The Chance that a Convex Body Is Lattice-Point Free: A Relative of Buffon's Needle Problem

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ABSTRACT: Given a convex body $K \subset \mathbf{R}^d$, what is the probability that a randomly chosen congruent copy, K^*, of K is lattice-point free, that is, $K^* \cap \mathbf{Z}^d = \emptyset$? Here $\mathbf{Z}^d$ is the usual lattice of integer points in $\mathbf{R}^d$. Luckily, the underlying probability is well defined since integer translations of K can be factored out. The question came up in connection with integer programming. We explain what the answer is for convex bodies of large enough volume. © 2006 Wiley Periodicals, Inc. Random Struct. Alg., 30, 414–426, 2007

1. INTRODUCTION

Let $\mathbf{Z}^d$ denote the integer lattice in the d-dimensional Euclidean space $\mathbf{R}^d$. A random copy, L, of $\mathbf{Z}^d$ is just $L = L_{\rho,t} = \rho(\mathbf{Z}^d + t)$ where $t \in [0,1)^d$ is a translation vector and $\rho \in SO(d)$ is a rotation of $\mathbf{R}^d$ around the origin. We can, of course, replace $[0,1)^d$ by any other basis parallelotope of $\mathbf{Z}^d$. Setting

$$\mathcal{L} = \{L_{\rho,t} : \rho \in SO(d), t \in [0,1)^d\},$$

there is a probability measure Prob on $\mathcal{L}$, which is the product of the Lebesgue measure on $[0,1)^d$ and of the normalized Haar measure on $SO(d)$. The following question, which is a

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distant relative of Buffon's needle problem, emerged while investigating [2] the *randomized integer convex hull*, $I_L(K) = \text{conv}(K \cap L)$ of a convex body $K \subset \mathbf{R}^d$. What is the probability that $K \cap L = \emptyset$? Note that in the abstract, the same question is formulated slightly differently.

This probability is clearly zero if K is "large," for instance, if it contains a ball of radius $\sqrt{d}/2$. But it is not zero if K is "flat." We show first an upper bound for the probability in question. Let $\mathcal{K}^d$ denote the set of all convex bodies (i.e., convex compact sets with nonempty interior) in $\mathbf{R}^d$.

Theorem 1.1. *For every $d \geq 2$ there exist positive constants $c_1(d)$ and $c_2(d)$ such that for every $K \in \mathcal{K}^d$ with $\text{Vol}\, K \geq c_2(d)$,*

$$\text{Prob}[K \cap L = \emptyset] \leq \frac{c_1(d)}{\text{Vol}\, K}.$$

Our next theorem shows that this result is the best possible apart from the constants c_i. We need a definition. Given a unit vector $t \in S^{d-1}$, the *width* of $K \in \mathcal{K}^d$ in direction t is defined as

$$w(K, t) = \max\{t(x - y) : x, y \in K\},$$

and the *width*, or *geometric width* of K is

$$w(K) = \min\{w(K, t) : t \in S^{d-1}\}.$$

Theorem 1.2. *For every $d \geq 2$ there exist positive constants $b_1(d), b_2(d)$, and w_d such that for every $K \in \mathcal{K}^d$ with $\text{Vol}\, K \geq b_2(d)$ and $w(K) \leq w_d$*

$$\text{Prob}[K \cap L = \emptyset] \geq \frac{b_1(d)}{\text{Vol}\, K}.$$

The constant w_d is not too small: we can take it to be $1/(2d^{3/2})$ for instance. What Theorems 1.1 and 1.2 state is that $\text{Prob}[K \cap L = \emptyset]$ is of order $1/\text{Vol}\, K$ for convex bodies K with large volume and $w(K) \leq w_d$. It is not clear (at least for the author) for which convex body of volume V the probability in question is the largest.

Using Vinogradov $\ll$ notation these results can be formulated more concisely as

$$\text{Prob}[K \cap L = \emptyset] \ll \frac{1}{\text{Vol}\, K}$$

for every $K \in \mathcal{K}^d$ of large volume and as

$$\text{Prob}[K \cap L = \emptyset] \gg \frac{1}{\text{Vol}\, K}$$

for every $K \in \mathcal{K}^d$ of large volume and small geometric width. Theorems 1.1 and 1.2 imply the following.

Corollary 1.3. *For every $d \geq 2$, as $V \to \infty$,*

$$\frac{1}{V} \ll \sup\{\text{Prob}[K \cap L = \emptyset] : K \in \mathcal{K}^d,\ \text{Vol}\, K = V\} \ll \frac{1}{V}.$$

The planar case of both Theorems is proved in [2]. So we assume, from now on, that $d \geq 3$. The paper is organized as follows. The next section explains the application of the above results for the randomized integer convex hull. In Section 3 notation, terminology, and some basic observations are described. Sections 4 and 5 contain the proofs of Theorems 1.1 and 1.2.

