Maximum Weight Independent Sets and Matchings in Sparse Random Graphs. Exact Results using the Local Weak Convergence Method

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Abstract

Let G(n,c/n) and $G_r(n)$ be an n-node sparse random graph and a sparse random r-regular graph, respectively, and let $\mathcal{I}(n,r)$ and $\mathcal{I}(n,c)$ be the sizes of the largest independent set in G(n,c/n) and $G_r(n)$. The asymptotic value of $\mathcal{I}(n,c)/n$ as $n \to \infty$, can be computed using the Karp-Sipser algorithm when $c \le e$. For random cubic graphs, r = 3, it is only known that $.432 \le \liminf_n \mathcal{I}(n,3)/n \le \limsup_n \mathcal{I}(n,3) \le .4591$ with high probability (w.h.p.) as $n \to \infty$, as shown in [FS94] and [Bol81], respectively.

In this paper we assume in addition that the nodes of the graph are equipped with non-negative weights, independently generated according to some common distribution, and we consider instead the maximum weight of an independent set. Surprisingly, we discover that for certain weight distributions, the limit $\lim_n \mathcal{I}(n,c)/n$ can be computed exactly even when c > e, and $\lim_n \mathcal{I}(n,r)/n$ can be computed exactly for some $r \geq 2$. For example, when the weights are exponentially distributed with parameter 1, $\lim_n \mathcal{I}(n,2e)/n \approx .5517$, and $\lim_n \mathcal{I}(n,3)/n \approx .6077$. Our results are established using the recently developed local weak convergence method further reduced to a certain local optimality property exhibited by the models we consider. Using the developed technique we show in addition that in the unweighted case $\lim_n \mathcal{I}(n,4)/n \geq .3533$, which is a new lower bound. We also prove that in any (non-random) graph with degree 3 and large girth, the size of the maximum independent set is at least .3923n - o(n), improving the previous bound (7/18)n - o(n) in [HS82]. Finally, we extend our results to maximum weight matchings in G(n,c/n) and $G_r(n)$. For the case of exponential distributions, we compute the corresponding limits for every c > 0 and every $r \geq 2$.

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1 Introduction

Two models of random graphs considered in this paper are a sparse random graph G(n, c/n) and a sparse random regular graph $G_r(n)$. The first is a graph on n nodes $\{0, 1, \ldots, n-1\} \equiv [n]$, where each potential undirected edge $(i, j), 0 \leq i < j \leq n-1$ is present in the graph with probability c/n, independently for all n(n-1)/2 edges. Here c>0 is a fixed constant, independent of n. A random r-regular graph $G_r(n)$ is obtained by fixing a constant integer $r \geq 2$ and considering a graph selected uniformly at random from the space of all r-regular graphs on n nodes (graphs in which every node has degree r). A set of nodes V in a graph $G_r(n)$ is defined to be an independent set if no two nodes of V are connected by an edge. Let $\mathcal{I}(n,c)$ and $\mathcal{I}(n,r)$ denote the maximum cardinality of an independent set in G(n,c/n) and $G_r(n)$ respectively. Suppose the nodes of a graph are equipped with some non-negative weights $W_i, 0 \leq i \leq n-1$ which are generated independently according to some common distribution $F_w(t) = \mathbb{P}\{W_i \leq t\}, t \geq 0$. Let $\mathcal{I}_w(n,c),\mathcal{I}_w(n,r)$ denote maximum weight of an independent set in G(n,c/n) and $G_r(n)$ respectively.

A matching is a set of edges A in a graph G such that every node is incident to at most one edge in A. Let $\mathcal{M}(n,c)$ and $\mathcal{M}(n,r)$ denote the maximum cardinality of a matching in G(n,c/n) and $G_r(n)$, respectively. It is known that $G_r(n), r \geq 3$ has a full matching w.h.p., that is $\mathcal{M}(n,r) = n/2$ ($\lfloor n/2 \rfloor$ for odd n) w.h.p. [JLR00]. If the edges of the graph are equipped with some non-negative random weights, then we consider instead the maximum weight of a matching $\mathcal{M}_w(n,c)$ and $\mathcal{M}_w(n,r)$ in graphs $G(n,c/n), G_r(n)$, respectively. The computation of $\mathcal{I}_w(n,c), \mathcal{I}_w(n,r), \mathcal{M}_w(n,c), \mathcal{M}_w(n,r)$ in the limit as $n \to \infty$ is the main subject of the present paper.

The asymptotic values of $\mathcal{I}(n,c)$ for $c \leq e$ and $\mathcal{M}(n,c)$ for all c were obtained by Karp and Sipser using a simple greedy type algorithm in [KS81]. Yet it is an open problem to compute

the corresponding limit for for independent sets for the case c > e or in random regular graphs or even to show that the limit exists [Ald].

The developments in this paper show that, surprisingly, proving the existence and the computation of the limits $\lim_n \mathcal{I}(n,\cdot)/n$, $\lim_n \mathcal{M}(\cdot)/n$ is easier in the weighted case than in the unweighted case, at least for certain weight distributions. In particular, we compute the limits for independent sets in $G_r(n)$, r = 2, 3, 4 and G(n, c/n), $c \leq 2e$, when the node weights are exponentially distributed, and we compute the limits for matchings in $G_r(n)$ and G(n, c/n) for all r, c, when the edge weights are exponentially distributed. It was shown by the first author [Gam02] that the limit $\lim_n \mathcal{M}(r,c)/n$ exists for every weight distribution with bounded support, though the non-constructive methods employed prevented the computation of the limits.

Our method of proof is based on a powerful local weak convergence method developed by Aldous [Ald92], [Ald01], Aldous and Steele [AS03], Steele [Ste02], further empowered by a certain local optimality observation derived in this paper. Local weak convergence is a recursion technique based on fixed points of distributional equations, which allows one to compute limits of some random combinatorial structures, see Aldous and Bandyopadhyay [AB] for a recent survey on applications of distributional equations and Aldous and Steele [AS03] for a survey on the local weak convergence method. In particular, the method is used to compute maximum weight matching on a random tree, when the weights are exponentially distributed. The tree structure was essential in [AS03] for certain computations and the approach does not extend directly to graphs like G(n,c/n) with c>1, where the convenience of a tree structure is lost due to the presence of a giant component. It was conjectured in [AS03] that a some long-range independence property might be helpful to deal with this difficulty. The present paper partially answers this qualitative conjecture in a positive way. We introduce a certain operator T acting on the space of distribution functions. We prove a certain local optimality property stating that, for example, for independent sets, whether a given node i belongs to the maximum weight independent set is asymptotically independent from the portion of the graph outside a constant size neighborhood of i, iff T^2 has a unique fixed point distribution. Moreover, when T^2 does have the unique fixed point, the size of the extremal object (say maximum weight independent set) can be derived from a fixed point of an operator T. The computations of fixed points is tedious, but simple in principle and the groundwork for that was already done in [AS03]. We hope that the long-range independence holds in other random combinatorial structures as well.

The issue of long-range independence of random combinatorial objects is addressed in a somewhat different, statistical physics context in Mossel [Mos03], Brightwell and Winkler [BW03], Rozikov and Suhov [RS03], Martin [Mar03], where independent sets (hard-core model) are considered on infinite regular trees, weighted by the Gibbs measure. The long-range interaction between nodes at a large distance is investigated with respect to this measure using the notion of reconstruction. It would be interesting to investigate the connections between the two models.

In a different setting Talagrand [Tal03] proves a certain long-range independence property for the random assignment problem, where the usual min-weight matching is replaced by a partition function on the space of feasible matchings. He uses a rigorous mathematical version of the cavity method, which originated in physics, to prove that the spins (edges of the matching) are asymptotically independent as the size of the problem increases. The particular form of the longrange independence is similar to the one we obtain, refer to Theorem 5 below, and in fact the cavity method, which is based on "knocking" out certain spins from the system and analyzing the relative change of the size of the extremal object, has some similarity with the local weak convergence method, which is also based on considering extremal objects (say independent sets) with one or several nodes excluded. It seems worth investigating whether there is a formal connection between the two methods. Finally, we refer the reader to Hartmann and Weigt [HW01] who derive the same result as Karp and Sipser for independent sets using non-rigorous arguments from statistical physics.

The rest of the paper is organized as follows. In the following section we describe some prior results on maximum independent sets and matchings in random graphs. Our main theorems are given in Section 3. The operator T, fixed point equations and long-range independence issues are discussed in Section 4. The main results are proven in Section 5. Some conclusions are in Section 6

We finish this section with some notational conventions. $\text{Exp}(\mu), \text{Pois}(\lambda), \text{Be}(p)$ denote respectively exponential, Poisson and Bernoulli distributions with parameters $\mu, \lambda > 0, 0 \le p \le 1$.

2 Prior work and open questions

It is known and simple to prove that $\mathcal{I}(n,r) = \mathcal{I}(n,c) = \Theta(n)$ w.h.p. for any constants $r \geq 1, c > 0$. Moreover, it is known that, w.h.p., $6\log(3/2) - 2 = .432... \leq \liminf_n \mathcal{I}(n,3)/n \leq \limsup_n \mathcal{I}(n,3)/n \leq .4591$. The lower bound is due to Frieze and Suen [FS94], and the upper bound is due to Bollobas [Bol81]. The upper bound is generalized for any $r \geq 0$ and uses a very ingenious construction of random regular graphs via matching and random grouping, [Bol80], [JLR00].

It is natural to expect that the following is true, which unfortunately remains only a conjecture, appearing in several places, most recently in [Ald] and [AS03].

Conjecture 1 For every c > 0 and $r \ge 3$ the limits

$$\lim_{n \to \infty} \frac{\mathbb{E}[\mathcal{I}(n,c)]}{n}, \qquad \lim_{n \to \infty} \frac{\mathbb{E}[\mathcal{I}(n,r)]}{n}$$

exist.

The existence of these limits also implies the convergence to the same limits w.h.p. by applying Azuma's inequality, see [JLR00] for the statement and the applicability of this inequality.

The limit $\lim_n \mathbb{E}\mathcal{I}(n,c)/n$ is known to exist for $c \leq e$ and can be computed using the Karp-Sipser [KS81] algorithm for maximum matching $\mathcal{M}(n,c)$. We describe the algorithm first in high level terms, and then state the result and its implications to maximum independent sets. The algorithm proceeds in two stages. In the first stage any leaf v in the graph G(n,c/n) is selected. The edge incident to this leaf is selected into a matching, and all the other edges incident to the parent of v are deleted. This is repeated until no leaves are left. In the second stage simply the largest matching M^* in the remaining graph $G^* \subset G$ is selected and added to the matching constructed in first stage. It is a simple exercise to prove that the resulting matching is optimal. Karp and Sipser prove the following result about the size $\mathcal{M}(KS)$ of the obtained matching.

Theorem 1 (Karp and Sipser [KS81]) The constructed matching is optimal, $\mathcal{M}(KS) = \mathcal{M}(n, c)$ with probability one. Moreover,

1. The maximum matching satisfies

(1)
$$\lim_{n \to \infty} \frac{\mathbb{E}[\mathcal{M}(n,c)]}{n} = 1 - \frac{\gamma^*(c) + \gamma^{**}(c) + c\gamma^*(c)\gamma^{**}(c)}{2},$$

where $\gamma^*(c)$ is the smallest solution of the equation $x = \exp(-c \exp(-cx))$ and $\gamma^{**}(c) = \exp(-c\gamma^*(c))$.

2. When $c \leq e$, the equation $x = \exp(-c\exp(-cx))$ has a unique solution $\gamma(c)$, and $|G^*| = o(n)$. That is the matching constructed in the first stage is asymptotically optimal as $n \to \infty$.

The different behavior for $c \leq e$ and c > e is called e-cutoff phenomena. Theorem 1 was strengthened later by by Aronson, Frieze and Pittel [APF98], who obtain bounds on convergence (1). It is the second part of the theorem above that can be used for the analysis of $\mathcal{I}(n,c)$. Observe, that in the first stage of the Karp-Sipser algorithm for every selected leaf if one takes a parent of this leaf instead of an edge between the leaf and its parent, one obtains a node set which is an edge-cover set in the graph $G \setminus G^*$ constructed in the first stage. That is every edge in $G \setminus G^*$ is incident to at least one node in this cover. It is a simple exercise to prove that this is in fact a minimum node cover in $G \setminus G^*$. Its complement is a maximum independent set in $G \setminus G^*$. Since $|G^*| = o(n)$ when $c \leq e$, one obtains then the following result.

Corollary 1 When $c \leq e$:

(2)
$$\lim_{n \to \infty} \frac{\mathbb{E}[\mathcal{I}(n,c)]}{n} = \frac{2\gamma(c) + \gamma^2(c)}{2}.$$

The Karp-Sipser algorithm hinges strongly on working with leaves and thus is not applicable to random regular graphs. Moreover, if the edges or the nodes of the graph G(n, c/n) are equipped with weights then the Karp-Sipser algorithm clearly can produce a strictly suboptimal solution and cannot be used in our setting of weighted nodes and edges. Also, when the edges of $G_r(n)$ are equipped with weights, the problem of computing maximum weight matching becomes non-trivial, as opposed to the unweighted case when the full matching exists w.h.p.

In a somewhat different domain of extremal combinatorics the following result was established by Hopkins and Staton [HS82]. A girth of a graph is the size of the smallest cycle. It is shown in [HS82] that the size of a largest independent set in an n-node graph with largest degree 3 and large girth is asymptotically at least (7/18)n - o(n). The techniques we employ in this paper allow us to improved this lower bound.

