

# Tight upper tail bounds for cliques

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## Abstract

With  $\xi_k = \xi_k^{n,p}$  the number of copies of  $K_k$  in the usual (Erdős-Rényi) random graph  $G(n, p)$ ,  $p \geq n^{-2/(k-1)}$  and  $\eta > 0$ , we show when  $k > 1$

$$\Pr(\xi_k > (1 + \eta)\mathbf{E}\xi_k) < \exp \left[ -\Omega_{\eta,k} \min\{n^2 p^{k-1} \log(1/p), n^k p^{\binom{k}{2}}\} \right].$$

This is tight up to the value of the constant in the exponent.

## 1 Introduction

Let  $G(n, p)$  be the Erdős-Rényi random graph on  $n$  vertices, in which every edge occurs independently with probability  $p$ , and let  $H$  be a fixed graph with  $v_H = |V(H)|$  and  $e_H = |E(H)|$ . A *copy* of  $H$  in  $G(n, p)$  is any subgraph of  $G(n, p)$  isomorphic to  $H$ . It has been a long studied question (e.g. [5, 6, 11, 12, 13, 15, 17]) to estimate, for  $\eta > 0$  and  $\xi_H = \xi_H^{n,p}$  the number of copies of  $H$  in  $G(n, p)$ ,

$$\Pr(\xi_H > (1 + \eta)\mathbf{E}\xi_H). \quad (1)$$

To avoid irrelevancies, let us declare at the outset that we always assume  $p \geq n^{-1/m_H}$ , where, as usual (e.g. [10, p.6]),

$$m_H = \max\{e_K/v_K : K \subseteq H\} \quad (2)$$

(so  $n^{-1/m_H}$  is a threshold for “ $G \supseteq H$ ”; see [10, Theorem 3.4]); in particular, when  $H = K_k$  we assume  $p \geq n^{-2/(k-1)}$ . For smaller  $p$  the problem is not

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very interesting (e.g. for bounded  $\eta$  the probability in (1) is easily seen to be  $\Theta(\min\{n^{v_K} p^{e_K} : K \subseteq H, e_K > 0\})$ ; see [10, Theorem 3.9] for a start), and we will not pursue it here.

Janson and Ruciński [12] offer a nice overview of the methods used prior to 2002 to obtain upper bounds on the probability in (1), by far the more challenging part of the problem. To get an idea of the difficulty, note that even for the case that  $H$  is a triangle, only quite poor upper bounds were known until a breakthrough result of Kim and Vu [15], who used, *inter alia*, the “polynomial concentration” machinery of [14] to show, for  $p > n^{-1} \log n$ ,

$$\exp_p[O_\eta(n^2 p^2)] < \Pr(\xi_H > (1 + \eta)\mathbf{E}\xi_H) < \exp[-\Omega_\eta(n^2 p^2)]. \quad (3)$$

(The easy lower bound, seemingly first observed in [17], is, for example, the probability of containing a complete graph on something like  $(1 + \eta)^{1/3} np$  vertices. Of course the subscript  $\eta$  in the lower bound is unnecessary if, for example,  $\eta \leq 1$ , which is what we usually have in mind.) Polynomial concentration was also used by Vu [16] to show that if  $H$  is strictly balanced and  $\mathbf{E}\xi_H \leq \log n$ , then

$$\Pr(\xi_H > (1 + \eta)\mathbf{E}\xi_H) < \exp[-\Omega_\eta(\mathbf{E}\xi_H)]. \quad (4)$$

The result of [15] was vastly extended in a beautiful paper of Janson, Oleszkiewicz and Ruciński [11], where it was shown that for any  $H$  and  $\eta$ ,

$$\exp_p[O_{H,\eta}(M_H(n, p))] < \Pr(\xi_H > (1 + \eta)\mathbf{E}\xi_H) < \exp[-\Omega_{H,\eta}(M_H(n, p))], \quad (5)$$

thus determining the probability (1) up to a factor  $O(\log(1/p))$  in the exponent for constant  $\eta$ . A definition of  $M$  is given in Section 10; for now we just mention that (for  $p \geq n^{-2/(k-1)}$ )  $M_{K_k}(n, p) = n^2 p^{k-1}$ . (It should also be noted that in the limited range where it applies, the upper bound in (4) is better than the one in (5).)