2. APPLICATION: THE RANDOMIZED INTEGER CONVEX HULL

For $K \in \mathcal{K}^d$ define the function $u : K \to \mathbf{R}$ by

$$u(x) = \mathrm{Vol}\,(K \cap (x - K)),$$

that is, $u(x)$ is the volume of the so-called Macbeath region, which is the intersection of K with K reflected around the point $x \in K$. Information on properties of the Macbeath region and $u(x)$ is available in [3, 6, 10] or [1]. We also set

$$K(u \leq t) = \{x \in K : u(x) \leq t\}.$$

For $D > 1$ define $\mathcal{K}_D = \mathcal{K}_D^d$ as the set of all $K \in \mathcal{K}^d$ for which $R/r \leq D$, where R and r denote the radii of the circumscribed and inscribed ball of K. In [2] we showed that the expected number, $E(f_0(I_L(K))$, of vertices of the randomized integer convex hull of a $K \in \mathcal{K}_d$ satisfies

$$\mathrm{Vol}\,K(u \leq 1) \ll E(f_0(I_L(K)) \ll \mathrm{Vol}\,K(u \leq 1)$$

as $\mathrm{Vol}\,K$ goes to infinity. It is known, see [3] for instance, that

$$(\log \mathrm{Vol}\,K)^{d-1} \ll \mathrm{Vol}\,K(u \leq 1) \ll (\mathrm{Vol}\,K)^{(d-1)/(d+1)},$$

where the implied constants depend only on d. Moreover, these estimates are best possible: the lower bound is reached for polytopes and the upper bound for smooth convex bodies.

Given $K \in \mathcal{K}_d$ and $L \in \mathcal{L}$, the missed volume is

$$M(K,L) = \mathrm{Vol}\,(K \setminus I_L(K)).$$

The expected missed volume is then the expectation of $M(K,L)$ over $L \in \mathcal{L}$:

$$M(K) := EM(K,L).$$

We proved in [2] that, for $K \in \mathcal{K}_D$ in the planar case

$$\int_K \frac{dx}{1+u(x)} \ll M(K) \ll \int_K \frac{dx}{1+u(x)}.$$

For $d \geq 3$ Theorems 1.1 and 1.2 provide an identical upper bound and a weaker lower bound for $M(K)$. To state the results we introduce some new terminology. The function $v : K \to \mathbf{R}$ is defined as

$$v(x) = \min\{\mathrm{Vol}\,K \cap H : x \in H,\ H \text{ is a halfspace}\}.$$

Given $x \in K$ the set $C(x) = K \cap H$ is a *minimal cap* if H is a halfspace, $x \in H$, and $\mathrm{Vol}\, K \cap H = v(x)$. Assume $t \in S^{d-1}$ is the unit normal vector of the bounding hyperplane of H. We write $w(x)$ for the width of $C(x)$ in the direction of t:

$$w(x) = w(C(x), t) = \max\{t(y - z) : y, z \in C(x)\}.$$

The minimal cap of x need not be unique, in which case let $w(x)$ be the supremum of the widths of the minimal caps of x. Finally, for $K \in \mathcal{K}_D$ write K_0 for the set of those $x \in K$ for which $w(x) \le w_d$ where w_d comes from Theorem 1.2.

Theorem 2.1. *If $d \ge 2$ and $D > 1$ and $K \in \mathcal{K}_D$ with $\mathrm{Vol}\, K \to \infty$, then*

$$\int_{K_0 \cap K(u \ge 1)} \frac{dx}{u(x)} \ll M(K) \ll \int_K \frac{dx}{1 + u(x)},$$

where the constants implied by the $\ll$ notation depend only on d and D.

Most likely, the upper and lower bounds are of the same order for every $K \in \mathcal{K}_D$. This is known for $d = 2$ but the proof (see [2]) is very technical. Yet using this theorem one can determine the order of magnitude of $M(K)$ for smooth convex bodies,

$$(\mathrm{Vol}\, K)^{(d-1)/(d+1)} \ll M(K) \ll (\mathrm{Vol}\, K)^{(d-1)/(d+1)},$$

and for polytopes,

$$(\log \mathrm{Vol}\, K)^d \ll M(K) \ll (\log \mathrm{Vol}\, K)^d.$$

In both cases the implied constants depend on K as well. The proofs of Theorem 2.1 and of the inequalities just stated follow those in [2] and are omitted.

3. PREPARATIONS

For $u \in \mathbf{R}^d$, $u \neq 0$ and $\nu > 0$ define

$$S(u, \nu) = \{x \in \mathbf{R}^d \ -\nu \le ux \le \nu\},$$

which is just a slab orthogonal to u and of width $2\nu/|u|$. Here $|u|$ stands for the Euclidean norm of the vector $u \in \mathbf{R}^d$. Given a vector $a = (a_1, \dots, a_d)$ in $\mathbf{R}^d$ with all $a_i > 0$ we define

$$Oct(a) = \mathrm{conv}\{\pm a_1 e_1, \cdots \pm a_d e_d\},$$

where $e_1, \dots, e_d$ is the standard basis of $\mathbf{R}^d$. Clearly, $Oct(a)$ is the octahedron with half-axes a_i in direction e_i.