3 Main results

We begin by introducing the key technique for our analysis – recursive distributional equations and its fixed point solutions. This technique was introduced by Aldous [Ald92], [Ald01] in the context of solving $\zeta(2)$ limit conjecture for random minimal assignment problem, and was further developed in Aldous and Steele [AS03], Steele [Ste02], Aldous and Bandyopadhyay [AB], Gamarnik [Gam02]. Let W be a non-negative random variable with a distribution function $F_w(t) = \mathbb{P}(W \leq t)$. We consider four operators $T = T_{\mathcal{I},r}, T_{\mathcal{I},c}, T_{\mathcal{M},r}, T_{\mathcal{M},c}$ acting on the space of distribution functions $F(t), t \geq 0$, where c > 0 is a fixed constant and $r \geq 2$ is a fixed integer.

1. Given W distributed according to F_w (we write simply $W \sim F_w$), and given a distribution function F = F(t), let $B_1, B_2, \ldots, B_r \sim F$ be generated independently. Then $T_{\mathcal{I},r}: F \to F'$, where F' is the distribution function of B' defined by

(3)
$$B' = \max(0, W - \sum_{1 \le i \le r} B_i).$$

2. Under the same setting as above, let $B_1, \ldots, B_m \sim F$, where m is a random variable distributed according to a Poisson distribution with parameter c, independently from W, B_i . Then $T_{\mathcal{I},c}: F \to F'$, where F' is the distribution function of B' defined by

(4)
$$B' = \max(0, W - \sum_{1 \le i \le m} B_i),$$

when $m \ge 1$ and B' = W when m = 0. For simplicity we will identify the sum above with zero when m = 0.

3. Let $W_1, \ldots, W_r \sim F_w$, $B_1, \ldots, B_r \sim F$. Then $T_{\mathcal{M},r} : F \to F'$, where F' is the distribution function of B' defined by

(5)
$$B' = \max_{1 \le i \le r} (0, W_i - B_i).$$

4. Finally, let $W_1, \ldots, W_m \sim F_w, B_1, \ldots, B_m \sim F$, where $m \sim \text{Pois}(c)$, independent from W_i, B_i . Then $T_{\mathcal{M},c}: F \to F'$, where F' is the distribution function of B' defined by

(6)
$$B' = \max_{1 \le i \le m} (0, W_i - B_i),$$

when $m \ge 1$ and B' = 0 when m = 0. Again, for simplicity, we assume that max expression above is zero when m = 0.

A distribution function F is defined to be a fixed point distribution of an operator T if T(F) = F.

We now state the main result of this paper. Recall, that a distribution function F(t) is defined to be continuous (atom free) if for every x in its support $\lim_{\epsilon \to 0} (F(x+\epsilon) - F(x-\epsilon)) = 0$. Equivalently, for $B \sim F$ and every x, $\mathbb{P}(B = x) = 0$. We use $1\{\cdot\}$ to denote the indicator function.

Theorem 2 Let F_w be a continuous non-negative distribution function. For $r \geq 1$ if the operator $T^2_{\mathcal{I},r-1}$ has a unique fixed point distribution function F^* , then, w.h.p.

(7)
$$\lim_{n} \frac{\mathcal{I}_{w}(n,r)}{n} = \mathbb{E}[W \, 1\{W - \sum_{1 \le i \le r} B_{i} > 0\}],$$

where $W \sim F_w$, $B_i \sim F^*$, and W, B_i are independent. When $G_r(n)$ is replaced by G(n, c/n), the same result holds for $T = T_{\mathcal{I},c}$, except the sum in the right-hand side of (7) is $\sum_{1 \leq i \leq m} B_i$ and $m \sim \text{Pois}(c)$.

Finally, the similar results hold for $\mathcal{M}_w(n,r)$ and $\mathcal{M}_w(n,c)$ in $G_r(n)$ and G(n,c), for $T=T_{\mathcal{M},r-1}$ and $T=T_{\mathcal{M},c}$, respectively, whenever the corresponding operator T is such that T^2 has the unique fixed point distribution F^* . The corresponding limits are

(8)
$$\lim_{n} \frac{\mathcal{M}_{w}(n,r)}{n} = \frac{1}{2} \mathbb{E} \left[\sum_{1 \le i \le r} W_{i} 1\{W_{i} - B_{i} = \max_{1 \le j \le r} (W_{j} - B_{j}) > 0 \} \right],$$

where $W_i \sim F_w$, $B_i \sim F^*$, and

(9)
$$\lim_{n} \frac{\mathcal{M}_{w}(n,c)}{n} = \frac{1}{2} \mathbb{E}\left[\sum_{i \le m} W_{i} 1\{W_{i} - B_{i} = \max_{j \le m} (W_{j} - B_{j}) > 0\}\right],$$

where $W_i \sim F_w, B_i \sim F^*, m \sim \text{Pois}(c)$.

For $G = G_r(n), G(n, c/n)$ with the weights on nodes given by a distribution function F_w , let $\mathcal{IN}_w(n,r), \mathcal{IN}_w(n,c)$ denote the cardinality of the maximum weight independent set in G which achieves the maximum weight. In case F_w is continuous, the maximum weight independent set is uniquely defined, so $\mathcal{IN}_w(n,r), \mathcal{IN}_w(n,c)$ are well-defined as well. Clearly, $\mathcal{I}(n,r) \geq \mathcal{IN}_w(n,r), \mathcal{I}(n,c) \geq \mathcal{IN}_w(n,c)$. $\mathcal{MN}_w(n,r)$ and $\mathcal{MN}_w(n,c)$ are defined similarly. When $T_{\mathcal{I},r}^2$ or $T_{\mathcal{I},r}^2$ have the unique fixed point it is also possible to compute asymptotically $\mathcal{IN}_w(n,r), \mathcal{IN}_w(n,c), \mathcal{MN}_w(n,r), \mathcal{MN}_w(n,c)$.

Corollary 2 Under the setting of Theorem 2, w.h.p.

(10)
$$\lim_{n} \frac{\mathcal{IN}_{w}(n,r)}{n} = \mathbb{E}[1\{W - \sum_{1 \le i \le r} B_i > 0\}],$$

where $W \sim F_w$, $B_i \sim F^*$, W, B_i independent, and F^* is the unique fixed point distribution of $T^2 = T_{\mathcal{T} r-1}^2$, and

(11)
$$\lim_{n} \frac{\mathcal{MN}_{w}(n,r)}{n} = \frac{1}{2} \mathbb{E}[1\{\max_{1 \le j \le r} (W_{j} - B_{j}) > 0\}],$$

where W_i , $\sim F_w$, $B_i \sim F^*$, W_i , B_i independent, and F^* is the unique fixed point distribution of $T^2 = T^2_{\mathcal{M},r-1}$ and W, B_i are independent,

Similar results hold for $T = T_{\mathcal{I},c}, T_{\mathcal{M},c}$ where again in the sum $\sum_{1 \leq j \leq r}$ we substitute r with random $m \sim \text{Pois}(c)$.

The Theorem 2 is the core result of this paper. It will allow us to obtain several interesting corollaries, which we state below.

Theorem 3 Suppose the weights of the nodes and edges of the graphs $G = G_r(n)$ and G = G(n, c) are distributed as Exp(1). Then

1. $T_{\mathcal{I},r-1}^2$ has a unique fixed point distribution F^* iff $r \leq 4$. In this case, w.h.p.

(12)
$$\lim_{n} \frac{\mathcal{I}_{w}(n,r)}{n} = \frac{(1-b)(r-rb+2b+2)}{4},$$

where b is the unique solution of $b = 1 - (\frac{1+b}{2})^{r-1}$. In particular, w.h.p.

(13)
$$\lim_{n} \frac{\mathcal{I}_{w}(n,2)}{n} = \frac{2}{3}, \qquad \lim_{n} \frac{\mathcal{I}_{w}(n,3)}{n} \approx .6077, \qquad \lim_{n} \frac{\mathcal{I}_{w}(n,4)}{n} \approx .4974,$$

2. $T_{\mathcal{I},c}^2$ has a unique fixed point distribution F^* iff $c \leq 2e$. In this case, w.h.p.

(14)
$$\lim_{n} \frac{\mathcal{I}_{w}(n,c)}{n} = (1-b)(1 + \frac{c(1-b)}{4}),$$

where b is the unique solution of $1 - b = e^{-\frac{c}{2}(1-b)}$. In particular, when c = 2e, this limit is $\approx .5517$.

3. $T^2_{\mathcal{M},r-1}$ has a unique fixed point F^* for every $r \geq 2$. Moreover, w.h.p.

$$\lim_{n} \frac{\mathcal{M}_{w}(n,r)}{n} = r(b^{r-1}+1) \int_{0}^{\infty} t e^{-t} (1-e^{-t}(1-b))^{r-1} dt - r \int_{0}^{\infty} t e^{-t} (1-e^{-t}(1-b))^{2r-2} dt,$$

where b is the unique solution of $b = 1 - \frac{1 - b^r}{r(1 - b)}$

4. $T_{\mathcal{M},c}^2$ has a unique fixed point F^* for all c>0. Moreover, for every c>0 w.h.p.

(16)
$$\lim_{n} \frac{\mathcal{M}_{w}(n,c)}{n} = \frac{c}{2} (e^{cb-c} + 1) \int_{0}^{\infty} t e^{-t-c(1-b)e^{-t}} dt - \frac{c}{2} \int_{0}^{\infty} t e^{-t-2c(1-b)e^{-t}+c(1-b)^{2}e^{-2t}} dt,$$

where b is the unique solution of $1 - e^{-cb} = c(1 - b)^2$.

The result above generalizes to the case when Exp(1) is replaced by $\text{Exp}(\mu)$ for any $\mu > 0$, since for any $\alpha > 0$ and $W \sim \text{Exp}(1)$, $\alpha W \sim \text{Exp}(1/\alpha)$. The expression in (15) involving integrals is similar to the one found in [AS03] for maximum weight matching on a tree. It is a pleasant surprise, though, that the answers for independent sets are derived in closed form.

Part 2 of Theorem 3 leads to an interesting phase transition behavior. For $F_w = \text{Exp}(1)$ our result says that the value of c = 2e is a phase transition point for the operator $T_{\mathcal{I},c}^2$ where for $c \leq 2e$ the operator has a unique fixed point distribution, but for c > 2e the fixed point distribution is not unique. Contrast this with e-cutoff phenomena described above. It turns out (see Theorems 9, 10 below) that this phase transition is directly related to some long range independence/dependence property of the maximum weight independent sets in the underlying graph G(n, c/n). Curiously, no such phase transition occurs for maximum weight matchings.

As a sanity check, let us show directly the validity of (12) for the case r = 1. The answer given by the formula is 3/4. It is, though, easy to compute $\mathcal{I}_w(n,1)$ exactly. The graph is a collection of n/2 isolated edges. For each edge we simply select the the incident node with larger

weight, which by memoryless property of Exp is 1/2+1=3/2. The limit $\mathcal{I}_w(n,1)/n \to 3/4$ then checks.

Note that $G_2(n)$ can be represented as a collection of disjoint cycles on n elements. W.h.p. these cycles have decreasing lengths, starting from $\Theta(n^{\frac{1}{2}})$. Theorems 2 and 3 also hold also for a simpler model of a 2-regular graph -n-cycle. The n nodes $0, 1, \ldots, n-1$ and edges $(0, 1), \ldots, (n-2, n-1), (n-1, 0)$ of this cycle are assumed to have weights distributed according to some distribution function F_w . Let $I_w(n, \text{cycle})$ and $M_w(n, \text{cycle})$ denote respectively the maximum weight of an independent set and the maximum weight of a matching.

Corollary 3 Suppose $T_{\mathcal{I},2}^2$ has a unique fixed point. Then (7), (8) hold when r=2 and $\mathcal{I}_w(n, \text{cycle})$ replaces $\mathcal{I}_w(n,2)$ and $\mathcal{M}_w(n, \text{cycle})$ replaces $\mathcal{M}_w(n,2)$. When $F_w = \text{Exp}(1)$, w.h.p.

(17)
$$\lim_{n} \frac{\mathcal{I}_{w}(n, \text{cycle})}{n} = \lim_{n} \frac{\mathcal{M}_{w}(n, \text{cycle})}{n} = \frac{2}{3}.$$

We can also compute the cardinality of independent sets which achieve the maximum weight, when $F_w = \text{Exp}(1)$.

Corollary 4 When $F_w = \text{Exp}(1)$, w.h.p.

(18)
$$\liminf_{n} \frac{\mathcal{I}(n,2)}{n} \ge \lim_{n} \frac{\mathcal{I}\mathcal{N}_{w}(n,2)}{n} = \frac{4}{9}, \qquad \liminf_{n} \frac{\mathcal{I}(n,3)}{n} \ge \lim_{n} \frac{\mathcal{I}\mathcal{N}_{w}(n,3)}{n} \approx .3923,$$
$$\liminf_{n} \frac{\mathcal{I}(n,4)}{n} \ge \lim_{n} \frac{\mathcal{I}\mathcal{N}_{w}(n,4)}{n} \approx .3533.$$

The last part of the corollary above implies a new lower bound on the size of the largest independent sets in a random 4-regular graph.

Corollary 5 The cardinality of the largest independent set in $G_4(n)$ is at least .3533n - o(n).

We note, however, that for r=3 our lower bound .3923n-o(n) is weaker than .432n-o(n) established in [FS94]. Yet, it is still very useful since the approach allows us the improve the following result of Hopkins and Staton [HS82]. Let $\mathcal{G}(n,r,d)$ denote the class of all (non-random) graphs on n nodes, with maximum degree r and girth at least d. For any $G \in \mathcal{G}(n,r,d)$ let $\mathcal{I}(G)$ denote the size of the largest independent set in G. Hopkins and Staton proved that $\lim\inf_{n,d}\min_{G\in\mathcal{G}(n,3,d)}\mathcal{I}(G)/n\geq 7/18\approx .3887$. Our techniques allow us to obtain the following improvement.

