \mathcal{K}_k^{(2)}(C_k)|, \\ &= \sum_{C_k \in \Pi_k} |\mathcal{K}_k^{(3)}(\mathcal{H}^Q) \cap \mathcal{K}_k^{(2)}(C_k)|, \\ &> (|\bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))| - 6\sqrt{\delta}k^3 \frac{N^k}{\lambda^{\binom{k}{2}}}) \frac{(\alpha - 2\delta)^{\binom{k}{3}}(1-\delta_k)}{1+\theta_{\lambda l,k}(\epsilon_2)}, \end{aligned}$$

the right hand side of which is equal to

$$\alpha^{\binom{k}{3}} |\bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))| (1 - \frac{6\sqrt{\delta}k^3 N^k}{\lambda^{\binom{k}{2}} |\bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))|}) \frac{(1-\delta_k)(1-2\delta/\alpha)^{\binom{k}{3}}}{1+\theta_{\lambda l,k}(\epsilon_2)}.$$

Using the fact that we chose ϵ_2 to satisfy (89) and using (125) and (126), we see the above quantity is larger than

$$\alpha^{\binom{k}{3}} |\bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))| (1 - \frac{12\sqrt{\delta}k^3}{\delta_k}) \frac{(1-\delta_k)(1-2\delta/\alpha)^{\binom{k}{3}}}{1+\delta}.$$

Since we chose δ to satisfy (78), we have that the above inequality is larger than

$$(\alpha^{\binom{k}{3}} - 2\delta_k) |\bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))|.$$

Since $|\mathcal{K}_k^{(3)}(\mathcal{H}) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))| \geq |\mathcal{K}_k^{(3)}(\mathcal{H}^Q) \cap \mathcal{K}_k^{(2)}(\Pi_k)|$, Proposition 7.2.4 is proved by the inequalities above. $\Box$

Now all that remains is to prove Lemma 7.3.14.

7.5. **Proof of Lemma 7.3.14.**

Recall that we are trying to show

$$|\mathcal{K}_k^{(2)}(\mathcal{D}_k) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))| < 6\sqrt{\delta}k^3 N^k / \lambda^{\binom{k}{2}}. \tag{140}$$

For the proof of Lemma 7.3.14, we heavily use the following hierarchy already stated in (105).

$$\alpha \gg \delta_k \gg \delta > \frac{1}{\lambda} \gg \epsilon_1, \frac{1}{(l\lambda)^k} \gg \epsilon_2 \gg \frac{1}{T_0} \gg \epsilon \gg \frac{1}{N}.$$

The proof of Lemma 7.3.14 is easy once we have established some supplemental propositions. Indeed, in the upcoming Proposition 7.5.2, we show

$$|\mathcal{K}_k^{(2)}(2\mathcal{D}_k) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))| < \epsilon_1 k^2 N^k. \tag{141}$$

In the upcoming Propositions 7.5.3 and 7.5.4, we show

$$|\mathcal{K}_k^{(2)}(\mathcal{RD}_k) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))| < 3\delta k^3 \frac{N^k}{\lambda^{\binom{k}{2}}} \tag{142}$$

and

$$|\mathcal{K}_k^{(2)}(\mathcal{DD}_k) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))| < 2\sqrt{\delta} k^3 \frac{N^k}{\lambda^{\binom{k}{2}}}. \tag{143}$$

Using the bounds in (142) and (143) and the easy identity $3\mathcal{D}_k = \mathcal{RD}_k \cup \mathcal{DD}_k$, we infer the inequality

$$|\mathcal{K}_k^{(2)}(3\mathcal{D}_k) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))| < 5\sqrt{\delta} k^3 \frac{N^k}{\lambda^{\binom{k}{2}}}. \tag{144}$$

The inequality (140) easily follows from (141), (144) and (82) and the identity $\mathcal{D}_k = 2\mathcal{D}_k \cup 3\mathcal{D}_k$. This proves Lemma 7.3.14.

Thus, we need only prove Propositions 7.5.2, 7.5.3 and 7.5.4 to complete the proof of Lemma 7.3.14. We begin by proving Proposition 7.5.2, and need the following definition.

Definition 7.5.1. Let x_i, y_j, $\alpha_{x_iy_j}$, be such that $1 \le i < j \le k$, $1 \le x_i, y_j \le t$, and $1 \le \alpha_{x_iy_j} \le l_{x_iy_j}$. An edge $e = \{v_{x_i}, v_{y_j}\} \in P^{x_iy_j}_{\alpha_{x_iy_j}}$ is called a *bad edge* if $P^{x_iy_j}_{\alpha_{x_iy_j}}$ is not a $(\lambda l, \epsilon_2, 2)$-cylinder.

Let $E_{bad} \subset G$ denote the set of all bad edges of G. Note that statement (ii) of Definition 6.1.2 implies

$$|E_{bad}| \le \epsilon_1 t^2 \binom{k}{2} \frac{m^2}{\lambda}. \tag{145}$$

Proposition 7.5.2.

$$|\mathcal{K}_k^{(2)}(2\mathcal{D}_k) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))| < \epsilon_1 k^2 N^k.$$

Proof of Proposition 7.5.2.
It is a straightforward observation that

$$\mathcal{K}_k^{(2)}(2\mathcal{D}_k) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s)) = \{X \in \bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s)) : [X]^2 \cap E_{bad} \ne \emptyset\}. \tag{146}$$

Let $e = \{v_i, v_j\} \in G,\ v_i \in V_i, v_j \in V_j,\ 1 \le i < j \le k$, be an arbitrary edge of G. Trivially,

$$|\{X \in \mathcal{K}_k^{(2)}(G) : v_i, v_j \in X\}| \le N^{k-2}. \tag{147}$$

The fact that $mt = N$, (145) and (147) combine to imply

$$|\{X \in \mathcal{K}_k^{(2)}(G) : [X]^2 \cap E_{bad} \ne \emptyset\}| < \epsilon_1 \binom{k}{2} \frac{N^k}{\lambda},$$

therefore

$$|\{X \in \bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s)) : [X]^2 \cap E_{bad} \ne \emptyset\}| < \epsilon_1 \binom{k}{2} \frac{N^k}{\lambda} < \epsilon_1 k^2 N^k.$$

Proposition 7.5.2 then easily follows from our observation in (146). □

Proposition 7.5.3.

$$|\mathcal{K}_k^{(2)}(\mathcal{RD}_k) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))| < 3\delta k^3 \frac{N^k}{\lambda^{\binom{k}{2}}}.$$

Proof of Proposition 7.5.3.