The Löwner–John theorem (see [5]) states that, given a convex body K in $\mathbf{R}^d$, there is a pair (E, E') of ellipsoids such that $E \subset K \subset E'$, E and E' are concentric, and E arises from E' by shrinking by a factor of $1/d$. We will need a similar result with octahedra replacing the ellipsoids:

Lemma 3.1. *Given a convex body K in $\mathbf{R}^d$, there is a positive vector $a \in \mathbf{R}^d$ such that a congruent copy, K^*, of K satisfies*

$$Oct(a) \subset K^* \subset Oct(d^{3/2} a).$$

Proof. Let (E, E') be the Löwner–John ellipsoid pair for K; let $a_1 \leq a_2 \leq \cdots \leq a_d$ denote the lengths of the half axes of E. Then the ellipsoid $\sum_1^d (x_i/a_i)^2 \leq 1$ contains a congruent copy, K^*, of K. It is trivial to check that $Oct(a) \subset K^* \subset Oct(d^{3/2}a)$. We remark that $2a_1 \leq w(K)$ since the width of E (which is $2a_1$) is at most the width of K because $E \subset K$. ∎

A random element $\rho \in SO(d)$ takes a fixed orthonormal basis $b_1, \ldots, b_d$ of $\mathbf{R}^d$ to another orthonormal basis $\rho b_1, \ldots, \rho b_d$. For simpler notation we write $[d] = \{1, 2, \ldots, d\}$ and we let λ denote the usual rotation invariant $(d-1)$ dimensional measure on S^{d-1} normalized so that $\lambda(S^{d-1}) = 1$. It will be convenient to denote by Prob_ρ the normalized Haar measure on $SO(d)$ since it is a probability measure and we often want to talk about the probability of an event.

Lemma 3.2. *Under the above conditions,*

$$\text{Prob}_\rho[Oct(a) \subset \rho S(u, \nu)] = \lambda \left\{ f \in S^{d-1} : |f_i| \leq \frac{\nu}{a_i |u|} \ \forall i \in [d] \right\}.$$

Proof. Fix an orthonormal basis $b_1, \ldots, b_d$ with $b_1 = u/|u|$ and let $\rho b_1 = f = (f_1, \ldots, f_d)$. Then $\rho S(u, \nu) = S(f, \nu/|u|)$. Here $S(f, \nu/|u|)$ contains $Oct(a)$ if and only if

$$\pm a_i e_i \in S(f, \nu/|u|) \ \forall i \in [d].$$

This is the same as $|a_i e_i f| = a_i |f_i| \leq \nu/|u|$. ∎

As f is a unit vector the probability in the lemma is positive if and only if

$$1 = \sum_1^d f_i^2 < \sum \nu^2 / \left(a_i^2 |u|^2\right).$$

This condition is equivalent to $|u|^2/\nu^2 < \sum a_i^{-2}$, which implies that if the probability in the Lemma is positive, then some a_i must be "small."

Let us consider a vector $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbf{R}^d$ such that $\alpha_i > 0$ for all $i \in [d]$ and $\alpha_i > 1$ for at least one $i \in [d]$. In this case,

$$A = \{f \in S^{d-1} : |f_i| \leq \alpha_i \ \forall i \in [d]\}$$

is nonempty. We have the following estimates.

Lemma 3.3. *With the above notation,*

$$\prod_{i:\alpha_i < 1} \alpha_i \ll \lambda(A) \ll \prod_{i:\alpha_i < 1} \alpha_i.$$

Proof. We only give a sketch of the proof, which goes by induction on d. The case $d = 2$ is clear. For the case $d - 1 \to d$, assume that α_d is the smallest component of α and define $\alpha^* = (\alpha_1, \ldots, \alpha_{d-1})$ and write A^* for the corresponding set in S^{d-2}. The induction hypothesis can be used for A^*. Simple arguments finish the proof; details are left to the reader. ∎

The *lattice width* $W(K)$ of a convex body $K \in \mathcal{K}$ is, by definition,

$$W(K) = \min_{z \in \mathbf{Z}^d,\, z \neq 0} \max\{z(x-y) : x, y \in K\}.$$

If the minimum is reached on $z \in \mathbf{Z}^d$, then z is called the lattice width direction of K. Clearly, such a z is a primitive vector, that is, the g.c.d. of the components of z is 1. We shall denote by $\mathbf{P}$ the set of all primitive vectors in $\mathbf{Z}^d$. Note that $0 \notin \mathbf{P}$. We will need the so-called Flatness Theorem, which is due to Khintchine [9], cf. [8] as well.