Theorem 4

$$\liminf_{n,d} \min_{G \in \mathcal{G}(n,3,d)} \frac{\mathcal{I}(G)}{n} \ge .3923.$$

An important implication of the uniqueness of the fixed point distribution of T^2 for the types of T described above, is that the uniqueness implies a certain long-range independence properties of the structures we consider. The following theorem makes this notion precise. While the theorem is not used directly in this paper, we believe it is interesting by itself. Below G is again one of the graphs $G_r(n)$ or G(n,c). Let E denote the (random) edge set of G.

Theorem 5 Let T be one of the four operators (3),(4),(5),(6) with respect to some continuous distribution function F_w , and let $C_w(n) = \mathcal{I}_w(n,r), \mathcal{I}_w(n,c), \mathcal{M}_w(n,r)$ or $\mathcal{M}_w(n,c)$. Denote by $\mathcal{O}(n)$ the subset of [n] or E (depending on a context), which achieves $C_w(n)$. Select two elements i, j of [n] or E uniformly at random. If T^2 has a unique fixed point distribution, then

(19)
$$\mathbb{P}(i, j \in \mathcal{O}(n)) \to \mathbb{P}(i \in \mathcal{O}(n)) \mathbb{P}(j \in \mathcal{O}(n)),$$

as $n \to \infty$.

Recall that for each of the four objects $|\mathcal{C}(n)| = \Theta(n)$ w.h.p. As a result the values $\mathbb{P}(v \in \mathcal{O}(n))$ do not vanish and the first part of the theorem above does have a non-trivial content. In fact we will show a much stronger result stating that, when T^2 has a unique fixed point, the event $v \in \mathcal{O}(n)$ is almost independent of the entire graph G outside a depth-d graph-theoretic neighborhood of v, when d is a sufficiently large constant integer.

4 Fixed points of the operator T^2 and the long-range independence

4.1 Maximum weight independent sets and matchings in trees. Fixed points of T^2 and the bonus function

We start by analyzing operator T – one of the four operators introduced in the previous section. Given two distribution functions F_1 , F_2 defined on $[0, \infty)$, we say that F_2 stochastically dominates F_1 and write $F_1 \prec F_2$ if $F_1(t) \geq F_2(t)$ for every $t \geq 0$. A sequence of distribution functions F_n is defined to converge weakly to a distribution function F (written $F_n \Rightarrow F$) if $\lim_n F_n(t) = F(t)$ for every t which is a point of continuity of F.

Lemma 6 The operators $T = T_{\mathcal{I},r}, T_{\mathcal{I},c}, T_{\mathcal{M},r}, T_{\mathcal{I},c}$ are continuous with respect to the weak convergence. That is, given a sequence of distributions $F, F_s, s = 1, 2, \ldots$, if $F_s \Rightarrow F$ then $T(F_s) \Rightarrow T(F)$.

Proof: The proof is almost immediate for $T = T_{\mathcal{I},r}$. Since the summation, subtraction and max operations are continuous functions then the assertion holds. Here we use the fact that for any continuous function $f, F_n \Rightarrow F$ implies $f(F_n) \Rightarrow f(F)$ (see Continuous Mapping Theorem in [Dur96]). The proof for the case $T = T_{\mathcal{I},c}$ is slightly more subtle since we are dealing with sum of randomly many elements. Let $X_1^{(n)}, X_2^{(n)}, \ldots, X_m^{(n)}$ be distributed independently according to F_n and let m be random variable with Pois(c) distribution. Let X_1, \ldots, X_m be distributed according to F. Define $X^{(n)} = \max(0, W - \sum_{1 \leq i \leq m} X_i^{(n)})$ and $X = \max(0, W - \sum_{1 \leq i \leq m} X_r)$. Fix arbitrary $\epsilon > 0$ and m_0 such that m does not exceed m_0 with probability at least $1 - \epsilon$. For any fixed $t \geq 0$

$$|\mathbb{P}(X^{(n)} \le t) - \mathbb{P}(X^{(n)} \le t | m \le m_0) \mathbb{P}(m \le m_0)| \le \epsilon,$$

and

$$|\mathbb{P}(X \le t) - \mathbb{P}(X \le t | m \le m_0) \mathbb{P}(m \le m_0)| \le \epsilon,$$

Now, we have $|\mathbb{P}(X^{(n)} \leq t | m \leq m_0) - \mathbb{P}(X \leq t | m \leq m_0)| \leq \epsilon$ for sufficiently large n since $F_n \Rightarrow F$ and m is conditioned to be at most m_0 . Combining, we obtain $|\mathbb{P}(X^{(n)} \leq t) - \mathbb{P}(X \leq t)| \leq 3\epsilon$ for sufficiently large n. This completes the proof for $T = T_{\mathcal{I},c}$. The proofs for $T = T_{\mathcal{M},r}, T = T_{\mathcal{M},c}$ are similar.

Let 0 denote (for simplicity) the distribution function of a random variable X which is zero w.p.1. Let $W_r = \max_{1 \le i \le r} W_i$ where $W_i \sim F_w$ are independent. Let also $W_c = \max_{1 \le i \le m} W_i$, where $W_i \sim F_w$ are independent and $m \sim \operatorname{Pois}(c)$. Denote by $F_{w,r}$ and $F_{w,c}$ the distribution functions of W_r and W_c , respectively.

Proposition 1 Fix $r \geq 1, c > 0$, a distribution function $F_w(t), t \geq 0$ and $T = T_{\mathcal{I},r}$ or $T_{\mathcal{I},c}$. As $s \to \infty$, the two sequences of distributions $T^{2s}(0), T^{2s}(F_w)$ weakly converge respectively to some distribution functions F_{**}, F^{**} which are fixed points of the operator T^2 . For any distribution function $F_0 = F_0(t), t \geq 0, T^{2s}(0) \prec T^{2s}(F_0) \prec T^{2s}(F_w)$ and $T^{2s+1}(F_w) \prec T^{2s+1}(F_0) \prec T^{2s+1}(0)$ for all $s = 1, 2, \ldots$

If the operator T is such that T^2 has a unique fixed point (and $F_{**} = F^{**} \equiv F^*$), then for any distribution function $F_0 = F_0(t), t \geq 0, T^s(F_0), s = 1, 2, \ldots$ converges to F^* as $s \to \infty$. In particular, $T^s(0), T^s(F_w) \to F^*$. Moreover, F^* is also the unique fixed point of T.

When $T = T_{\mathcal{M},r}$ or $T = T_{\mathcal{M},c}$ the same result holds with $F_{w,r}$ and $F_{w,c}$ respectively replacing F_w .

Proof: Let $F = F(t), t \ge 0$ be any distribution function corresponding to any non-negative random variable. It follows immediately from the definitions that

(20)
$$T_{\mathcal{I},r}(F) \prec F_w, \qquad T_{\mathcal{I},c}(F) \prec F_w, \qquad T_{\mathcal{M},r}(F) \prec F_{w,r}, \qquad T_{\mathcal{M},c}(F) \prec F_{w,c}.$$

Observe that the each of the four operators T above is anti-monotone. That is if $F_1 \prec F_2$ for some distribution functions F_1, F_2 , then $T(F_1) \succ T(F_2)$. Applying this twice we obtain $T^2(F_1) \prec T^2(F_2)$. Then $0 \prec T^2(0) \prec T^4(0) \prec \cdots$, and this sequence weakly converges to some function $F_{**} = F(t)_{**} \leq 1$, since the values of $T^{2s}(0)$ at each fixed t are decreasing and are bounded below by 0. Note that $F_{**}(t)$ is a non-decreasing function of t, since this is the case for each distribution $T^{2s}(0)$. Finally, from (20) we have that $\lim_{t\to\infty} F_{**}(t) = 1$. Thus $F_{**}(t)$ is a distribution function. As $T^{2s}(0) \Rightarrow F_{**}$ and by Lemma 6, T is continuous, we have $T^2(F_{**}) = F_{**}$.

We now fix $T = T_{\mathcal{I},r}$. The proof of the other three cases is very similar. From (20), by taking $F = T_{\mathcal{I},r}(F_w)$ we obtain $T^2(F_w) \prec F_w$. Then, by monotonicity of T^2 we obtain $F_w \succ T^2(F_w) \succ \cdots \succ T^{2s}(F_w)$ and the sequence $T^{2s}(F_w)$ converges weakly to some function F^{**} . We repeat the arguments above to show that F^{**} is actually a distribution function. Again applying Lemma 6, we conclude $T^2(F^{**}) = F^{**}$.

Suppose now T^2 has a unique fixed point $F^* = F^{**} = F_{**}$. For any distribution function F_0 we have $0 \prec T(F_0), T^2(F_0) \prec F_w$, where again we use (20). Applying the monotonicity $T^{2s}(0) \prec T^{2s+1}(F_0), T^{2s+2}(F_0) \prec T^{2s}(F_w)$. But since $T^{2s}(0), T^{2s}(F_w) \Rightarrow F^*$, then $T^s(F_0) \Rightarrow F^*$. In particular, by taking $F_0 = 0$ and $F_0 = F_w$, we obtain $T^s(0), T^s(F_w) \to F^*$. Finally, we obtain $T^{2s}(T(F^*)) \to F^*$. But $T^{2s}(T(F^*)) = T(T^{2s}(F^*)) = T(F^*)$. Thus $T(F^*) = F^*$. Clearly it is the unique fixed point of T since any fixed point of T is also a fixed point of T^2 .

We now switch to analyzing the maximum weight independent set problem on a tree. The derivation here repeats the development in [AS03] for maximum weight matching in random trees. We highlight important differences where appropriate.

Suppose we have a (non-random) finite tree H with nodes $0, 1, \ldots, h = |H| - 1$, with a fixed root 0. The nodes of this tree are equipped with some (non-random) weights $W_0, W_1, \ldots, W_h \geq 0$. For any node $i \in H$, let H(i) denote the subtree rooted at i consisting of all the descendants of i. In particular, H(0) = H. Let $\mathcal{I}_{H(i)}$ denote the maximum weight of an independent set in H(i) and let $B_{H(i)} = \mathcal{I}_{H(i)} - \sum_{j} \mathcal{I}_{H(j)}$, where the sum runs over nodes j which are children of i. If i has no children then $B_{H(i)}$ is simply $\mathcal{I}_{H(i)} = W_i$. Observe, that $B_{H(i)}$ is also a difference between $\mathcal{I}_{H(i)}$ and the maximum weight of an independent set in H(i), which is not allowed to use node i. Clearly, $0 \leq B_{H(i)} \leq W_i$. The value $B_{H(i)}$ was considered in [AS03] in the context of maximum weight matchings and was referred to as a bonus of a node i in tree H(i). W.l.g. denote by $1, \ldots, m$ the children of the root node 0.

Lemma 7

(21)
$$B_{H(0)} = \max(0, W_0 - \sum_{1 \le i \le m} B_{H(i)}).$$

Moreover, if $W_0 > \sum_{1 \leq i \leq m} B_{H(i)}$ (that is if $B_{H(0)} > 0$) then the maximum weight independent set must contain node 0. If $W_0 < \sum_{1 \leq i \leq m} B_{H(i)}$ then the maximum weight independent set does not contain the node 0.

Remark : There might be several independent sets in H which achieve \mathcal{I}_H . The second part of the lemma refers to any independent set achieving maximum weight. Also, the statement of the lemma applies as well to every node i with respect to its tree H(i). That is let j_1, \ldots, j_l be the children of a node i. Then the lemma claims $B_{H(i)} = \max(0, W_0 - \sum_{1 \le i \le l} B_{H(j_i)})$.

Proof: Consider an independent set $V \subset H$ which achieves the maximum weight. We take an arbitrary such in case there are many. If $0 \in V$ then $i \notin V$ for $1 \le i \le m$. Then V is obtained by taking maximum independent sets V_i in H(i) such that $i \notin V(i)$. By definition the weight of such V_i is $\mathcal{I}_{H(i)} - B_{H(i)}$. On the other hand, if $0 \notin V$, then a maximum weight of an independent set in H is obtained simply as $\sum_{1 \le i \le m} \mathcal{I}_{H(i)}$. We conclude that

(22)
$$\mathcal{I}_{H} = \max(W_0 + \sum_{1 \le i \le m} (\mathcal{I}_{H(i)} - B_{H(i)}), \sum_{1 \le i \le m} \mathcal{I}_{H(i)}).$$

Recall that $B_H = \mathcal{I}_H - \sum_{1 \leq i \leq m} \mathcal{I}_{H(i)}$. Subtracting $\sum_{1 \leq i \leq m} \mathcal{I}_{H(i)}$ from both sides of (22) we obtain (21). The second part of the lemma follows directly from the discussion above.

A similar development is possible for maximum weight matching. Suppose the edges of the tree H are equipped with weights $W_{i,j}$. Let $\mathcal{M}_{H(i)}$ denote the maximum weight of a matching in H(i), and let $B_{H(i)}$ denote the difference between $\mathcal{M}_{H(i)}$ and the maximum weight of a matching in H(i) which is not allowed to include any edge in H(i) incident to i. Again $1, 2, \ldots, m$ are assumed to be the children of the root 0.

Lemma 8

(23)
$$B_H = \max(0, \max_{1 \le i \le m} (W_{0,i} - B_{H(i)})).$$

Moreover, if $W_{0,i} - B_{H(i)} > W_{0,i'} - B_{H(i')}$ for all $i' \neq i$ and $W_{0,i} - B_{H(i)} > 0$, then every maximum weight matching contains edge (0,i). If $W_{0,i} - B_{H(i)} < 0$ for all i = 1, ..., m, then every maximum weight matching does not contain any edge incident to 0.

