Clique percolation

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September 8, 2008

Abstract

Derényi, Palla and Vicsek introduced the following dependent percolation model, in the context of finding communities in networks. Starting with a random graph G generated by some rule, form an auxiliary graph G' whose vertices are the k-cliques of G, in which two vertices are joined if the corresponding cliques share k-1 vertices. They considered in particular the case where G=G(n,p), and found heuristically the threshold function p=p(n) above which a giant component appears in G'. Here we give a rigorous proof of this result, as well as many extensions. The model turns out to be very interesting due to the essential global dependence present in G'.

1 Cliques sharing vertices

Fix $k \geq 2$ and $1 \leq \ell \leq k-1$. Given a graph G, let $G^{k,\ell}$ be the graph whose vertex set is the set of all copies of K_k in G, in which two vertices are adjacent if the corresponding copies of K_k share at least ℓ vertices. Starting from a random graph G = G(n,p), our aim is to study percolation in the corresponding graph $G_p^{k,\ell}$, i.e., to find for which values of p there is a 'giant' component in $G_p^{k,\ell}$, containing a positive fraction of the vertices of $G_p^{k,\ell}$.

For $\ell = k-1$, this question was proposed by Derényi, Palla and Vicsek [10], motivated by the study of 'communities' in real-world networks, but independent of the motivation, we consider it to be an extremely natural question in the theory of random graphs. Indeed, it is perhaps the most natural example of dependent percolation arising out of the model G(n, p).

As we shall see in a moment, it is not too hard to guess the answer; simple heuristic derivations based on the local analysis of $G_p^{k,\ell}$ were given in [10] and by Palla, Derényi and Vicsek [17]. (For a survey of related work see [16].) Note,

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 $^{^{\}ddagger} Research$ supported in part by NSF grants DMS-0505550, CNS-0721983 and CCF-0728928, and ARO grant W911NF-06-1-0076

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however, that $G_p^{k,\ell}$ may well have many more than n^2 edges, so $G_p^{k,\ell}$ is not well approximated by a graph with independence between different edges: there is simply not enough information in G(n,p). Thus it is not surprising that it requires significant work to pass from local information about $G_p^{k,\ell}$ to global information about the giant component. Nonetheless, it turns out to be possible to find exactly the threshold for percolation, for all fixed k and ℓ .

Given 0 , let

$$\mu = \mu(n, p) = {\binom{k}{\ell} - 1} {\binom{n}{k - \ell}} p^{{\binom{k}{2} - {\binom{\ell}{2}}}}, \tag{1}$$

so μ is $\binom{k}{\ell} - 1$ times the expected number of K_k s containing a given copy of K_ℓ . Intuitively, this corresponds to the average number of new K_ℓ s reached in one step from a given K_ℓ , so we expect percolation if and only if $\mu > 1$. Since $\binom{k}{2} - \binom{\ell}{2} = \ell(k-\ell) + \binom{k-\ell}{2} = (k-\ell)(k+\ell-1)/2$, we have $\mu = \Theta(1)$ if and only if

$$p = \Theta\left(n^{-\frac{2}{k+\ell-1}}\right);\tag{2}$$

we shall focus our attention on p in this range.

In addition to finding the threshold for percolation, we shall also describe the asymptotic proportion of K_k s in the giant component in terms of the survival probability of a certain branching process. Set $M = \binom{k}{\ell} - 1$. Given $\lambda > 0$, let Z_{λ} have a Poisson distribution with mean λ/M . Let $\mathfrak{X}(\lambda) = (X_t)_{t=0}^{\infty}$ be the Galton–Watson branching process which starts with a single particle in X_0 , in which each particle in X_t has children in X_{t+1} independently of the other particles and of the history, and in which the distribution of the number of children of a given particle is given by MZ_{λ} . Let $\rho = \rho(\lambda)$ denote the probability that $\mathfrak{X}(\lambda)$ does not die out. Then a simple calculation shows that ρ satisfies the equation

$$\rho = 1 - \exp\left(-(\lambda/M)(1 - (1 - \rho)^M)\right).$$

From standard branching process results, ρ is the largest solution to this equation, $\rho(\lambda)$ is a continuous function of λ , and $\rho(\lambda) > 0$ if and only if λ , the expected number of children of each particle, is strictly greater than 1.

Let $\mathfrak{X}'(\lambda)$ denote the union of $\binom{k}{\ell}$ independent copies of the branching process $\mathfrak{X}(\lambda)$ described above, and let $\sigma = \sigma(\lambda)$ denote the survival probability of $\mathfrak{X}'(\lambda)$, so $\sigma = 1 - (1 - \rho)^{\binom{k}{\ell}}$. Our main result is that when $\mu = \Theta(1)$, the largest component of $G_p^{k,\ell}$ contains whp a fraction $\sigma(\mu) + o(1)$ of the vertices of $G_p^{k,\ell}$, where μ is defined by (1). Here, as usual, an event holds with high probability, or whp, if its probability tends to 1 as $n \to \infty$.

Let $\nu = \binom{n}{k} p^{\binom{k}{2}}$ denote the expected number of copies of K_k in G(n,p), i.e., the expected number of vertices of $G_p^{k,\ell}$. Let us write $C_i(G)$ for the number of vertices in the *i*th largest component of a graph G.

Theorem 1. Fix $1 \le \ell < k$, and let p = p(n) be chosen so that $\mu = \Theta(1)$, where μ is defined by (1). Then, for any $\varepsilon > 0$, who we have

$$(\sigma(\mu) - \varepsilon)\nu \le C_1(G_p^{k,\ell}) \le (\sigma(\mu) + \varepsilon)\nu$$

and $C_2(G_p^{k,\ell}) \le \varepsilon \nu$.

It is well known that $|G_p^{k,\ell}|$ is concentrated around its mean ν whenever $\nu \to \infty$, so Theorem 1 simply says that the largest component of $G_p^{k,\ell}$ contains a fraction $\sigma(\mu) + o(1)$ of the vertices whp. The extension to the case where $\mu \to 0$ or $\mu \to \infty$ is essentially trivial, and will be discussed in Subsection 1.3.

We shall prove Theorem 1 in two stages, considering the subcritical case in the next subsection, and the supercritical case in Subsection 1.2. Very roughly speaking, to handle the subcritical case (and to prove the upper bound on the giant component in the supercritical case) we shall show approximate domination of a suitable component exploration in $G_p^{k,\ell}$ by the branching process $\mathfrak{X}'(\lambda)$, $\lambda = (1+\varepsilon)\mu$. Due to the dependence in the model, we have to be very careful exactly how we explore $G_p^{k,\ell}$ to make this argument work. For the upper bound we first show (by approximate local coupling with the branching process) that roughly the right number of vertices are in large components, even if p is reduced slightly, i.e., even if we omit some edges. Then we use a multi-round 'sprinkling' argument, putting back the omitted edges in several rounds, and showing that it is very likely that the sprinkled edges join these large components. The details of both arguments turn out to be less simple that one might like.

1.1 The subcritical case

We shall start by considering the subcritical case, proving the following much stronger form of Theorem 1 in this case.

Theorem 2. Let $1 \le \ell \le k-1$ and $\varepsilon > 0$ be given. There is a constant $C = C(k,\ell,\varepsilon)$ such that, if p = p(n) is chosen so that $\mu \le 1-\varepsilon$ for all large enough n, then $C_1(G_p^{k,\ell}) \le C \log n$ whp.

Proof. Since the event $C_1(G^{k,\ell}) > C \log n$, considered as a property of the underlying graph G, is an increasing event, we may assume without loss of generality that $\mu = 1 - \varepsilon$ for every n. Thus (2) holds.

Fixing a set V_0 of k vertices of G = G(n, p), we shall show that, given that V_0 forms a complete graph in G, the probability that the corresponding component $C(V_0)$ of $G_p^{k,\ell}$ has size more than $C \log n$ is at most n^{-k-1} , provided C is large enough. Since the probability that V_0 forms a complete graph in G is $p^{\binom{k}{2}}$, while there are $\binom{n}{k}$ possibilities for V_0 , it then follows that $\mathbb{P}(C_1(G_p^{k,\ell}) \geq C \log n) \leq \binom{n}{k} p^{\binom{k}{2}} n^{-k-1} = o(1)$.

From now on we condition on V_0 forming a K_k in G = G(n, p). The strategy is to show domination of a natural component exploration process by the branching process described earlier. We shall show essentially that the average number of new K_{ℓ} s reached from a given K_{ℓ} in G via K_k s in G is at most $\mu + o(1)$, though there will be some complications.

In outline, our exploration of the component $C(V_0) \subset G_p^{k,\ell}$ proceeds as follows. At each stage we have a set V_t of reached vertices of G, starting with V_0 ; we also keep track of a set E of reached edges, initially the edges spanned

by V_0 . At the end of stage t of our exploration, E will consist of all edges of $G[V_t]$. Within V_t , every K_ℓ is labelled as either 'tested' or 'untested'. We start with all $\binom{k}{\ell}$ K_ℓ s in V_0 marked as untested. The exploration stops when there are no untested K_ℓ s.

As long as there are untested K_{ℓ} s, we proceed as follows. Pick one, S, say. One by one, test each set K of k vertices with $S \subset K \not\subset V_t$ to see whether all edges induced by K are present in G. If so, we add any new vertices to V_t , i.e., we set $V_{t+1} = V_t \cup V(K)$. We now add all edges of K not present in V_t to E; we call these edges regular. Any new $K_{\ell}s$ formed in E are marked as untested. Note that any such K_{ℓ} must contain at least one vertex of $V_{t+1} \setminus V_t$, and hence must lie entirely inside K.

Next, we test all edges between V_t and $V(K) \setminus V_t$ to see if they are present in G, adding any edge found to E, and marking any new K_ℓ s formed as untested. Edges added during this operation are called *exceptional*. At this point, we have revealed the entire subgraph of G induced by V_{t+1} , i.e., we have $E = E(G[V_{t+1}])$. We then continue through our list of possible sets K containing S, omitting any set K contained in the now larger set V_{t+1} . Once we have considered all possible $K \supset S$, we mark S as tested, and continue to another untested K_ℓ , if there is one.

The algorithm described above can be broken down into a sequence of steps of the following form. At the *i*th step, we test whether all edges in a certain set A_i are present in G = G(n, p); the future path of the exploration depends only on the answer to this question, not on which particular edges are missing if the answer is no. Although this is wasteful from an algorithmic point of view, it is essential for the analysis. We write A_i for the event $A_i \subset E(G)$. After *i* steps, we will have 'uncovered' a set E_i of edges (called E above). The set E_i consists of the edges spanned by V_0 together with the union of those sets A_j for which A_j holds.

The event that the algorithm reaches a particular state, i.e., receives a certain sequence of answers to the first i questions, is of the form $\mathcal{U} \cap \mathcal{D}$, where $\mathcal{U} = \{E_i \subset E(G)\}$ is an up-set, and \mathcal{D} is a down-set, formed by the intersections of various \mathcal{A}_j^c . The key point is that \mathcal{U} is a principal up-set, so $\mathcal{U} \cap \mathcal{D}$ may be regarded as a down-set \mathcal{D}' in the product probability space $\Omega' = \{0,1\}^{E(K_n) \setminus E_i}$ with the appropriate measure. Hence, for any A_{i+1} disjoint from E_i , the conditional probability that \mathcal{A}_{i+1} holds given the current state of the algorithm is

$$\mathbb{P}(\mathcal{A}_{i+1} \mid \mathcal{U} \cap \mathcal{D}) = \mathbb{P}(\mathcal{A}'_{i+1} \mid \mathcal{D}') \le \mathbb{P}(\mathcal{A}'_{i+1}) = p^{|A_{i+1}|}, \tag{3}$$

where \mathcal{A}'_{i+1} is the event in Ω' corresponding to \mathcal{A}_{i+1} , and the inequality follows from Harris's Lemma [13] applied in Ω' .

Let us write X_i for the number of new K_{ℓ} s found as a result of adding regular edges when testing the *i*th K_{ℓ} , S_i , say; we shall deal with exceptional edges separately in a moment. Recall that we add regular edges when we find a new K_k with at most $k - \ell$ and at least 1 vertex outside the current vertex set V_t .

Let $\eta > 0$ be a constant such that $(1 + \eta)\mu \leq (1 - \varepsilon/2)$. When testing S_i , there are at most $\binom{n}{k-\ell}$ possibilities for new K_k s with $k-\ell$ vertices outside the

current V_t . Given the history, by (3) each such K_k is present with probability at most $p^{\binom{k}{2}-\binom{\ell}{2}}$, so the number of such K_k s we find is stochastically dominated by the Binomial distribution Bi $\binom{n}{k-\ell}, p^{\binom{k}{2}-\binom{\ell}{2}}$, and hence, for n large, by a Poisson distribution with mean $(1+\eta/2)\binom{n}{k-\ell}p^{\binom{k}{2}-\binom{\ell}{2}}$. [Here we use the fact that a Poisson distribution with mean $-N\log(1-\pi)$ dominates a Binomial Bi (N,π) , which, as pointed out to us by Svante Janson, follows immediately from the same statement for N=1.]

For $1 \leq j \leq k-\ell-1$, we may also find new K_k s containing S_i together with j other vertices of the current set V_t , and hence with only $k-\ell-j$ vertices outside V_t . Assuming $|V_t| \leq k(\log n)^{100k^3} \leq (\log n)^{101k^3}$, say, the number of possibilities for a fixed j is crudely at most $(\log n)^{101k^3}jn^{k-\ell-j}$, and each of these tests succeeds with probability at most $p^{\binom{k}{2}-\binom{\ell+j}{2}}$. A simple calculation shows that $n^{k-\ell-j}p^{\binom{k}{2}-\binom{\ell+j}{2}}$ is at most $n^{-\delta}$ for some $\delta>0$, so the expected number of K_k s of this type is at most $n^{-\delta/2}$, say. Moreover, the distribution of the number found is stochastically dominated by a Poisson distribution with mean $2n^{-\delta/2}$.

Each K_k we find consisting of $k-\ell-j$ new vertices and $\ell+j$ old vertices, $j \geq 0$, generates $\binom{k}{\ell} - \binom{\ell+j}{\ell} \leq M$ new K_{ℓ} s, where $M = \binom{k}{\ell} - 1$. It follows that, given the history, the conditional distribution of X_i/M is stochastically dominated by a Poisson distribution with mean

$$(1+\eta/2)\binom{n}{k-\ell}p^{\binom{k}{2}-\binom{\ell}{2}}+2n^{-\delta/2}=(1+\eta/2)\mu/M+o(1), \qquad (4)$$

which is at most $(1 + \eta)\mu/M < (1 - \varepsilon/2)/M$ if n is large enough.