First, we estimate $|\mathcal{RD}_3|$. From (71), we have that

$$\sum\{\mu_{C_3} : C_3 \text{ is not a } (\delta, r_1) - \text{fully regular triad of } \mathcal{P}_0\} < \delta \left(\frac{kN}{m\lambda}\right)^3,$$

where we recall

$$\mu_{C_3} = \frac{|\mathcal{K}_3^{(2)}(C_3)|}{m^3}.$$

It therefore follows that

$$\sum\nolimits_{C_3 \in \mathcal{RD}_3} |\mathcal{K}_k^{(2)}(C_3)| < \delta \left(\frac{kN}{\lambda}\right)^3. \tag{148}$$

Since all regular-defective triads C_3 are $(\lambda l, \epsilon_2, 3)$-cylinders, we apply Fact 2.1.7 to (148) to obtain

$$|\mathcal{RD}_3| \left(\frac{m}{\lambda l}\right)^3 (1 - \theta_{3,\lambda l}(\epsilon_2)) < \delta \left(\frac{kN}{\lambda}\right)^3.$$

Since it follows from (87) that $\theta_{3,\lambda l}(\epsilon_2) < 1/2$, and given the fact that $mt = N$, we obtain from the above inequality

$$|\mathcal{RD}_3| < 2\delta k^3 t^3 l^3. \tag{149}$$

Now we estimate $|\mathcal{RD}_k|$. For each regular-defective triad C_3, there are no more than

$$t^{k-3} l^{\binom{k}{2}-3} \tag{150}$$

regular-defective cylinders C_k satisfying that C_3 is a triad of C_k. (149) and (150) combine to yield

$$|\mathcal{RD}_k| < 2\delta k^3 t^k l^{\binom{k}{2}}. \tag{151}$$

To complete the proof of Proposition 7.5.3, we need to estimate $|\mathcal{K}_k^{(2)}(\mathcal{RD}_k)|$. Since each regular-defective cylinder C_k is a $(\lambda l, \epsilon_2, k)$-cylinder, we apply Fact 2.1.7 to each $C_k \in \mathcal{RD}_k$ to obtain

$$|\mathcal{K}_k^{(2)}(C_k)| < (1 + \theta_{k,\lambda l}(\epsilon_2)) \frac{m^k}{(\lambda l)^{\binom{k}{2}}}. \tag{152}$$

Since we have in (88) that $\theta_{k,\lambda l}(\epsilon_2) < 1/2$, (151) and (152) combine to yield

$$\begin{aligned} |\mathcal{K}_k^{(2)}(\mathcal{RD}_k)| &= \textstyle\sum_{C_k \in \mathcal{RD}_k} |\mathcal{K}_k^{(2)}(C_k)|, \\ &< 2\delta k^3 t^k l^{\binom{k}{2}} (1 + \theta_{k,\lambda l}(\epsilon_2)) \frac{m^k}{(\lambda l)^{\binom{k}{2}}}, \\ &< 3\delta k^3 \frac{N^k}{\lambda^{\binom{k}{2}}}. \end{aligned}$$

Thus

$$|\mathcal{K}_k^{(2)}(\mathcal{RD}_k) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))| \leq |\mathcal{K}_k^{(2)}(\mathcal{RD}_k)| < 3\delta k^3 \frac{N^k}{\lambda^{\binom{k}{2}}}$$

which completes the proof of Proposition 7.5.3.

Proposition 7.5.4.

$$|\mathcal{K}_k^{(2)}(\mathcal{DD}_k) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s))| < 2\sqrt{\delta} k^3 \frac{N^k}{\lambda^{\binom{k}{2}}}.$$

Proof of Proposition 7.5.4.

Proposition 7.5.4 follows from the following inequality:

$$|\{X \in \bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s)) : [X]^3 \cap \mathcal{K}_3^{(2)}(\mathcal{DD}_3) \neq \emptyset\}| < 2\sqrt{\delta} k^3 \frac{N^k}{\lambda^{\binom{k}{2}}}. \tag{153}$$

Thus, it suffices to prove the inequality in (153) above. To that end, we begin by partitioning $\mathcal{DD}_3$ into two classes, $\mathcal{DD}_3^{(1)}$ and $\mathcal{DD}_3^{(2)}$, where

$$\mathcal{DD}_3^{(1)} = \{C_3 \in \mathcal{DD}_3 : |\mathcal{K}_3^{(2)}(C_3) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s))| < \sqrt{\delta} |\mathcal{K}_3^{(2)}(C_3)|\}$$

and

$$\mathcal{DD}_3^{(2)} = \mathcal{DD}_3 \setminus \mathcal{DD}_3^{(1)} = \{C_3 \in \mathcal{DD}_3 : |\mathcal{K}_3^{(2)}(C_3) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s))| \geq \sqrt{\delta} |\mathcal{K}_3^{(2)}(C_3)|\}.$$

Set $\mathcal{K}_3^{(2)}(\mathcal{DD}_3^{(i)}) = \bigcup_{C_3 \in \mathcal{DD}_3^{(i)}} \mathcal{K}_3^{(2)}(C_3)$, $i = 1, 2$. To prove the inequality in (153), we show that

$$|\{X \in \bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s)) : [X]^3 \cap \mathcal{K}_3^{(2)}(\mathcal{DD}_3^{(1)}) \neq \emptyset\}| < \sqrt{\delta} k^3 \frac{N^k}{\lambda^{\binom{k}{2}}} \tag{154}$$

and

$$|\{X \in \bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s)) : [X]^3 \cap \mathcal{K}_3^{(2)}(\mathcal{DD}_3^{(2)}) \neq \emptyset\}| < \sqrt{\delta} k^3 \frac{N^k}{\lambda^{\binom{k}{2}}}. \tag{155}$$

To facilitate the proofs of the inequalities in (154) and (155), we define the following sets: for $B = \{a, b, c\} \in [k]^3$,

$$\mathcal{DD}_3^{(1)}(B)\mathcal{DD}_3^{(1)}(B) = \{C_3 = P^{x_a y_b}_{\alpha_{x_a y_b}} \cup P^{x_a z_c}_{\alpha_{x_a z_c}} \cup P^{y_b z_c}_{\alpha_{y_b z_c}} \in \mathcal{DD}_3^{(1)}\}$$

and

$$\mathcal{DD}_3^{(2)}(B)\mathcal{DD}_3^{(2)}(B) = \{C_3 = P_{\alpha_{x_a y_b}}^{x_a y_b} \cup P_{\alpha_{x_a z_c}}^{x_a z_c} \cup P_{\alpha_{y_b z_c}}^{y_b z_c} \in \mathcal{DD}_3^{(2)}\}.$$

Thus, $\mathcal{DD}_3^{(1)} = \bigcup_{B\in[k]^3} \mathcal{DD}_3^{(1)}(B)$ and $\mathcal{DD}_3^{(2)} = \bigcup_{B\in[k]^3} \mathcal{DD}_3^{(2)}(B)$.

We begin our proof of (153) by establishing the inequality in (154). By definition of $\mathcal{DD}_3$, for each $C_3 \in \mathcal{DD}_3^{(1)}$, $\mathcal{K}_3^{(2)}(C_3)$ intersects $\bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s))$ very little. The inequality in (154) is just the union of all these small intersections, and hence is small itself. We now show the details.