While it seems natural to expect that the lower bound in (5) is “usually” the truth (see Section 10 for a precise guess), the only progress in this direction until quite recently was [13], which established the upper bound  $\exp[-\Omega(M_H(n, p) \log^{1/2}(1/p))]$  for  $H = K_4$  or  $C_4$  (the 4-cycle) and *some* values of  $p$ .

The  $\log(1/p)$  gap was finally closed for the case  $H = K_3$  by Chatterjee [5] and, independently, the present authors [6]. More precisely, [5] showed that for a suitable  $C$  depending on  $\eta$  and  $p > Cn^{-1} \log n$ ,

$$\Pr(\xi_{K_3} > (1 + \eta)\mathbf{E}\xi_{K_3}) < p^{\Omega_\eta(n^2 p^2)},$$

while [6] showed, somewhat more generally, that for  $p > n^{-1}$ ,

$$\exp[-O_\eta(f(3, n, p))] < \Pr(\xi_{K_3} > (1 + \eta)\mathbf{E}\xi_{K_3}) < \exp[-\Omega_\eta(f(3, n, p))],$$

where  $f(k, n, p) := \min\{n^2 p^{k-1} \log(1/p), n^k p^{\binom{k}{2}}\}$ . (In what follows we will often abbreviate  $f(k, n, p) = f(k, n)$ .)

In this paper we considerably extend the method of [6] to settle the problem for general cliques and a bit more.

**Theorem 1.1.** *Assume  $H$  on  $k$  vertices has minimum degree at least  $k - 2$  (that is, the complement of  $H$  is a matching). Then for all  $\eta > 0$  and  $p \geq n^{-2/(k-1)}$ ,*

$$\Pr(\xi_H \geq (1 + \eta)\mathbf{E}(\xi_H)) \leq \exp[-\Omega_{\eta, H}(f(k, n, p))].$$

**Theorem 1.2.** *For  $H = K_k$  and for all  $p \geq n^{-2/(k-1)}$ ,*

$$\Pr(\xi_H \geq 2\mathbf{E}(\xi_H)) \geq \exp[-O_H(f(k, n, p))].$$

*Remarks.* 1. We are most interested in the “nonpathological” range where  $f(k, n, p) = n^2 p^{k-1} \log(1/p)$ , so when  $p \geq n^{-2/(k-1)} (\log n)^{2/[(k-1)(k-2)]}$  (or a bit less). It may be helpful to think mainly of this range as we proceed.

2. Though mainly concerned with the case  $H = K_k$  in Theorem 1.1, we prove the more general statement for inductive reasons. For noncliques the bound of Theorem 1.1 is not usually tight; more precisely: it is tight (up to the constant in the exponent) if  $p = \Omega(1)$  or if  $\Delta := \Delta_H = k - 1$  and  $p = \Omega(n^{-1/\Delta})$ , in which cases our upper bound agrees with the lower bound in (5); it is not tight if  $\Delta = k - 2$  and  $p = o(1)$  (see the proof of Lemma 2.4) or if  $H \neq K_k$  and  $p < n^{-c/\Delta}$  for some fixed  $c > 1$  (see the proof of Lemma 2.5; in fact  $p = o(n^{-1/\Delta})$  is probably enough here—which would complete this little story—but we don’t quite show this).

In the next section we show that Theorem 1.1 follows from an analogous assertion for  $k$ -partite graphs; most of the paper (Sections 3-8) is then concerned with this modified problem. Section 9 gives the proof of Theorem 1.2 and Section 10 contains a few concluding remarks.

## 2 Reduction

For the rest of the paper we set  $t = \log(1/p)$  and take  $H$  to be a graph with vertices  $v_1, v_2, \dots, v_k$ . We define  $G = G(n, p, H)$  to be the random graph with vertex set  $V = V_1 \cup \dots \cup V_k$ , where the  $V_i$ 's are disjoint  $n$ -sets and  $\Pr(xy \in E(G)) = p$  whenever  $x \in V_i$  and  $y \in V_j$  for some  $v_i v_j \in E(H)$ , these choices made independently. We define a *copy* of  $H$  in  $G$  to be a set of vertices  $\{x_1, \dots, x_k\}$  with  $x_i \in V_i$  and  $x_i x_j \in E(G)$  whenever  $v_i v_j \in E(H)$ ; use  $X_H^{n,p}$  for the number of such copies; and set  $\Psi(H, n, p) = \mathbf{E}(X_H^{n,p}) = n^k p^{e_H}$ . When there is no danger of confusion we will often use  $X_H^n$ —or, for typographical reasons  $X(H, n)$ —for  $X_H^{n,p}$  and  $\Psi(H, n)$  for  $\Psi(H, n, p)$ .