Turning to exceptional edges, we claim that the jth exceptional edge added creates at most $\binom{k-1+j}{\ell-1}$ new K_ℓ s; all we shall use about this bound is that it depends only on j, k and ℓ , not on n. Indeed, we add exceptional edges immediately after adding a K_k that includes a certain set N of new vertices. At this point, the degree in E (the uncovered edges) of every vertex in N is exactly k-1. We now add one or more exceptional edges joining N to V_t . Any such edge e has one end, x, say, in N. If e is the jth exceptional edge in total, then just after adding e the vertex x has degree at most k-1+j. Any new K_ℓ s involving e consist of x together with $\ell-1$ neighbours of x, so there are at most $\binom{k-1+j}{\ell-1}$ such K_ℓ s.

Assuming $|V_t| \leq k(\log n)^{100k^3}$, the number of potential exceptional edges associated to a new K_k is at most $(k-\ell)|V_t| = O^*(1)$, where, as usual, $g_1(n) = O^*(g_2(n))$ means that there is a constant a such that $g_1(n) = O(g_2(n)(\log n)^a)$. It follows that, for fixed r, the probability that we find at least r such edges at a given step is $O^*(p^r)$. Furthermore, the probability that we find j exceptional edges in total during the first $(\log n)^{100k^3}$ steps is $O^*(p^j)$, since there are $O^*(1)$ possibilities for the set of at most j steps at which we might find them. Let us choose a constant J so that $p^J \leq n^{-100k^3}$ (here, $p^J \leq n^{-k-2}$ would do; the stronger bound is useful later), and let $\mathcal B$ be the 'bad' event that we find

more than J exceptional edges in the first $(\log n)^{100k^3}$ steps. Then we have $\mathbb{P}(\mathcal{B}) = O^{\star}(p^J) = O^{\star}(n^{-100k^3}) = o(n^{-99k^3})$.

As long as \mathcal{B} does not hold, we create at most $J' = \sum_{j \leq J} {k-1+j \choose \ell-1} = O(1)$ new K_{ℓ} s when adding exceptional edges in the first $(\log n)^{100k^3}$ steps; let us note for later that we also create at most $\sum_{j \leq J} {k-1+j \choose k-1} K_k$ s when adding exceptional edges. We view our exploration as a set of branching processes: we start one process for each of the initial K_{ℓ} s. Whenever we add a K_{ℓ} in the normal way, we view it as a child of the K_{ℓ} we were testing. When we add a K_{ℓ} as a result of adding an exceptional edge, we view it as the root of a new process. As long as $|V_{\ell}| \leq k(\log n)^{100k^3}$ holds, from (4) the branching processes we construct are stochastically dominated by independent copies \mathfrak{X}_{ℓ} of the Galton–Watson process $\mathfrak{X}(\lambda)$ described earlier, where $\lambda = (1 + \eta)\mu < (1 - \varepsilon/2)$. If \mathcal{B} does not hold, then we start in total at most $J'' = {k \choose \ell} + J' = O(1)$ processes in the first $(\log n)^{100k^3}$ steps.

Recall that the offspring distribution in $\mathfrak{X}(\lambda)$ is given by MZ_{λ} , where Z_{λ} has a Poisson distribution with mean λ/M , so $\mathbb{E}(MZ_{\lambda}) = \lambda$. Here, $\lambda = (1 + \eta)\mu < 1 - \varepsilon/2$. Since MZ_{λ} has an exponential upper tail, it follows from standard branching process results that there is a constant a > 0 such that the probability that the total size of \mathfrak{X}_i exceeds m+1 is at most $\exp(-am)$ for any $m \geq 0$. Taking C large enough, it follows that with probability $1 - o(n^{-k-1})$, each of $\mathfrak{X}_1, \ldots, \mathfrak{X}_{J''}$ has size at most $(C/J'') \log n$. If this event holds and \mathcal{B} does not hold, then our exploration dies having reached a total of at most $C \log n$ vertices. Hence, the probability that $C(V_0)$ contains more than $C \log n \leq (\log n)^{100k^3}$ vertices of G = G(n, p) is $o(n^{-k-1}) + o(n^{-99k^3}) = o(n^{-k-1})$.

At this point the proof of Theorem 2 is almost complete: we have shown that whp, any component of $G_p^{k,\ell}$ involves K_k s meeting at most $C \log n$ vertices of G = G(n,p). To complete the proof, it is an easy exercise to show that if $p \leq n^{-\delta}$ for some $\delta > 0$, then whp any $C \log n$ vertices of G(n,p) span at most $C' \log n$ copies of K_k , for some constant C'. Alternatively, note that the number of K_k s found involving new vertices is at most the final number of vertices reached, while all other K_k s are formed by the addition of exceptional edges, and if \mathcal{B} does not hold, then, arguing as for the bound on the number of K_ℓ s formed by adding exceptional edges, the number of K_k s so formed is bounded by a constant.

In the proof above, subcriticality only came in at the end, where we used it to show that the branching processes \mathfrak{X}_i were very likely to die; in the supercritical case, the proof gives a domination result that we shall state in a moment. For this, the order in which we test the K_{ℓ} s matters – we proceed in rounds, in round 0 testing the $\binom{k}{\ell}$ initial K_{ℓ} s, and then in round $i \geq 1$ testing all K_{ℓ} s created during round i. Let $H = H(G_p^{k,\ell})$ be the bipartite incidence graph corresponding to $G_p^{k,\ell}$: the vertex classes are V_1 , the set of all K_{ℓ} s in $G_p^{k,\ell}$, and V_2 , the set of all K_{ℓ} s. Two vertices are joined if one of the corresponding complete graphs is contained in the other. Given a vertex $v_0 \in V_1$ of H, let $N_i = N_i(v_0)$ denote the number of K_{ℓ} s whose graph distance in H from v_0 is

at most 2i + 1. If v_0 is the vertex of H corresponding to the complete subgraph on V_0 , then after i rounds of the above algorithm we have certainly reached all N_i K_{ℓ} s within distance 2i + 1 of v_0 .

The domination argument in the proof of Theorem 2 thus also proves the lemma below, in which J'' is a constant depending only on k and ℓ , $\mathfrak{X}_1, \ldots, \mathfrak{X}_{J''}$ are independent copies of our Galton–Watson branching process as above, and $M_{\leq t}(\mathfrak{X}_1, \ldots, \mathfrak{X}_{J''})$ denotes the total number of particles in the first t generations of $\mathfrak{X}_1, \ldots, \mathfrak{X}_{J''}$.

Lemma 3. Let $\eta > 0$ be fixed, let p = p(n) satisfy (2), and let V_0 be a fixed set of k vertices of G = G(n, p). Condition on V_0 spanning a complete graph in G, and let v_0 be the corresponding vertex of H. Then we may couple the random sequence N_1, N_2, \ldots with J'' independent copies \mathfrak{X}_i of $\mathfrak{X}((1 + \eta)\mu)$ so that with probability $1 - o(n^{-99k^3})$ we have $N_t \leq M_{\leq t} = M_{\leq t}(\mathfrak{X}_1, \ldots, \mathfrak{X}_{J''})$ for all t such that $M_{\leq t} \leq (\log n)^{100k^3}$.

We finish this subsection by presenting a consequence of a much simpler version of the domination argument above. If we are prepared to accept a larger error probability, we may abandon the coupling the first time an exceptional edge appears. As shown above, the probability that we find any exceptional edges within $O^*(1)$ steps is at most $n^{-\delta}$ for some $\delta > 0$. Abandoning our coupling if this happens, we need only consider the original $\binom{k}{\ell}$ branching processes, one for each copy of K_{ℓ} in V_0 . In other words, we may compare our neighbourhood exploration process with the branching process $\mathfrak{X}'(\lambda)$, $\lambda = (1+\eta)\mu$, which starts with $\binom{k}{\ell}$ particles in generation 0, and in which, as in $\mathfrak{X}(\lambda)$, the offspring distribution for each particle is given by M times a Poisson distribution with mean λ/M .

Lemma 4. Let $\eta > 0$ be fixed, let p = p(n) satisfy (2), and let V_0 be a fixed set of k vertices of G = G(n, p). Condition on V_0 spanning a complete graph in G, and let v_0 be the corresponding vertex of H. Then there is a constant $\delta > 0$ such that we may couple the random sequence N_1, N_2, \ldots with $\mathfrak{X}' = \mathfrak{X}'((1+\eta)\mu)$ so that, with probability at least $1 - n^{-\delta}$, we have $N_t \leq M_t$ for all t such that $M_{\leq t} \leq (\log n)^{100k^3}$, where M_t is the number of particles in generation t of \mathfrak{X}' , and $M_{\leq t} = M_0 + M_1 + \cdots + M_t$.

In the next subsection we shall show that when $\mu > 1$, the graph $G_p^{k,\ell}$ does contain a giant component, and moreover that this giant component is of about the right size; Lemma 4 will essentially give us the upper bound, but we have to work a lot more for the lower bound.

1.2 The supercritical case

Recall that $|G_p^{k,\ell}|$, the number of K_k s in G(n,p), is certainly concentrated about its mean $\nu = \binom{n}{k} p^{\binom{k}{2}}$. For the moment, we concentrate on the case where (2) holds; we return to larger p later.

One bound in Theorem 1 is easy, at least in expectation: Lemma 4 gives an upper bound on the expected size of the giant component. In fact, it gives much more, namely an upper bound on the expected number of vertices in 'large' components. It is convenient to measure the size of a component by the number of K_{ℓ} s rather than the number of K_{k} s.

Let $N_{\geq a}(G_p^{k,\ell})$ denote the number of vertices of $G_p^{k,\ell}$ whose component in the bipartite graph H contains at least a vertices of V_2 , i.e., at least a copies of K_{ℓ} .

Lemma 5. Let p = p(n) be chosen so that μ is constant, and let $\varepsilon > 0$ be fixed. For any $\omega = \omega(n)$ tending to infinity we have $\mathbb{E}(N_{\geq \omega}(G_p^{k,\ell})) \leq (\sigma(\mu) + \varepsilon)\nu$ if n is large enough.

Proof. We may assume without loss of generality that $\omega \leq \log n$. From standard branching process results, for any fixed λ , the probability that $\mathfrak{X}'(\lambda)$ contains at least a particles but does not survive forever tends to 0 as $a \to \infty$. Thus, $\mathbb{P}(|\mathfrak{X}'(\lambda)| \geq \omega) = \sigma(\lambda) + o(1)$.

Fix a set V_0 of k vertices of G = G(n, p), and condition on V_0 forming a K_k in G, which we denote v_0 . Let $\lambda = (1 + \eta)\mu$ where, for the moment, $\eta > 0$ is constant. Since $\omega \leq \log n$, Lemma 4 tells us that the probability π that the component of v_0 in H contains at least ω K_ℓ s is at most $\mathbb{P}(|\mathfrak{X}'((1 + \eta)\mu)| \geq \omega) + o(1) = \sigma((1 + \eta)\mu) + o(1)$. Letting $\eta \to 0$ and using continuity of σ , it follows that $\pi \leq \sigma(\mu) + o(1)$ as $n \to \infty$. Since $\mathbb{E}(N_{\geq \omega})$ is simply $\pi\binom{n}{k}p^{\binom{k}{2}}$, this proves the lemma.

As in Bollobás, Janson and Riordan [5], for example, a simple variant of Lemma 4 also gives us a second moment bound.

Lemma 6. Let p = p(n) be chosen so that μ is constant, and let $\varepsilon > 0$ be fixed. For any $\omega = \omega(n)$ tending to infinity we have $\mathbb{E}(N_{\geq \omega}(G_p^{k,\ell})^2) \leq (\sigma(\mu)^2 + \varepsilon)\nu^2$.

Proof. The expected number of pairs of overlapping K_k s in G = G(n, p) is

$$\sum_{i=1}^{k-1} \binom{n}{k} \binom{k}{i} \binom{n-k}{k-i} p^{2\binom{k}{2} - \binom{i}{2}},$$

which, by a standard calculation, is $o(\nu^2)$. Hence, it suffices to bound the expected number of pairs of vertex disjoint K_k s each in a 'large' component. We may do so as in the proof of Lemma 5, using a variant of Lemma 4 in which we start with two disjoint K_k s, and explore from each separately, abandoning each exploration if it reaches size at least $\log n$, and abandoning both if they meet, an event of probability o(1).

Let us turn to our proof of the heart of Theorem 1, namely the lower bound. In proving this we may assume that $\mu > 1$ is constant. We start with a series of simple lemmas.

Let V_0 be a set of k vertices of G = G(n, p), and let $\mathcal{A} = \mathcal{A}(V_0)$ be the event that V_0 spans a K_k in G. Let $\mathcal{Q} = \mathcal{Q}(V_0)$ be the event that $G_p^{k,\ell}$ contains a tree

T with $\lceil (\log n)^{5k^3} \rceil$ vertices, one of which, v_0 , is the clique corresponding to V_0 , with the following additional property: ordering the vertices of T so that the distance from the root v_0 is increasing, each corresponding K_k meets the union of all earlier K_k s in exactly ℓ vertices. Equivalently, the union of the cliques in T contains exactly $k + (k - \ell)(|T| - 1)$ vertices of G.

Recall that $\mu = \mu(n, p)$ is defined by (1).

Lemma 7. Fix $\varepsilon > 0$, and let $p = \Theta(n^{-\frac{2}{k+\ell-1}})$ be chosen so that μ is a constant greater than 1. If n is large enough, then $\mathbb{P}(\mathcal{Q} \mid \mathcal{A}) \geq \sigma(\mu) - \varepsilon$.

Proof. Throughout we condition on $\mathcal{A} = \mathcal{A}(V_0)$, writing v_0 for the corresponding vertex of $G_p^{k,\ell}$. We start by marking all $\binom{k}{\ell}$ copies of K_ℓ in V_0 as untested; we shall then explore part of the component of $G_p^{k,\ell}$ containing the vertex v_0 corresponding to V_0 . At the *i*th step in our exploration, we consider an untested copy S_i of K_ℓ , and test for the presence of certain K_k s consisting of S_i plus exactly $k-\ell$ 'new' vertices not so far reached in our exploration. For each such K_k we find, we mark the $M=\binom{k}{\ell}-1$ new K_ℓ s created as untested; having found all such K_k s, we mark S_i as tested. We abandon our exploration if there is no untested S_i left, or if we reach more than $(\log n)^{5k^3}$ K_k s. Note that the total number of vertices reached is exactly $|V_0|$ plus $k-\ell$ times the number of K_k s found, so if we find more than $(\log n)^{5k^3}$ K_k s, then $\mathcal{Q}(V_0)$ holds.