Theorem 3.4 (Flatness Theorem). *If $C \in \mathcal{K}^d$ and $C \cap \mathbf{Z}^d = \emptyset$, then $W(C) \leq W_d$, where W_d is a constant depending only on d.*

4. PROOF OF THEOREM 1.1

Assume $K \in \mathcal{K}^d$ with $\operatorname{Vol} K = V$ large. Lemma 3.1 implies the existence of an $a = (a_1, \dots, a_d) \in \mathbf{R}^d$ with $0 < a_1 \leq a_2 \leq \cdots \leq a_d$ such that $V \ll \prod_1^d a_i$ and such that a congruent copy, K^*, of K contains $Oct(a)$. Here we may and do assume that

$$a_1 \leq \frac{a_2}{2} \leq \cdots \leq \frac{a_d}{2^{d-1}}.$$

This can be achieved by keeping a_d the same and replacing a_i by $a_{i+1}/2$ if $a_i > a_{i+1}/2$ recursively for $i = d-1, d-2, \dots, 1$. Clearly, this does not influence the validity of $V \ll \prod_1^d a_i$.

Now we begin the proof. First

$$\operatorname{Prob}[K \cap L = \emptyset] = \operatorname{Prob}[K^* \cap L = \emptyset] \leq \operatorname{Prob}[Oct(a) \cap L = \emptyset].$$

By the Flatness Theorem, $Oct(a) \cap L = \emptyset$ implies that the lattice width (in the lattice L) of $Oct(a)$ is at most W_d, which implies, in turn, that $Oct(a) \subset \rho S(u, W_d/2)$ for some $\rho \in SO(d)$ with suitable $u \in \mathbf{P}$, that is,

$$\operatorname{Prob}[Oct(a) \cap L = \emptyset] \leq \sum_{u \in \mathbf{P}} \operatorname{Prob}_\rho[Oct(a) \subset \rho S(u, W_d/2)].$$

The geometric width of $Oct(a)$ is

$$2\left(\sum_1^d \frac{1}{a_i^2}\right)^{-1/2} \geq 2\left(\sum_{i=1}^d \frac{1}{(2^{i-1}a_1)^2}\right)^{-1/2} > a_1\sqrt{3}.$$

Since $\rho S(u, W_d/2)$ cannot contain a set of width larger than $W_d/|u|$, we have

$$a_1\sqrt{3} < \frac{W_d}{|u|}.$$

In other words, the sum over $u \in \mathbf{P}$ is to be restricted to u with $|u| \leq \frac{W_d}{a_1\sqrt{3}}$. Let $\mathbf{P}^*$ denote the set of these $u \in \mathbf{P}$.

Given such a $u \in \mathbf{P}^*$, let $i = i(u)$ be the smallest index j with

$$\frac{W_d}{a_j|u|\sqrt{3}} < 1.$$

We have seen that $i(u) > 1$. Thus, using Lemmas 3.2 and 3.3, we get for a fixed $u \in \mathbf{P}^*$ that

$$\begin{aligned}\text{Prob}[Oct(a) \subset \rho S(u, W_d/2)] &= \lambda \left\{ f \in S^{d-1} : |f_j| \le \frac{W_d}{2a_j|u|}, j \in [d] \right\} \\ &\ll \prod_{j=i(u)}^{d} \frac{W_d}{2a_j|u|} \ll \prod_{j=2}^{d} \frac{1}{2a_j|u|} \\ &\ll \frac{|u|^{-(d-1)}}{a_2 \cdots a_d}.\end{aligned}$$

This shows that

$$\sum_{u \in \mathbf{P}^*} \text{Prob}[Oct(a) \subset \rho S(u, W_d/2)] \ll \frac{1}{a_2 \ldots a_d} \sum_{u \in \mathbf{P}^*} |u|^{-(d-1)}.$$

The last sum can be estimated from above by standard methods: instead of summing over $u \in \mathbf{P}^*$, we can sum over all $u \in \mathbf{Z}^d \cap B$ where B is the ball centered at the origin and having radius $\frac{W_d}{a_1\sqrt{3}}$. This sum, in turn, differs little from the integral $\int_B |x|^{-d+1}dx$. Thus, we have

$$\sum_{u \in \mathbf{P}^*} |u|^{-(d-1)} \le \sum_{u \in \mathbf{Z}^d \cap B} |u|^{-(d-1)} \ll \int_B |x|^{-d+1}dx \ll \frac{1}{a_1}.$$

This implies now that

$$\sum_{u \in \mathbf{P}^*} \text{Prob}[Oct(a) \subset \rho S(u, W_d/2)] \ll \frac{1}{a_1 \ldots a_d} \ll \frac{1}{V}. \qquad \blacksquare$$