The next two propositions show an equivalence between  $G(n, p)$  and  $G$  with regard to upper tails for subgraph counts. In each we set  $\alpha = |\text{Aut}(H)| n^k / (kn)_k \sim k^{-k} |\text{Aut}(H)|$  (where as usual  $(a)_b = a(a-1) \cdots (a-b+1)$ ).

**Proposition 2.1.** *For  $\eta > 0$  and  $\varepsilon = \eta/(2 + \eta)$ ,*

$$\Pr(X_H^{n,p} \geq (1 + \varepsilon)\Psi(H, n, p)) > \frac{\alpha\varepsilon}{1 - \alpha + \alpha\varepsilon} \Pr(\xi_H^{kn,p} \geq (1 + \eta)\mathbf{E}(\xi_H^{kn,p}))$$

We omit the proof of Proposition 2.1 since it is a straightforward generalization of the case  $H = K_3$  proved in [6].

**Proposition 2.2.** *For any  $\varepsilon > 0$  there is a  $C = C_{\varepsilon, H}$  such that for  $p > Cn^{-1/m_H}$ ,*

$$\Pr(X_H^{n,p} \geq (1 + \varepsilon)\Psi(H, n, p)) < 2 \Pr(\xi_H^{kn,p} \geq (1 + \alpha\varepsilon/2)\mathbf{E}(\xi_H^{kn,p})).$$

(See (2) for  $m_H$ .)

*Proof.* We may choose  $G^* = G(kn, p)$  by first choosing  $G = G(n, p, H)$  and then letting

$$E(G^*) = E(G) \cup S$$

where  $\Pr(xy \in S) = p$  whenever  $x \neq y$ ,  $x \in V_i$  and  $y \in V_j$  for some  $v_i v_j \notin E(H)$ , these choices made independently. Write  $\xi$  and  $X$  for the numbers of copies of  $H$  in  $G^*$  and  $G$  respectively (thus  $\xi = \xi_H^{kn,p}$  and  $X = X_H^{n,p}$ ), and set  $\xi^* = \xi - X$ . Since  $\mathbf{E}X = \alpha\mathbf{E}\xi$ , we have, using Harris' Inequality,

$$\Pr(\xi > (1 + \frac{\alpha\varepsilon}{2})\mathbf{E}\xi) \geq \Pr(X > (1 + \varepsilon)\mathbf{E}X) \Pr(\xi^* > \mathbf{E}\xi^* - \frac{\alpha\varepsilon}{2}\mathbf{E}\xi); \quad (6)$$

so we need to say that the second probability on the right is at least  $1/2$ . This is standard, but we summarize the argument for completeness.

A result of Janson from [9] (see [10, (2.14)]) gives

$$\Pr(\xi^* \leq \mathbf{E}\xi^* - t) < \exp[-\frac{t^2}{2\bar{\Delta}}], \quad (7)$$

with

$$\bar{\Delta} = \sum_{\sigma \sim \tau}^* \mathbf{E}I_\sigma I_\tau \leq \sum_{\sigma \sim \tau} \mathbf{E}I_\sigma I_\tau, \quad (8)$$

where (recycling notation a little)  $H_1, \dots$  are the copies of  $H$  in  $K_{kn}$ ;  $I_\sigma = \mathbf{1}_{\{H_\sigma \subseteq G^*\}}$ ; “ $\sigma \sim \tau$ ” means  $H_\sigma$  and  $H_\tau$  share an edge (so  $\sigma \sim \sigma$ ); and  $\sum^*$  means we sum only over  $\sigma, \tau$  for which  $H_\sigma, H_\tau$  cannot appear in  $G$ .