Fix $B \in [k]^3$, and let $C_3 \in \mathcal{DD}_3^{(1)}(B)$. As a member of $\mathcal{DD}_3^{(1)}$, C_3 satisfies

$$|\mathcal{K}_3^{(2)}(C_3) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s))| < \sqrt{\delta}|\mathcal{K}_3^{(2)}(C_3)|.$$

Since C_3 is a $(\lambda l, \epsilon_2, 3)$-cylinder, we may apply Fact 2.1.7 along with the inequality in (87) to further conclude

$$|\mathcal{K}_3^{(2)}(C_3) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s))| < \sqrt{\delta}\frac{m^3}{(\lambda l)^3}(1 + \theta_{3,\lambda l}(\epsilon_2)) < 2\sqrt{\delta}\frac{m^3}{(\lambda l)^3}. \tag{156}$$

Clearly, $|\mathcal{DD}_3^{(1)}(B)| \le t^3 l^3$, thus it follows from (156) that

$$|\bigcup_{C_3\in\mathcal{DD}_3(B)} (\mathcal{K}_3^{(2)}(C_3) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s)))| < 2\sqrt{\delta}\frac{N^3}{\lambda^3}. \tag{157}$$

Our next goal is to count the set of copies $X \in \mathcal{K}_k^{(2)}(G)$ such that $[X]^3 \cap (\bigcup_{C_3\in\mathcal{DD}_3^{(1)}(B)} (\mathcal{K}_3^{(2)}(C_3) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s)))) \neq \emptyset$. We first appeal to Fact 7.2.2. Since G is a (λ, ϵ, k)-cylinder, we may apply Fact 7.2.2 to conclude that all but $6k\epsilon N^3$ triangles $\{v_{i_1}, v_{i_2}, v_{i_3}\} \in \mathcal{K}_3^{(2)}(G(B))$ satisfy that

$$|\{X \in \mathcal{K}_k^{(2)}(G) : v_{i_1}, v_{i_2}, v_{i_3} \in X\}| < (1 + \theta''_{k,\lambda}(\epsilon))\frac{N^{k-3}}{\lambda^{\binom{k}{2}-3}}. \tag{158}$$

With ϵ satisfying (95), we conclude from (158) above that all but $6k\epsilon N^3$ triangles $\{v_{i_1}, v_{i_2}, v_{i_3}\} \in \mathcal{K}_3^{(2)}(G(B))$ satisfy

$$|\{X \in \mathcal{K}_k^{(2)}(G) : v_{i_1}, v_{i_2}, v_{i_3} \in X\}| < 2\frac{N^{k-3}}{\lambda^{\binom{k}{2}-3}}. \tag{159}$$

As a result of (157) and (159), we conclude that

$$|\{X \in \mathcal{K}_k^{(2)}(G) : [X]^3 \cap (\bigcup_{C_3\in\mathcal{DD}_3^{(1)}(B)} (\mathcal{K}_3^{(2)}(C_3) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s)))) \neq \emptyset\}|$$

$$< 6k\epsilon N^k + 4\sqrt{\delta}\frac{N^k}{\lambda^{\binom{k}{2}}}. \tag{160}$$

Summing (160) over all $\binom{k}{3}$ sets $B \in [k]^3$, we easily infer from (160) that

$$|\{X \in \bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s)) : [X]^3 \cap \mathcal{K}_3^{(2)}(\mathcal{DD}_3^{(1)}) \neq \emptyset\}|$$

has upper bound

$$6k\binom{k}{3}\epsilon N^k + 4\sqrt{\delta}\binom{k}{3}\frac{N^k}{\lambda^{\binom{k}{2}}} < 5\sqrt{\delta}\binom{k}{3}\frac{N^k}{\lambda^{\binom{k}{2}}}$$

where the last inequality follows from our choice of ϵ in (94). Thus, (154) is proved.

We now prove the inequality in (155). To that effect, for fixed $B \in [k]^3$, we show

$$|\mathcal{DD}_3^{(2)}(B)| \leq 2\sqrt{\delta}t^3l^3. \tag{161}$$

The inequality in (155) will easily follow from (161). Indeed, with (161) and each $C_3 \in \mathcal{DD}_3^{(2)}(B)$ by Fact 2.1.7 satisfying $|\mathcal{K}_3^{(2)}(C_3)| < \frac{m^3}{(\lambda l)^3}(1 + \theta_{3,\lambda l}(\epsilon_2))$, we have

$$\begin{aligned} |\mathcal{K}_3^{(2)}(\mathcal{DD}_3^{(2)}(B))| &= |\bigcup_{C_3 \in \mathcal{DD}_3^{(2)}(B)} \mathcal{K}_3^{(2)}(C_3)|, \\ &< 2\sqrt{\delta}\frac{N^3}{\lambda^3}(1 + \theta_{3,\lambda l}(\epsilon_2)), \\ &< 3\sqrt{\delta}\frac{N^3}{\lambda^3} \end{aligned}$$

where the last inequality follows from (87). Applying Fact 7.2.2 to $G(B)$ as before, we have

$$|\{X \in \mathcal{K}_k^{(2)}(G) : [X]^3 \cap \mathcal{K}_3^{(2)}(\mathcal{DD}_3^{(2)}(B)) \neq \emptyset\}|$$

has upper bound

$$6k\epsilon N^k + 3\sqrt{\delta}\frac{N^k}{\lambda^{\binom{k}{2}}}(1 + \theta''_{k,\lambda l}(\epsilon)) < 5\sqrt{\delta}\frac{N^k}{\lambda^{\binom{k}{2}}}$$

where the last inequality follows from (94). Thus, over all $B \in [k]^3$, we have

$$|\{X \in \mathcal{K}_k^{(2)}(G) : [X]^3 \cap \mathcal{K}_3^{(2)}(\mathcal{DD}_3^{(2)}) \neq \emptyset\}| < 5\sqrt{\delta}\binom{k}{3}\frac{N^k}{\lambda^{\binom{k}{2}}},$$

from which we infer

$$|\{X \in \bigcup_{s=1}^{r_k} \mathcal{K}_k^{(2)}(Q(s)) : [X]^3 \cap \mathcal{K}_3^{(2)}(\mathcal{DD}_3^{(2)}) \neq \emptyset\}| < \sqrt{\delta}k^3\frac{N^k}{\lambda^{\binom{k}{2}}}.$$

Thus, as promised, the inequality in (155) is established.

What remains to be shown is that for all $B \in [k]^3$, (161) holds. On the contrary, assume that there exists $B \in [k]^3$ such that

$$|\mathcal{DD}_3^{(2)}(B)| \geq \lceil 2\sqrt{\delta}t^3l^3 \rceil. \tag{162}$$

Our goal is to show that the assumption in (162) leads to a contradiction with our hypothesis that $\mathcal{H}(B)$ is (α, δ, r)-regular with respect to $G(B)$. That is, we show that from the set $\mathcal{DD}_3^{(2)}(B)$ being large as in (162), we can construct a "witness against the regularity of $\mathcal{H}(B)$ with respect to $G(B)$".