5. PROOF OF THEOREM 1.2

This proof is more difficult than the previous one. We first show that it is enough to prove the theorem when K is an octahedron: Lemma 3.1 implies that for every $K \in \mathcal{K}^d$ with $\text{Vol}\, K = V$ large there is $a = (a_1, \ldots, a_d) \in \mathbf{R}^d$ with $0 < a_1 \le \cdots \le a_d$ with $\prod a_i \ll V$ such that a congruent copy, K^*, of K is contained in $Oct(a)$. (The a_i here are equal to what was $d^{3/2}a_i$ in Lemma 3.1.) It follows from the remark at the end of the proof of Lemma 3.1 that $2a_1 \le d^{3/2}w(K)$. We may assume, again, that

$$0 < a_1 \le \frac{a_2}{2} \le \cdots \le \frac{a_d}{2^{d-1}},$$

by keeping a_1 the same and replacing, recursively, a_{i+1} by $2a_i$ if $a_{i+1} < 2a_i$. It is clear that

$$\text{Prob}[K \cap L = \emptyset] = \text{Prob}[K^* \cap L = \emptyset] \ge \text{Prob}[Oct(a) \cap L = \emptyset].$$

Set $\delta = 0.48$. For fixed $u \in \mathbf{P}$ we define

$$E(u) = \{\rho \in SO(d) : Oct(a) \subset \rho S(u, \delta)\}.$$

The slab $S(u, \delta)$ is a little smaller than the slab between two consecutive lattice hyperplanes orthogonal to u. This fact allows us to get rid of translations:

Claim 5.1. *If $\rho \in E(u)$, then a positive fraction of all translations $t \in [0, 1)^d$ have the property that $Oct(a)$ is between two consecutive lattice hyperplanes, orthogonal to ρu, in the lattice $L = \rho(\mathbf{Z}^d + t)$.*

Proof. Of course we can consider all translations $t \in B$ for an arbitrary basis parallelotope B of $\mathbf{Z}^d$, not only for $B = [0, 1)^d$. We choose B so that the associated basis contains u. As $Oct(a) \subset \rho S(u, \delta)$, $Oct(a)$ lies between two consecutive L-lattice hyperplanes orthogonal to ρu for at least 4% (as $2\delta = 0.96$) of translations $t \in B$ because only the u-component of t matters. ∎

We want to estimate, from below, the measure of $\bigcup_{u \in \mathbf{P}} E(u) \subset SO(d)$. Setting first

$$\mathbf{P}^* = \left\{ u \in \mathbf{P} : 2.1 \le \frac{1}{a_1|u|} \le 2.3 \right\}$$

and

$$\mathbf{P}(u) = \{v \in \mathbf{P}^* : |v| \ge |u|, v \ne u\},$$

we have

$$\begin{aligned}\mathrm{Prob}_\rho\left[\bigcup_{u \in \mathbf{P}} E(u)\right] &\ge \mathrm{Prob}_\rho\left[\bigcup_{u \in \mathbf{P}^*} E(u)\right] \\ &\ge \sum_{u \in \mathbf{P}^*}\left(\mathrm{Prob}_\rho[E(u)] - \sum_{v \in \mathbf{P}(u)} \mathrm{Prob}_\rho[E(u) \cap E(v)]\right).\end{aligned}$$

Our next target is to prove that $\sum_{u \in \mathbf{P}^*} \mathrm{Prob}_\rho[E(u)] \ll 1/V$ and that $\sum_{u \in \mathbf{P}^*} \sum_{v \in \mathbf{P}(u)} \mathrm{Prob}_\rho[E(u) \cap E(v)]$ is much smaller than $1/V$.

Remark 1. We need the condition $w(K) \le w_d$ since we need to have some nonempty $E(u)$. So we need some $u \in \mathbf{P}$ such that $\rho S(u, \delta)$ contains $Oct(a)$, that is, a_1 must be smaller than $\delta/|u|$ for some $u \in \mathbf{P}$. As we have seen, $2a_1 \le d^{3/2} w(K)$, we can take $w_d = 1/(2d^{3/2})$ implying $a_1 \le 1/4$. With this choice there are several primitive vectors satisfying the requirement.

Remark 2. We mention in passing that in the planar case there is no ρ in $E(u) \cap E(v)$ since the intersection of the two slabs has area less than 1 and so it cannot contain $Oct(a)$ or K.