But (very wastefully),

$$\begin{aligned} \bar{\Delta} &< n^{v_H} \sum \{n^{v_H - v_K} p^{2e_H - e_K} : K \subseteq H, e_K > 0\} \\ &< n^{2v_H} p^{2e_H} \sum \{n^{-v_K} (Cn^{-1/m_H})^{-e_K} : K \subseteq H, e_K > 0\} \\ &= O(C^{-1} \mathbf{E}^2 \xi), \end{aligned}$$

where  $C$  is the constant from (9), which may be taken large compared to the implied constant in “ $O(\cdot)$ .” Thus, using (7) with the above bound on  $\bar{\Delta}$  and  $t = (\alpha\varepsilon/2)\mathbf{E}\xi$ , we find that the second probability on the right side of (6) is at least  $1 - \exp[-\Omega((\alpha\varepsilon)^2 C)] > 1/2$ . ■

According to Proposition 2.1, Theorem 1.1 will follow from the corresponding  $k$ -partite statement, *viz.*

**Theorem 2.3.** *If  $H$  has minimum degree at least  $k - 2$ , then*

(a) *for all  $\varepsilon > 0$ ,*

$$\Pr(X_H^{n,p} \geq (1 + \varepsilon)\Psi(H, n, p)) < \exp[-\Omega_{H,\varepsilon}(f(k, n, p))];$$

(b) *for any  $\tau \geq 1$ ,*

$$\Pr(X_H^{n,p} \geq 2\tau\Psi(H, n, p)) < \exp[-\Omega_H(f(k, n\tau^{1/k}, p))].$$

Note that (b) for a given  $H$  follows from (a), since (noting that  $\tau\Psi(H, n) = \Psi(H, n\tau^{1/k})$  and using (a) for the second inequality)

$$\begin{aligned}\Pr(X_H^n \geq 2\tau\Psi(H, n)) &\leq \Pr(X_H^{n\tau^{1/k}} \geq 2\Psi(H, n\tau^{1/k})) \\ &\leq \exp\left[-\Omega_H\left(f(k, n\tau^{1/k}, p)\right)\right].\end{aligned}$$

We include (b) because it will be needed for induction; that is, for a given  $H$  we just prove (a), occasionally appealing to earlier cases of (b).

We have formulated the theorem for all  $p$  so that the inductive parts of the proof don't require checking that  $p$  falls in some suitable range. Note, however, that for the proof we can assume (for our choice of positive constants  $C$  and  $c$  depending on  $H$  and  $\varepsilon$ )

$$p > Cn^{-2/(k-1)}, \quad (9)$$

since for smaller  $p$  ( $> n^{-1/m_H}$ ) the theorem is trivial, and

$$p < c, \quad (10)$$

since above this the desired bound is given by (5). As detailed in the next two lemmas, (5), together with some auxiliary results from [11], also allows us to ignore certain other cases of Theorem 2.3(a).

**Lemma 2.4.** *If  $\Delta_H \leq k - 2$  then*

$$\Pr(X_H^{n,p} \geq (1 + \varepsilon)\Psi(H, n, p)) \leq p^{\Omega_{H,\varepsilon}(n^2 p^{k-1})}.$$

*Proof.* By Proposition 2.2, it is enough to show

$$\Pr(\xi_H^{n,p} \geq (1 + \varepsilon)\mathbf{E}(\xi_H^{n,p})) \leq p^{\Omega_{H,\varepsilon}(n^2 p^{k-1})}; \quad (11)$$

but this follows from (5), which since  $M_H(n, p) \geq n^2 p^{\Delta_H}$  (see [11, Lemma 6.2]), bounds the left side of (11) by

$$\exp[-\Omega_{H,\varepsilon}(n^2 p^{\Delta_H})] \leq \exp[-\Omega_{H,\varepsilon}(n^2 p^{k-1}t)].$$

■

**Lemma 2.5.** *For any  $H \neq K_k$  on  $k$  vertices and  $\gamma > 0$ , if  $p < n^{-(1+\gamma)/(k-1)}$  then*

$$\Pr(X_H^n \geq (1 + \varepsilon)\Psi(H, n)) < p^{\Omega_{H,\varepsilon,\gamma}(n^2 p^{k-1})}.$$