The exploration above corresponds to the construction of $\binom{k}{\ell}$ random rooted trees whose vertices are the S_i , in which the children of S_i are the new K_{ℓ} s created when testing S_i . The number of children of S_i is MX_i , where X_i is the number of K_k s we find when testing S_i . Let $0 < \eta < 1$ be a constant to be chosen later. Let Z_1, Z_2, \ldots be a sequence of iid Poisson random variables with mean $(1 - \eta)\mu/M < \mu/M$. Our aim is to show that as long as we have found at most $(\log n)^{5k^3}$ copies of K_k in total, the conditional distribution of X_i given the history may be coupled with Z_i so that $X_i \geq Z_i$ holds with probability $1 - o(n^{-\delta})$, for some $\delta > 0$. The Galton–Watson branching process $\mathfrak{X}'((1 - \eta)\mu)$ defined by Z_1, Z_2, \ldots is supercritical, and so survives forever with probability $\sigma((1 - \eta)\mu)$. It then follows that $\mathcal{Q}(V_0)$ holds with probability at least $\sigma((1 - \eta)\mu) - o(1)$. Using continuity of σ and choosing η small enough, the conclusion of the lemma follows.

In order to establish the coupling above, we must be a little careful with the details of our exploration. At step i, before testing S_i , we will have a certain set V_i of reached vertices, consisting of all vertices of all K_k s found so far, and a certain set $D_i \supset V_i$ of 'dirty' vertices. The remaining vertices are 'clean'; we write C_i for the set of these vertices. At the start, V_0 is our initial set of k vertices, while $D_0 = V_0$ and $C_0 = V(G) \setminus V_0$.

We test S_i as follows: for each $v \in C_i$, let $\mathcal{E}_{v,i}$ be the event that all ℓ possible edges joining v to S_i are present in G = G(n,p). First, for every vertex $v \in C_i$, we test whether $\mathcal{E}_{v,i}$ holds, writing W_i for the set of $v \in C_i$ for which $\mathcal{E}_{v,i}$ does hold. We then look for copies of $K_{k-\ell}$ inside $G[W_i]$, writing N_i for the maximum number of vertex disjoint copies. Taking a particular set of N_i disjoint copies, we then add each of the corresponding K_k s to our component, defining V_{i+1}

appropriately. We then set $D_{i+1} = D_i \cup W_i$, and $C_{i+1} = V(G) \setminus D_{i+1}$.

The structure of the algorithm guarantees the following: given the state at time i, all we know about the edges between V_i and C_i is that certain sets of ℓ edges are not all present: more precisely, we know exactly that none of the events $\mathcal{E}_{v,j}$ holds, for $v \in C_i$ and j < i. Let $n_i = |W_i|$, a random variable. Having found W_i , it follows that the edges within W_i are untested, so each is present with its unconditional probability, and $G[W_i]$ has the distribution of the random graph $G(n_i, p)$. Let $\eta' > 0$ be a very small constant to be chosen below. Let \mathcal{E}_i be the event that $n_i \geq (1 - 2\eta')np^{\ell}$.

We shall show in a moment that \mathcal{E}_i holds with very high (conditional) probability, given the history; first, let us see how this enables us to complete the proof. If \mathcal{E}_i does hold, then the conditional expected number of $K_{k-\ell}$ s in $G[W_i]$ is exactly $\binom{n_i}{k-\ell}p^{\binom{k-\ell}{2}}$. Provided we choose η' small enough, this expectation is at least $(1-\eta/2)\tau$, where $\tau=(np^\ell)^{k-\ell}p^{\binom{k-\ell}{2}}/(k-\ell)!\sim \mu/M$. Since $\tau=\Theta(1)$, by a result of Bollobás [2], the number N_i' of $K_{k-\ell}$ s in $G[W_i]$ is asymptotically Poisson with mean τ . Indeed, N_i' may be coupled with a Poisson distribution Z with mean $(1-\eta)\mu/M$ so that $N_i' \geq Z$ holds with probability $1-o(n^{-\delta})$. Furthermore, by the first moment method, with probability $1-o(n^{-\delta})$, the graph $G[W_i]$ does not contain two $K_{k-\ell}$ s sharing a vertex, so $N_i = N_i'$.

It remains only to prove that \mathcal{E}_i does indeed hold with high conditional probability. Recall that at the start of stage i, all we know about the edges between C_i and V_i is that none of the events $\mathcal{E}_{v,j}$, $v \in C_i$, j < i holds. This information may be regarded as a separate condition \mathcal{F}_v for each $v \in C_i$, where $\mathcal{F}_v = \bigcap_{j < i} \mathcal{E}_{v,j}^c$ depends only on edges between v and V_i . Given this information, the events $\mathcal{E}_{v,i}$ are independent, and each holds with probability $r = \mathbb{P}(\mathcal{E}_{v,i} \mid \mathcal{F}_v)$. Now $\mathcal{E}_{v,i}$ is an up-set and \mathcal{F}_v is a down-set, so $r \leq \mathbb{P}(\mathcal{E}_{v,i}) = p^{\ell}$. Hence, whatever $|C_i|$ is, the conditional probability that $n_i \geq 2p^{\ell}n$ is exponentially small. Since $|C_{i+1}| = |C_i| - n_i$, and we stop after at most $(\log n)^{5k^3}$ steps, we may thus assume in what follows that $|C_i| \geq n - o(n)$.

Regarding the sets S_j , $j \leq i$, as fixed, and forgetting our present conditioning, if all we assume about the edges from v to V_i is that $\mathcal{E}_{v,i}$ holds, i.e., that all edges from v to S_i are present, then each $\mathcal{E}_{v,j}$, j < i, has conditional probability $p^{|S_j \setminus S_i|} \leq p$. Recalling that we abandon our exploration after at most $(\log n)^{5k^3}$ steps, it follows that

$$\mathbb{P}(\mathcal{F}_v^c \mid \mathcal{E}_{v,i}) = \mathbb{P}\left(\bigcup_{j < i} \mathcal{E}_{v,j} \mid \mathcal{E}_{v,i}\right) \le \sum_{j < i} \mathbb{P}(\mathcal{E}_{v,j} \mid \mathcal{E}_{v,i}) \le ip \le (\log n)^{5k^3} p \le \eta',$$

if n is large enough. Hence $\mathbb{P}(\mathcal{F}_v \mid \mathcal{E}_{v,i}) \geq 1 - \eta'$. In other words, $\mathbb{P}(\mathcal{F}_v \cap \mathcal{E}_{v,i}) \geq (1 - \eta')\mathbb{P}(\mathcal{E}_{v,i})$. This trivially implies that $\mathbb{P}(\mathcal{F}_v \cap \mathcal{E}_{v,i}) \geq (1 - \eta')\mathbb{P}(\mathcal{F}_v)\mathbb{P}(\mathcal{E}_{v,i})$, i.e., that $\mathbb{P}(\mathcal{E}_{v,i} \mid \mathcal{F}_v) \geq (1 - \eta')\mathbb{P}(\mathcal{E}_{v,i}) = (1 - \eta')p^{\ell}$.

It follows that n_i stochastically dominates a Binomial distribution with parameters $|C_i|$ and $(1 - \eta')p^{\ell}$. Since $|C_i| \ge n - o(n)$, we get the required lower bound on n_i , completing the proof.

Let N denote the number of K_k s in G = G(n,p) for which the corresponding event \mathcal{Q} holds, and let $N' = N_{\geq (\log n)^{5k^3}}(G_p^{k,\ell})$ be the number of K_k s in large components of $G_p^{k,\ell}$, that is, components containing at least $(\log n)^{5k^3}$ copies of K_ℓ . If V_0 spans a K_k for which \mathcal{Q} holds, then by definition the corresponding component of $G_p^{k,\ell}$ contains a tree with at least $(\log n)^{5k^3}$ vertices; furthermore, exploring this tree from the root, for each new vertex we find $M = {k \choose \ell} - 1 \geq 1$ new K_ℓ s. Hence the component is large, so $N \leq N'$.

Lemma 8. Fix $\varepsilon > 0$, and let $p = \Theta(n^{-\frac{2}{k+\ell-1}})$ be chosen so that μ is a constant greater than 1. Then

$$(\sigma(\mu) - \varepsilon)\nu \le N \le N' \le (\sigma(\mu) + \varepsilon)\nu$$

holds whp.

Proof. Fixing a set V_0 of k vertices of G = G(n, p), recall that $\mathcal{A} = \mathcal{A}(V_0)$ is the event that V_0 spans a K_k in G. We have $\mathbb{E}(N) = \binom{n}{k} \mathbb{P}(\mathcal{A}) \mathbb{P}(\mathcal{Q} \mid \mathcal{A})$, which is at least $(\sigma(\mu) - o(1))\binom{n}{k}p^{\binom{k}{2}} = (\sigma(\mu) - o(1))\nu$ by Lemma 7. As noted above, $N \leq N'$ always holds. Thus $\mathbb{E}(N^2) \leq \mathbb{E}((N')^2)$. But $\mathbb{E}((N')^2) \leq (\sigma(\mu)^2 + o(1))\nu^2$ by Lemma 6. Hence $\mathbb{E}(N^2) \leq (1 + o(1))\mathbb{E}(N)^2$, which implies that N is concentrated around its mean. Furthermore, $\mathbb{E}(N) \leq \mathbb{E}(N') \sim \sigma(\mu)\nu$, so we have $\mathbb{E}(N) \sim \sigma(\mu)\nu$, and the result follows.

Remark 9. It is perhaps interesting to note that there is an alternative proof of the bounds on N' given in Lemma 8, using the a sharp-threshold result of Friedgut [12] instead of the second moment method. Let us briefly outline the argument. Let \mathcal{U} be the event that the number $N' = N_{\geq (\log n)^{5k^3}}(G_p^{k,\ell})$ of K_k s in large components satisfies $N' \geq (\sigma(\mu) - \varepsilon)\nu$. In the light of the expectation bound given by Lemma 5, it suffices to prove that \mathcal{U} holds whp.

We view \mathcal{U} as an event in the probability space G(n,p), in which case it is clearly increasing and symmetric. We shall consider $\mathbb{P}_{p'}(\mathcal{U})$, the probability that G(n,p') has the property \mathcal{U} . When we do so, we keep the definition of \mathcal{U} fixed, i.e., the definition of \mathcal{U} refers (via μ and ν) to p, not to p'.

Fix $\eta > 0$ such that $\sigma(\mu - \eta) > \sigma(\mu) - \varepsilon/4$. Applying Lemma 7 with p' reduced by an appropriate constant factor, we find that $\mathbb{E}_{p'}(N') \geq \mathbb{E}_{p'}(N) \geq (\sigma(\mu - \eta) - \varepsilon/4)\binom{n}{k}(p')^{\binom{k}{2}}$, which is at least $(\sigma(\mu) - 3\varepsilon/4)\nu$ if we choose p' correctly. Since N' is bounded by the total number of K_k s, which is very unlikely to be much larger than its mean ν , it follows that $\mathbb{P}_{p'}(\mathcal{U})$ is bounded away from zero.

Since p/p' is a constant larger than 1, if \mathcal{U} has a sharp threshold, we have $\mathbb{P}_p(\mathcal{U}) \to 1$ as required. Otherwise, Theorem 1.2 of Friedgut [12] applies. We conclude that there is a constant C such that $\mathbb{P}_p(\mathcal{U} \mid \mathcal{E}) \to 1$, where \mathcal{E} is the event that a fixed copy of K_C is present in G = G(n,p). Of course, conditioning on \mathcal{E} is equivalent to simply adding the edges of K_C to G. Hence, whp, G(n,p) has the property that after adding a particular copy of K_C to G, the event \mathcal{U} holds. But the expected number of K_k s in $G \cup K_C$ that share at least ℓ vertices

with a K_k in $G \cup K_C$ not present in G turns out to be less that $n^{-\delta}\nu$ for some $\delta > 0$. Hence, $G \cup K_C$ contains at most $n^{-\delta/2}\nu$ such K_k s whp. Whenever this holds, removing the edges of K_C from G splits existing components into at most $n^{-\delta/2}\nu$ new components. It follows that G has whp at most $n^{-\delta/2}\nu(\log n)^{5k^3}$ fewer K_k s in large components that $G \cup K_C$. Since $G \cup K_C$ has property \mathcal{U} whp, it follows that G has the same property with a slightly increased ε whp.

At this point, we have shown that whp we have the 'right' number of K_k s in 'large' components; it remains to show that in fact almost all such K_k s are in a single giant component. In the special case $k=2, \ell=1$, i.e., when $G_n^{k,\ell}$ is simply G(n,p), there are many simple ways of showing this, most of them based on 'sprinkling' of one form or another: following the original approach of Erdős and Rényi [11] to the study of the giant component of G(n,p), one chooses p' slightly smaller than p, and views G(n, p) as obtained from G(n, p') by 'sprinkling' in a few extra edges. Using independence of the sprinkled edges from G(n, p'), it is easy to show that who the sprinkled edges join up almost all large components of G(n, p') into a single giant component. Unfortunately, most of these approaches do not carry over to the present setting; the essential problem is that, depending on the parameters, $G_p^{k,\ell}$ may well have many more vertices than G(n,p). In fact, it may have many more than n^2 vertices. Approaches such as forming an auxiliary graph on the large components, joining two if they are connected by sprinkled edges, and then comparing this graph to G(n', p')for suitable n' and p', do not seem to work: here n' is much larger than n, and there is not nearly enough independence for such a comparison to be possible. For the same reason, we cannot count cuts between largish components, and estimate the number not joined by sprinkled edges: we may have many more than 2^{n^2} cuts, while the probability that a given cut is not joined will certainly be at least 2^{-n^2} .

Fortunately, we can get another version of the sprinkling argument to work: the key result is the following rather ugly lemma. In stating this we write p_0 for $n^{-\frac{2}{k+\ell-1}}$, so $\mu(n,p) = \Theta(1)$ is equivalent to $p = \Theta(p_0)$. We write ν_0 for $\nu(p_0) = \binom{n}{k} p_0^{\binom{k}{2}}$.