Proof: The proof is very similar to the one of Lemma 7. If a maximum weight matching contains edge (0, i) then its weight is $W_{0,i}$ plus maximum weight of a matching in H(i) with node i excluded, plus the sum of the maximum weights of matchings in $T_{H(j)}$, $1 \le j \le m, j \ne i$. If a maximum weight matching contains none of the edges $(0, i), 1 \le i \le m$, then its weight is simply sum of maximum weight of matchings in H(i) for all i = 1, ..., m. We obtain

(24)
$$\mathcal{M}_{H} = \max(\max_{1 \le i \le m} (W_{0,i} + \mathcal{M}_{H(i)} - B_{H(i)} + \sum_{j \ne i} \mathcal{M}_{H(j)}), \sum_{1 \le i \le m} \mathcal{M}_{H(j)}).$$

Subtracting $\sum_{1 \leq j \leq m} \mathcal{M}_{H(j)}$ from both sides of (24) we obtain (23). The proof of the second part follows immediately from the discussion above.

4.2 Long-range independence

We now consider trees H of specific types. Given integers $r \geq 3, d \geq 2$ let $H_r(d)$ denote an r-regular finite tree with depth d. The root node 0 has degree r-1, all the nodes at distance $\geq 1, \leq d-1$ from the root have outdegree r-1, and all the nodes at distance d from 0 are leaves. (Usually, in the definition of an r-regular tree, the root node is assumed to have degree r, not r-1. The slight distinction here is done for convenience.) Also, given a constant c>0, a Poisson tree H(c,d) with parameter c and depth d is constructed as follows. The root node has a degree which is a random variable distributed according to Pois(c) distribution. All the children of 0 have outdegrees which are also random, distributed according to Pois(c). In particular, the children of 0 have total degrees 1+Pois(c). Similarly, children of children of 0 also have outdegree Pois(c), etc. We continue this process until either the process stops at some depth d' < d, where no nodes in level d' have any children, or until we reach level d. In this case all the children of the nodes in level d are deleted and the nodes in level d become leaves. We obtain a tree with depth $\leq d$. We call this a depth-d Poisson tree.

Let $H = H_r(d)$ or H(c,d). Suppose the nodes and the edges of H are equipped with weights $W_i, W_{i,j}$, which are generated at random independently using a distribution function F_w . Fix any infinite sequences $\bar{w}=(w_1,w_2,\ldots)\in[0,\infty)^{\infty}$ and $\bar{b}=(b_1,b_2,\ldots)\in\{0,1\}^{\infty}$. For every $i=1,2,\ldots,d$ let $i1,i2,\ldots,ij_i$ denote the nodes of H in level i (if any exist for H(c,d)). When $H = H_r(d), j_i = (r-1)^i$, of course. Let $(\mathcal{I}|(\bar{b},\bar{w}))$ denote the maximum weight of an independent set V in H such that the nodes dj with $b_i = 1$ are conditioned to be in V, nodes dj with $b_i = 0$ are conditioned not to be in V, and the weights of nodes dj are conditioned to be equal to w_i for $j = 1, \ldots, j_d$. That is we are looking for maximum weight of an independent set among those which contain depth d leaves with $b_i = 1$, do not contain depth d leaves with $b_i = 0$, and with the weights of the leaves deterministically set by \bar{w} . For brevity we call it the maximum weight of an independent set with boundary condition (b, \bar{w}) . For the case H = H(c, d), the boundary condition is simply absent when the tree does not contain any nodes in the last level d. $(\mathcal{I}_{H(ij)}|(\bar{b},\bar{w}))$ are defined similarly for the subtrees H(ij) spanned by nodes ij in level i: given again b, \bar{w} , let $(\mathcal{M}|(b,\bar{w}))$ and $(\mathcal{M}_{H(ij)}|(b,\bar{w}))$ denote, respectively, the maximum weight of a matching E in H and H_{ij} , such that the edges incident to nodes dj are conditioned to be in E when $b_j = 1$, edges incident to nodes dj are conditioned not to be in D when with $b_j = 0$, and the weights of the edges incident to nodes dj are conditioned to be equal to $w_j, j = 1, \ldots, j_d$ (of course, we refer to edges between nodes in levels d-1 and d as there is only one edge per each node in level d).

For the case of independent sets, let $(B|(\bar{b},\bar{w}))$ denote the bonus of the root node 0 given the boundary condition (\bar{b},\bar{w}) . Namely,

$$(B|(\bar{b},\bar{w})) = (\mathcal{I}|(\bar{b},\bar{w})) - \sum_{1 \le j \le j_1} (\mathcal{I}_{H(1j)}|(\bar{b},\bar{w})).$$

For the case of matchings, let $(B|(\bar{b},\bar{w}))$ also denote the bonus of the root node 0 given the boundary condition (\bar{b},\bar{w}) . Namely,

$$(B|(\bar{b},\bar{w})) = (\mathcal{M}|(\bar{b},\bar{w})) - \sum_{1 \le j \le j_1} (\mathcal{M}_{H(1j)}|(\bar{b},\bar{w})).$$

It should be always clear from the context whether B is taken with respect to independent sets or matchings.

The following theorem establishes the crucial long-range independence property for the maximum weight independent sets and matchings in trees $H = H_r(d), H(c, d)$ when the corresponding operator T^2 has a unique fixed point. It establishes that the distribution of the bonus $(B|(\bar{b}, \bar{w}))$ of the root asymptotically is independent of the boundary condition (\bar{b}, \bar{w}) as d becomes large. Recall our convention that 0 denotes the distribution of a random variable which is zero to one with probability one.

Theorem 9 Given a distribution function F_w and a regular tree $H = H_r(d)$ let $T = T_{\mathcal{I},r}$. Then for every $t \geq 0$ and \bar{b}, \bar{w}

(25)
$$T^{d-1}(0)(t) \le \mathbb{P}((B|(\bar{b},\bar{w})) \le t) \le T^{d-1}(F_w)(t)$$

when d is odd, and

(26)
$$T^{d-1}(F_w)(t) \le \mathbb{P}((B|(\bar{b},\bar{w})) \le t) \le T^{d-1}(0)(t)$$

when d is even. Suppose in addition the operator $T^2 = T_{\mathcal{I},r-1}^2$ has the unique fixed distribution F^* . Then

(27)
$$\sup_{\bar{b},\bar{w}} \left| \mathbb{P}((B|(\bar{b},\bar{w})) \le t) - F^*(t) \right| \to 0,$$

as $d \to \infty$. Similar assertion holds for $T = T_{\mathcal{M},r}$ and for H = H(c,d) with $T = T_{\mathcal{I},c}$ and $T = T_{\mathcal{M},c}$. For the cases $T = T_{\mathcal{M},r}$ and $T_{\mathcal{M},c}$, F_w is replaced with $F_{w,r}$ and $F_{w,c}$ respectively.

Before we prove the proposition above, which is essentially the key result of the paper, let us compare it with the developments in ([AS03]). In that paper maximum weight matching is considered on an n-node tree, drawn independently and uniformly from the space of all n^{n-2} labelled trees. The notion of a bonus is introduced and the recursion (23) is derived. However, since a tree structure is assumed to begin with, there is no need to consider the boundary conditions (\bar{b}, \bar{w}) . Here we avoid the difficulty of the non-tree structure by proving the long-range independence property via the uniqueness of fixed points of T^2 .

Proof: We prove (25) and (26) for independent sets in $H = H_r(d)$. The proof of (27) will then be almost immediate. The proofs for other cases is similar. We will indicate the differences where appropriate. The proof of (25) proceeds by induction in d. Suppose d = 1. Then we have

$$(B|(\bar{b}, \bar{w})) = \max(0, W_0 - \sum_{i:1 \le i \le r-1, \bar{b}_{1i}=1} \bar{w}_{1i}) \le W_0.$$

Then $(B|(\bar{b},\bar{w})) \prec F_w$. Trivially $0 \prec (B|(\bar{b},\bar{w}))$. This establishes the bound for the case d=1. Suppose the assumption holds for d-1 and d is odd. We have

(28)
$$(B|(\bar{b}, \bar{w})) = \max(0, W_0 - \sum_{i:1 \le i \le r-1} (B_{1i}|(\bar{b}, \bar{w}))).$$

By the inductive assumption the distribution of each of $(B_{1i}|(\bar{b},\bar{w}))$ dominates $T^{d-2}(F_w)$ and is dominated by $T^{d-2}(0)$. Applying this to (28) we obtain (25). The case d is even is considered similarly. This completes the induction and establishes (25) and (26). Now if T^2 has the unique fixed point distribution F^* then by Proposition 1 $T^d(0)$, $T^d(F_w)$ converge to F^* as $d \to \infty$, and we obtain (27). The proof for the case of Poisson tree H(c,d) instead of $H_r(d)$ is pretty much identical. For the proof of the same result for maximum weight matching we just use $F_{w,r}$ and $F_{w,c}$ instead of F_w , just as we did in the proof of Proposition 1.

While it is not important for the further results in this paper, it is interesting that the uniqueness of the solution $T^2(F) = F$ is the tight condition for (27), as the following theorem indicates.

Theorem 10 Suppose the operator T^2 has more than one fixed point distributions F^* . Then for every such F^*

(29)
$$\liminf_{d} \sup_{\bar{b},\bar{w}} \left| \mathbb{P}((B|(\bar{b},\bar{w})) \le t) - F^*(t) \right| > 0.$$

Proof: As usual we start with $T = T_{\mathcal{I},r}$. The proofs for other cases are similar, we highlight the differences where appropriate. Let F_{**} and F^{**} be distributions introduced in Proposition 1. The non-uniqueness of the fixed point of T^2 implies using Proposition 1 that $F_{**} \neq F^{**}$. For every node j in layer d (last layer) of $H_r(d)$ set $b_j = 0, w_j = 0$. In particular, the bonus B_j of each such node is zero. Then for every node in layer d-1 its bonus is given by the distribution $T(0) = F_w$, the bonus of each node in layer d-2 has distribution $T^2(0)$, etc. The root node 0 has bonus with distribution $T^d(0)$. When d is an even number diverging to infinity, from Proposition 1, $T^d(0)$ converges weakly to F_{**} . Thus the distribution of $(B|(\bar{b},\bar{w}))$ converges to F_{**} . If F^* , a fixed point of T^2 , is distinct from F_{**} , then we obtain that (29) holds. Suppose, on the other hand, $F^* = F_{**}$. We claim that $T(F_{**}) \neq F_{**}$. Assuming this is the case we consider the same boundary condition $\bar{b} = \bar{w} = 0$ but take d to be odd integer diverging to infinity. Then the bonus of the root 0 converges in distribution to $T(F_{**}) \neq F_{**}$ and (29) is shown again.

Assume $T(F_{**}) = F_{**}$. Recall from the first part of Proposition 1 that for every distribution $F_0, T^{2s+1}(F_0) \prec T^{2s+1}(0)$. Taking $F_0 = T(F^{**})$ we obtain $F^{**} = T^{2s+2}(F^{**}) \prec T^{2s+1}(0)$. Taking $s \to \infty, F^{**} \prec T(F_{**}) = F_{**}$. But from Proposition 1 $F_{**} \prec F^{**}$, and, as a result $F_{**} = F^{**}$ implying (again using Proposition 1) T^2 has the unique fixed point. We obtained a contradiction. \Box

5 Applications to maximum weight independent sets and matchings in $G_r(n)$ and G(n, c/n)

5.1 Long-range independence in $G_r(n)$, G(n, c/n)

The goal of the current section is to demonstrate that Theorem 9 allows us to reduce the computation of the maximum weight independent set and the maximum weight matching in random graphs to a much simpler problem of finding those in trees. We highlight this key message of the paper as the following local optimality property: if the operator T^2 corresponding to a maximum weight combinatorial object (independent set or matching) in a sparse random graph has a unique fixed point, then for a randomly selected node (edge) of the graph, the event "the node (edge) belongs to the optimal object" and the distribution of the node (edge) weight, conditioned that it does, asymptotically depends only on the constant size neighborhood of the node and is independent from the rest of the graph. In other words, when T^2 has a unique fixed point, the maximum weight independent sets and matchings exhibit a long-range independence property.

Our hope is that similar local optimality can be established for other random combinatorial structures.

[Proof of Theorem 2]. Again we start by proving the result for $\mathcal{I}_w(n,r)$. The Proof: proofs for other three objects is similar and we highlight some differences in the end. Let $V_r \subset G_r(n)$ denote the independent set which achieves the maximum weight $\mathcal{I}_w(n,r)$. It is unique by continuity of the distribution F_w . Consider a randomly selected node of the graph G, which, w.l.g., we may assume is node 0. By symmetry we have $\mathbb{E}[\mathcal{I}_w(n,r)] = n\mathbb{E}[W_0 1\{0 \in V_r\}].$ Let us fix a large positive integer d, which is a constant independent from n, and let H(d) denote the depth-d neighborhood of 0. That is H is the collection of nodes in G which are connected to 0 by paths with length $\leq d$. It is well known that, w.h.p. as $n \to \infty$, H is a depth-d rregular tree, [JLR00], except in this case, unlike in Subsection 4.2, the root node has outdegree r and the remaining non-leaf nodes have outdegree r-1. Let ∂H denote the leaves of this tree (level d). Fix any binary vector b with dimension $|\partial H| + |G \setminus H|$, and any non-negative vector \bar{w} also with dimension $|\partial H| + |G \setminus H|$. Assume the vector b is such that if two nodes in $\partial H \cup (G \setminus H)$ are connected by an edge, only one of these two nodes can have the corresponding component of b equal to 1. That is, the nodes marked 1 by b correspond to some independent set in $\partial H \cup (G \setminus H)$. Consider the problem of finding the maximum weight independent set V_r in G when the weights of the nodes in $\partial H \cup (G \setminus H)$ are conditioned to be \bar{w} , nodes $i \in \partial H \cup (G \setminus H)$ with the corresponding component of \bar{b} equal to one are conditioned to belong to V_r and the remaining nodes in $\partial H \cup (G \setminus H)$ are conditioned not to belong to V_r . Notation-wise, we consider the value $(\mathcal{I}_w(n,r)|(b,\bar{w}))$. In particular, we need to select to maximum weight independent set in the tree H, which is consistent with conditioning (b, \bar{w}) . Naturally, the consistency needs to be checked only across the boundary ∂H .