*Proof.* By Lemma 2.4 we may assume  $\Delta := \Delta_H = k - 1$  (and will write  $\Delta$  in place of  $k - 1$  in this section). By Proposition 2.2 it's enough to show

$$\Pr(\xi_H^{n,p} \geq (1 + \vartheta)\mathbf{E}(\xi_H^{n,p})) < p^{\Omega_{\vartheta,H}(n^2 p^\Delta)},$$

which, in view of (5) and the definition of  $M_H(n, p)$ , will follow if we show that, for any  $K \subseteq H$ ,  $n^{v_K} p^{e_K} = \Omega((n^2 p^\Delta t)^{\alpha_K^*})$ , or, more conveniently,

$$n^{v_K - 2\alpha_K^*} p^{e_K - \Delta\alpha_K^*} = \Omega(t^{\alpha_K^*}). \quad (12)$$

We need one easy observation from [11] (see their Lemma 6.1):

$$e_K \leq \Delta(v_K - \alpha_K^*).$$

Then, noting that

$$e_K - \Delta\alpha_K^* < 0 \quad (13)$$

(since  $e_K < \Delta v_K/2 \leq \Delta\alpha_K^*$ ) and using our upper bound on  $p$ , we find that the left side of (12) is at least

$$\begin{aligned} n^{v_K - 2\alpha_K^* - (1+\gamma)(e_K - \Delta\alpha_K^*)/\Delta} &\geq n^{v_K - 2\alpha_K^* - (v_K - 2\alpha_K^*) + \gamma(\Delta\alpha_K^* - e_K)/\Delta} \\ &= n^{\gamma(\Delta\alpha_K^* - e_K)/\Delta}, \end{aligned}$$

which (again using (13)) gives (12). ■

### 3 Large deviations

This section collects a few standardish large deviation basics that will be used throughout the paper. It's perhaps worth noting that these elementary inequalities are the only “machinery” we will need.

We use  $B(m, \alpha)$  for a random variable with the binomial distribution  $\text{Bin}(m, \alpha)$ . The next lemma, which is easily derived from [2, Theorem A.1.12] and [10, Theorem 2.1] respectively (for example), will be used repeatedly, eventually without explicit mention.

**Lemma 3.1.** *There is a fixed  $C > 0$  so that for any  $\lambda \leq 1$ ,  $K > 1 + \lambda$ ,  $m$  and  $\alpha$ ,*

$$\Pr(B(m, \alpha) \geq Km\alpha) < \min\{(e/K)^{Km\alpha}, \exp[-C\lambda^2 Km\alpha]\}. \quad (14)$$

*Remark.* We may assume  $Km\alpha \geq 1$ . Thus, if  $em\alpha^c < 1$  then  $e/K < \alpha^{1-c}$  and the bound in (14) is at most  $\alpha^{(1-c)Km\alpha}$ .

The next lemma, an immediate consequence of Lemma 3.1 (and the above Remark), will also be used repeatedly, usually following a preliminary application of Lemma 3.1 to justify the assumption  $enq^c < 1$ .

**Lemma 3.2.** *Fix  $c < 1$  and assume  $enq^c < 1$ . If  $S \subseteq V_i$  is random with  $\Pr(x \in S) \leq q \ \forall x \in V_i$ , these events independent, then for any  $T$ ,*

$$\Pr(|S| \geq T) < q^{(1-c)T}.$$

We also need the following inequality, which is an easy consequence of, for example, [3, Lemma 8.2].

**Lemma 3.3.** *Suppose  $w_1, \dots, w_m \in [0, z]$ . Let  $\xi_1, \dots, \xi_m$  be independent Bernoullis,  $\xi = \sum \xi_i w_i$ , and  $\mathbf{E}\xi = \mu$ . Then for any  $\eta > 0$  and  $\lambda \geq \eta\mu$ ,*

$$\Pr(\xi > \mu + \lambda) < \exp[-\Omega_\eta(\lambda/z)].$$

## 4 Outline

In this section we list the steps in the proof of Theorem 2.3(a), filling in some definitions as we go along. The proof proceeds by induction on (say)  $k^2 + e_H$ , so that in proving the statement for  $H$  we may assume its truth for all graphs with either fewer than  $k$  vertices or with  $k$  vertices and fewer than  $e_H$  edges. The case  $k = 2$  is trivial and  $k = 3$  is the main result of [6], so we assume throughout that  $k \geq 4$ .