For convenience of notation in what follows, set

$$\rho = \lceil 2\sqrt{\delta} t^3 l^3 \rceil.$$

Recall that each $C_3 \in \mathcal{DD}_3^{(2)}(B)$ satisfies

$$\begin{aligned} |\mathcal{K}_3^{(2)}(C_3) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s))| &\geq \sqrt{\delta} |\mathcal{K}_3^{(2)}(C_3)|, \\ &> \sqrt{\delta} \frac{m^3}{(\lambda l)^3}(1 - \theta_{3,\lambda l}(\epsilon_2)) \end{aligned} \tag{163}$$

where the last inequality follows from Fact 2.1.7. To construct the witness, consider the following. Fix any $C_3 \in \mathcal{DD}_3^{(2)}(B)$. For each $s \in [r_k]$, $C_3 \cap Q(s)$ is a 3-partite subgraph of C_3. Over all r_k of the subgraphs $C_3 \cap Q(s)$, we have an r_k-tuple of 3-partite subgraphs of C_3. Collecting all these subgraphs of C_3 and then letting C_3 run over the entire set $\mathcal{DD}_3^{(2)}(B)$ of ρ small triads, we have a ρr_k-tuple of 3-partite subgraphs of $G(B)$. Thus, our witness is the ρr_k-tuple $(C_3 \cap Q(s); C_3 \in \mathcal{DD}_3^{(2)}(B),\ s \in [r_k])$. To be formal, we assign an arbitrary numbering to this ρr_k-tuple. Let $\phi : \mathcal{DD}_3^{(2)}(B) \times [r_k] \longrightarrow [\rho r_k]$ be an arbitrary bijection. Consider the ρr_k-tuple of 3-partite subgraphs of $G(B)$ given by

$$\overrightarrow{Q'_B} = (Q'_B(z) : 1 \leq z \leq \rho r_k)$$

where for each $z \in [\rho r_k]$,

$$Q'_B(z) = C_3 \cap Q(s) \tag{164}$$

where $C_3 \in \mathcal{DD}_3^{(2)}(B)$, $s \in [r_k]$ and $\phi((C_3, s)) = z$. Note that

$$\mathcal{K}_3^{(2)}(Q'_B(z)) = \mathcal{K}_3^{(2)}(C_3) \cap \mathcal{K}_3^{(2)}(Q(s)).$$

Due to the inequality in (163),

$$\begin{aligned} |\bigcup_{z=1}^{\rho r_k} \mathcal{K}_3^{(2)}(Q'_B(z))| &= |\bigcup_{C_3 \in \mathcal{DD}_3^{(2)}(B)} \bigcup_{s=1}^{r_k} (\mathcal{K}_3^{(2)}(C_3) \cap \mathcal{K}_3^{(2)}(Q(s)))|, \\ &= |\bigcup_{C_3 \in \mathcal{DD}_3^{(2)}(B)} (\mathcal{K}_3^{(2)}(C_3) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s)))|, \\ &> \lceil 2\sqrt{\delta} t^3 l^3 \rceil \sqrt{\delta} \frac{m^3}{(\lambda l)^3}(1 - \theta_{3,\lambda l}(\epsilon_2)), \\ &\geq 2(1 - \theta_{3,\lambda l}(\epsilon_2)) \delta \frac{N^3}{\lambda^3}. \end{aligned} \tag{165}$$

Combining the inequality in (87) with Fact 2.1.7 ,(165) and (95), we conclude

$$2(1 - \theta_{3,\lambda l}(\epsilon_2)) \delta \frac{N^3}{\lambda^3} > \frac{18}{10} \delta \frac{N^3}{\lambda^3} > (1 + \theta_{3,l}(\epsilon)) \delta \frac{N^3}{\lambda^3} > \delta |\mathcal{K}_3^{(2)}(G(B))|.$$

We thus conclude

$$|\bigcup_{z=1}^{\rho r_k} \mathcal{K}_3^{(2)}(Q'_B(z))| > \delta |\mathcal{K}_3^{(2)}(G(B))|.$$

Since $\rho r_k = \lceil 2\sqrt{\delta} t^3 l^3 \rceil r_k \le 2 r_k T_0^3 L_0^3 \le r$, (cf., (97)), it follows from the (α, δ, r)-regularity of $\mathcal{H}(B)$ with respect to $G(B)$ that

$$|\mathcal{H} \cap \bigcup_{z=1}^{\rho r_k} \mathcal{K}_3^{(2)}(Q'_B(z))| > (\alpha - 2\delta) |\bigcup_{z=1}^{\rho r_k} \mathcal{K}_3^{(2)}(Q'_B(z))|,$$

or equivalently,

$$\frac{|\mathcal{H} \cap \bigcup_{z=1}^{\rho r_k} \mathcal{K}_3^{(2)}(Q'_B(z))|}{|\bigcup_{z=1}^{\rho r_k} \mathcal{K}_3^{(2)}(Q'_B(z))|} > \alpha - 2\delta. \tag{166}$$

Note that by definition in (164),

$$\begin{aligned} \frac{|\mathcal{H} \cap \bigcup_{z=1}^{\rho r_k} \mathcal{K}_3^{(2)}(Q'_B(z))|}{|\bigcup_{z=1}^{\rho r_k} \mathcal{K}_3^{(2)}(Q'_B(z))|} &= \frac{|\mathcal{H} \cap (\bigcup_{C_3 \in \mathcal{DD}_3^{(2)}(B)} (\mathcal{K}_3^{(2)}(C_3) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s))))|}{|\bigcup_{C_3 \in \mathcal{DD}_3^{(2)}(B)} (\mathcal{K}_3^{(2)}(C_3) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s)))|}, \\ &= \frac{|\bigcup_{C_3 \in \mathcal{DD}_3^{(2)}(B)} (\mathcal{H} \cap (\mathcal{K}_3^{(2)}(C_3) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s)))))|}{|\bigcup_{C_3 \in \mathcal{DD}_3^{(2)}(B)} (\mathcal{K}_3^{(2)}(C_3) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s)))|}. \end{aligned}$$

The crucial observation here is that with $\mathcal{H}^Q = \mathcal{H} \cap \bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s))$, it follows that for each $C_3 \in \mathcal{DD}_3^{(2)}(B)$,

$$\mathcal{H} \cap (\mathcal{K}_3^{(2)}(C_3) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s)) = \mathcal{H}^Q \cap (\mathcal{K}_3^{(2)}(C_3) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s))$$

holds. Thus, we may rewrite (166) as

$$\frac{|\mathcal{H}^Q \cap (\bigcup_{C_3 \in \mathcal{DD}_3^{(2)}(B)} (\mathcal{K}_3^{(2)}(C_3) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s))))|}{|\bigcup_{C_3 \in \mathcal{DD}_3^{(2)}(B)} (\mathcal{K}_3^{(2)}(C_3) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s)))|} > \alpha - 2\delta. \tag{167}$$

On the other hand, the following fact (which we prove momentarily) leads to a contradiction with (167).