Let $(B|(\bar{b},\bar{w}))$ and $(B_i|(\bar{b},\bar{w})), 1 \leq i \leq r$ denote the bonus of the node 0 and the bonuses of its neighboring nodes $1,2,\ldots,r$, respectively. By Lemma 7, we have $(B|(\bar{b},\bar{w})) = \max(0,W_0 - \sum_{1\leq i\leq r}(B_i|(\bar{b},\bar{w})))$. Since the distribution function F_w of W_0 is continuous, then F^* is continuous as well and, as a result, $W_0 - \sum_{1\leq i\leq r}(B_i|(\bar{b},\bar{w})) = 0$ with probability zero. Then applying the second part of Lemma 7, node 0 belongs to the maximum weight independent set if and only if

 $B = W_0 - \sum_{1 \le i \le r} (B_i | (\bar{b}, \bar{w})) > 0$. Therefore

$$\frac{\mathbb{E}[\mathcal{I}_w(n,r)]}{n} = \sum_{\bar{b},\bar{w}} \mathbb{E}[W_0 1\{W_0 - \sum_{1 \le i \le r} (B_i | (\bar{b},\bar{w})) > 0\}] \mathbb{P}(\bar{b},\bar{w}).$$

But by Theorem 10, for every \bar{b}, \bar{w} the distribution of $(B_i|(\bar{b}, \bar{w}))$ converges to F^* , the unique fixed point of $T^2_{\mathcal{I},r-1}$ as d becomes large. We conclude

$$\lim_{n} \frac{\mathbb{E}[\mathcal{I}_{w}(n,r)]}{n} = \mathbb{E}[W_{0}1\{W_{0} - \sum_{1 \le i \le r} B_{i} > 0\}],$$

where $B_i \sim F^*$. This completes the proof for the maximum weight independent set in $G_r(n)$. When the graph G(n, c/n) is considered the proof is very similar, we just use the fact that H – depth d neighborhood of 0 approaches in distribution a Poisson tree [Spe94]. The proofs for maximum weight matchings are similar, we use Lemma 8 instead of Lemma 7.

Proof : [Proof of Corollary 2] In fact we have proved this result already en route of proving Theorem 2 above. We have shown that given \bar{b}, \bar{w} and conditioned on H being a tree a fixed node 0 belongs to the maximum weight independent set iff $(B|(\bar{b},\bar{w})) = W_0 - \sum_{1 \leq i \leq r} (B_i|(\bar{b},\bar{w})) > 0$. Repeating the proof of Theorem 2 $\mathcal{IN}_w(n,r)/n = \mathbb{E}[1\{0 \in V_r\}] \to \mathbb{E}[1\{W_0 - \sum_{1 \leq i \leq r} B_i\}]$, where $B_i \sim F^*$.

Proof: [Proof of Theorem 5] The essential ingredients for the proof of this results are already established in the proof of Theorem 2 above. We start with $\mathcal{I}_w(n,r)$. The proofs for other cases are very similar and we omit them. Let $\mathcal{O} = \mathcal{O}(n,r) \subset [n]$ be the independent set achieving the maximum weight. Fix two nodes $i, j \in [n]$ and arbitrary $\epsilon > 0$. Let H = H(i, d) denote the depth-d graph-theoretic neighborhood of i in the graph $G = G_r(n)$ and let ∂H denote the boundary of H - the nodes of H at distance d from i. Let \mathcal{E}_T denote the event that H is a (r-regular) tree. From the theory of random regular graphs [JLR00], $\mathbb{P}(\mathcal{E}_T) \to 1$ as $n \to \infty$. As a result $\mathbb{P}(\mathcal{E}_T) \geq 1 - \epsilon$ for all $n \geq n_0$ for some $n_0 = n_0(d)$ (note the dependence on d). Fix any realization of $(G \setminus H) \cup \partial H$ together with the realization of the weights \bar{w} of nodes in $G \setminus H \cup \partial H$ and indicators b of whether the nodes belong to \mathcal{O} . As far as deciding which nodes of $H \setminus \partial H$ are in \mathcal{O} and in particular whether node i belongs to \mathcal{O} only the restriction of \bar{w}, b to ∂H is relevant. Conditioning on the event \mathcal{E}_T denote by $(B|(b,\bar{w}))$ the bonus of i in H (for completeness define $(B|(\bar{b},\bar{w}))$ to be zero when the event \mathcal{E}_T does not hold). Applying Lemma 7 and the continuity of F_w , we have $i \in \mathcal{O}$ iff $(B|(\bar{b},\bar{w})) > 0$. Applying Theorem 9 $|\mathbb{P}((B|(\bar{b},\bar{w})) > 0) - \mathbb{P}(B > 0)| < \epsilon$ for all $d \geq d_0(\epsilon)$ for some $d_0(\epsilon)$, and for any \bar{w}, \bar{b} , where $B \sim F^*$ and F^* is the unique fixed point of $T^2 = T_{\mathcal{I},r-1}^2$. Thus

$$|\mathbb{P}(i \in \mathcal{O}) - \mathbb{P}(B > 0)| \leq \sum_{\hat{G}\bar{w},\bar{b}} |\mathbb{P}(i \in \mathcal{O}|\mathcal{E}_T, \hat{G}, \bar{w}, \bar{b}) - \mathbb{P}(B > 0)|\mathbb{P}(\hat{G}, \bar{w}, \bar{b})$$

$$+ |\mathbb{P}(i \in \mathcal{O}|\bar{\mathcal{E}}_T) - \mathbb{P}(B > 0)|\mathbb{P}(\bar{\mathcal{E}}_T)$$

$$\leq 2\epsilon,$$
(30)

whenever $d \geq d_0(\epsilon)$ and $n \geq n_0(d)$, where \hat{G} denotes generically a realization of the subgraph $(G \setminus H) \cup \partial H$. Observe that since $|H| \leq 1 + r + (r-1)^2 + \cdots + (r-1)^d$ then $\mathbb{P}(j \notin H) \geq 1 - \epsilon$

for all $n \geq n_1(d)$ for some $n_1(d)$. Then

$$|\mathbb{P}(i, j \in \mathcal{O}) - \mathbb{P}(i \in \mathcal{O})\mathbb{P}(j \in \mathcal{O})|$$

$$(31) \leq |\mathbb{P}(i \in \mathcal{O}|\mathcal{E}_T, j \notin H, j \in \mathcal{O})\mathbb{P}(\mathcal{E}_T, j \notin H, j \in \mathcal{O}) - \mathbb{P}(i \in \mathcal{O})\mathbb{P}(j \in \mathcal{O})|$$

$$(32) + \mathbb{P}(i \in \mathcal{O}, \mathcal{E}_T, j \in H \cap \mathcal{O}) + \mathbb{P}(i \in \mathcal{O}, \bar{\mathcal{E}}_T, j \in \mathcal{O})$$

The event $j \notin H, j \in \mathcal{O}$ is completely described by the realizations $\hat{G}, \bar{w}, \bar{b}$. Since $|\mathbb{P}((B|(\bar{b}, \bar{w})) > 0) - \mathbb{P}(B > 0)| < \epsilon$ for all $d \geq d_0(\epsilon)$ then

(33)
$$|\mathbb{P}(i \in \mathcal{O}|\mathcal{E}_T, j \notin H, j \in \mathcal{O}) - \mathbb{P}(B > 0)| < \epsilon,$$

for all $d \geq d_0(\epsilon)$. Also $\mathbb{P}(j \in \mathcal{O}) = \mathbb{P}(\mathcal{E}_T, j \notin H, j \in \mathcal{O}) + \mathbb{P}(\mathcal{E}_T, j \in H \cap \mathcal{O}) + \mathbb{P}(\bar{\mathcal{E}}_T, j \in \mathcal{O})$. But $\mathbb{P}(\mathcal{E}_T, j \in H \cap \mathcal{O}) \leq \mathbb{P}(j \in H) \leq \epsilon$ for all $n \geq n_1(d)$ and $\mathbb{P}(\bar{\mathcal{E}}_T, j \in \mathcal{O}) \leq \mathbb{P}(\bar{\mathcal{E}}_T) \leq \epsilon$ for all $n \geq n_0(d(\epsilon))$. As a result

$$(34) |\mathbb{P}(j \in \mathcal{O}) - \mathbb{P}(\mathcal{E}_T, j \notin H, j \in \mathcal{O})| \le 2\epsilon$$

whenever $n \ge \max(n_0(d(\epsilon)), n_1(d(\epsilon)))$. Combining (33), (30) and (34) we obtain

$$\mathbb{P}(i \in \mathcal{O}|\mathcal{E}_T, j \notin H, j \in \mathcal{O})\mathbb{P}(\mathcal{E}_T, j \notin H, j \in \mathcal{O}) - \mathbb{P}(i \in \mathcal{O})\mathbb{P}(j \in \mathcal{O})$$

$$\leq (\mathbb{P}(i \in \mathcal{O}) + 3\epsilon)(\mathbb{P}(j \in \mathcal{O}) + 2\epsilon) - \mathbb{P}(i \in \mathcal{O})\mathbb{P}(j \in \mathcal{O})$$

$$\leq 5\epsilon + \epsilon^2 < 6\epsilon$$

Similarly we show

$$\mathbb{P}(i \in \mathcal{O}|\mathcal{E}_T, j \notin H, j \in \mathcal{O})\mathbb{P}(\mathcal{E}_T, j \notin H, j \in \mathcal{O}) - \mathbb{P}(i \in \mathcal{O})\mathbb{P}(j \in \mathcal{O}) \ge -6\epsilon.$$

We conclude that the value in (31) is bounded by 6ϵ . Each summand in (32) is bounded by ϵ since $\mathbb{P}(j \in H)$ when $n \geq n_1(d(\epsilon))$ and $\mathbb{P}(\bar{\mathcal{E}}_T) \leq \epsilon$ when $n \geq n_0(d(\epsilon))$. We conclude that whenever $n \geq \max(n_0(d(\epsilon)), n_1(d(\epsilon))), |\mathbb{P}(i, j \in \mathcal{O}) - \mathbb{P}(i \in \mathcal{O})\mathbb{P}(j \in \mathcal{O})| \leq 8\epsilon$. This completes the proof of the theorem.

5.2 Computation of limits. Exponentially distributed weights

We prove Theorem 3 in this subsection. Thanks to Theorem 2, we can focus on proving the uniqueness and computing the fixed points of the operator T^2 . As usual we start with maximum weight independent set in $G_r(n)$. The analysis of other cases is similar and will follow immediately. The calculations are similar to the ones in [AS03] performed for maximum weight matching in random trees. The difference is that we have to compute the fixed point of the operator T^2 and not just T.

Proof: [Proof of Theorem 3]

• Independent sets in $G_r(n)$. Let F^* denote any fixed point distribution of $T^2 = T^2_{\mathcal{I},r-1}$ (at least one exists by Proposition 1), and let $B \sim F^*$. Then $B = \max(W - \sum_{1 \leq i \leq r-1} \hat{B}_i), W \sim$

 $\operatorname{Exp}(1), \hat{B}_i \sim T(F^*)$. Similarly, if $\hat{B} \sim T(F^*)$, then $\hat{B} = \max(W - \sum_{1 \leq i \leq r-1} B_i), W \sim \operatorname{Exp}(1), B_i \sim F^*$. Then for any $t \geq 0$

$$\mathbb{P}(B > t) = \mathbb{P}(W > \sum_{1 \le i \le r-1} \hat{B}_i + t) = e^{-t} \mathbb{P}(W > \sum_{1 \le i \le r-1} \hat{B}_i),$$

and similarly

$$\mathbb{P}(\hat{B} > t) = e^{-t}\mathbb{P}(W > \sum_{1 \le i \le r-1} B_i),$$

where we use the memoryless property of the exponential distribution. Let $b = \mathbb{P}(B = 0)$, $\hat{b} = \mathbb{P}(\hat{B} = 0)$ where $B \sim F^*$ and $\hat{B} \sim T(F^*)$. Our next goal is computing b and \hat{b} . From above we obtain

(35)
$$\mathbb{P}(B > t) = e^{-t}(1 - b), \qquad \mathbb{P}(\hat{B} > t) = e^{-t}(1 - \hat{b}),$$

implying

(36)
$$\mathbb{P}(B > t | B > 0) \sim \operatorname{Exp}(1), \qquad \mathbb{P}(\hat{B} > t | \hat{B} > 0) \sim \operatorname{Exp}(1).$$

Then

(37)
$$b = \mathbb{P}(W - \sum_{1 \le i \le r-1} \hat{B}_i \le 0) = \int_0^\infty e^{-t} \mathbb{P}(\sum_{1 \le i \le r-1} \hat{B}_i \ge t) dt$$

In order to compute $\mathbb{P}(\sum_{1 \leq i \leq r-1} \hat{B}_i \geq t)$ we condition on $j \leq r-1$ terms \hat{B}_i out of r-1 being equal to zero, and the rest positive. This occurs with probability $\binom{r-1}{j} \hat{b}^j (1-\hat{b})^{r-1-j}$. When j < r-1, the sum of r-1-j non-zero terms \hat{B}_i has an Erlang distribution with parameter r-1-j (sum of r-1-j independent random variables distributed as Exp(1)). The density function of this distribution is $f(z) = \frac{z^{r-2-j}}{(r-2-j)!}e^{-z}$ and the tail probability is

(38)
$$\mathbb{P}(\cdot > t) = \int_{t}^{\infty} \frac{z^{r-2-j}}{(r-2-j)!} e^{-z} dz = \sum_{0 \le i \le r-2-j} \frac{t^{i}}{i!} e^{-t}$$

and

$$\int_0^\infty e^{-t} \mathbb{P}(\cdot \cdot > t) dt = \sum_{0 \le i \le r - 2 - j} \frac{1}{2^{i+1}} = 1 - \frac{1}{2^{r-1-j}}.$$

Combining with (37) and interchanging integration and summation, we obtain

(39)
$$b = \sum_{0 \le j \le r-1} {r-1 \choose j} \hat{b}^j (1-\hat{b})^{r-1-j} (1 - \frac{1}{2^{r-1-j}}) = 1 - (\frac{1+\hat{b}}{2})^{r-1}.$$

Similar calculations lead to $\hat{b} = 1 - (\frac{1+\hat{b}}{2})^{r-1}$. Combining

(40)
$$b = f(f(b)), \quad \text{where } f(x) = 1 - (\frac{1+x}{2})^{r-1}$$

Lemma 11 The equation (40) has a unique solution b^* within the range $b \in [0, 1]$ iff $r \le 4$. In this case b^* is also the unique solution of f(b) = b.