Most of the proof (Lemmas 4.1-4.3) consists of identifying certain anomalies, for example vertices of unusually high degree, and bounding the number of copies of  $H$  in which they appear. The remaining copies are then easily handled (in Lemma 4.4) using Lemma 3.3.

Here and throughout we use  $C$  and  $C_\varepsilon$  for (positive) constants depending on (respectively)  $H$  and  $(H, \varepsilon)$ , different occurrences of which will usually denote different values. Similarly, we use  $\Omega$  and  $\Omega_\varepsilon$  as shorthand for  $\Omega_H$  and  $\Omega_{H, \varepsilon}$ . We say an event  $E$  occurs *with large probability* (w.l.p.) if  $\Pr(E) > 1 - \exp[-\Omega_\varepsilon(n^2 p^{k-1} t)]$ , and write “ $\alpha <^* \beta$ ” for “w.l.p.  $\alpha < \beta$ ” (where  $\varepsilon$  is as in the statement of the theorem). Note that (9) (with a suitable  $C$ ) guarantees that an intersection of, for example,  $n^5$  w.l.p. events is itself a w.l.p. event, a fact we will sometimes use without mention in what follows.

By Lemma 2.4 we may assume  $\Delta_H = k - 1$ . We reorder the vertices of  $H$  so that  $k - 1 = d(v_1) \geq d(v_2) \geq \dots \geq d(v_k)$  and if  $d(v_2) = k - 2$  then



$v_2 \not\sim v_3$ . We set  $A = V_1, B = V_2, C = V_3$  and *always take  $a, b$  and  $c$  to be elements of  $A, B$  and  $C$  respectively*. For disjoint  $X, Y \subseteq V$  we use  $\nabla(X, Y)$  for the set of edges with one end in each of  $X$  and  $Y$ , and  $\nabla(X)$  for the set of edges with one end in  $X$ . We use  $N(x)$  for the neighborhood of (set of vertices adjacent to) a vertex  $x$ .

For  $K \subseteq H$  with vertex set  $\{v_i : i \in T\}$  ( $T \subseteq [k]$ ), define a *copy of  $K$  in  $G$*  ( $= G(n, p, H)$ ) to be a set of vertices  $\{x_i : i \in T\}$  with  $x_i \in V_i$  and  $x_i x_j \in E(G)$  whenever  $v_i v_j \in E(K)$ . For  $x_1, x_2, \dots, x_l$  vertices belonging to distinct  $V_i$ 's we use  $w_K(x_1, \dots, x_l)$  for the number of copies of  $K$  containing  $x_1, \dots, x_l$ ; when  $K = H$  we call this the *weight* of  $\{x_1, \dots, x_l\}$ . We use  $H_S = H - \{v_i : i \in S\}$  ( $S \subset [k]$ ), and abbreviate  $H_{\{i\}} = H_i$ ,  $w_{H_S}(\cdot) = w_S(\cdot)$ ,  $w_{\{i\}} = w_i$  and  $w_\emptyset(\cdot) (= w_H(\cdot)) = w(\cdot)$ .

Set  $\vartheta = .05\varepsilon$  and define  $\delta$  by  $(1 + \delta)^k = 2$ . For  $x \in V$  and  $i \in [k]$ , let  $d_i(x) = |N(x) \cap V_i|$ , and set  $d(x) = \max\{d_i(x) : i \in [k]\}$ . Say a vertex  $x$  is *high degree* if  $d(x) > (1 + \delta)np$ , and a copy of  $H$  is *type one* if contains a high degree vertex from  $A, B$  or  $C$ .

**Lemma 4.1.** *W.l.p.  $G$  contains less than  $7\vartheta\Psi(H, n)$  type one copies of  $H$ .*

Let  $A', B', C'$  denote the subsets of  $A, B, C$  respectively of vertices which are not high degree. For vertices  $x, y \in G$  let  $d_j(x, y) = |N(x) \cap N(y) \cap V_j|$  and  $d(x, y) = \max_{j \geq 4} d_j(x, y)$ . A pair of vertices  $(x, y)$  is *high degree* if  $d(x, y) > np^{3/2}$ . For  $k > 4$  a copy of  $H$  is *type two* if it contains a high degree pair  $(x, y)$  belonging to either  $A' \times C'$  or  $B' \times C'$ ; for  $k = 4$  we don't need this, and simply declare that there are no copies of type two.