Lemma 10. Fix constants $\varepsilon > 0$ and A > 0, let G_0 be any graph on [n], and let C_1, C_2, \ldots, C_r list all components of the corresponding graph $G_0^{k,\ell}$ that contain one or more $K_k s$ in G_0 with property \mathcal{Q} . Suppose that

- 1. between them the C_i contain at least $2\varepsilon\nu_0$ copies of K_k in G_0 ,
- 2. no single C_i contains all but $\varepsilon \nu_0$ copies of K_k in G_0 ,
- 3. G_0 contains at most $A\nu_0$ copies of K_k ,
- 4. for $1 \le s < k$ we have

$$Z_s \le An^{2k-s}p_0^{2\binom{k}{2}-\binom{s}{2}},$$

where Z_s is the number of pairs of $K_k s$ in G_0 sharing exactly s vertices, and

5. no vertex of G_0 lies in more than ν_0/\sqrt{n} copies of K_k in G_0 .

Set $\gamma = (\log n)^{-4}$, let $G = G(n, \gamma p_0)$ be a random graph on the same vertex set as G_0 , and let $G_1^{k,\ell} \supset G_0^{k,\ell}$ be the graph $G^{k,\ell}$ derived from $G_1 = G_0 \cup G$. Then, for any fixed i, the probability that there is some j such that C_i and C_j are contained in a common component of $G_1^{k,\ell}$ is at least c, for some constant $c = c(A, \varepsilon) > 0$ depending only on A and ε .

In other words, roughly speaking, and ignoring all the conditions for a moment, sprinkling in extra edges with density γp_0 is enough to give any given 'large' component of $G_0^{k,\ell}$ at least probability c of joining up with another such component, for some c > 0 that does not depend on n.

We shall prove Lemma 10 later; first, we show that Theorem 1 follows.

Proof of Theorem 1. Let p = p(n) be chosen so that $\mu = \mu(n, p)$ is constant and $\mu > 1$. It suffices to show that for any $\varepsilon > 0$,

$$C_1(G_p^{k,\ell}) \ge N_0 = (\sigma(\mu) - 2\varepsilon)\nu = (\sigma(\mu) - 2\varepsilon) \binom{n}{k} p^{\binom{k}{2}}$$
 (5)

holds whp: letting $\varepsilon \to 0$, (5) implies that $C_1(G_p^{k,\ell}) \geq (\sigma(\mu) - o_p(1))\nu$, while Lemma 5 immediately implies that $\mathbb{E}(C_1(G_p^{k,\ell})) \leq (\sigma(\mu) + o(1))\nu$. Together, these two statements imply that $C_1(G_p^{k,\ell}) = (\sigma(\mu) + o_p(1))\nu$, which is what the first statement of Theorem 1 claims. For the second, we simply observe that the same argument gives $C_1(G_p^{k,\ell}) + C_2(G_p^{k,\ell}) = (\sigma(\mu) + o_p(1))\nu$.

To establish (5), let us choose p' < p so that $(\sigma(\mu(p')) - \varepsilon/3)(p'/p)^{\binom{k}{2}} \ge \sigma(\mu) - \varepsilon$. From continuity of σ , we can choose such a p' with $p - p' = \Theta(p_0)$. By Lemma 8, applied with p' in place of p and $\varepsilon/3$ in place of ε , whp at least $N_1 = N_0 + \varepsilon \binom{k}{n} p^{\binom{k}{2}}$ copies of K_k in G(n, p') have property \mathcal{Q} ; let V_1, \ldots, V_{N_1} be (the vertex sets of) N_1 such copies.

Let $T = (\log n)^3$, and let H_1, \ldots, H_T be independent copies of $G(n, \gamma p_0)$ that are also independent of $G_0 = G(n, p')$, with the vertex sets of H_1, \ldots, H_T and of G_0 identical. Set $G_t = G_0 \cup \bigcup_{i=1}^t H_i$, and note that G_T has the distribution of the random graph G(n, p'') for some p''. Since $p' + T\gamma p_0 \leq p$ if n is large enough, we have $p'' \leq p$ if n is large enough, so we may couple G_T and G(n, p) so that the latter contains the former. Hence, it suffices to prove that whp there is a single component of $G^{k,\ell}(G_T)$ containing at least N_0 of the k-cliques V_1, \ldots, V_{N_1} .

As the reader will have guessed, we shall sprinkle in edges in T rounds, applying Lemma 10 successively with each pair (G_{t-1}, H_t) in place of (G_0, G) , and $\varepsilon' = \varepsilon \nu / \nu_0 = \varepsilon (p/p_0)^{\binom{k}{2}}$ in place of ε . As noted above, by Lemma 8, whp G_0 contains at least $N_1 = (\sigma(\mu) - \varepsilon)\nu$ copies of K_k with property \mathcal{Q} . We may assume that $\varepsilon < \sigma(\mu)/3$, in which case $N_1 \geq 2\varepsilon\nu = 2\varepsilon'\nu_0$. Since the event that V_i has property \mathcal{Q} is increasing, and $G_0 \subset G_t$ for all t, whp the first assumption of Lemma 10 holds for G_0 and hence for all G_t .

If the second assumption fails at some point, then we are done: G_t and hence $G_T \supset G_t$ already contains a single component containing at least $N_1 - \varepsilon' \nu_0 = N_1 - \varepsilon \nu = N_0$ copies of K_k , as required. The remaining assumptions are downset conditions, bounding the number of copies of certain subgraphs in G_t from above. Standard results tell us that G(n, p) satisfies these conditions who if we choose A large enough; it follows that who G_T and hence every G_t does too.

From the comments above, we may assume that the conditions of Lemma 10 are satisfied at each stage. Suppose that after t rounds, i.e., t applications of Lemma 10, the sets V_1, \ldots, V_{N_1} are now contained in r = r(t) components C_1, \ldots, C_r of $G_t^{k,\ell}$. By Lemma 10, each C_i has a constant probability c > 0 of joining up with some other C_j in each round, so after $(\log n)^2$ further rounds, the probability that a particular C_i has not joined some other C_j is at most $(1-c)^{(\log n)^2} = o(n^{-2})$. It follows that with probability $1-o(n^{-1})$, after $(\log n)^2$ rounds every C_i has joined some other C_j . If this holds, the number r' of components containing V_1, \ldots, V_{N_1} is now at most r/2. Hence, after $\log r \leq \log n$ sets of $(\log n)^2$ rounds, either an assumption is violated, or there is a single component containing all V_i . But as shown above, there is only one assumption that can be violated with probability bounded away from zero, and if this assumption is violated at some stage, we are already done.

It remains only to prove Lemma 10.

Proof of Lemma 10. We assume without loss of generality that i=1. Let $a=\lceil (\log n)^{5k^3} \rceil$. Since C_1 contains a K_k with property \mathcal{Q}, C_1 contains at least a distinct copies of K_ℓ , each lying in a K_k in C_1 . Let S_1, \ldots, S_a be a such copies.

From Assumptions 1 and 2, C_2, \ldots, C_r between them contain at least $\varepsilon\nu_0$ copies of K_k . The set $V_0 = V(S_1) \cup \cdots \cup V(S_a)$ has size $O(a) = O^*(1)$, and so, using Assumption 5, meets at most $o(\nu_0)$ copies of K_k in G_0 . It follows that we may find $b = \varepsilon\nu_0/3$ copies D_1, \ldots, D_b of K_k in C_2, \ldots, C_r such that each D_j is vertex disjoint from V_0 . (We round b up to the nearest integer, but omit this irrelevant distraction from the formulae.)

It suffices to show that with probability bounded away from zero, there is a path of K_k s in $G_1^{k,\ell}$ joining some S_i to some D_j . We shall do this using the second moment method. For this, it helps to count only paths with a simple form.

By a potential k-path we mean a sequence V_1, \ldots, V_k of sets of k vertices of G_0 with the following properties: V_1 contains some S_i , all other vertices of $\bigcup_{t=1}^k V_t$ lie outside V_0 (and hence outside S_i), V_k coincides with some D_j , and for $2 \le t \le k$, V_t consists of $k - \ell$ vertices outside $\bigcup_{1 \le s < t} V_s$ together with ℓ vertices of V_{t-1} , not all of which lie in $\bigcup_{s < t-1} V_s$.

A potential k-path starting at S_i and ending at D_j contains exactly $k(k - \ell) - k$ vertices outside $S_i \cup D_j$: starting with S_i we add $k - \ell$ new vertices for each set V_t in the path, but this count includes the vertices of D_j . It follows that the number of potential k-paths joining S_i to D_j is $\Theta(n^{k(k-\ell)-k})$, so the total number of potential k-paths is $\Theta(abn^{k(k-\ell)-k})$.

A potential k-path $(V_t)_{t=1}^k$ joining S_i to D_j is a k-path if all edges contained in each V_t but not in S_i or D_j are present in $G = G(n, \gamma p_0)$. Note that any potential k-path contains exactly $r = k(\binom{k}{2} - \binom{\ell}{2}) - \binom{k}{2}$ such edges: for each t there are $\binom{k}{2} - \binom{\ell}{2}$ edges spanned by V_t but by no earlier V_s , but this count includes all edges of D_j .

Let X denote the number of k-paths. If any k-path is present, then some S_i is joined to some D_j in the graph $G_1^{k,\ell}$ formed from $G_0 \cup G$, so it suffices to show that $\mathbb{P}(X > 0)$ is bounded away from zero.

Since each potential k-path is present with probability exactly $(\gamma p_0)^r$, we have

$$\begin{split} \mathbb{E}(X) &= \Theta\left(abn^{k(k-\ell)-k}(\gamma p_0)^{k(\binom{k}{2}-\binom{\ell}{2})-\binom{k}{2}}\right) \\ &= \Theta\left(ab\left(n^k p_0^{\binom{k}{2}}\right)^{-1}\left(n^{k-\ell} p_0^{\binom{k}{2}-\binom{\ell}{2}}\right)^k \gamma^r\right). \end{split}$$

Now the bracket raised to the power k in the last line above is $\Theta(1)$ by definition of p_0 , while $b = \varepsilon \nu_0/3 = \Theta(n^k p_0^{\binom{k}{2}})$. Thus we have $\mathbb{E}(X) = \Theta(a\gamma^r)$. Since, crudely, $r \leq k \binom{k}{2} \leq k^3/2$, while $a \geq (\log n)^{3k^3}$ and $\gamma = (\log n)^{-4}$, we have $\mathbb{E}(X) \to \infty$.

It remains to estimate the second moment of X. For this, it turns out to be easier to consider a related random variable Y.

A potential free k-path is defined exactly as a potential k-path, except that we omit the condition that V_k coincides with some D_j . It is easy to see that the fraction of potential free k-paths that are potential k-paths is exactly $b/\binom{n-|V_0|}{k} = \Theta(b/n^k) = \Theta(p_0^{\binom{k}{2}})$.

A free k-path is a potential free k-path in which all edges except those contained in the starting set S_i are present in $G = G(n, \gamma p_0)$. Note that there are $r' = r + \binom{k}{2}$ such edges, so each potential free k-path is an actual free k-path with probability $(\gamma p_0)^{r+\binom{k}{2}}$. Let Y denote the number of free k-paths. It follows that

$$\mathbb{E}(Y) = \Theta\left(\mathbb{E}(X)p_0^{-\binom{k}{2}}(\gamma p_0)^{\binom{k}{2}}\right) = \Theta\left(\mathbb{E}(X)\gamma^{\binom{k}{2}}\right) = \Theta(a\gamma^{r'}). \tag{6}$$

For $0 \le s \le k$, let Z_s denote the number of ordered pairs of copies of K_k in G_0 sharing exactly s vertices, and let $Z_s' \le Z_s$ denote the number of such pairs lying entirely outside V_0 . Let X_s denote the number of ordered pairs of k-paths whose destinations (final sets V_k) share exactly s vertices, and Y_s the number of ordered pairs of free k-paths with this property. Among ordered pairs (P_1, P_2) of potential free k-paths whose destinations share s vertices, the fraction of pairs in which P_1 and P_2 are also potential k-paths is exactly $Z_s'\left(\binom{n-|V_0|}{k}\binom{k}{s}\binom{n-|V_0|}{k-s}\right)^{-1} = \Theta(Z_s'n^{-(2k-s)})$. Moreover, this statement remains true if we restrict our attention to pairs (P_1, P_2) with a certain number of common edges. Indeed, under any sensible assumption on (P_1, P_2) , the pair (V_k, V_k')

of destinations of a random pair (P_1, P_2) is uniform on all pairs of k-sets in $[n] \setminus V_0$ sharing s vertices.

Given a pair of paths with destinations sharing s vertices, for both paths to be present as free k-paths requires the presence of $2\binom{k}{2}-\binom{s}{2}$ more edges in G than required by their presence as k-paths. It follows that

$$\mathbb{E}(X_s)/\mathbb{E}(Y_s) = \Theta\left(Z_s' n^{-(2k-s)} (\gamma p_0)^{-2\binom{k}{2} + \binom{s}{2}}\right).$$

By Assumption 4 we have $Z_s = O(n^{2k-s}p_0^{2\binom{k}{2}-\binom{s}{2}})$ for $1 \le s \le k-1$. This also holds for Z_k by Assumption 3, and hence also for $Z_0 \le Z_k^2$. Hence, for $0 \le s \le k$,

$$\mathbb{E}(X_s)/\operatorname{\mathbb{E}}(Y_s) = O\left(\gamma^{-2\binom{k}{2}+\binom{s}{2}}\right) = O\left(\gamma^{-2\binom{k}{2}}\right).$$

Since $\mathbb{E}(X^2) = \sum_{s=0}^k \mathbb{E}(X_s)$ and $\mathbb{E}(Y^2) = \sum_{s=0}^k \mathbb{E}(Y_s)$, it follows immediately that $\mathbb{E}(X^2)/\mathbb{E}(Y^2) = O(\gamma^{-2\binom{k}{2}})$. We claim that $\mathbb{E}(Y^2) = O(\mathbb{E}(Y)^2)$. Recalling from (6) that $\mathbb{E}(X)/\mathbb{E}(Y) = \Theta(\gamma^{-\binom{k}{2}})$, it then follows that $\mathbb{E}(X^2) = O(\mathbb{E}(X)^2)$, and hence that $\mathbb{P}(X > 0)$ is bounded away from zero.

To evaluate $\mathbb{E}(Y^2)$, we could argue from the fact that free k-paths are balanced in a certain sense, but rather than make this precise, it turns out to be easier to simply use our coupling results from Subsection 1.1.