Fact 7.5.5. *For each* $C_3 \in \mathcal{DD}_3^{(2)}(B)$,

$$\frac{|\mathcal{H}^Q \cap (\mathcal{K}_3^{(2)}(C_3) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s)))|}{|\mathcal{K}_3^{(2)}(C_3) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s))|} \le \alpha - 2\delta.$$

We prove Fact 7.5.5 immediately after we produce the promised contradiction to (167).

Indeed, by Fact 7.5.5, for each $C_3 \in \mathcal{DD}_3^{(2)}(B)$,

$$|\mathcal{H}^Q \cap (\mathcal{K}_3^{(2)}(C_3) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s)))| \le (\alpha - 2\delta) |\mathcal{K}_3^{(2)}(C_3) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s))|. \tag{168}$$

Summing (168) over all $C_3 \in \mathcal{DD}_3^{(2)}(B)$ yields

$$\sum_{C_3 \in \mathcal{DD}_3^{(2)}(B)} |\mathcal{H}^Q \cap (\mathcal{K}_3^{(2)}(C_3) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s)))| \le (\alpha - 2\delta) \sum_{C_3 \in \mathcal{DD}_3^{(2)}(B)} |\mathcal{K}_3^{(2)}(C_3) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s))|. \tag{169}$$

Note that for each $C_3, C_3' \in \mathcal{DD}_3^{(2)}(B)$, $C_3 \neq C_3'$, $\mathcal{K}_3^{(2)}(C_3) \cap \mathcal{K}_3^{(2)}(C_3') = \emptyset$. That is to say, the sets of triangles $\mathcal{K}_3^{(2)}(C_3)$ over distinct $C_3 \in \mathcal{DD}_3^{(2)}(B)$ are pairwise disjoint. Thus, it follows that

$$\sum_{C_3 \in \mathcal{DD}_3^{(2)}(B)} |\mathcal{H}^Q \cap (\mathcal{K}_3^{(2)}(C_3) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s)))| = |\bigcup_{C_3 \in \mathcal{DD}_3^{(2)}(B)} \mathcal{H}^Q \cap (\mathcal{K}_3^{(2)}(C_3) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s)))|,$$

and

$$\sum_{C_3 \in \mathcal{DD}_3^{(2)}(B)} |\mathcal{K}_3^{(2)}(C_3) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s))| = |\bigcup_{C_3 \in \mathcal{DD}_3^{(2)}(B)} \mathcal{K}_3^{(2)}(C_3) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s))|.$$

Thus, (169) may be rewritten as

$$|\bigcup_{C_3 \in \mathcal{DD}_3^{(2)}(B)} \mathcal{H}^Q \cap (\mathcal{K}_3^{(2)}(C_3) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s)))| \leq (\alpha - 2\delta) |\bigcup_{C_3 \in \mathcal{DD}_3^{(2)}(B)} \mathcal{K}_3^{(2)}(C_3) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s))|,$$

or equivalently,

$$\frac{|\mathcal{H}^Q \cap (\bigcup_{C_3 \in \mathcal{DD}_3^{(2)}(B)} (\mathcal{K}_3^{(2)}(C_3) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s))))|}{|\bigcup_{C_3 \in \mathcal{DD}_3^{(2)}(B)} (\mathcal{K}_3^{(2)}(C_3) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s)))|} \leq \alpha - 2\delta,$$

which is a direct contradiction with (167). This contradiction confirms that the assumption we made in (162) was false, hence, our proof of (161) is complete.

Proof of Fact 7.5.5.

Indeed, fix $C_3 \in \mathcal{DD}_3^{(2)}(B)$. Recall that as an element of $\mathcal{DD}_3$, C_3 satisfies that $\mathcal{H}_{C_3}^Q$ is (δ, r_1)-fully regular with respect to C_3, but $d_{\mathcal{H}_{C_3}^Q}(C_3) = \frac{|\mathcal{H}_{C_3}^Q \cap \mathcal{K}_3^{(2)}(C_3)|}{|\mathcal{K}_3^{(2)}(C_3)|} \leq \alpha - 3\delta$. As an element of $\mathcal{DD}_3^{(2)}(B)$,

$$\begin{aligned} |\mathcal{K}_3^{(2)}(C_3) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s))| &\geq \sqrt{\delta} |\mathcal{K}_3^{(2)}(C_3)|, \\ &> \delta |\mathcal{K}_3^{(2)}(C_3)|. \end{aligned}$$

Note that $r_k \leq r_1$ follows from (85). We submit to $\mathcal{H}_{C_3}^Q$ the "witness" $(C_3 \cap Q(s); s \in [r_k])$. We conclude from the (δ, r_1)-full regularity of $\mathcal{H}_{C_3}^Q$ that

$$\left| \frac{|\mathcal{H}_{C_3}^Q \cap (\mathcal{K}_3^{(2)}(C_3) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s)))|}{|\mathcal{K}_3^{(2)}(C_3) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s))|} - d_{\mathcal{H}_{C_3}^Q}(C_3) \right| < \delta,$$

which is equivalent to

$$\left| \frac{|\mathcal{H}^Q \cap (\mathcal{K}_3^{(2)}(C_3) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s)))|}{|\mathcal{K}_3^{(2)}(C_3) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s))|} - d_{\mathcal{H}_{C_3}^Q}(C_3) \right| < \delta.$$

Thus, with $d_{\mathcal{H}^Q_{C_3}}(C_3) \le \alpha - 3\delta$, we conclude

$$\frac{|\mathcal{H}^Q \cap (\mathcal{K}_3^{(2)}(C_3) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s)))|}{|\mathcal{K}_3^{(2)}(C_3) \cap \bigcup_{s=1}^{r_k} \mathcal{K}_3^{(2)}(Q(s))|} \le \alpha - 2\delta.$$

□

8. The Upper Bound of Theorem 3.1.1

We begin this section by reviewing Theorem 3.1.1.

8.1. **Review of Theorem 3.1.1.**

Let us recall the of our work in Theorem 3.1.1 (i.e. Setup 1).

Setup:

For a given integer k, set of nonnegative reals $\{\alpha_B : B \in [k]^3\}$, constants δ, λ, r, and ϵ, suppose triple system $\mathcal{H}$ and underlying graph G satisfy the following:

(i) $\mathcal{H}$ is a k-partite 3-cylinder with k-partition $(V_1, \dots, V_k)$, $|V_1| = \dots = |V_k| = n$.
(ii) $G = \bigcup_{1 \le i < j \le k} G^{ij}$ is an underlying (λ, ϵ, k)-cylinder.
(iii) For all $B \in [k]^3$, $\mathcal{H}(B)$ is (α_B, δ, r)-fully regular with respect to $G(B)$.

Recall the Statement of Theorem 3.1.1.