**Lemma 4.2.** *W.l.p.  $G$  contains less than  $2\vartheta\Psi(H, n)$  type two copies of  $H$ .*

Set  $s = \min\{t, n^{k-2}p^{\binom{k-1}{2}}\}$ , the two regimes corresponding to the two ranges of  $f(k, n, p)$  ( $= n^2 p^{k-1} s$ ). Define  $w^*(\cdot)$  in the same way as  $w(\cdot)$ , but with the count restricted to copies of  $H$  that are not type one or two. Set

$$\zeta = \begin{cases} 3^{k-2}\Psi(H, n, p)/(n^2 p^{k-1} s) & \text{if } k \geq 5 \\ 225\Psi(H, n, p)/(n^2 p^3 s) & \text{if } k = 4 \end{cases} \quad (15)$$

and (in either case) say  $ab \in \nabla(A, B)$  is *heavy* if  $w^*(a, b) > \zeta$ . Finally, say a copy of  $H$  is *type three* if it is not type one or two and contains a heavy edge, and *type four* if it is not type one, two or three.

**Lemma 4.3.** *W.l.p.  $G$  contains less than  $4\vartheta\Psi(H, n)$  type three copies of  $H$ .*

**Lemma 4.4.** *With probability at least  $1 - \exp[-\Omega_\varepsilon(f(k, n, p))]$   $G$  contains less than  $(1 + 2\vartheta)\Psi(H, n)$  type four copies of  $H$ .*

Of course Theorem 2.3(a) (for  $k \geq 4$ ) follows from Lemmas 4.1-4.4; these are proved in the next four sections.

## 5 Proof of Lemma 4.1

For  $i \in [3]$  set  $D_1(i) = \{x \in V_i : d(x) > np^{2/5}\}$  and  $D_2(i) = \{x \in V_i : np^{2/5} \geq d(x) > (1 + \delta)np\}$ , and for  $j \in [2]$  set  $S_j(i) = \sum\{d(x) : x \in D_j(i)\}$ . We will show

**Proposition 5.1.** *For all  $1 \leq i \leq 3$ ,*

$$w.l.p. \quad \forall x \in D_j(i), \quad w(x)/d(x) < \begin{cases} 2n^{k-2}p^{e_H-(k-1)} & \text{if } j = 1 \\ 2n^{k-2}p^{e_H-k+2(k-1)/5} & \text{if } j = 2 \end{cases}$$

and

**Proposition 5.2.** *For all  $1 \leq i \leq 3$ ,*

$$w.l.p. \quad S_j(i) < \begin{cases} \vartheta n^2 p^{k-1} & \text{if } j = 1 \\ kn^2 p^{k-1} t & \text{if } j = 2. \end{cases} \quad (16)$$

The lemma follows since the number of type one copies of  $H$  is at most

$$\begin{aligned} \sum_{x: \text{high degree}} w(x) &<^* \sum_{i=1}^3 (S_1(i) \cdot 2n^{k-2}p^{e_H-(k-1)} + S_2(i) \cdot 2n^{k-2}p^{e_H-k+2(k-1)/5}) \\ &<^* 3(2\vartheta\Psi(H, n) + 2k\Psi(H, n)p^{2(k-1)/5-1}t) \\ &< 7\vartheta\Psi(H, n), \end{aligned}$$

using Propositions 5.1 and 5.2 for the first and second inequalities. ■

*Proof of Proposition 5.1.* Fix  $i$  and condition on  $\nabla(V_i)$  (thus determining  $D_1(i)$  and  $D_2(i)$ ). If  $d_H(v_i) = k - 1$ , then for any  $x \in D_1(i)$ , induction gives

$$\Pr(w(x) \geq 2\Psi(H_i, d(x))) < \exp[-\Omega(f(k-1, d(x)))],$$

whence (noting  $\Psi(H_i, \cdot) = \Psi(H_1, \cdot)$ )

$$\begin{aligned} \Pr(\exists x \in D_1(i) : w(x) \geq 2\Psi(H_1, d(x))) &< n \exp[-\Omega(f(k-1, np^{2/5}))] \\ &< p^{n^2 p^{k-1}}. \end{aligned} \quad (17)$$