We may evaluate Y, and hence Y^2 , as follows. Start with our set V_0 of 'reached' vertices, namely $V_0 = \bigcup_{i=1}^a V(S_i)$. Also, mark S_1, \ldots, S_a as untested copies of K_ℓ . Now explore as in the proofs of Theorem 2 and Lemma 3, except that we only look for new vertices outside V_0 ; note that our edge probability is now γp_0 rather than $\Theta(p_0)$, so the corresponding branching process is strongly subcritical. We stop the exploration after k 'rounds', in the terminology of Lemma 3; of course it may well die earlier.

We consider three cases. Firstly, let \mathcal{A} be the event that in the exploration just described, we find no exceptional edges. Since $|V_0| = O^*(1)$, and the total size of the relevant branching processes is also $O^*(1)$ whp, we have $\mathbb{P}(\mathcal{A}^c) = O^*(\gamma p_0) = O(n^{-\delta})$ for some $\delta > 0$ depending only on k and ℓ . When \mathcal{A} holds, we obtain a coupling of our exploration with a independent copies of the branching process $\mathfrak{X}(\lambda)$, where $\lambda = \mu(\gamma p_0) = \Theta(\gamma^{\binom{k}{2}-\binom{\ell}{2}})$. If \mathcal{A} holds, the number of K_k s reached in the final round is equal to N_k/M , where N_k is the number of particles in generation k of the combined branching process, and we divide by $M = \binom{k}{\ell} - 1$ since we add M copies of K_ℓ for each K_k we find.

Now from standard branching process results, $\mathbb{E}(N_k^2) = \Theta(a^2 \lambda^{2k}) = \Theta(a^2 \gamma^{2r'})$, recalling that $r' = k(\binom{k}{2} - \binom{\ell}{2})$ is the number of edges of G_0 in a free k-path. It follows that $\mathbb{E}(Y^2 1_{\mathcal{A}}) = O(a^2 \gamma^{2r'})$.

We claim that there is a constant K such that the chance of finding more than K exceptional edges is $o(n^{-10k^3})$. To see this, first note that the probability that a Poisson random variable with mean at most 1 exceeds $\log n$ is of order $(\log n)^{-\log n} = o(n^{-20k^3})$. Hence, with probability $1 - o(n^{-10k^3})$, the first k generations of $a + \log n$ copies of $\mathfrak{X}(\lambda)$ contain at most $(a + \log n)k(\log n)^k = 0$

 $O^*(1)$ particles – simply crudely bound the number of children of each particle by $\log n$. Now arguing as in the proof of Theorem 2, given that we have reached $O^*(1)$ vertices, the chance of finding an exceptional edge is at most $n^{-\delta}$ for some $\delta > 0$. Hence, the chance of finding K such edges within the first $O^*(1)$ steps is $O^*(n^{-\delta K})$ which is $o(n^{-10k^3})$ if we pick K large enough. But if we find no more than K exceptional edges within $O^*(1)$ steps, and the first k generations of $a+K \leq a+\log n$ branching processes have total size $O^*(1)$, then (recalling that we stop after k rounds), our coupling succeeds, with a+K branching processes as the upper bound.

Let \mathcal{B} be the event that we do find more than K exceptional edges, so $\mathbb{P}(\mathcal{B}) = o(n^{-10k^3})$. The number of pairs of free k-paths present in the complete graph on K_n is easily seen to be at most n^{2k^3} , so we have $\mathbb{E}(Y^2 1_{\mathcal{B}}) \leq \mathbb{P}(\mathcal{B})n^{2k^3} = o(n^{-8k^3}) = o(a^2\gamma^{2r'})$.

Finally, let $\mathcal{C} = (\mathcal{A} \cup \mathcal{B})^c$. If \mathcal{C} holds then, as above, with very high probability we have reached $O^*(1)$ vertices in our exploration. The picture given by our exploration may be complicated by the exceptional edges, but $O^*(1)$ vertices in any case contain $O^*(1)$ (pairs of) free k-paths, so we have $\mathbb{E}(Y^21_{\mathcal{C}}) = O^*(\mathbb{P}(\mathcal{C})) = O^*(\mathbb{P}(\mathcal{A}^c)) = o(1)$.

Putting it all together, $\mathbb{E}(Y^2) = \mathbb{E}(Y^2 1_{\mathcal{A}}) + \mathbb{E}(Y^2 1_{\mathcal{B}}) + \mathbb{E}(Y^2 1_{\mathcal{C}}) = O(a^2 \gamma^{2r'})$. From (6) we thus have $\mathbb{E}(Y^2) = O(\mathbb{E}(Y)^2)$. As noted earlier it follows that $\mathbb{E}(X^2) = O(\mathbb{E}(X)^2)$, and thus that $\mathbb{P}(X > 0)$ is bounded away from 0, as required.

1.3 Far from the critical point

In the previous subsections we focused on the 'approximately critical' case, where p is chosen so that the expected number of other K_k s adjacent to (i.e., sharing at least ℓ vertices with) a given K_k is of order 1. In more standard percolation contexts, one can make this assumption without loss of generality; using monotonicity it follows that the fraction of vertices in the largest component tends to 0 or 1 outside this range of p.

Here we do not have such simple monotonicity, because the number of vertices of $G_p^{k,\ell}$ changes as p varies. However, it is still easy to deduce results for values of p outside the range $p = \Theta(p_0)$ from those for p inside this range.

For $p = o(p_0)$, this is essentially trivial; since the property of G corresponding to $G^{k,\ell}$ containing a component of size at least $C \log n$ is monotone, Theorem 2 together with concentration of the number of K_k s trivially implies that the largest component of $G_p^{k,\ell}$ contains who a fraction o(1) of the vertices of $G_p^{k,\ell}$, as long as $\nu = \nu(p) = \binom{n}{k} p^{\binom{k}{2}}$, the expected number of vertices of $G_p^{k,\ell}$, grows faster than $\log n$. When ν grows slower than $\log n$ (or indeed than \sqrt{n}), by estimating the expected number of cliques sharing one or more vertices it is very easy to check that who $G_p^{k,\ell}$ contains no edges, and thus no giant component (as long as ν does tend to infinity).

To handle the case $p/p_0 \to \infty$, we use a slightly different argument. Let N denote the number of pairs of vertex disjoint cliques in G(n,p) that lie in the

same component of $G_p^{k,\ell}$. Let $p = \Theta(p_0)$. Since the expected number of pairs of cliques in G(n,p) sharing one or more vertices is $o(\nu^2)$, Theorem 1 shows that $\mathbb{E}_p(N) \geq \left(\sigma(\mu(p))^2 - o(1)\right)\nu^2$, considering only pairs in the giant component. Fix two disjoint sets V_1 , V_2 of k vertices of G(n,p), and let π_p be the probability that V_1 and V_2 are joined in $G_p^{k,\ell}$ given that V_1 and V_2 are cliques in G(n,p). Then we have $\mathbb{E}_p(N) = \binom{n}{k}\binom{n-k}{k}p^{2\binom{k}{2}}\pi_p \sim \nu^2\pi_p$. Hence, whenever $\mu(p) = \Theta(1)$, we have $\pi_p \geq \sigma(\mu(p))^2 - o(1)$.

Now π_p is the probability of an increasing event (in the product space corresponding to the $\binom{n}{2} - 2\binom{k}{2}$ possible edges outside V_0 , V_1), and is hence an increasing function of p. Since $\sigma(\mu) \to 1$ as $\mu \to \infty$, it follows that $\pi_p \to 1$ if $p/p_0 \to \infty$. Thus, the expected number of unconnected pairs of cliques in $G_p^{k,\ell}$ is $o(\nu^2)$ whenever $p/p_0 \to \infty$. Since the number of cliques is concentrated around ν , it follows that whp almost all vertices of $G_p^{k,\ell}$ lie in a single component.

1.4 Near the critical point

Derényi, Palla and Vicsek [10] suggest that for $\ell = k-1$, 'at the critical point', i.e., when $p = ((k-1)n)^{-1/(k-1)}$, the largest component in $G_p^{k,\ell}$ contains roughly n vertices of $G_p^{k,\ell}$, i.e., roughly n k-cliques. This is based both on computer experiments, and on the heuristic that at the critical point, the giant component in random graphs is roughly 'treelike'. This latter heuristic seems extremely weak: there is no reason why a treelike structure in $G_p^{k,\ell}$ cannot contain many more than n k-cliques. Indeed, one would not expect whether or not two k-cliques share a single vertex to play much role in the component structure of $G_n^{k,\ell}$.

It would be interesting to know whether the observation of [10] is in fact correct, but there are several problems. Firstly, the question is not actually that natural: why chose exactly this value of p? In G(n,p), it is natural to take p=1/(n-1) (or p=1/n; it turns out not to matter) as 'the' critical probability, since in this case one has at the beginning a very good approximation by an exactly critical branching process. However, in general there is a scaling window within which, for example, the largest and second largest components are comparable in size. For G(n,p) the window is $p=n^{-1}+O(n^{-4/3})$; see Bollobás [3] and Łuczak [14]; see also the book [4]. For other random graph models, establishing the behaviour of the largest component in and around the scaling window can be very difficult; see, for example, Ajtai, Komlós and Szemerédi [1], Bollobás, Kohayakawa and Łuczak [6], and Borgs, Chayes, van der Hofstad, Slade and Spencer [7, 8, 9].

In general, one would expect that inside the scaling window, the largest component would have size of order $N^{2/3}$, where N is the 'volume', which here would presumably be $\nu = \mathbb{E}(|G_p^{k,\ell}|)$. Note that this need not contradict the experimental results of Derényi, Palla and Vicsek [10]: it may simply be that their choice of p is (slightly) outside the window.

Unfortunately, due to the dependence in the model, it seems likely to be extremely difficult to establish results about the scaling window, or about the

behaviour at $p = ((k-1)n)^{-1/(k-1)}$. The problem is that there are o(1) errors in the branching process approximation discussed above that appear right from the beginning. On the one hand, for $\ell = k - 1$, as soon as we find a new K_k sharing k-1 vertices with an earlier K_k , there is a probability of order p that a single extra 'exceptional' edge is present forming a K_{k+1} , and thus forming extra K_{k-1} s from which we need to explore at the next step. In the other direction, after even one step of our exploration, we have tested whether any vertex v not so far reached is joined to all vertices in certain K_{k-1} s. The negative information that v is not so joined reduces the probability that v is joined to any new K_{k-1} slightly; in fact by a factor of $1-\Theta(p)$ for each K_{k-1} previously tested. To study the scaling window, or the behaviour at $p = ((k-1)n)^{-1/(k-1)}$ or at $\mu(p) = 1$, say, one would presumably need to understand the net effect of these positive and negative deviations from the branching process to an accuracy much higher than the size of each effect. This seems a tall order even for the first few steps in the branching process, let alone when the component has grown to size $\Theta(N^{2/3})$ or even $\Theta(n)$.

2 Variants

In the rest of the paper we consider several variants of the clique percolation problem discussed above. In most cases where we can prove results, the proofs are minor modifications of those above, so to avoid trying the reader's patience too far we shall only briefly indicate the changes.

2.1 Oriented cliques

Given $n \geq 2$ and $0 \leq p \leq 1$, let $\overrightarrow{G}(n,p)$ be the random directed graph on [n] in which each of the n(n-1) possible directed edges is present with probability p, independently of the others. Thus doubled edges are allowed: edges \overrightarrow{vw} and \overrightarrow{wv} may both be present (though this will turn out to be irrelevant), and the simple graph underlying $\overrightarrow{G}(n,p)$ has the distribution of $G(n, 2p-p^2)$.

Let \overrightarrow{H} be a fixed orientation of K_k ; for the moment we shall take \overrightarrow{H} to be $\overrightarrow{K_k}$, that is, K_k with a linear order: $V(\overrightarrow{K_k}) = [k]$ and $E(\overrightarrow{K_k}) = \{\overrightarrow{ij} : 1 \le i < j \le k\}$. Given a directed graph \overrightarrow{G} , let $V = V_{\overrightarrow{H}}(\overrightarrow{G})$ denote the set of all copies of \overrightarrow{H} in \overrightarrow{G} . To be totally formal, we may take V to be the set of all subsets of $\binom{k}{2}$ edges of \overrightarrow{G} that form a graph isomorphic to \overrightarrow{H} . If a given set S of k vertices of \overrightarrow{G} contains double edges, then it may span several copies of \overrightarrow{H} , while if S spans no double edges it spans at most one copy of \overrightarrow{H} . (For orientations \overrightarrow{H} with automorphisms, the latter statement would not be true if we considered injective homomorphisms from \overrightarrow{H} . This is the reason for the somewhat fussy definition of a 'copy' of \overrightarrow{H} .)

For $1 \leq \ell \leq k-1$, let $\overrightarrow{G}^{k,\ell} = \overrightarrow{G}^{k,\ell}_{\overrightarrow{H}}$ be the graph formed from \overrightarrow{G} as follows: let the vertex set of $\overrightarrow{G}^{k,\ell}$ be $V = V_{\overrightarrow{H}}(\overrightarrow{G})$, and join two vertices if the corresponding

copies of \overrightarrow{H} share at least ℓ vertices. Note that two copies may share k vertices (if double edges are involved); this will turn out to be irrelevant. Our aim now is to study the emergence (as p varies) of a giant component in $\overrightarrow{G}_p^{k,\ell}$, the graph $\overrightarrow{G}_p^{k,\ell}$ defined on the copies of \overrightarrow{H} in $\overrightarrow{G}(n,p)$.

2.1.1 Linearly ordered cliques

We start by restricting our attention to $\overrightarrow{H} = \overrightarrow{K_k}$. With $\ell = k - 1$, the study of this model was proposed by Palla, Farkas, Pollner, Derényi and Vicsek [18], who predicted a critical point of $p = (nk(k-1))^{-1/(k-1)}$. As we shall see, this prediction is correct.

Let us consider the component exploration in $\overrightarrow{G}_p^{k,\ell}$ analogous to that in $G_p^{k,\ell}$ described in Section 1. The typical case is that we are looking for new \overrightarrow{K}_k s containing a given \overrightarrow{K}_ℓ , say S, consisting of S together with $k-\ell$ new vertices. As before, we expect to find a roughly Poisson number of such new \overrightarrow{K}_k s, but now the mean is slightly different: in addition to choosing a set N of $k-\ell$ new vertices, we must consider the $k!/\ell!$ linear orders on $S \cup N$ consistent with the order we already have on S. Given N and such an order, the probability that this particular \overrightarrow{K}_k is present is then $p^{\binom{k}{2}-\binom{\ell}{2}}$ as before. As in the undirected case, each new \overrightarrow{K}_k we find typically gives rise to $M = \binom{k}{\ell} - 1$ new \overrightarrow{K}_ℓ s to explore from in the next step.