Theorem 8.1.1. *For all integers $k \ge 4$, for all sets of nonnegative reals $\{\alpha_B : B \in [k]^3\}$, for all $\beta > 0$, there exists $\delta > 0$ such that for all integers $\lambda > \frac{1}{\delta}$, there exist r and ϵ so that the following holds: whenever triple system $\mathcal{H}$ and underlying cylinder G satisfy the conditions of the Setup with parameters k, $\{\alpha_B : B \in [k]^3\}$, δ, λ, r and ϵ, then*

$$\frac{\Pi_{B\in[k]^3}\alpha_B}{\lambda^{\binom{k}{2}}} n^k(1-\beta) \le \left|\mathcal{K}_k^{(3)}(\mathcal{H})\right| \le \frac{\Pi_{B\in[k]^3}\alpha_B}{\lambda^{\binom{k}{2}}} n^k(1+\beta). \tag{170}$$

We proved the lower bound in (170) in Theorem 3.2.1. What remains to be shown for the proof of Theorem 8.1.1 is the upper bound of (170). We mentioned earlier that the upper bound in (170) follows from the lower bound in (170). For that purpose, we recall Theorem 3.2.1 as a summary of the lower bound in (170).

Theorem 8.1.2. *For all integers $k \ge 4$, for all sets of nonnegative reals $\{\alpha_B : B \in [k]^3\}$, for all $\beta > 0$, there exists $\delta > 0$ such that for all integers $\lambda > \frac{1}{\delta}$, there exist r and ϵ so that the following holds: whenever triple system $\mathcal{H}$ and underlying cylinder G satisfy the conditions of the Setup with parameters k, $\{\alpha_B : B \in [k]^3\}$, δ, λ, r and ϵ, then*

$$\left|\mathcal{K}_k^{(3)}(\mathcal{H})\right| \ge \frac{\Pi_{B\in[k]^3}\alpha_B}{\lambda^{\binom{k}{2}}} n^k(1-\beta). \tag{171}$$

To complete the proof of Theorem 8.1.1, it suffices to prove the following proposition, which establishes the upper bound in (170).

Proposition 8.1.3. *For all integers $k \geq 4$, for all sets of nonnegative reals $\{\alpha_B : B \in [k]^3\}$, for all $\beta > 0$, there exists $\delta > 0$ such that for all integers $\lambda > \frac{1}{\delta}$, there exist r and ϵ so that the following holds: whenever triple system $\mathcal{H}$ and underlying cylinder G satisfy the conditions of the Setup with parameters k, $\{\alpha_B : B \in [k]^3\}$, δ, λ, r and ϵ, then*

$$\left|\mathcal{K}_k^{(3)}(\mathcal{H})\right| \leq \frac{\Pi_{B\in[k]^3}\alpha_B}{\lambda^{\binom{k}{2}}} n^k(1+\beta). \tag{172}$$

Before we begin the proof of Proposition 8.1.3, we first sketch the easy idea behind it. For each $\mathcal{D} \subseteq [k]^3$, consider k-partite 3-cylinder $\mathcal{H}_{\mathcal{D}} = \bigcup_{B\in[k]^3} \mathcal{H}_{\mathcal{D}}(B)$, where for each $B \in [k]^3$,

$$\mathcal{H}_{\mathcal{D}}(B) = \begin{cases} \mathcal{H}(B) & \text{if } B \notin \mathcal{D} \\ \mathcal{K}_3^{(2)}(G(B)) \setminus \mathcal{H}(B) & \text{if } B \in \mathcal{D} \end{cases} \tag{173}$$

Clearly, $\mathcal{H} = \mathcal{H}_{\emptyset}$. We apply Theorem 8.1.2 to each $\mathcal{H}_{\mathcal{D}}$, $\mathcal{D} \neq \emptyset$, to conclude a lower bound on $|\mathcal{K}_k^{(3)}(\mathcal{H}_{\mathcal{D}})|$. Since $\bigcup_{\mathcal{D}\subseteq[k]^3} \mathcal{K}_k^{(3)}(\mathcal{H}_{\mathcal{D}}) = \mathcal{K}_k^{(2)}(G)$, and $|\mathcal{K}_k^{(2)}(G)| \sim \frac{n^k}{\lambda^{\binom{k}{2}}}$, we are able to infer an upper bound on $\left|\mathcal{K}_k^{(3)}(\mathcal{H})\right|$, and show it is given by (172).

8.2. **Proof of Proposition 8.1.3.**

We now give the definitions of the constants involved in Proposition 8.1.3.

Definitions of the Constants:

Let $k \geq 4$ be a given integer, let $\{\alpha_B : B \in [k]^3\}$ be a given set of nonnegative reals, and let $\beta > 0$ be a given constant. Define auxiliary positive constants

$$\begin{aligned} \beta' &= \beta'(k, \{\alpha_B : B \in [k]^3\}, \beta), && (174) \\ \theta &= \theta(k, \{\alpha_B : B \in [k]^3\}, \beta) && (175) \end{aligned}$$

to satisfy

$$\frac{\theta + \beta'}{\Pi_{B\in[k]^3}\alpha_B} - \beta' \leq \beta. \tag{176}$$

We now produce the constant δ promised by Proposition 8.1.3. Given the set $\{\alpha_B : B \in [k]^3\}$, define for every $\mathcal{D} \subseteq [k]^3$ auxiliary sets $\{\alpha_B^{(\mathcal{D})} : B \in [k]^3\}$ according to the following rule: for every $B \in [k]^3$,

$$\alpha_B^{(\mathcal{D})} = \begin{cases} \alpha_B & \text{if } B \notin \mathcal{D} \\ 1 - \alpha_B & \text{if } B \in \mathcal{D}. \end{cases} \tag{177}$$

For $k \geq 4$ given above, for $\mathcal{D} \subseteq [k]^3$ and set $\{\alpha_B^{(\mathcal{D})} : B \in [k]^3\}$, and constant β' in (174) given above, let

$$\delta_{\mathcal{D}} = \delta_{\mathcal{D}}(k, \{\alpha_B^{(\mathcal{D})} : B \in [k]^3\}, \beta')$$

be that constant guaranteed by Theorem 8.1.2. Set

$$\delta = min\{\delta_{\mathcal{D}} : \mathcal{D} \subseteq [k]^3\}. \tag{178}$$

This concludes our definition of the promised constant δ.