Figures 1 and 2 below show the graphs of f(f(b)) for the cases r = 4 and r = 8. The first corresponds to the case of the unique solution. In the second case there are more than one solution.

Proof: Note that for every $r \ge 1$ the equation x = f(x) has exactly one solution in [0, 1] since f is a strictly decreasing function and $f(0) = 1 - 1/2^{r-1} > f(1) = 0$. This solution is also a solution to x = f(f(x)).

We now prove the uniqueness for $2 \le r \le 4$ and non-uniqueness for r > 4. Let $r \le 4$. We claim that for all $x \in [0,1]$.

(41)
$$\frac{df(f(x))}{dx} = \frac{(r-1)^2}{4} \left(\frac{1+f(x)}{2}\right)^{r-2} \left(\frac{1+x}{2}\right)^{r-2} < 1.$$

This would imply that f(f(x)) - x is a strictly decreasing function and is equal to zero in at most one point. (41) is equivalent to

(42)
$$(1+f(x))(1+x) < \frac{2^{\frac{2r-2}{r-2}}}{(r-1)^{\frac{2}{r-2}}}.$$

Let us check the validity of this inequality at the end points x = 0, 1. Since f(0) < 1, f(1) = 0, the left hand side in both points is at most 2 which is strictly smaller than the right-hand side for r = 2, 3, 4, as it is easily checked. It remains to check the inequality at the points where the derivative of the function g(x) = (1 + f(x))(1 + x) vanishes. We have

$$\dot{g}(x) = 1 + \dot{f}(x) + f(x) + x\dot{f}(x)
= 1 - \frac{r-1}{2} \left(\frac{1+x}{2}\right)^{r-2} + 1 - \left(\frac{1+x}{2}\right)^{r-1} - x\frac{r-1}{2} \left(\frac{1+x}{2}\right)^{r-2}
= 2 - \left(\frac{1+x}{2}\right)^{r-1} - \frac{(1+x)(r-1)}{2} \left(\frac{1+x}{2}\right)^{r-2}
= 2 - r \left(\frac{1+x}{2}\right)^{r-1}$$

Thus $\dot{g}(x) = 0$ in exactly one point $x = 2(\frac{2}{r})^{\frac{1}{r-1}} - 1$. We check that (42) holds in this point for r = 2, 3, 4.

We now prove the non-uniqueness of the solution to (40) when $r \geq 5$. Let b^* denote the unique fixed point of $f(b^*) = b^*$. Clearly $f(f(b^*)) = b^*$. We claim that

$$\left. \frac{df(f(x))}{dx} \right|_{x=b^*} > 1.$$

This implies the result since we get that for ϵ sufficiently small, $f(f(b^* - \epsilon)) < b^* - \epsilon$. But f(f(0)) > 0. Therefore there exists a different fixed point of (40) in the interval $(0, b^*)$. To show (43) note that

$$\frac{df(f(x))}{dx}\Big|_{x=b^*} = \frac{(r-1)^2}{4} \left(\frac{1+f(b^*)}{2}\right)^{r-2} \left(\frac{1+b^*}{2}\right)^{r-2} = \frac{(r-1)^2}{4} \left(\frac{1+b^*}{2}\right)^{2r-4}.$$

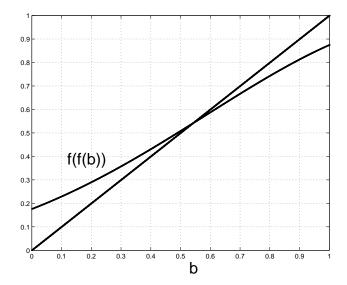


Figure 1: f(f(b)) = b has one solution when r = 4.

We need to show that

(44)
$$\frac{(r-1)^2}{4} \left(\frac{1+b^*}{2}\right)^{2r-4} > 1,$$

which is equivalent to $b^* > 2(\frac{2}{r-1})^{\frac{1}{r-2}} - 1 \equiv b(r)$. We claim that $b(r) < f(b(r)) = 1 - (\frac{1+b(r)}{2})^{r-1}$. Since $b^* = f(b^*)$ and f is a decreasing function, this would imply $b(r) < b^*$ or (44). Note, that $(1+b(r))/2 = (\frac{2}{r-1})^{\frac{1}{r-2}}$. Thus we need to check that

$$\begin{split} 2(\frac{2}{r-1})^{\frac{1}{r-2}} - 1 < 1 - (\frac{2}{r-1})^{\frac{r-1}{r-2}} &<=> 2(\frac{2}{r-1})^{\frac{1}{r-2}} < 2 - (\frac{2}{r-1})^{\frac{r-1}{r-2}} \\ <=> (\frac{2}{r-1})^{\frac{1}{r-2}} < 1 - \frac{2^{\frac{1}{r-2}}}{(r-1)^{1+\frac{1}{r-2}}} \\ <=> (\frac{2}{r-1})^{\frac{1}{r-2}} < \frac{r-1}{r} \\ <=> 2 < (r-1)(1-\frac{1}{r})^{r-2}. \end{split}$$

Note that $g(r) = (1 - \frac{1}{r})^{r-2}$ is a strictly growing function of r since $\log(g(r)) = (r - 2)\log(1 - \frac{1}{r})$. Therefore, it suffices to verify that for r = 5 we have $2 < 4(1 - \frac{1}{5})^3 \approx 2.048$.

Applying Lemma 11 $b = \hat{b}$ is the unique solution of f(b) = b. When r = 2, we obtain b = 1/3. When r = 3 we obtain $b = 1 - (1+b)^2/4$ or $b = 2\sqrt{3} - 3 \approx .4641$. When r = 4 we find numerically that $b \approx .5419$.

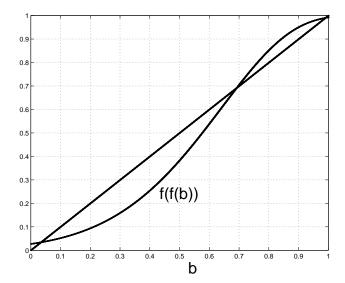


Figure 2: f(f(b)) = b has more than one solution when r = 8.

Recall, that the distribution F^* of B is given by $\mathbb{P}(B=0)=b$ and $\mathbb{P}(B>t)=(1-b)e^{-t}$. Applying (7) of Theorem 2 and generating $B_1,\ldots,B_r\sim F^*$ independently, we obtain

(45)
$$\lim_{n} \frac{\mathcal{I}_{w}(n,r)}{n} = \int_{0}^{\infty} t e^{-t} \mathbb{P}(t > \sum_{1 \le i \le r} B_{i}) dt$$

We again condition on j out of r terms B_i to be equal to zero and the rest positive. This occurs with probability $\binom{r}{j}b^j(1-b)^{r-j}$. W.l.g. assume that the non-zero terms are B_1, \ldots, B_{r-j} . We have

(46)
$$\int_0^\infty t e^{-t} \mathbb{P}(t > \sum_{1 \le i \le r-i} B_i) dt = 1 - \int_0^\infty t e^{-t} \mathbb{P}(\sum_{1 \le i \le r-i} B_i > t) dt.$$

Repeating the calculations (38), we obtain that $\mathbb{P}(\sum_{1 \leq i \leq r-j} B_i > t) = \sum_{0 \leq i \leq r-1-j} \frac{t^i}{i!} e^{-t}$. Then the expression of the right-hand side of (46) becomes

$$(47) 1 - \int_0^\infty t e^{-t} \sum_{0 \le i \le r-1-j} \frac{t^i}{i!} e^{-t} dt = 1 - \sum_{0 \le i \le r-1-j} \frac{i+1}{2^{i+2}} = \frac{r-j+2}{2^{r-j+1}}.$$

Combining, we obtain

$$\lim_{n} \frac{\mathcal{I}_{w}(n,r)}{n} = \sum_{0 \le j \le r} {r \choose j} b^{j} (1-b)^{r-j} \frac{r-j+2}{2^{r-j+1}} = \frac{1}{2} 2 \sum_{0 \le j \le r} {r \choose j} b^{j} (\frac{1}{2} - \frac{b}{2})^{r-j} + \frac{1}{2} \sum_{0 \le j \le r} {r \choose j} b^{j} (\frac{1}{2} - \frac{b}{2})^{r-j} (r-j).$$

The first summand is simply $(\frac{1}{2} + \frac{b}{2})^r$. We compute the second summand using the following probabilistic argument. Rewrite the expression as

$$\left(\frac{1}{2} + \frac{b}{2}\right)^r \sum_{0 \le j \le r} {r \choose j} \frac{b^j \left(\frac{1}{2} - \frac{b}{2}\right)^{r-j}}{\left(\frac{1}{2} + \frac{b}{2}\right)^r} (r - j).$$

The sum above is simply the number of successes in r Bernoulli trials with the probability of success equal to $(\frac{1}{2} - \frac{b}{2})/(\frac{1}{2} + \frac{b}{2}) = (1-b)/(1+b)$. Namely, it is r(1-b)/(1+b). We obtain

(48)
$$\lim_{n} \frac{\mathcal{I}_{w}(n,r)}{n} = \left(\frac{1}{2} + \frac{b}{2}\right)^{r} \left(1 + \frac{r(1-b)}{2+2b}\right) = \frac{(1+b)^{r-1}}{2^{r+1}} (2+2b+r-rb).$$

Recall from Lemma 11 that $(1+b)^{r-1}/2^{r-1} = 1-b$. This proves (12) Plugging the corresponding value of b for r=2,3,4, we obtain $\lim_n \frac{\mathcal{I}_w(n,2)}{n} = \frac{2}{3}$, $\lim_n \frac{\mathcal{I}_w(n,3)}{n} = .6077$ and $\lim_n \frac{\mathcal{I}_w(n,4)}{n} = .5632$. This concludes the proofs for the case of maximum weight independent set in $G_3(n)$ and $G_4(n)$. Before we continue with other cases, it is convenient to prove Corollary 4, as the proof is almost immediate from above.

Proof : [Proof of Corollary 4] We have established above that $T_{\mathcal{I},r}^2$ has a unique fixed point iff $r \leq 4$. Applying (10) we need to compute $\mathbb{E}[1\{W - \sum_{1 \leq i \leq r} B_i > 0\}] = \mathbb{P}(W > \sum_{1 \leq i \leq r} B_i)$, where $W \sim \text{Exp}(1)$, $B_i \sim F^*$. Instead of computing this quantity directly, note that the probability above is exactly 1 - b, if we the summation above was up to r - 1 not r. Repeating the computations up to (39), we obtain

$$\mathbb{P}(W > \sum_{1 \le i \le r} B_i) = (\frac{1+b}{2})^r.$$

Plugging the obtained values of b for r = 2, 3, 4, we obtain (18).

• Independent sets in G(n, c/n).

Let F^* denote now any fixed point of $T^2 = T_{\mathcal{I},c}^2$. We introduce again $b = F^*(0) = \mathbb{P}(W - \sum_{1 \leq i \leq m} \hat{B}_i \leq 0)$, where $\hat{B}_i \sim T(F^*)$ and $m \sim \text{Pois}(c)$. Similarly, $\hat{b} = T(F^*)(0) = \mathbb{P}(W - \sum_{1 \leq i \leq m} B_i \leq 0)$, where $B_i \sim F^*$. Repeating the computations done for $G_r(n)$, we obtain similarly to (39) that conditioning on m = k

$$b = 1 - (\frac{1 + \hat{b}}{2})^k.$$

Then $b = \sum_{k \geq 0} \frac{c^k}{k!} e^{-c} (1 - (\frac{1+\hat{b}}{2})^k) = 1 - e^{-\frac{c(1-\hat{b})}{2}}$. Similarly, $\hat{b} = 1 - e^{-\frac{c(1-b)}{2}}$. Thus, b must satisfy $1 - b = \exp(-\frac{c}{2}\exp(-\frac{c}{2}(1-b)))$. Recall, however, that by the second part of Theorem 1, the equation above has a unique solution iff $c \leq 2e$. In this case b is also the unique solution of $1 - b = \exp(-\frac{c}{2}(1-b))$. We now apply (7) of Theorem 2 to compute $\lim_n \frac{\mathcal{I}_w(n,c)}{n}$, where we substitute $m \sim \operatorname{Pois}(c)$ for r. In order to shortcut the computations, we use (45) and (48). We have

$$\lim_{n} \frac{\mathcal{I}_{w}(n,c)}{n} = \sum_{k>0} \frac{c^{k}}{k!} e^{-c} \int_{0}^{\infty} t e^{-t} \mathbb{P}(t > \sum_{1 \le i \le k} B_{i}) dt,$$

where $B_i \sim F^*$. Recall, though, from (48), that

$$\int_0^\infty t e^{-t} \mathbb{P}(t > \sum_{1 \le i \le k} B_i) dt = (\frac{1+b}{2})^k (1 + \frac{k(1-b)}{2(1+b)}).$$

Combining

$$\lim_{n} \frac{\mathcal{I}_{w}(n,c)}{n} = e^{-\frac{c(1-b)}{2}} + \frac{(1-b)}{2(1+b)} \frac{c(1+b)}{2} e^{-\frac{c(1-b)}{2}} = (1 + \frac{c(1-b)}{4})e^{-\frac{c(1-b)}{2}}.$$

Using $b = 1 - \exp(-\frac{c(1-b)}{2})$, we obtain (14).