Let $\overrightarrow{\mu} = \overrightarrow{\mu}(k, \ell, p)$ be given by

$$\overrightarrow{\mu} = \binom{k}{\ell} - 1 \frac{k!}{\ell!} \binom{n}{k-\ell} p^{\binom{k}{2} - \binom{\ell}{2}}.$$

The proof of Theorem 1 goes through *mutatis mutandis* to give the result below. One can also obtain analogues of the undirected results for the cases $\overrightarrow{\mu} \to 0$ and $\overrightarrow{\mu} \to \infty$; we omit these for brevity.

Theorem 11. Fix $1 \le \ell < k$ and let p = p(n) be chosen so that $\overrightarrow{\mu} = \Theta(1)$. Then, for any $\varepsilon > 0$, who we have

$$(\sigma(\overrightarrow{\mu}) - \varepsilon)\nu \le C_1(\overrightarrow{G}_p^{k,\ell}) \le (\sigma(\overrightarrow{\mu}) + \varepsilon)\nu,$$

where $\nu = \binom{n}{k} k! p^{\binom{k}{2}}$ is the expected number of copies of $\overrightarrow{K_k}$ in $\overrightarrow{G}(n,p)$.

Note that the function σ appearing here is the same function as in Theorem 1, but now evaluated at $\overrightarrow{\mu}$ rather than at μ . In particular, $\sigma(\overrightarrow{\mu}) > 0$ if and only if $\overrightarrow{\mu} > 1$, and the critical point is given by the solution to $\overrightarrow{\mu} = 1$. In the special case $\ell = k - 1$, we have $\overrightarrow{\mu}(k, \ell, p) = (k - 1)knp^{k-1}$, so the critical point is exactly as predicted by Palla, Farkas, Pollner, Derényi and Vicsek [18].

As the proof really follows that of Theorem 1 very closely, we only briefly describe the differences. The argument in Subsection 1.1 is essentially unmodified; it is still true that the first O(1) 'exceptional' edges give rise to the addition of

O(1) extra $\overrightarrow{K_{\ell}}$ s, arguing as before using the total degree of new vertices, rather than in- or out-degree, say.

For the lower bound, we can argue much of the time using the underlying undirected graph G rather than $\overrightarrow{G} = \overrightarrow{G}(n,p)$. Indeed, when exploring from a $\overrightarrow{K_\ell}$ S_i , say, we let W_i be the set of 'clean' vertices joined in G to every vertex of S_i . We then look for undirected k-cliques in $G[W_i]$. Arguing as before, the number we find can be coupled to agree (up to a negligible error term) with a Poisson distribution with the appropriate mean, now $(1-\eta)\binom{n}{k-\ell}(2p-p^2)\binom{k}{2}-\binom{\ell}{2}$. Moreover, as before, we may assume that the k-cliques we find are vertex disjoint. Only at this point do we check the orientations of the $\binom{k}{2}-\binom{\ell}{2}$ new edges involved in each k-clique; the probability that we find one of the $k!/\ell!$ orientations that gives a $\overrightarrow{K_k}$ extending S_i is $(k!/\ell!)(1/2)^{\binom{k}{2}-\binom{\ell}{2}}+o(1)$, so the number of such $\overrightarrow{K_k}$ s that we do find may be closely coupled to a Poisson distribution with mean $\overrightarrow{\mu}$ as required.

Finally, the argument joining up large components goes through with only trivial modifications to the definitions.

2.1.2 Cliques with arbitrary orientations

We now turn out attention to the phase transition in the graph $\overrightarrow{G}_p^{k,\ell}$ defined on the copies of \overrightarrow{H} in $\overrightarrow{G}(n,p)$, where \overrightarrow{H} is some non-transitive orientation of K_k . Perhaps surprisingly, it turns out that something genuinely new happens in this case.

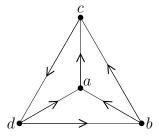


Figure 1: An orientation \overrightarrow{H} of K_4 .

Let k=4, $\ell=3$, let \overrightarrow{H} be the orientation of K_4 shown in Figure 1, and let $\overrightarrow{G}_p^{k,\ell}$ be defined as before. When exploring a component of $\overrightarrow{G}_p^{k,\ell}$, suppose that we have found a certain copy of \overrightarrow{H} , and are looking for new copies containing a particular subgraph S of order 3. There are now four separate cases, although one can combine them in pairs. First suppose the vertex set of S is $\{b, c, d\}$, so S is an oriented triangle. If we find a vertex v joined to b, c and d, there are six combinations of orientations of vb, vc and vd that lead to a copy of \overrightarrow{H} : either two edges are oriented towards v and one away, in which case v plays the role of a in the new copy of \overrightarrow{H} , or two are oriented away from v and one towards,

in which case v plays the role of b. The same holds if $V(S) = \{a, c, d\}$, since S is again an oriented triangle.

In the other two cases, S is a linearly ordered triangle, and either v sends edges to the top two vertices of S and receives an edge from the bottom one, and so plays the role of d, or v sends an edge to the top vertex and receives edges from the bottom two, playing the role of c.

Suppose more generally that H is an orientation of K_k in which no two vertices are equivalent (the orientation in Figure 1 has this property). Let $M(\overrightarrow{H})$ be the k-by-k matrix whose ijth entry is the number of ways of orienting the edges from a new vertex v to $[k] \setminus \{j\}$ such that $\overrightarrow{H} - j \cup \{v\}$ forms a graph isomorphic to \overrightarrow{H} with v playing the role of vertex i. For example, with \overrightarrow{H} as in Figure 1, numbering the vertices in the order a, b, c, d, we have

$$M = \begin{pmatrix} 3 & 3 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \tag{7}$$

Let us say that a copy in \overrightarrow{G} of a subgraph of \overrightarrow{H} induced by k-1 vertices is of $type\ j$ if it is formed by omitting the vertex j. Also, let us say that a copy of \overrightarrow{H} found in our exploration by adding a new vertex v to a subgraph of \overrightarrow{H} with k-1 vertices is of $type\ i$ if the new vertex corresponds to vertex i of \overrightarrow{H} . Then, towards the start of our exploration, the expected number of type i copies of \overrightarrow{H} we reach from a type j subgraph is $M_{ij}np^{k-1}$. When we continue the exploration, each type i copy of \overrightarrow{H} gives rises to one new subgraph of each type other than i, and this gives us our branching process approximation.

For the formal statement, less us pass to the general case $1 \le \ell \le k-1$. For simplicity, the reader may prefer to consider only graphs \overrightarrow{H} such that all $\binom{k}{\ell}$ sets of $k-\ell$ vertices of \overrightarrow{H} are non-equivalent, so that when we extend an ℓ vertex graph to a graph isomorphic to \overrightarrow{H} we can identify which $k-\ell$ vertices of [k] the new vertices correspond to. In general, we may resolve ambiguous cases arbitrarily. (One could instead collapse the corresponding types in the branching process, but this complicates the description.) Let M be the $\binom{k}{\ell}$ -by- $\binom{k}{\ell}$ matrix defined as follows: given two ℓ -element subsets A and B of [k], let S be the subgraph of \overrightarrow{H} induced by the vertices in A, and consider a set N of $k-\ell$ 'new' vertices joined to each other and to all vertices in A. Let M_{BA} be the number of ways of orienting these new edges so that $A \cup N$ forms a copy of \overrightarrow{H} , and the new vertices correspond to $[k] \setminus B$. For $\ell = k-1$, this generalizes the definition above.

Let $\mathfrak{X}=\mathfrak{X}_{\overrightarrow{H}}$ be the multi-type Galton–Watson branching process in which each particle has a type from $\binom{[k]}{\ell}$, started with one particle of each type, in which children of a particle of type A are generated as follows: first generate independent Poisson random variables Z_B , $B\in \binom{[k]}{\ell}$, with $\mathbb{E}(Z_B)=M_{BA}\binom{n}{k-\ell}p^{\binom{k}{2}-\binom{\ell}{2}}$. Then generate $\sum_{B\neq A'}Z_B$ children of each type A'. Let

 $\sigma_{\overrightarrow{H}} = \sigma_{\overrightarrow{H}}(p)$ denote the survival probability of \mathfrak{X} . The proof of Theorem 11 extends very easily to prove the following result.

Theorem 12. Fix $1 \le \ell < k$ and an orientation \overrightarrow{H} of K_k , and let p = p(n) be chosen so that $n^{k-\ell}p^{\binom{k}{2}-\binom{\ell}{2}} = \Theta(1)$. Then, for any $\varepsilon > 0$, who we have

$$(\sigma_{\overrightarrow{H}}(\mu) - \varepsilon)\nu \le C_1(\overrightarrow{G}_p^{k,\ell}) \le (\sigma_{\overrightarrow{H}}(\mu) + \varepsilon)\nu,$$

where $\nu = \binom{n}{k} (k!/\operatorname{aut}(\overrightarrow{H})) p^{\binom{k}{2}}$ is the expected number of copies of \overrightarrow{H} in $\overrightarrow{G}(n,p)$.

Theorem 12 is rather unwieldy, but it is not too hard to extract the critical point. Indeed, in \mathfrak{X} the expected number of type-B children of a particle of type A is $X_{BA}\binom{n}{k-\ell}p^{\binom{k}{2}-\binom{\ell}{2}}$, where $X_{BA}=(J-I)M$, with I the identity matrix and J the matrix with all entries 1. From elementary branching process results, the critical value of p is thus given by the solution to

$$\lambda \binom{n}{k-\ell} p^{\binom{k}{2} - \binom{\ell}{2}} = 1,$$

where λ is the maximum eigenvalue of $X = (X_{BA})$.

Note that this is consistent with Theorem 11: taking \overrightarrow{H} to be $\overrightarrow{K_k}$, that is, K_k with a transitive order, it is easy to check that $M_{BA} = (k - \ell)!$ for every $A, B \in {[k] \choose \ell}$. Indeed, we must choose one of the $(k - \ell)!$ possible orders on the new vertices. Then the relative order of the new and old vertices is determined by the fact that the new vertices should play the role of $[k] \setminus B$ in the resulting $\overrightarrow{K_k}$. It follows that X is the ${k \choose \ell}$ -by- ${k \choose \ell}$ matrix with all entries equal to ${k \choose \ell} - 1$, so

$$\lambda = \binom{k}{\ell} \left(\binom{k}{\ell} - 1 \right) (k - \ell)! = \left(\binom{k}{\ell} - 1 \right) \frac{k!}{\ell}.$$

To give a non-trivial application of Theorem 12, let \overrightarrow{H} be the orientation of K_4 shown in Figure 1. Then M is given by (7), so we have

$$X = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 3 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 2 & 2 \\ 3 & 3 & 2 & 2 \\ 6 & 6 & 1 & 1 \\ 6 & 6 & 1 & 1 \end{pmatrix}$$

It follows that λ , which may be found as twice the maximum eigenvalue of a 2-by-2 matrix, is equal to $2(2+\sqrt{13})$, so the critical p is $\left((4+2\sqrt{13})n\right)^{-1/3}$.

2.2 Cliques joined by edges

In this subsection we return to unoriented graphs, and consider another natural notion of adjacency for copies of K_k in a graph G: given a parameter $1 \le \ell \le k^2$,

two K_k s are considered adjacent if they are vertex disjoint and there are at least ℓ edges of G from one to the other. (One could omit the disjointness condition; much of the time this will make little difference. Insisting on this condition simplifies the picture slightly.) Let $\widetilde{G}^{k,\ell}(G)$ be the corresponding graph on the copies of K_k in G, and let $\widetilde{G}_p^{k,\ell} = \widetilde{G}^{k,\ell}(G(n,p))$ be the graph obtained in this way from G(n,p). For this notion of adjacency, the most natural special case to consider is $\ell=1$; the other extreme case, $\ell=k^2$, of course corresponds to considering copies of K_{2k} sharing k vertices.

It turns out that we can fairly easily determine the percolation threshold in $\widetilde{G}_p^{k,\ell}$ for those parameters (k,ℓ) for which, near the threshold, there are 'not too many' copies of K_k in G(n,p); more precisely, there are o(n) copies. This always includes the case $\ell=1$.

Let $\mu' = \mu'(n, k, \ell, p)$ be given by

$$\mu' = \binom{n}{k} \binom{k^2}{\ell} p^{\binom{k}{2} + \ell},\tag{8}$$

and, as before, let $\nu = \nu(n, k, p) = \binom{n}{k} p^{\binom{k}{2}}$ be the expected number of copies of K_k in G(n, p), so $\nu = \mathbb{E} |\widetilde{G}_p^{k,\ell}|$. Let $\mathfrak{X}_0(\lambda)$ denote a Galton-Watson branching process in which the offspring distribution is Poisson with mean λ , started with a single particle, and let $\sigma_0(\lambda)$ denote the survival probability of $\mathfrak{X}_0(\lambda)$. Note that $\sigma_0(\lambda)n$ is the asymptotic size (number of vertices) in the largest component of $G(n, \lambda/n)$.

The following result is analogous to Theorem 1, but, in part due to the extra assumption on ν , much simpler.

Theorem 13. Fix $k \geq 3$ and $1 \leq \ell \leq k-2$. Let p = p(n) be chosen so that $\mu' = \Theta(1)$ and $\nu = o(n)$. Then, for any $\varepsilon > 0$,

$$(\sigma_0(\mu') - \varepsilon)\nu \le C_1(\widetilde{G}_p^{k,\ell}) \le (\sigma_0(\mu') + \varepsilon)\nu \tag{9}$$

holds whp.

Note that there is a choice of p=p(n) satisfying the conditions of Theorem 13 if and only if $\ell < k/2$. Indeed, the main force of Theorem 13 is to establish that in this case, the threshold for percolation in $\widetilde{G}_p^{k,\ell}$ is at the solution p_0 to $\mu'(p)=1$, which satisfies $p_0=\Theta(n^{-\frac{2k}{k(k-1)+2\ell}})$, with the constant given by (8). As in Section 1, the proof of Theorem 13 will give an $O(\log n)$ bound in the subcritical case, as well as an $o(\nu)$ bound on the 2nd largest component in the supercritical case. The former applies also for $\ell \geq k/2$, but only under the assumption that $\nu=o(n)$, i.e., well below what is presumably the critical point in this case. One can also extrapolate to the highly supercritical case as in Subsection 1.3. Here one needs the condition $\nu=o(n)$ only for the starting value of p, and the conclusion is that for $1\leq \ell < k/2$ and any p with $p/p_0 \to \infty$ one has, as expected, almost all vertices of $\widetilde{G}_p^{k,\ell}$ in a single component.