Let $\lambda > \frac{1}{\delta}$ be a given integer. For $\mathcal{D} \subseteq [k]^3$, let

$$\begin{aligned} r_{\mathcal{D}} &= r_{\mathcal{D}}(k, \{\alpha_B^{(\mathcal{D})} : B \in [k]^3\}, \beta', \lambda), \\ \epsilon_{\mathcal{D}} &= \epsilon_{\mathcal{D}}(k, \{\alpha_B^{(\mathcal{D})} : B \in [k]^3\}, \beta', \lambda) \end{aligned}$$

be those constants guaranteed to exist by Theorem 8.1.2. Set

$$r = max\{r_{\mathcal{D}} : \mathcal{D} \subseteq [k]^3\}. \tag{179}$$

We now determine the value ϵ guaranteed to exist by Proposition 8.1.3. For this, we need to recall Fact 2.1.7 of Section 2. Recall that Fact 2.1.7 states that for all integers k and λ and suitably small constants ϵ, there exists a function $\theta_{k,\lambda}(\epsilon)$, $\theta_{k,\lambda}(\epsilon) \to 0$ as $\epsilon \to 0$, which satisfies the conclusion of Fact 2.1.7. Therefore, for k and λ given above, let ϵ' be a positive constant so that the function $\theta_{k,\lambda}(\epsilon')$ guaranteed to exist by Fact 2.1.7 satisfies

$$\theta_{k,\lambda}(\epsilon') \leq \theta \tag{180}$$

where θ is that constant given in (175). Set

$$\epsilon = min\left\{\epsilon', min\{\epsilon_{\mathcal{D}} : \mathcal{D} \subseteq [k]^3\}\right\}. \tag{181}$$

This concludes our definitions of the constants guaranteed by Proposition 8.1.3. We now proceed to the proof of Proposition 8.1.3.

Proof of Proposition 8.1.3.

Let $k \geq 4$ be a given integer, let $\{\alpha_B : B \in [k]^3\}$ be a set of nonnegative reals, and let $\beta > 0$ be a given constant. Let $\delta > 0$ be given in (178). Let $\lambda > \frac{1}{\delta}$ be a given integer, and let r and ϵ be given in (179) and (181) respectively. Suppose $\mathcal{H}$ is a triple system and G is an underlying graph which with the parameters k, $\{\alpha_B : B \in [k]^3\}$, δ, λ, r and ϵ satisfy the hypothesis of the Setup. Our goal is to show

$$\left|\mathcal{K}_k^{(3)}(\mathcal{H})\right| \leq \frac{\Pi_{B\in[k]^3}\alpha_B}{\lambda^{\binom{k}{2}}} n^k(1+\beta).$$

Our strategy is to use the lower bound in (171) for each k-partite 3-cylinder $\mathcal{H}_{\mathcal{D}}$ defined in (173), where $\mathcal{D} \subseteq [k]^3$, $\mathcal{D} \neq \emptyset$.

For $\mathcal{D} \subseteq [k]^3$ and $B \in [k]^3$, recall that in (177) of the Definitions of the Constants, we defined

$$\alpha_B^{(\mathcal{D})} = \begin{cases} \alpha_B & \text{if } B \notin \mathcal{D} \\ 1-\alpha_B & \text{if } B \in \mathcal{D}. \end{cases}$$

Note that if $\mathcal{D} = \emptyset$, $\alpha_B^{(\mathcal{D})} = \alpha_B$ for all $B \in [k]^3$, and if $\mathcal{D} = [k]^3$, $\alpha_B^{(\mathcal{D})} = 1 - \alpha_B$ for all $B \in [k]^3$. We also note the following easy fact about the numbers $\alpha_B^{(\mathcal{D})}$.

Fact 8.2.1.

$$\sum_{\mathcal{D}\subseteq[k]^3} \Pi_{B\in[k]^3}\alpha_B^{(\mathcal{D})} = 1.$$

Proof of Fact 8.2.1.

Since

$$\sum_{\mathcal{D}\subseteq[k]^3} \Pi_{B\in[k]^3}\alpha_B^{(\mathcal{D})} = \Pi_{B\in[k]^3}(\alpha_B^{(\emptyset)} + \alpha_B^{([k]^3)})$$

where for all $B \in [k]^3$, $\alpha_B^{(\emptyset)} = \alpha_B$ and $\alpha_B^{([k]^3)} = 1 - \alpha_B$, we see that Fact 8.2.1 follows. □

We also note the following (deeper) observations:

(a) For each $\mathcal{D} \subseteq [k]^3$, the k-partite 3-cylinder $\mathcal{H}_{\mathcal{D}} = \bigcup_{B \in [k]^3} \mathcal{H}_{\mathcal{D}}(B)$ has underlying (λ, ϵ, k)-cylinder $G = \bigcup_{1 \le i < j \le k} G^{ij}$.

(b) For each $\mathcal{D} \subseteq [k]^3$ and each $B \in [k]^3$, the (α_B, δ, r)-full regularity of $\mathcal{H}(B)$ with respect to $G(B)$ implies that $\mathcal{H}_{\mathcal{D}}(B)$ is $(\alpha_B^{(\mathcal{D})}, \delta, r)$-fully regular. In particular, for all $\mathcal{D} \subseteq [k]^3$, $\mathcal{H}_{\mathcal{D}}$ and G satisfy the conditions of the Setup with parameters k, $\{\alpha_B^{(\mathcal{D})} : B \in [k]^3\}$, δ, λ, r, ϵ.

(c) For each $\mathcal{D} \subseteq [k]^3$, by our choices of δ, r and ϵ in (178), (179) and (181) respectively, Theorem 8.1.2 applies to $\mathcal{H}_{\mathcal{D}}$ and G with the parameters k, $\{\alpha_B^{(\mathcal{D})} : B \in [k]^3\}$, β', δ, λ, r, ϵ. We therefore conclude that for each $\mathcal{D} \subseteq [k]^3$,

$$|\mathcal{K}_k^{(3)}(\mathcal{H}_{\mathcal{D}})| \ge \frac{\Pi_{B \in [k]^3} \alpha_B^{(\mathcal{D})}}{\lambda^{\binom{k}{2}}} n^k (1 - \beta'). \tag{182}$$

We now combine our observation in (c) above with Fact 8.2.1. For $\mathcal{D}_1, \mathcal{D}_2 \subseteq [k]^3$, $\mathcal{D}_1 \neq \mathcal{D}_2$, $\mathcal{K}_k^{(3)}(\mathcal{H}_{\mathcal{D}_1}) \cap \mathcal{K}_k^{(3)}(\mathcal{H}_{\mathcal{D}_2}) = \emptyset$. Moreover, $\bigcup_{\mathcal{D} \subseteq [k]^3} \mathcal{K}_k^{(3)}(\mathcal{H}_{\mathcal{D}}) = \mathcal{K}_k^{(2)}(G)$. Therefore,

$$\sum_{\mathcal{D} \subseteq [k]^3} |\mathcal{K}_k^{(3)}(\mathcal{H}_{\mathcal{D}})| = |\mathcal{K}_k^{(2)}(G)|. \tag{183}$$