We continue with the proof. By the choice of $\mathbf{P}^*$, $\frac{\delta}{a_1|u|} \geq \delta \cdot 2.1 > 1$ and also $\frac{\delta}{a_2|u|} < 1$ and we have, using Lemmas 3.2 and 3.3 again,

$$\sum_{u\in\mathbf{P}^*} \text{Prob}_\rho[E(u)] = \sum_{u\in\mathbf{P}^*} \lambda \left\{ f \in S^{d-1} : |f_j| \leq \frac{\delta}{a_j|u|}, j \in [d] \right\}$$
$$\gg \sum_{u\in\mathbf{P}^*} \prod_{j=2}^{d} \frac{\delta}{a_j|u|} \gg \sum_{u\in\mathbf{P}^*} \frac{|u|^{-(d-1)}}{a_2 \dots a_d}$$
$$\gg \frac{1}{a_2 \dots a_d} \sum_{u\in\mathbf{P}^*} |u|^{-(d-1)}.$$

The last sum can be estimated from below by the standard method, which uses the Möbius function $\mu(d)$ (see, for instance, [7] page 268, or [4], Lemma 1, for very similar computations):

$$\sum_{u\in\mathbf{P}^*} |u|^{-(d-1)} \gg \frac{1}{a_1}.$$

We omit the routine details.

So we get that

$$\sum_{u\in\mathbf{P}^*} \text{Prob}_\rho[E(u)] \gg \frac{1}{V}.$$

Our next target is to give an upper bound on $\sum_{v\in\mathbf{P}(u)} \text{Prob}_\rho[E(u) \cap E(v)]$ when $u \in \mathbf{P}^*$ is fixed. This will be done in several steps.

Assume $\rho \in E(u) \cap E(v)$ and let A be the two-dimensional plane spanned by u and v. Further, let γ denote the smaller angle between the lines of u and v. Fix an orthonormal basis $b_1, b_2, \dots, b_d$ with $b_1 = u/|u|$ and $b_2 \in A$, the rest of the b_i arbitrary. (Of course $b_1 \perp b_2$.) Suppose $\rho b_1 = f$ and $\rho b_2 = g$. Since $Oct(a)$ lies in both $\rho S(u, \delta)$ and $\rho S(v, \delta)$, its projection onto A lies in the parallelogram in Fig. 1.

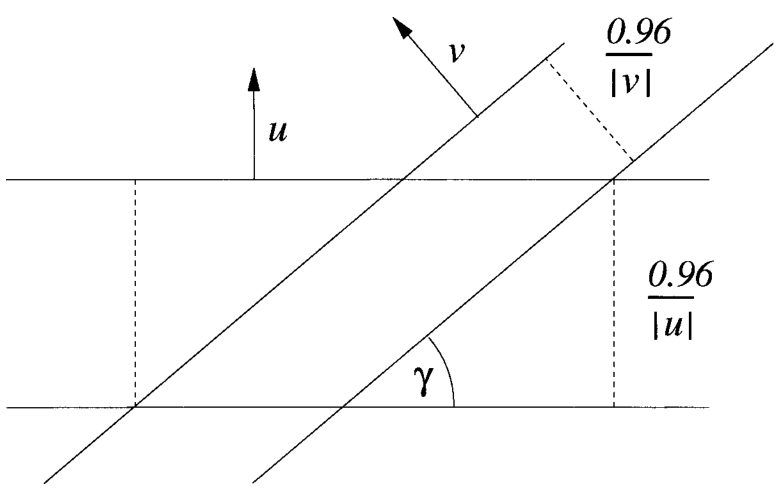

Fig. 1. The parallelogram in A.

The radius of the ball inscribed to the $(d-1)$-dimensional octahedron $Oct(a_2, \ldots, a_d)$ is

$$\left(\sum_2^d \frac{1}{a_i^2}\right)^{-1/2} \geq a_2\sqrt{3}.$$

Thus, the diameter of the parallelogram in Fig. 1 is at least $2a_2\sqrt{3}$, implying

$$2\sqrt{3}a_2 < \frac{2\delta}{\sin\gamma}\left(\frac{1}{|u|} + \frac{1}{|v|}\right) \leq \frac{4\delta}{|u|\sin\gamma}, \tag{1}$$

and hence

$$\sin\gamma < \frac{2\delta}{\sqrt{3}a_2|u|} \leq \frac{2\delta}{2\sqrt{3}a_1|u|} < 0.64.$$

The octahedron $Oct(a)$ lies in the slab $\rho S(u,\delta) \subset S(f, \delta/|u|)$ and also in the slab $\rho S(v,\delta) \subset S(g, 2\delta/|u|\sin\gamma)$, where $2\delta/|u|\sin\gamma$ comes from the fact that the width (in direction g) of the parallelogram in Fig. 1 is at most $4\delta/|u|\sin\gamma$, see (1). So we need to have

$$|f_i| \leq \frac{\delta}{a_i|u|} =: \alpha_i \,\forall i \in [d], \text{ and } |g_i| \leq \frac{2\delta}{a_i|u|\sin\gamma} =: \beta_i \,\forall i \in [d]. \tag{2}$$

Note that for $i = 1$ both inequalities are satisfied.