• Matchings in $G_r(n)$.

Let F^* denote any fixed point distribution of $T = T_{\mathcal{M},r}$ Then we have the following distributional identities $\hat{B} = \max_{1 \leq i \leq r-1} (0, W_i - B_i), W_i \sim \operatorname{Exp}(1), B_i \sim F^*, \hat{B} \sim T(F^*)$ and $B = \max_{1 \leq i \leq r-1} (0, W_i - \hat{B}_i), W_i \sim \operatorname{Exp}(1), \hat{B}_i \sim T(F^*), B \sim F^*$. Let $\hat{b} = \mathbb{P}(W - \hat{B} < 0)$, where $W \sim \operatorname{Exp}(1)$ and $\hat{B} \sim T(F^*)$, and let $b = \mathbb{P}(W - B < 0)$, where $B \sim F^*$. Then for any $t \geq 0$

$$(49) \ \mathbb{P}(B \le t) = \mathbb{P}(\max_{1 \le i \le r-1} (W_i - \hat{B}_i) \le t) = (1 - \mathbb{P}(W_1 > t + \hat{B}_1))^{r-1} = (1 - e^{-t}(1 - \hat{b}))^{r-1}.$$

In particular

(50)
$$\mathbb{P}(B=0) = \hat{b}^{r-1}, \quad d\mathbb{P}(B \le t) = (r-1)(1-\hat{b})e^{-t}(1-e^{-t}(1-\hat{b}))^{r-2}dt, \ t > 0$$

(We note as above that $\mathbb{P}(W_1 = \cdot) = 0$, since W_1 has a continuous distribution). Then

$$b = \int_{0}^{\infty} e^{-t} \mathbb{P}(B > t) dt$$

$$= \int_{0}^{\infty} e^{-t} (1 - (1 - e^{-t} (1 - \hat{b}))^{r-1}) dt$$

$$= 1 - \int_{0}^{\infty} e^{-t} (1 - e^{-t} (1 - \hat{b}))^{r-1} dt$$

$$= 1 - \int_{0}^{\infty} e^{-t} (1 - e^{-t} (1 - \hat{b}))^{r-1} dt$$

$$= 1 - \int_{0}^{\infty} (1 - e^{-t} (1 - \hat{b}))^{r-1} d(-e^{-t})$$

$$= 1 - \int_{0}^{1} (1 - z(1 - \hat{b}))^{r-1} dz$$

$$= 1 - \frac{(1 - z(1 - \hat{b}))^{r}}{r(-(1 - \hat{b}))} \Big|_{0}^{1}$$

$$= 1 - \frac{1 - \hat{b}^{r}}{r(1 - \hat{b})}.$$
(51)

Similarly, we obtain

$$\hat{b} = 1 - \frac{1 - b^r}{r(1 - b)},$$

and combining we conclude that b must be a fixed point of the equation f(f(x)) = x, where

$$f(x) = 1 - \frac{1 - x^r}{r(1 - x)}.$$

Lemma 12 For every $r \geq 2$ the equation f(f(x)) = x has a unique solution x^* in the range [0,1], which is the unique solution of the equation f(x) = x.

Proof: Note that $(1-x^r)/(1-x) = 1 + x + \cdots + x^{r-1}$ and therefore f(x) is a strictly decreasing function with f(0) = 1 - 1/r, f(0) = 0. Therefore f(x) = x has exactly one solution x^* . We now prove that f(f(x)) > x for all $x < x^*$ and f(f(x)) < x for all $x > x^*$. This would complete the proof of the lemma. We need to show that for $x < x^*$

$$f(f(x)) = 1 - \frac{1 - f^r(x)}{r(1 - f(x))} > x,$$

which is equivalent to $1-f^r(x) < r(1-f(x))(1-x)$. But since f is a decreasing function and $f(x^*) = x^*$, then f(x) > x for all $x < x^*$ and therefore $1-f^r(x) < 1-x^r = r(1-f(x))(1-x)$. Similarly, when $x > x^*$ we have $f(x) < x^*$ and then $1-f^r(x) > 1-x^r = r(1-f(x))(1-x)$, resulting in

$$f(f(x)) = 1 - \frac{1 - f^r(x)}{r(1 - f(x))} < x.$$

We conclude that $b = \hat{b}$ is determined as the unique solution of

(52) $b = 1 - \frac{1 - b^r}{r(1 - b)},$

and the unique fixed point of T^2 is the distribution given by (49) with $b = \hat{b}$ given above. Now, using (8) of Theorem 2 and (50) we have

$$\lim_{n} \frac{\mathcal{M}_{w}(n,r)}{n} = \mathbb{E}\left[\sum_{1 \leq i \leq r} W_{i} 1\{W_{i} - B_{i} = \max_{1 \leq j \leq r} (0, W_{j} - B_{j})\}\right],$$

$$= r \mathbb{E}\left[W_{1} 1\{W_{1} - B_{1} > \max_{2 \leq j \leq r} (0, W_{j} - B_{j})\}\right],$$

$$= r \int_{0}^{\infty} t e^{-t} b^{r-1} \mathbb{P}(t > \max_{2 \leq j \leq r} (W_{j} - B_{j})) dt$$

$$+ r \int_{0}^{\infty} \int_{0}^{t} t e^{-t} (r - 1)(1 - b) \times$$

$$\times e^{-z} (1 - e^{-z} (1 - b))^{r-2} \mathbb{P}(t - z > \max_{2 \leq j \leq r} (W_{j} - B_{j})) dt dz,$$
(54)

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where the summands (53) and (54) corresponds to conditioning on $B_1 = 0$ and $B_1 = z > 0$. We now compute the integrals in these summands. We have

$$\begin{split} \int_0^\infty t e^{-t} b^{r-1} \mathbb{P}(t > \max_{2 \le j \le r} (W_j - B_j)) dt &= \int_0^\infty t e^{-t} b^{r-1} \mathbb{P}^{r-1} (t > W_2 - B_2) dt \\ &= \int_0^\infty t e^{-t} b^{r-1} (1 - \mathbb{P}(W_2 > t + B_2))^{r-1} dt \\ &= \int_0^\infty t e^{-t} b^{r-1} (1 - e^{-t} \mathbb{P}(W_2 > B_2))^{r-1} dt \\ &= \int_0^\infty t e^{-t} b^{r-1} (1 - e^{-t} (1 - b))^{r-1} dt \end{split}$$

Similarly, we obtain that the integral in (54) is equal to

$$= \int_{0}^{\infty} \int_{0}^{t} t e^{-t} (r-1)(1-b)e^{-z} (1-e^{-z}(1-b))^{r-2} (1-e^{-t}(1-b)))^{r-1} dt dz$$

$$= \int_{0}^{\infty} \int_{0}^{t} t e^{-t} (r-1)(1-b)(1-e^{-z}(1-b))^{r-2} (1-e^{-t}(1-b)))^{r-1} dt d(-e^{-z})$$

$$= \int_{0}^{\infty} t e^{-t} (r-1)(1-b)(1-e^{-t}(1-b)))^{r-1} dt \int_{0}^{e^{-t}} (1-w(1-b))^{r-2} dw$$

$$= \int_{0}^{\infty} t e^{-t} (1-e^{-t}(1-b)))^{r-1} dt ((1-w(1-b))^{r-1} \Big|_{e^{-t}}^{0}$$

$$= \int_{0}^{\infty} t e^{-t} (1-e^{-t}(1-b)))^{r-1} dt - \int_{0}^{\infty} t e^{-t} (1-e^{-t}(1-b)))^{2r-2} dt$$

Substituting to summands in (53) and (54), we obtain (15).

• Matchings in G(n, c/n).

The derivation is very similar to the one for $\mathcal{M}_w(n,r)$. We introduce b and \hat{b} exactly as above. The equation (51) becomes

$$b = \sum_{m \ge 0} \frac{c^m}{m!} e^{-c} \left(1 - \frac{1 - b^{m+1}}{(m+1)(1-\hat{b})}\right)$$

$$= 1 - \frac{1}{c(1-\hat{b})} \sum_{m \ge 0} \frac{c^{m+1}}{(m+1)!} e^{-c} + \frac{1}{c(1-\hat{b})} \sum_{m \ge 0} \frac{c^{m+1}\hat{b}^{m+1}}{(m+1)!} e^{-c}$$

$$= 1 - \frac{1}{c(1-\hat{b})} + \frac{e^{-c(1-\hat{b})}}{c(1-\hat{b})},$$

Then 1-b is a fixed point of the equation f(f(x))=x, where $f(x)=\frac{1-e^{-cx}}{cx}$. That is

$$\frac{1 - e^{c\frac{1 - e^{-cx}}{cx} - c}}{c\frac{1 - e^{-cx}}{cx}} = x.$$

First we note that x=0 does not satisfy f(f(x))=x, so we assume x>0. The expression above then becomes $e^{-cx}+cx^2-1=0$. The function $e^{-cx}+cx^2-1$ is a strictly convex function which is equal to zero at x=0, has derivative -c<0 at x=0 and diverges to infinity as x diverges to infinity. Therefore its graph has exactly one intersection with the horizontal axis in $x \in (0,\infty)$. Note that at x=1 the value is $e^{-c}+c-1>0$ (for every c>0) therefore there exists exactly solution to $e^{-cx}+cx^2-1=0$ in $x \in (0,1)$. We conclude that b is uniquely determined by the equation $e^{-c(1-b)}+cb^2-1=0$. The remainder of the calculation is done just like for $G_r(n)$ by conditioning first on specific values of r and recalling m=r with probability $\frac{c^r}{r}!e^{-c}$. We omit the fairly straightforward calculations. \square

5.3 Extensions: Cycles and (non-random) low degree graphs

In this section we first prove Corollary 3. We then focus on proving Theorem 4, thereby improving the previous bound of Hopkins and Staton [HS82].

Proof: [Proof of Corollary 3] The result is essentially established en route of proving Theorems 2 and 3. Recall that in the proof of Theorem 2 the only place we used the randomness of our regular graph $G_r(n)$ was to say that a constant depth-d neighborhood H(d) of a randomly selected node $i \in [n]$ is a depth-d r-regular tree w.h.p. In the case of a cycle any constant depth-d neighborhood of any node is w.p.1 a path of length 2d with the selected node in the middle. This is depth-d 2-regular tree. The answer for the maximum weight matching on a cycle is the same as for independent set since we may simply assume the weights are assigned to nodes which are to the left for each edge.

Proof : [Proof of Theorem 4] As in the proof of Corollary 3, we recall that the only place the randomness of $G_r(n)$ was used in the proof of Theorem 2 and Theorem 3 was the fact that a constant depth-d neighborhood of a randomly chosen node $i \in [n]$ is w.h.p. a depth d r-regular tree. In order to continue the proof we introduce a class of almost r-regular (non-random) graphs with large girth. Given positive integers d, R let $G_r(n, d, R)$ denote the class of n-node graphs with girth at least d and such that all but at most R nodes have degree r and the remaining nodes have degree less than r. Given $G \in \mathcal{G}_r(n, d, R)$ a random node $i \in [n]$ its depth-d neighborhood H(i, d) is w.h.p. (with respect to randomness of the choice of i) a depth-d r-regular tree. Repeating the proofs of Theorem 2 and 3 and Corollary 4 we conclude that for the case r = 3 for every constant R and for all sufficiently large n, d, the expected maximum weight of an independent set in G is given by (13) with r = 3, that is $\approx .6077n$. Also by Corollary 4, the expected cardinality of the independent set achieving this weight is given by (18) with r = 3, that is $\approx .3923n$, also for all n and d sufficiently large. Then the maximum cardinality of an independent set in G is at least .3923n - o(n) and this is a non-probabilistic bound since we consider a maximum cardinality independent set in a fixed graph G. Thus for every constant R

$$\liminf_{d} \liminf_{n} \min_{G \in \mathcal{G}_3(n,d,R)} \frac{\mathcal{I}(G)}{n} \ge .3923...$$

To finish the proof we need to obtain similar lower bound for the class $\mathcal{G}(n,3,d)$ – all graphs with degree at most r=3 and girth at least d. Suppose we are given any such n-node graph G. Consider any two nodes i_1, i_2 which have degree < 3 such that $i_2 \notin H(i_1, d)$ (if any such pair exist). Connect i_1, i_2 by an edge. This operation can only decrease the size of the largest

independent set in G. We claim that in addition the resulting graph still has girth at least d. In fact, if the added edge becomes a part of a cycle with length less than d, then in the original graph the nodes i_1, i_2 were connected by path of length at most d-2, contradicting the condition $i_2 \notin H(i_1, d)$. Therefore the resulting graph still has girth at least d. We continue this operation for every such pair i_1, i_2 until we either do not have any nodes with degree less than 3 at all or all such nodes belong to H(i, d) for some node i. Since $|H(i, d)| \le 1 + 3 + 3^2 + \dots 3^d \equiv R$, we obtain a graph $G' \in \mathcal{G}_r(n, d, R)$. Then

$$\liminf_{d} \liminf_{n} \min_{G \in \mathcal{G}(n,3,d)} \frac{\mathcal{I}(G)}{n} \geq \liminf_{d} \liminf_{n} \min_{G \in \mathcal{G}_{3}(n,d,R)} \frac{\mathcal{I}(G)}{n} \geq .3923n.$$

This completes the proof of the theorem.