Since G is a (λ, ϵ, k)-cylinder, we apply Fact 2.1.7 to conclude

$$\left|\mathcal{K}_k^{(2)}(G)\right| \le \frac{n^k}{\lambda^{\binom{k}{2}}}(1 + \theta_{k,\lambda}(\epsilon)) \le \frac{n^k}{\lambda^{\binom{k}{2}}}(1 + \theta) \tag{184}$$

where the last inequality follows from our choice of ϵ in (181). Combining the inequalities in (182) and (184) in (183) yields

$$|\mathcal{K}_k^{(3)}(\mathcal{H}_{\emptyset})| + \sum_{\substack{\mathcal{D} \subseteq [k]^3 \\ \mathcal{D} \neq \emptyset}} \frac{\Pi_{B \in [k]^3} \alpha_B^{(\mathcal{D})}}{\lambda^{\binom{k}{2}}} n^k (1 - \beta') \le \frac{n^k}{\lambda^{\binom{k}{2}}}(1 + \theta)$$

or equivalently, with $\mathcal{H} = \mathcal{H}_{\emptyset}$,

$$\left|\mathcal{K}_k^{(3)}(\mathcal{H})\right| \frac{\lambda^{\binom{k}{2}}}{n^k(1 - \beta')} \le \frac{1 + \theta}{1 - \beta'} - \sum_{\substack{\mathcal{D} \subseteq [k]^3 \\ \mathcal{D} \neq \emptyset}} \Pi_{B \in [k]^3} \alpha_B^{(\mathcal{D})}. \tag{185}$$

Employing the equality in Fact 8.2.1 in (185) yields

$$\begin{aligned} \left|\mathcal{K}_k^{(3)}(\mathcal{H})\right| \frac{\lambda^{\binom{k}{2}}}{n^k(1 - \beta')} &\le \frac{1 + \theta}{1 - \beta'} - 1 + \Pi_{B \in [k]^3} \alpha_B^{(\emptyset)}, \\ &= \Pi_{B \in [k]^3} \alpha_B \left(1 + \frac{\theta + \beta'}{(1 - \beta')\Pi_{B \in [k]^3} \alpha_B}\right) \end{aligned}$$

or equivalently,

$$\left|\mathcal{K}_k^{(3)}(\mathcal{H})\right| \leq \frac{\Pi_{B\in[k]^3}\alpha_B}{\lambda^{\binom{k}{2}}} n^k \left(1-\beta'+\frac{\theta+\beta'}{\Pi_{B\in[k]^3}\alpha_B}\right). \tag{186}$$

Since we chose constants β' and θ to satisfy the inequality in (176), we conclude from (186) that

$$\left|\mathcal{K}_k^{(3)}(\mathcal{H})\right| \leq \frac{\Pi_{B\in[k]^3}\alpha_B}{\lambda^{\binom{k}{2}}} n^k(1+\beta)$$

and hence our proof of Proposition 8.1.1 is complete. □

9. Concluding Remarks

In this section, we make brief remarks concerning Theorem 3.1.1. To begin, we consider the following slight generalization of The Counting Lemma. To discuss this statement, we define an appropriate context.

Setup:

For a given integer $k \geq 3$, for a fixed triple system $\mathcal{J}_0$ on vertex set $[k]$, for given constants α, δ, λ, r and ϵ, suppose $\mathcal{H}$ is a triple system and G is an underlying graph satisfying the following:

(*i*) $\mathcal{H}$ is a k-partite 3-cylinder with k-partition $(V_1,\dots,V_k)$, $|V_1|=\ldots=|V_k|=m$.
(*ii*) G is an underlying (λ,ϵ,k)-cylinder of $\mathcal{H}$.
(*iii*) for all $B\in\mathcal{J}_0$, $\mathcal{H}(B)$ is (α,δ,r)-fully regular with respect to the triad $G(B)$ (cf., Definition 2.2.7).

Denote by $\binom{\mathcal{H}}{\mathcal{J}_0}$ the set of copies of the hypergraph $\mathcal{J}_0$ in the hypergraph $\mathcal{H}$. We define the following special set $\mathcal{S}_{\mathcal{J}_0}(\mathcal{H})\subseteq\binom{\mathcal{H}}{\mathcal{J}_0}$ of copies of $\mathcal{J}_0$ in $\mathcal{H}$ as

$$\mathcal{S}_{\mathcal{J}_0}(\mathcal{H})=\left\{\mathcal{J}\in\binom{\mathcal{H}}{\mathcal{J}_0} : [V(\mathcal{J})]^2\subseteq G\right\}.$$

Then the following may be proved using Theorem 3.1.1.

Theorem 9.0.2. *For all integers $k\geq 4$ and triple systems $\mathcal{J}_0\subseteq[k]^3$, for all $\alpha,\beta>0$, there exists a constant $\delta>0$ so that for all integers $\lambda\geq\frac{1}{\delta}$, there exists an integer r and $\epsilon>0$ so that whenever triple system $\mathcal{H}$ and $G=\bigcup_{1\leq i<j\leq k}G^{ij}$ satisfy the conditions of the Setup above with constants k, α, δ, λ, r and ϵ and triple system $\mathcal{J}_0$, then*

$$\frac{\alpha^{|\mathcal{J}_0|}}{\lambda^{\binom{k}{2}}}m^k(1-\beta)\leq|\mathcal{S}_{\mathcal{J}_0}(\mathcal{H})|\leq\frac{\alpha^{|\mathcal{J}_0|}}{\lambda^{\binom{k}{2}}}m^k(1+\beta).$$

Note that when $\mathcal{J}_0=K_k^{(3)}$, Theorem 9.0.2 reduces to Theorem 3.1.1. An application of Theorem 9.0.2 appears in [3].

The following theorem in [7] was proved using the Hypergraph Regularity Lemma of [2] in tandem with Theorem 3.1.1. Let a set of triple systems $\{\mathcal{F}_i\}_{i\in I}$ be given and define $\mathcal{P}_n$ to be the set of all 3-uniform hypergraphs on vertex set $[n]$ not containing any

$\mathcal{F} \in \{\mathcal{F}_i\}_{i\in I}$ as a subhypergraph. Denote by $\mathrm{ex}(n, \{\mathcal{F}_i\}_{i\in I})$ the maximum size of a triple system in $\mathcal{P}_n$. Then

$$|\mathcal{P}_n| = 2^{\mathrm{ex}(n,\{\mathcal{F}_i\}_{i\in I})+o(n^3)}. \tag{187}$$

Note that (187) extends the result of Theorem 1.1.4 for triple systems. A recent extension of (187) in [4] provides a similar identity for $|\mathcal{P}'_n|$, where $\mathcal{P}'_n$ is defined as the set of triple systems on vertex set $[n]$ containing no $\mathcal{F} \in \{\mathcal{F}_i\}_{i\in I}$ as an *induced* subhypergraph.

Finally, in the upcoming paper [9], V. Rödl and J. Skokan developed a method for counting copies of $K_5^{(4)}$, the hypergraph consisting of 5 points and 5 quadruples, in an appropriate but quite technical environment. We remark that their environment is a variation of Setup 1, the environment which hosted Theorem 3.1.1. An alternative (and hopefully simpler) method for proving Theorem 3.1.1 is currently being explored in [8].

Graph Terminology

Graph Notation

Hypergraph Terminology

Hypergraph Notation

Partitions

Miscellaneous Notation

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