After these remarks, we turn to the proof of Theorem 13.

Proof. We start with the upper bound. Let us call a copy of K_k in G(n,p) isolated if it shares no vertices with any other copies of K_k . Let N and M denote the number of isolated and non-isolated copies of K_k in G(n,p). By a standard calculation, the probability that a given copy of K_k is not isolated is $(1+o(1))k\nu/n=o(1)$, so $\mathbb{E}(M)=o(\nu)$, and whp we have $M=o(\nu)$. More precisely, we may choose some $\omega=\omega(n)\to\infty$ so that the event \mathcal{B} that $M\geq\nu/\omega$ has probability o(1). Since the number of copies of K_k in G(n,p) is concentrated about its mean, choosing ω suitably, the event \mathcal{A} that $|N-\nu|\leq\nu/\omega$ also holds whp.

Let S_1, S_2, \ldots, S_N list the vertex sets of all isolated copies of K_k in G(n, p), and T_1, \ldots, T_M those of all non-isolated copies. We condition on N, M, and the sequences (S_i) and (T_i) . We assume that $A \setminus \mathcal{B}$ holds; we may do so since $\mathbb{P}(A \setminus \mathcal{B}) = 1 - o(1)$. Let \mathcal{E} denote one of the specific events we condition on, and let E^+ denote the set of all edges lying within some S_i or T_i , and E^- the set of all $\binom{n}{2} - |E^+|$ remaining potential edges of G. Let us call a non-empty set $F \subset E^-$ forbidden if by adding zero of more edges of E^+ to F one can form a K_k ; we write \mathcal{F} for the collection of forbidden sets. The event \mathcal{E} may be represented as the intersection of an up-set condition \mathcal{U} , that every edge in E^+ is present in G(n, p), and a down-set condition \mathcal{D} , that no forbidden set is present in E^- . Note that \mathcal{D} may be regarded as a down-set in $\{0,1\}^{E^-}$.

For the moment, we condition only on \mathcal{U} . To be pedantic (while, at the same time, committing the common abuse of using the same notation for a random variable and its possible values), we fix sequences (S_i) and (T_i) consistent with $\mathcal{A} \setminus \mathcal{B}$, and condition on the event $\mathcal{U} = \mathcal{U}((S_i), (T_i))$. Since we are conditioning only on the presence of a fixed set of edges, every edge of E^- is present independently with probability p. Let H be the auxiliary graph with vertex set [N] in which i and j are joined if S_i and S_j are joined by at least ℓ edges. The probability p' of this event satisfies

$$p' = \binom{k^2}{\ell} p^{\ell} + O(p^{\ell+1}) \sim \mu' / \nu \sim \mu' / N.$$

Since, given \mathcal{U} , H has exactly the distribution of G(N, p'), it follows from the classical result of Erdős and Rényi [11] that whp the largest component of H has order within $\varepsilon N/2$ of $\sigma_0(\mu')N$. Note that this corresponds to the desired number of K_k s in the largest component C of $\widetilde{G}_p^{k,\ell}$. The problem is that we have not yet conditioned on \mathcal{D} , or allowed for the possible presence of non-isolated K_k s in C.

To prove the upper bound in (9) we must account for the non-isolated K_k s. Let us say that S_i and T_j form a bad pair if they are joined by ℓ edges in G(n,p). Given \mathcal{U} , the probability of this event is exactly p', so the expected number of bad pairs (S_i,T_j) is $p'NM=o(p'N^2)=o(N)$. Similarly, T_i and T_j form a bad pair if they are vertex disjoint, and joined by at least ℓ edges. The expected number of bad pairs (T_i,T_j) is at most $p'M^2=o(N)$. Let $H'\supset H$ be the graph on [N+M] defined in the natural way: two vertices are joined if the corresponding copies of K_k are disjoint and joined by at least ℓ edges. We

have shown that $E(H') \setminus E(H)$, which is exactly the number of bad pairs, has expectation o(N).

It is well known that for λ fixed, the giant component in $G(m, \lambda/m)$ is stable upwards, in the sense that adding $o_p(m)$ vertices and edges cannot increase its size by more than $o_p(m)$. Indeed, this follows from the qualitative form of the distribution of the small components: for details, see, for example, Theorem 3.9 of Bollobás, Janson and Riordan [5], where the corresponding result is proved for a more general model. (This result also shows 'downwards stability', which we do not need here. Downwards stability is much harder to prove: Luczak and McDiarmid [15] established this for the Erdős–Rényi model; in [5], their argument is extended to the more general model considered there.) Applying this stability result to H, we deduce that, given \mathcal{U} , we have $C_1(H') = C_1(H) + o_p(N)$. For any n', we have

$$\mathbb{P}(C_1(H') \ge n' \mid \mathcal{E}) = \mathbb{P}(C_1(H') \ge n' \mid \mathcal{U} \cap \mathcal{D}) \le \mathbb{P}(C_1(H') \ge n' \mid \mathcal{U}),$$

where the inequality is from Harris's Lemma applied in $\{0,1\}^{E^-}$. Since H' is exactly the graph $\widetilde{G}_p^{k,\ell}$, the upper bound in (9) follows.

Turning to the lower bound, we may now ignore the complications due to non-isolated K_k s, and confine our attention to H. However, we must now show that conditioning on \mathcal{D} , which tends to decrease $C_1(H)$, does not do so too much. We shall use the same type of argument as in the proof of Lemma 7: exploring H step by step, we shall show that conditioning on \mathcal{D} does not decrease the probability of finding an edge in H by showing that finding an edge in H would not decrease the probability of \mathcal{D} much. There will be some complications due, for example, to the possible presence of K_k s made up of edges in G = G(n, p) corresponding to edges in H.

As before, we shall condition on (S_i) and (T_i) , assuming that $\mathcal{A} \setminus \mathcal{B}$ holds, i.e., that $N \sim \nu$ and $M = o(\nu)$. In fact, we shall impose a further condition. Let \mathcal{B}' be the event that there is a vertex of G(n,p) in more than $(\log n)^2$ copies of K_k , noting that whether or not \mathcal{B}' holds is determined by the sequences (S_i) and (T_i) . Since $\nu = \binom{n}{k} p^{\binom{k}{2}} = o(n)$, it is easy to check that $\mathbb{P}(\mathcal{B}') = o(1)$: we omit the standard calculation which is based on the fact that K_k is strictly balanced, so having found a moderate number of K_k s containing a given vertex v does not significantly increase the chance of finding a further such K_k .

From now on we condition on the sequences (S_i) and (T_i) , assuming as we may that $\mathcal{A} \setminus (\mathcal{B} \cup \mathcal{B}')$ holds. Defining $\mathcal{U} = \mathcal{U}((S_i), (T_i))$ and $\mathcal{D} = \mathcal{D}((S_i), (T_i))$ as before, this is again equivalent to conditioning on $\mathcal{U} \cap \mathcal{D}$. As before, since we fix (S_i) and (T_i) , the event \mathcal{U} is simply the event that every edge in the fixed set $E^+ = \bigcup E(S_i) \cup \bigcup E(T_i)$ is present in G(n, p). Note for later that, since \mathcal{B}' does not hold, we have

$$d_{E^+}(v) \le k(\log n)^2 \tag{10}$$

for every $v \in V(G)$, where $d_{E^+}(v)$ is the number of edges of E^+ incident with v.

Let $f_1, f_2, ...$ be the $\binom{N}{2}$ possible edges of H, listed in an arbitrary order. We now describe an algorithm that reveals a subgraph H_0 of H. During step

 $r, 1 \leq r \leq {N \choose 2}$, we shall test whether f_r is present in H, except that if f_r , together with some previously discovered edges of H_0 , would form a cycle in H_0 , or would cause the degree of some vertex of H_0 to exceed $(\log n)^2$, then we omit step r. Step r consists of a series of sub-steps: in each we consider one of the ${k^2 \choose \ell}$ sets I of ℓ potential edges of G = G(n, p) whose presence would give rise to the edge f_r in H, and test whether all edges in I are present in G. If such a test succeeds, we add f_r to H_0 , and omit further tests for the same f_r , i.e., continue to step r+1.

Suppose that we have reached the tth sub-step of the algorithm described above, and let $I=I_t$ be the set of ℓ potential edges of G whose presence we are about to test for. We claim that, given the history, the conditional probability that all edges in I are is present is $(1+o(1))p^{\ell}$. More precisely, let E_t^+ be the union of all sets I_s , s < t, which we found to be present, and let $\mathcal{U}_t = \{E_t^+ \subset E(G)\}$. Also, let \mathcal{F}_t be the set of sets I_s , s < t, found to be absent, and let \mathcal{D}_t be the event that no $F \in \mathcal{F}_t$ is present in E(G). Recalling that we start by conditioning on $\mathcal{U} \cap \mathcal{D}$, the algorithm reaches its particular present state if and only if $\mathcal{U} \cap \mathcal{D} \cap \mathcal{U}_t \cap \mathcal{D}_t$ holds, so our precise claim is that for any $\eta > 0$, if n is large enough, then for any possible I_t , \mathcal{U}_t and \mathcal{D}_t we have

$$\mathbb{P}(I_t \subset E(G) \mid \mathcal{U} \cap \mathcal{D} \cap \mathcal{U}_t \cap \mathcal{D}_t) \ge (1 - \eta)p^{\ell}. \tag{11}$$

Before proving (11), let us see that the Theorem 13 follows. Let H_1 be the union of H_0 and all edges f_r which we omitted to test. Assuming (11), we always have

$$\mathbb{P}(f_r \in E(H_1) \mid E(H_1) \cap \{f_1, \dots, f_{r-1}\})$$

$$\geq (1 - \eta - o(1)) \binom{k^2}{\ell} p^{\ell} \sim (1 - \eta) p' \sim (1 - \eta) \mu' / N. \quad (12)$$

Indeed, if f_r is omitted, the conditional probability above is 1 by definition; otherwise, we apply (11) to the $\binom{k^2}{\ell}$ sub-steps associated to f_r . Now (12) tells us that for n large enough, H_1 stochastically dominates $G(N, (1-2\eta)\mu'/N)$, say. Taking η small enough, it follows that whp

$$C_1(H_1)/N \ge \sigma_0((1-2\eta)\mu') - \varepsilon/4 \ge \sigma_0(\mu') - \varepsilon/2. \tag{13}$$

If $\Delta(H) \leq (\log n)^2 - 1$, then we only omit step r if adding f_r would create a cycle, so in this case H_0 is the union of one spanning tree for each component of H, and all edges of H_1 join vertices of H_0 that are already joined by paths in H_0 . Hence $C_1(H_0) = C_1(H) = C_1(H_1)$. As noted earlier, given only \mathcal{U} , the graph H has exactly the distribution of G(N, p'). Since \mathcal{U} is a principal up-set, and \mathcal{D} is a down-set, it follows that the distribution of H given $\mathcal{U} \cap \mathcal{D}$, which is what we are considering here, is stochastically dominated by that of G(N, p'). Since $Np' \sim \mu' = \Theta(1)$, it follows that whp $\Delta(H) \leq (\log n)^2 - 1$, so whp $C_1(H) = C_1(H_1)$. Since $N \sim \nu$, this together with (13) gives the lower bound in (9), completing the proof of Theorem 13.

It remains only to prove (11). Let us start by observing that $E_t^+ \cup I_t$ cannot contain any forbidden set $F \in \mathcal{F}$, i.e., that the set $E^+ \cup E_t^+ \cup I_t$ contains no K_k other than $S_1, \ldots, S_N, T_1, \ldots, T_M$. This is true of $E^+ \cup E_t^+$ since we condition on \mathcal{D} , and we are assuming that the present state of the algorithm is a possible one. Suppose then that adding I_t to $E^+ \cup E_t^+$ creates a new copy K of K_k , and let the edge f_r we are testing be ij. Now E_t^+ contains edges between $S_{i'}$ and $S_{j'}$ if and only if we have already found the edge i'j' in H_0 . If K meets three or more of the $S_{i'}$, then $H_0 \cup f_r$ would contain a triangle, which is impossible by definition of the algorithm. This leaves only the case that K meets exactly two sets $S_{i'}$, which must be S_i and S_j . But then the only edges of $E_t^+ \cup I_t$ between S_i and S_j are those of I_t . Now K contains at least k-1 edges between these sets, while $|I_t| = \ell < k-1$, so there is no such K_k .

There are two types of conditioning in (11), that on $\mathcal{U} \cap \mathcal{U}_t$ and that on $\mathcal{D} \cap \mathcal{D}_t$. The first type is trivial, since $\mathcal{U} \cap \mathcal{U}_t$ is simply the event that every edge in $E^+ \cup E_t^+$ is present. Let $X = E(K_n) \setminus (E^+ \cup E_t^+)$. Then we may as well work in \mathbb{P}^X , the product probability measure on $\{0,1\}^X$ in which each edge is present with probability p. Let

$$\widetilde{\mathcal{F}}_t = \{ F \cap X : F \in \mathcal{F} \cup \mathcal{F}_t \}, \tag{14}$$

and let $\widetilde{\mathcal{D}} \subset \{0,1\}^X$ be the event that none of the 'forbidden' sets in $\widetilde{\mathcal{F}}_t$ is present, so $\mathbb{P}^X(\widetilde{\mathcal{D}}) = \mathbb{P}(\mathcal{D} \cap \mathcal{D}_t \mid \mathcal{U} \cap \mathcal{U}_t)$. Also, let $\mathcal{I}_t \subset \{0,1\}^X$ be the event that all edges in I_t are present, noting that $I_t \subset X$. Then (11) is equivalent to

$$\mathbb{P}^{X}(\mathcal{I}_{t} \mid \widetilde{\mathcal{D}}_{t}) \ge (1 - \eta)\mathbb{P}^{X}(\mathcal{I}_{t}). \tag{15}$$

The key idea is to split $\widetilde{\mathcal{F}}_t$ into two sets, $\widetilde{\mathcal{F}}'$ and $\widetilde{\mathcal{F}}''$, the first consisting of those F that intersect I_t , and the second those that do not. It turns out that we can ignore the ones that do not. More precisely, let $\widetilde{\mathcal{D}}'$ be the event that no $F \in \widetilde{\mathcal{F}}'$ is present, and $\widetilde{\mathcal{D}}''$ the event that no $F \in \widetilde{\mathcal{F}}''$ is present, so $\widetilde{\mathcal{D}}_t = \widetilde{\mathcal{D}}' \cap \widetilde{\mathcal{D}}''$.