5.4 Deterministic and Bernoulli weights

Are the results obtained above relevant to the case when the weight of each node and edge W=1with probability one or, in general, when the weights take some discrete values? Let us examine these questions with respect to our usual four operators T. We start with the case W=1. For $T = T_{\mathcal{I},r}$, the corresponding distributional equation is $B = \max(0, 1 - \sum_{1 \le i \le r-1} B_i)$. If $B_i = 0$, w.p.1, then B = 1, w.p.1, and vice verse. Thus B = 0 and B = 1 are two fixed points of T^2 , and T^2 does not have a unique fixed point distribution. There is a physical explanation for the lack of uniqueness, coming directly from the lack of long-range independence. Given a depth-d r-regular tree T with all the weights equal to unity, note that the boundary does carry a non-vanishing information about the root in the following sense. If all the leaves of the tree (boundary nodes) are conditioned to belong to the maximum weight independent set, then all the parents of leaves cannot be part of the set. Then the maximum independent set is obtained by selecting all the nodes in level d, not selecting nodes in level d-1, selecting all the nodes in level d-2, and so on. In the end whether the root is selected is fully determined by the parity of d. Thereby we do not have a long-range independence. Contrast this with the discussion in Brightwell and Winkler [BW03], where similar observation is used to show long-range dependence for Gibbs measures on infinite regular trees for the hard-core model. It is not hard to see that the similar lack of long-range independence holds for maximum weight matchings, where the weights are all 1.

The situation is different for $T = T_{\mathcal{I},c}$. Let F be a distribution function given by $F(t) = p, t \in [0,1), F(1) = 1$. Namely, F is simply a Bernoulli distribution with parameter p (Be(p)). If $B = \max(0, 1 - \sum_{1 \le i \le m} B_i)$, where $B_i \sim F$ and $m \sim \operatorname{Pois}(c)$, then B = 1 if $\sum_{1 \le i \le m} B_i = 0$, which occurs with probability $\sum_{k \ge 0} \frac{c^k}{k!} e^{-c} p^k = e^{-c(1-p)}$, and B = 0 otherwise. Thus T(F) is $\operatorname{Be}(p_1)$ where $p_1 = 1 - e^{-c(1-p)}$. Similarly $T^2(F)$ is $\operatorname{Be}(p_2)$, with with $p_2 = 1 - e^{-ce^{-c(1-p)}}$. In general, for $s = 1, 2, \ldots, T^{2s}(F)$ is $\operatorname{Be}(p_{2s})$ with $1 - p_{2s} = e^{-ce^{-c(1-p_{2s-2})}}$. By Proposition 1 we know that for $F = \operatorname{Be}(0)$ and $F = \operatorname{Be}(1)$, $T^{2s}(F)$ converges to some fixed point distributions F_{**} , F^{**} which by the argument above are $\operatorname{Be}(p_{**})$, $\operatorname{Be}(p^{**})$ with both $p = p_{**}$ and $p = p^{**}$ satisfying $1 - p = e^{-ce^{-c(1-p)}}$. Recall from Theorem 1, that the equation $x = e^{-ce^{-cx}}$ has a unique solution iff $c \le e$. By Proposition 1 this implies that when $c \le e$, $T^s(F_0)$ converges to $\operatorname{Be}(p^*)$ for any starting distribution F_0 . Again applying Proposition 1 we obtain that $T^s(F_0)$ converges to $\operatorname{Be}(p^*)$, T and T^2 have the same unique fixed point distribution $-\operatorname{Be}(p^*)$, where p^* is also the unique solution of $1 - p^* = e^{-c(1-p^*)}$.

It is a simple exercise to see that the same holds for $T = T_{\mathcal{M},c}$: T^2 has a unique fixed point iff $c \leq e$, in which case the fixed point distribution is also $Be(p^*)$. This is, of course, fully consistent with Theorem 1. We summarize these observations.

Proposition 2 Let the nodes and edges weights be equal to one, w.p.1. For every $r \geq 1$, $T_{\mathcal{I},r}^2, T_{\mathcal{M},r}^2$ have at least two fixed point distributions. $T_{\mathcal{I},c}^2, T_{\mathcal{M},c}^2$ have the unique fixed point distribution iff $c \leq e$ in which case the unique fixed point distribution is $Be(p^*)$, where p^* is the unique solution of $1 - p^* = e^{-c(1-p^*)}$.

Can we fully reproduce Theorem 1 for the case $c \leq e$? The problem with the case $F_w = 1$ as, generally, with non-continuous distributions F_w , is that the probability of $W - \sum_{1 \leq i \leq r} B_i = 0$ is no longer zero. As a result we do not have an exact condition purely in terms of $B = \max(0, W - \sum_{1 \leq i \leq r} B_i)$ for whether the root node belongs to say maximum weight independent set. One natural approach would be to approximate F_w with a continuous distribution. But the difficulty is the lack of closed form expression for the solution of fixed point of $T^2(F^*) = F^*$. Such a solution F^* can, though, also be approximated by computing $T^{2s}(0)$ and $T^{2s}(F_w)$ ($T^{2s}(F_{w,r}), T^{2s}(F_{w,c})$) for matching) for large s such that the differences between the two distributions is sufficiently small.

Suppose now the weights W_i of the nodes are distributed as Be(z) for some parameter $z \in [0, 1]$. For simplicity we will only consider the case r = 3 and $T = T_{\mathcal{I},3}$ and obtain a complete criteria for uniqueness.

Proposition 3 For r=3 the operator $T_{\mathcal{I},r-1}^2$ has a unique fixed point distribution iff $z \in [\frac{1}{4},1]$. In this case the fixed point distribution is Be(p) with $p=\frac{\sqrt{5-4z}-1}{2(1-z)}$.

Proof: Let F = Be(p) for any $x \in [0, 1]$. Then for $B' \sim T(F)$ we have $B' = \max(0, W - B_1 - B_2)$ where B_1, B_2 are independent and distributed as F. Then B' = 0 when W = 0 or when W = 1 and $B_1 + B_2 > 0$. This occurs with probability $z + (1 - z)(1 - p^2) = 1 - (1 - z)p^2$. Thus $B' \sim \text{Be}(1 - (1 - z)p^2)$. Repeating the development above for the case of the deterministic weight, we need to analyze the number of the solutions to the equation f(f(x)) = x where $f(x) = 1 - (1 - z)x^2$. First, the equation f(x) = x leads to the unique solution $x^* = \frac{-1 + \sqrt{5 - 4z}}{2(1 - z)}$ (it is simple to check that the solution is in [0, 1] for every $z \in [0, 1]$). Also x^* is a solution of $g(x) \equiv f(f(x)) = x$. We need to show that this is the unique solution of this equation iff z > 1/4. First we show that when $z \in 1/4$ $\dot{x}(x) > 1$. Since g(0) > 0 this would imply that there exists

First we show that when z < 1/4, $\dot{g}(x) > 1$. Since g(0) > 0 this would imply that there exists a solution of g(x) = x in the open interval $(0, x^*)$ and the case z < 1/4 would be resolved. We have $\dot{g}(x^*) = 4(1-z)^2 f(x^*) x^* = (2(1-z)x^*)^2$, where we use $f(x^*) = x^*$. Then the inequality $\dot{g}(x) > 1$ holds when $x^* > 1/(2-2z)$ which after as simple algebra is reduced to z < 1/4. Thus when z < 1/4 there are more than one fixed points of $T^2 = T_{L,2}^2$.

Suppose now $z \ge 1/4$. Let again x^* be the unique solution to $f(x^*) = x^*$ and consider $\dot{g}(x) = 4(1-z)^2 f(x) x$. We claim that when z > 1/4, $\dot{g}(x) < 1$ for all $x \in [0,1]$ and when z = 1/4, $\dot{g}(x) < 1$ for all $x \ne x^*$ and $\dot{g}(x^*) = 1$. This immediately implies that g(x) = x has at most one solution, meaning it has exactly one since $g(x^*) = x^*$. Note $\dot{g}(0) = 0 < 1$. We now prove that $\dot{g}(1) = 4(1-z)^2 z < 1$ and $\dot{g}(x) < 1$ for all x such that $\frac{d^2 g(x)}{dx^2} = 0$, except for $x = x^*$ for which we will show that $\dot{g}(x^*) = 1$.

We start with $\dot{g}(1)=4(1-z)^2z\equiv\phi(z)$. Note $\phi(0)=\phi(1)=0<1$. Then the maximum is achieved at the points z where $\dot{\phi}(z)=-8(1-z)z+4(1-z)^2=4(1-z)(1-3z)=0$. We have already considered the case z=1. Otherwise z=1/3 for which $\phi(1/3)=16/27<1$. Thus $\dot{g}(1)\leq \sup_{z\in[0,1]}\phi(z)<1$.

Consider now points x such that $\frac{d^2g(x)}{dx^2} = 4(1-z)^2[-2(1-z)x^2 + 1 - (1-z)x^2] = 0$, from which we obtain $x = 1/\sqrt{3-3z}$. Plugging this into

$$\dot{g}(x) = 4(1-z)^2(1-(1-z)x^2)x = 4(1-z)^2\frac{2}{3}\frac{1}{\sqrt{3-3z}} = (\frac{64(1-z)^3}{27})^{\frac{1}{2}}.$$

The last expression is smaller than unity whenever z > 1/4. When z = 1/4 the expression is equal to the unity. In this case, however, $x = 1/\sqrt{3-3z} = 2/3$ which is checked to be equal to $x^* = \frac{-1+\sqrt{5-4z}}{2(1-z)}$ when z = 1/4. This concludes the proof of the proposition.

6 Conclusions

We have derived in this paper the limits of maximum weight independent sets and matchings in sparse random graphs for some types of i.i.d weight distributions. Our method is based on a certain *local optimality* property which states loosely that in cases of certain distributions of the random weights, the optimal random combinatorial structure under the consideration exhibits a long-range independence and, as a result, the value which each node (edge) "contributes" to the optimal structure is almost completely determined by the constant depth neighborhood of the node (edge). We certainly believe that such local optimality holds for many other random combinatorial structures and it seems to be an interesting property to study by itself, not to mention its applications to studying random combinatorial structures.

References

- [AB] D. Aldous and A. Bandyopadhyay, A survey of max-type recursive distributional equations, Preprint.
- [Ald] D. Aldous, Some open problems, http://stat-www.berkeley.edu/users/aldous/ Research/problems.ps.
- [Ald92] _____, Asymptotics in the random assignment problem, Probab.Th.Rel.Fields (1992), no. 93, 507–534.
- [Ald01] _____, The $\zeta(2)$ limit in the random assignment problem, Random Structures and Algorithms (2001), no. 18, 381–418.
- [APF98] J. Aronson, B. Pittel, and A. Frieze, Maximum matchings in sparse random graphs: Karp-Sipser revisited, Random Structures and Algorithms 12 (1998), 11–178.
- [AS03] D. Aldous and J. M. Steele, *The objective method: Probabilistic combinatorial optimization and local weak convergence*, Discrete Combinatorial Probability, H. Kesten Ed., Springer-Verlag, 2003.

- [Bol80] B. Bollobas, A probabilistic proof of an asymptotic formula for the number of regular graphs, European J. Combinatorics 1 (1980), 311–316.
- [Bol81] _____, The independence ratio of regular graphs, Proc. Amer. Math. Soc. 83 (1981), no. 2, 433–436.
- [BW03] G.R. Brightwell and P. Winkler, Gibbs extremality for the hard-core model on a Bethe lattice, Preprint (2003).
- [Dur96] R. Durrett, Probability: theory and examples, Duxbury Press, second edition, 1996.
- [FS94] A. Frieze and S. Suen, On the independence number of random cubic graphs, Random Structures and Algorithms 5 (1994), 649–664.
- [Gam02] D. Gamarnik, Linear phase transition in random linear constraint satisfaction problems, Preprint, 2002.
- [HS82] G. W. Hopkins and W. Staton, Girth and independence ratio, Canad. Math. Bull. 25 (1982), 179–186.
- [HW01] A. K. Hartmann and M. Weigt, Statistical mechanics perspective on the phase transition of vertex covering of finite-connectivity random graphs, Theoretical Computer Science **265** (2001), 199–225.
- [JLR00] S. Janson, T. Luczak, and A. Rucinski, *Random graphs*, John Wiley and Sons, Inc., 2000.
- [KS81] R. Karp and M. Sipser, *Maximum matchings in sparse random graphs*, 22nd Annual Symposium on Foundations of Computer Science, 1981, pp. 364–375.
- [Mar03] J. Martin, Reconstruction thresholds on regular trees, Preprint (2003).
- [Mos03] E. Mossel, Survey: information flow on trees, Preprint (2003).
- [RS03] U. A. Rozikov and U. M. Suhov, A hard-core model on a cayley tree: an example of a loss network, Preprint (2003).
- [Spe94] Joel H. Spencer, Ten lectures on the probabilistic method, second ed., SIAM, 1994.
- [Ste02] J. M. Steele, *Minimal spanning trees for graphs with random edge lenghts*, Mathematics and Computer Science II. Algorithms, Trees, Combinatorics and Probabilities. Editors: B. Chauvin, P. Flajolet, D. Gardy, A. Mokkadem. (2002), 223–246.
- [Tal03] M. Talagrand, An assignment problem at high temperature, Annals of Probability **31** (2003), no. 2, 818–848.