Claim 5.2. *If $f \in S^{d-1}$ and $|f_i| \leq \alpha_i$ for $i = 2, 3, \ldots, d$, then $|f_1| \geq 1/\sqrt{2}$. Further, if $g \in S^{d-1}$ and $f \perp g$, then $|g_1| < 1/\sqrt{2}$.*

Proof. This is simple:

$$\sum_2^d f_i^2 \leq \sum_2^d \alpha_i^2 \leq \frac{\delta^2}{|u|^2}\left(\frac{1}{a_2^2} + \frac{1}{(2a_2)^2} + \ldots\right)$$
$$< \frac{\delta^2 \cdot 4}{3|u|^2 a_2^2} \leq \frac{\delta^2}{3|u|^2 a_1^2} < \frac{\delta^2 \cdot 2.3^2}{3} < \frac{1}{2}.$$

(Here the last but one inequality follows from the definition: $u \in \mathbf{P}^*$ if and only if $\frac{1}{a_1|u|}$ lies in $[2.1, 2.3]$.) This implies the first part of the claim since f is a unit vector. For the second part, assume $|g_1| \geq 1/\sqrt{2}$. Then $\sum_2^d g_i^2 \leq 1/2$ and since $\sum_2^d f_i^2 < 1/2$, the Cauchy–Schwarz inequality gives $|\sum_2^d f_i g_i| < 1/2$ and we can't have $f \perp g$. ∎

Now we return to estimating

$$\text{Prob}_\rho[E(u) \cap E(v)] \leq \lambda\{(f,g) \in S^{d-1} \times S^{d-1} f \perp g, \text{ satisfying (2)}\}.$$

For fixed f define $G_f = \{g \in S^{d-1} : g \perp f,\ |g_i| \leq \beta_i,\ i = 2, \ldots, d\}$ and $G_f^* = \{tg : g \in G_f,\ t \in [0,1]\}$. Let pr be projection from $\mathbf{R}^d$ onto the hyperplane $\{x \in \mathbf{R}^d : x_1 = 0\}$. G_f lies on a $(d-2)$-dimensional great circle of S^{d-1} and it is clear that

$$\text{Vol}_{d-2}\, G_f = (d-1)\text{Vol}_{d-1}\, G_f^* = \frac{d-1}{|f_1|}\text{Vol}_{d-1}\, \text{pr}\, G_f^*.$$

Now define the set $H = H(u,\gamma) \subset \mathbf{R}^{d-1}$ by

$$H = \{h \in S^{d-2} : |h_i| \le \sqrt{2}\beta_i,\ i = 2,\dots,d\}$$

and $H^* = \{th : h \in H,\ t \in [0,1]\}$. As we have seen, $g \in G_f$ implies $|g_1| < 1/\sqrt{2}$. Then $|\mathrm{pr}\, g| > 1/\sqrt{2}$ follows, showing that for each $g \in G_f$ the projection of the segment $[0,g]$ lies in H^*. In other words $\mathrm{pr}\, G_f^* \subset H^*$. Further, it is evident that

$$(d-1)\mathrm{Vol}_{d-1} H^* = \mathrm{Vol}_{d-2} H.$$

So we have

$$\mathrm{Vol}_{d-2} G_f \le \frac{1}{|f_1|}\mathrm{Vol}_{d-2} H \le \sqrt{2}\mathrm{Vol}_{d-2} H.$$

Thus, we have, using Lemma 3.2,

$$\begin{aligned}\mathrm{Prob}_\rho[E(u)\cap E(v)] &\le \lambda\{f \in S^{d-1} : |f_i| \le \alpha_i\ \forall i \in [d]\}\sqrt{2}\mathrm{Vol}_{d-2} H\\ &= \sqrt{2}\mathrm{Prob}_\rho[E(u)]\mathrm{Vol}_{d-2} H.\end{aligned}$$

We are going to estimate $\mathrm{Vol}_{d-2} H$ using Lemma (3.2). So our target is to bound the product of the $\sqrt{2}\beta_i = \frac{2\delta\sqrt{2}}{|u|a_i \sin\gamma}$ that are below 1.

For this end, fix $u \in \mathbf{P}^*$ and fix γ and consider $v \in \mathbf{P}(u)$ with angle γ between u and v. The sequence

$$\frac{2\delta\sqrt{2}}{|u|a_2 \sin\gamma} > \frac{2\delta\sqrt{2}}{|u|a_3 \sin\gamma} > \dots > \frac{2\delta\sqrt{2}}{|u|a_d \sin\gamma}$$

is decreasing. Its first element is larger than 1 by inequality (1). Let $i = i(v)$ be the largest index $j \in [d]$ with $\frac{2\delta\sqrt{2}}{|u|a_j \sin\gamma} > 1$. We classify the vectors in $v \in \mathbf{P}(u)$ according to $i(v)$: define

$$\mathbf{P}(u)_j = \{v \in \mathbf{P}(u) : i(v) = j\}.$$

Now we can use the previous estimate for $\mathrm{Prob}_\rho[E(u)\cap E(v)]$:

$$\begin{aligned}\sum_{v\in\mathbf{P}(u)_j} \mathrm{Prob}_\rho[E(u)\cap E(v)] &\le \sqrt{2}\mathrm{Prob}_\rho[E(u)] \sum_{v\in\mathbf{P}(u)_j} \mathrm{Vol}_{d-2} H\\ &\ll \mathrm{Prob}_\rho[E(u)] \sum_{v\in\mathbf{P}(u)_j} \prod_{i=j+1}^{d} (|u|a_i \sin\gamma)^{-1}\\ &= \mathrm{Prob}_\rho[E(u)] \sum_{v\in\mathbf{P}(u)_j} \frac{1}{(|u|\sin\gamma)^{d-j} a_{j+1}\dots a_d}.\end{aligned}$$

For simpler writing set $\gamma_j = \arcsin \frac{2\delta\sqrt{2}}{|u|a_j}$ for $j \in [d]$ and $\gamma_{d+1} = 0$ and $U = (2.1a_1)^{-1}$. The sum over $v \in \mathbf{P}(u)_j$ can be estimated from above by the integral (we omit the routine details) over all $x \in \mathbf{R}^d$ satisfying $|u| \le |x| \le U$ such that the angle between vectors x and u lies in $[\gamma_{j+1}, \gamma_j]$. So we have

$$\begin{aligned}\sum_{v\in\mathbf{P}(u)_j} \frac{1}{(|u|\sin\gamma)^{d-j}a_{j+1}\dots a_d} &\ll \int_{|u|}^{U}\int_{\gamma_{j+1}}^{\gamma_j} \frac{r^{d-1}(\sin\gamma)^{d-2}d\gamma\, dr}{(|u|\sin\gamma)^{d-j}a_{j+1}\dots a_d}\\ &\ll \frac{U^d-|u|^d}{|u|^{d-j}a_{j+1}\dots a_d}\int_{\gamma_{j+1}}^{\gamma_j}(\sin\gamma)^{j-2}d\gamma\\ &\ll \frac{U^d}{|u|^{d-j}a_{j+1}\dots a_d}\frac{1}{j-1}\left[\left(\frac{2\delta\sqrt{2}}{|u|a_j}\right)^{j-1}-\left(\frac{2\delta\sqrt{2}}{|u|a_{j+1}}\right)^{j-1}\right]\\ &\ll \frac{U^d}{|u|^{d-1}a_j^{j-1}a_{j+1}\dots a_d} \ll \frac{U^d}{|u|^{d-1}a_2a_3\dots a_d}.\end{aligned}$$

Here the integral of $(\sin\gamma)^{j-2}$ is estimated by substituting $t = \sin\gamma$ and ignoring the $(1-t^2)^{-1/2}$ factor, which is bounded since $\sin\gamma < 0.64$. Recall that $u \in \mathbf{P}^*$ implies that $\frac{1}{a_1|u|} \in [2.1, 2.3]$. Adding the above inequalities for $j = 2, 3, \dots, d$ we get that

$$\begin{aligned}\sum_{j=2}^{d}\sum_{v\in\mathbf{P}(u)_j}\mathrm{Prob}_\rho[E(u)\cap E(v)] &\ll \mathrm{Prob}_\rho[E(u)]\frac{U^d}{|u|^{d-1}a_2a_3\dots a_d}\\ &\ll \mathrm{Prob}_\rho[E(u)]\frac{U}{a_2a_3\dots a_d}\\ &\ll \frac{1}{V}\mathrm{Prob}_\rho[E(u)],\end{aligned}$$

since $U/|u| \le 2.3/2.1$ and $U = (2.1a_1)^{-1}$. So we have, replacing the implicit constant in $\ll$ by the explicit constant $c = c(d)$,

$$\sum_{j=2}^{d}\sum_{v\in\mathbf{P}(u)_j}\mathrm{Prob}_\rho[E(u)\cap E(v)] \le \frac{c}{V}\mathrm{Prob}_\rho[E(u)] \le \frac{1}{2}\mathrm{Prob}_\rho[E(u)],$$

since c/V becomes smaller than $1/2$ if V is large enough.

We can finish the proof now. For large enough V we have

$$\begin{aligned}\mathrm{Prob}_\rho\left[\bigcup_{u\in\mathbf{P}}E(u)\right] &\ge \sum_{u\in\mathbf{P}^*}\left(\mathrm{Prob}_\rho[E(u)] - \sum_{v\in\mathbf{P}(u)}\mathrm{Prob}_\rho[E(u)\cap E(v)]\right)\\ &\ge \frac{1}{2}\sum_{u\in\mathbf{P}^*}\mathrm{Prob}_\rho[E(u)] \gg \frac{1}{V}.\end{aligned}$$

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