We may rewrite (15) in any of the following forms, which are step-by-step trivially equivalent: (we drop the superscript X at this point, since the events we are now considering are in any case independent of edges outside X)

$$\mathbb{P}(\mathcal{I}_{t} \mid \widetilde{\mathcal{D}}' \cap \widetilde{\mathcal{D}}'') \geq (1 - \eta)\mathbb{P}(\mathcal{I}_{t}) \\
\mathbb{P}(\mathcal{I}_{t} \cap \widetilde{\mathcal{D}}' \cap \widetilde{\mathcal{D}}'') \geq (1 - \eta)\mathbb{P}(\mathcal{I}_{t})\mathbb{P}(\widetilde{\mathcal{D}}' \cap \widetilde{\mathcal{D}}'') \\
\mathbb{P}(\widetilde{\mathcal{D}}' \cap \widetilde{\mathcal{D}}'' \mid \mathcal{I}_{t}) \geq (1 - \eta)\mathbb{P}(\widetilde{\mathcal{D}}' \cap \widetilde{\mathcal{D}}'') \\
\mathbb{P}(\widetilde{\mathcal{D}}' \mid \mathcal{I}_{t} \cap \widetilde{\mathcal{D}}'')\mathbb{P}(\widetilde{\mathcal{D}}'' \mid \mathcal{I}_{t}) \geq (1 - \eta)\mathbb{P}(\widetilde{\mathcal{D}}' \mid \widetilde{\mathcal{D}}'')\mathbb{P}(\widetilde{\mathcal{D}}'') \\
\mathbb{P}(\widetilde{\mathcal{D}}' \mid \mathcal{I}_{t} \cap \widetilde{\mathcal{D}}'') \geq (1 - \eta)\mathbb{P}(\widetilde{\mathcal{D}}' \mid \widetilde{\mathcal{D}}''). \tag{16}$$

The only step which is not trivial from the definition of conditional probability is the last one: for this we note that by definition $\widetilde{\mathcal{D}}''$ depends only on edges of $X \setminus I_t$.

We shall prove (16) by simply ignoring the conditional probability on the right, showing that

$$\mathbb{P}(\widetilde{\mathcal{D}}' \mid \mathcal{I}_t \cap \widetilde{\mathcal{D}}'') \ge (1 - \eta).$$

This clearly implies (16), and hence, from the argument above, implies (11). Let $\widetilde{\mathcal{U}}'$ be the complement of $\widetilde{\mathcal{D}}'$, so our aim is to show that $\mathbb{P}(\widetilde{\mathcal{U}}' \mid \mathcal{I}_t \cap \widetilde{\mathcal{D}}'') \leq \eta$. Now \mathcal{I}_t is again an event of a very simple form, that a certain particular set I_t of edges is present. Since $\widetilde{\mathcal{U}}'$ is an up-set while $\widetilde{\mathcal{D}}''$ is a down-set, applying Harris's Lemma in $\{0,1\}^{X\setminus I_t}$, it follows that

$$\mathbb{P}(\widetilde{\mathcal{U}}' \mid \mathcal{I}_t \cap \widetilde{\mathcal{D}}'') \leq \mathbb{P}(\widetilde{\mathcal{U}}' \mid \mathcal{I}_t),$$

so it suffices to prove that $\mathbb{P}(\widetilde{\mathcal{U}}' \mid \mathcal{I}_t) \leq \eta$.

At this point we have eliminated all non-trivial conditioning; all that is left is counting. Indeed,

$$\mathbb{P}(\widetilde{\mathcal{U}}' \mid \mathcal{I}_t) \le \sum_{F' \in \widetilde{\mathcal{F}}'} p^{|F' \setminus I_t|}. \tag{17}$$

Recalling (14), there are two contributions to the sum above. The first is from sets $F' \in \widetilde{\mathcal{F}}'$ corresponding to sets $I_s \in \mathcal{F}_t$, i.e., to failed tests for previous I_s . By definition of $\widetilde{\mathcal{F}}'$, we have such an $F' \in \widetilde{\mathcal{F}}'$ if and only if $I_s \cap I_t \neq \emptyset$, in which case I_s and I_t correspond to the same potential edge f_r of H. But then there are at most $\binom{k^2}{\ell} - 1$ possibilities for I_s , and for each we have $|F' \setminus I_t| = |I_s \setminus I_t| \geq 1$, so the contribution to (17) is O(p) = o(1).

The remaining terms come from $F' = F \cap X$ with $F \in \mathcal{F}$ and with $F' \cap I_t \neq \emptyset$, i.e., with $F \cap I_t \neq \emptyset$. Recalling that F is a set of edges that, together with E^+ , would create a K_k , it thus suffices to show that

$$\sum_{K} p^{E(K)\setminus (E^+ \cup E_t^+ \cup I_t)} = o(1), \tag{18}$$

where the sum runs over all copies of K_k on V(G) containing at least one edge from I_t . Now H_0 has maximum degree at most $(\log n)^2$ by the definition of our algorithm. Hence $d_{E_t^+}(v) \leq \ell(\log n)^2$ for every $v \in V(G)$. Using (10) it follows that the graph G' on V(G) formed by the edges in $E^+ \cup E_t^+ \cup I_t$ has maximum degree at most $(k + \ell)(\log n)^2 + \ell \leq (\log n)^3$, say. This is all we shall need to prove (18).

Let Z_i be the contribution to the sum in (18) from copies K of K_k such that $K \cap G'$ has exactly i components (including trivial components of size 1). Since K must contain an edge of $I_t \subset E(G')$, we have $1 \leq i \leq k-1$. Let $\Delta = \Delta(G') \leq (\log n)^3$, and set

$$z_i = \ell n^{i-1} \Delta^{k-1-i} p^{\binom{k}{2} - \binom{k+1-i}{2}}.$$

It is easy to check that for $1 \leq i \leq k-1$ we have $Z_i \leq z_i$: there are ℓ choices for (one of the) edges of I_t to include, then picking vertices one by one, either n choices if we start a new component of $K \cap G'$, or at most Δ if we do not. Finally, if $K \cap G'$ has i components, then considering the case where these components are all complete, by convexity $E(K \cap G')$ is maximized if the components have sizes $k+1-i,1,1,\ldots,1$, so $|E(K) \setminus E(G')| \geq {k \choose 2} - {k+1-i \choose 2}$. For i=1 we may

improve our estimate slightly: since G' does not contain K_k , we gain at least a factor of p, so $Z_1 \leq pz_1$.

Now $Z_1 \leq pz_1 = p\ell \Delta^{k-2} \leq p(\log n)^{O(1)} = o(1)$, so this contribution to (18) is negligible. To handle the remaining cases, note that

$$z_{i+1}/z_i = n\Delta^{-1}p^{k-i},$$

so z_{i+1}/z_i increases as i increases. Hence the maximum of z_i for $2 \le i \le k-1$ is attained either at i=2 or at i=k-1. Now $z_2=\ell n\Delta^{k-3}p^{k-1}$, and it is easy to check that this is o(1). Indeed, since $\mu'=\Theta(1)$ we have $p=\Theta(n^{-\frac{2k}{k(k-1)+2\ell}})$, and since $2k(k-1)>k(k-1)+2\ell$ we have that np^{k-1} is a constant negative power of n.

At the other end of the range, $z_{k-1} = \ell n^{k-2} p^{\binom{k}{2}-1} = \Theta(\nu/(n^2 p)) = o(1/(np))$, and we have $np \to \infty$. Thus both z_2 and z_{k-1} are o(1), which gives $z_i = o(1)$ for $2 \le i \le k-1$. Thus

$$\sum_{K} p^{E(K)\setminus (E^{+}\cup E_{t}^{+}\cup I_{t})} = \sum_{i=1}^{k-1} Z_{i} \le pz_{1} + \sum_{i=2}^{k-1} z_{i} = o(1),$$

proving (18). This was all that remained to prove the theorem.

It is natural to wonder whether Theorem 13 can be extended. For $\ell=1$, the picture is complete: defining p_0 by $\mu'(p_0)=1$, since $\mu'=\Theta(\nu p^\ell)=\Theta(\nu p)$ we have $\nu=\Theta(1/p_0)=o(n)$ whenever $p=\Theta(p_0)$. As noted earlier, percolation in $\widetilde{G}_n^{k,\ell}$ for $p/p_0\to\infty$ follows by monotonicity arguments.

For general k and ℓ , the conditions of Theorem 13 can presumably be relaxed at least somewhat. Unfortunately, the proof we have given relies on $\nu = o(n)$, and hence on $\ell < k/2$.

2.3 Copies of general graphs

We conclude this paper by briefly considering the graph $G_H^{\ell}(p)$ obtained from G(n,p) by taking one vertex for each copy of some fixed graph H with |H|=k, and joining two vertices if these copies share at least ℓ vertices, where $1 \leq \ell \leq k-1$. Ones first guess might be that the results in Section 1 extend at least to regular graphs H without much difficulty, but this turns out to be very far from the truth. In fact, it seems that almost all cases are difficult to analyze.

We start with the most interesting end of the range, where $\ell = k - 1$, as in the original question of Derényi, Palla and Vicsek [10]. To keep things simple, let H be the cycle C_k . For k = 3, C_k is complete, so this case is covered in Section 1. The case of C_4 is already interesting: when moving from one copy of C_4 to another, we may change opposite vertices essentially independently of each other. The appropriate exploration is thus as follows: suppose we have reached a C_4 with vertex set $P_0 \cup P_1$, where each of P_0 and P_1 is a pair of opposite vertices. Furthermore, suppose we reached this C_4 from another C_4 containing P_0 . Then we continue by replacing P_0 by some other pair P_2 of common neighbours of P_1 .

Suppose that $p = \Theta(n^{-1/2})$; in particular, set $p = \lambda n^{-1/2}$. The number Z of common neighbours of P_1 outside P_0 has essentially a Poisson distribution with mean λ^2 . The number of choices for $P_2 \neq P_0$ is $\binom{Z+2}{2} - 1$, which has expectation

$$\mathbb{E}\left(\frac{(Z+2)(Z+1)}{2}-1\right) = \mathbb{E}\left(\frac{Z(Z-1)}{2}+2Z\right) = \lambda^4/2 + 2\lambda^2,$$

so we believe the critical point will be when this expectation is 1, i.e., at

$$p = p_0 = n^{1/2} \sqrt{\sqrt{6} - 2}. (19)$$

Of course, it is not clear that the branching process approximation we have implicitly described is a good approximation to the component exploration process. However, it is not hard to convince oneself that this is the case, at least at first. The key point is that when we have not yet reached many vertices, the chance of finding a new vertex adjacent to three or more reached vertices is very small. Hence the sets of common neighbours of two pairs P and P' are essentially independent, even if P and P' share a vertex. We have not checked the details, but we expect that it is not hard to show rigorously that p_0 is indeed the threshold in this case, although unforeseen complications are of course conceivable.

Taking things further, one might expect the argument above to work for C_6 , say, but in fact it breaks down after one step. Suppose we start from aubvcw and first replace a, b and c by other suitable vertices. Then we have sets A, B and C of candidates for a, b, and c. The problem is at the next step: the possibilities for u', v', w' associated to different triples $(a', b', c') \in A \times B \times C$ are far from independent: for triples (a', b', c') and (a', b', c''), the choices for u' are exactly the same. In fact, not only can we not prove a result for any C_k , $k \geq 5$, but we do not even have a conjecture as to the correct critical probability, although this is clearly of order $\Theta(n^{-1/2})$.

Although C_4 is the simplest non-complete example, cycles turn out not to be the easiest generalization: it is almost certainly not hard to adapt the outline argument above to complete bipartite graphs $K_{r,s}$. If s=r, then setting $p=\lambda n^{-1/r}$, and letting Z_r denote a Poisson random variable with mean λ^r , the critical point should be given by the solution to

$$\mathbb{E}\left(\binom{Z_r+r}{r}-1\right)=1,$$

generalizing (19). If $r \neq s$, the situation is a little different, as alternate steps in the exploration have different behaviour. Suppose that r < s, and set $p = \lambda n^{-(s+1)/(rs+s)}$. Then $np^r \to \infty$ and $np^s \to 0$, so on average a set of r vertices has many common neighbours, and so lies in many copies of $K_{r,s}$, while a typical set of s vertices has no common neighbours. Starting from a given $K_{r,s}$, with vertex classes R and S of sizes r = |R| and s = |S|, let T denote the set of common neighbours of R. Then $\mathbb{E}(|T \setminus S|) = (n-r)p^r \to \infty$, and |T| will be concentrated near np^r . Replacing S by any of the other $\binom{|T|}{s} - 1 \sim n^s p^{rs}/s!$

subsets S' of T of size s, since sets of size s have few common neighbours, the most likely way the exploration will continue is that some S' will have one common neighbour x outside R. Then for each vertex y of R we reach a new $K_{r,s}$ with R replaced by $R \setminus \{y\} \cup \{x\}$. For each S' we expect to find around $np^s \to 0$ such vertices x, so overall the average number of new choices for R' is $(1+o(1))n^sp^{rs}s!^{-1}np^sr$, and we expect the critical point to be given by $\lambda n^{-(s+1)/(rs+s)}$ where λ satisfies $\lambda^{rs+s} = s!/r$; we have not checked the details.

Finally, since the case $\ell=k-1$ seems too hard in general, one could consider the other extreme $\ell=1$. This is much easier, though also less interesting. If H is strictly balanced, it is very easy to see that the critical point occurs when $(k-1)\mu=1$, where μ is the expected number of copies of H containing a given vertex v. For non-balanced H things are a little more complicated: having found a 'cloud' of copies of H containing a single copy of the (for simplicity unique) densest subgraph H' of H, one next looks for a second cloud meeting the current cloud, and the critical point should be when the expected number of clouds meeting a given cloud is 1. This type of argument can probably be extended to $\ell=2$, at least if we impose the natural condition in this case that our copies of H should share an edge, rather than just two vertices. Beyond this, the whole question seems very difficult.

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