Similarly,

$$\begin{aligned}
\Pr(\exists x \in D_2(i) : w(x) \geq 2\Psi(H_1, np^{2/5})) &< n \Pr(X_{H_i}^{np^{2/5}} \geq 2\Psi(H_i, np^{2/5})) \\
&< n \exp[-\Omega(f(k-1, np^{2/5}))] \\
&< p^{n^2 p^{k-1}} \tag{18}
\end{aligned}$$

Note that, here and throughout, we omit the routine verifications of inequalities like those in the last lines of (17) and (18).

If  $d(v_i) = k - 2$ , then  $v_i \not\sim v_j$  for some  $j \in [k]$ . We partition  $V_j = P_1 \cup \dots \cup P_{\lfloor 1/p \rfloor}$  with each  $P_\ell$  of size at most  $(1 + \delta)np$ , and write  $w^\ell(x)$  for the number of copies of  $H$  containing  $x$  and meeting  $P_\ell$ . Noting that here  $\Psi(H_1, \cdot) = p^{-1}\Psi(H_i, \cdot)$  (and  $w(x) = \sum_\ell w^\ell(x)$ ), we have

$$\begin{aligned}
\Pr(w(x) \geq 2\Psi(H_1, d(x))) &< \Pr(\exists \ell \ w^\ell(x) \geq 2\Psi(H_i, d(x))) \\
&< p^{-1} \exp[-\Omega(f(k-1, d(x)))]
\end{aligned}$$

for a given  $x$ , so that

$$\begin{aligned}
\Pr(\exists x \in D_1(i) : w(x) \geq 2\Psi(H_1, d(x))) &< np^{-1} \exp[-\Omega(f(k-1, np^{2/5}))] \\
&< p^{n^2 p^{k-1}}, \tag{19}
\end{aligned}$$

and

$$\begin{aligned}
\Pr(\exists x \in D_2(i) : w(x) \geq 2\Psi(H_1, np^{2/5})) &< np^{-1} \Pr(X_{H_i}^{np^{2/5}} \geq 2\Psi(H_i, np^{2/5})) \\
&< np^{-1} \exp[-\Omega(f(k-1, np^{2/5}))] \\
&< p^{n^2 p^{k-1}}. \tag{20}
\end{aligned}$$

Finally, (17)-(20) imply that w.l.p.

$$\begin{aligned}
w(x)/d(x) &< 2\Psi(H_1, d(x))/d(x) = 2(d(x))^{k-1} p^{e_H - (k-1)} / d(x) \\
&\leq 2n^{k-2} p^{e_H - (k-1)} \quad \forall x \in D_1(i)
\end{aligned}$$

and

$$\begin{aligned}
w(x)/d(x) &< 2\Psi(H_1, np^{2/5})/d(x) = 2(np^{2/5})^{k-1} p^{e_H - (k-1)} / d(x) \\
&\leq 2n^{k-2} p^{e_H - k + 2(k-1)/5} \quad \forall x \in D_2(i).
\end{aligned}$$

■

*Proof of Proposition 5.2.* We bound  $|\nabla(D_j(i))|$ , which is, of course, an upper bound on  $S_j(i)$ . We first assert that, for any  $i \in [3]$ , w.l.p.

$$|D_1(i)| < \vartheta np^{k-7/5} \quad \text{and} \quad |D_2(i)| < np^{k-2}t. \quad (21)$$

This will follow from Lemmas 3.1 and 3.2 (so really two applications of Lemma 3.1), a combination we will see repeatedly. For a given  $i$  and  $j$  the events  $\{x \in D_j(i)\}$  ( $x \in V_i$ ) are independent with (using Lemma 3.1)

$$\Pr(x \in D_1(i)) < k \Pr(B(n, p) > np^{2/5}) < k(ep^{3/5})^{np^{2/5}} < p^{0.5np^{2/5}}$$

and

$$\Pr(x \in D_2(i)) < k \Pr(B(n, p) > (1 + \delta)np) < \exp[-\Omega(np)].$$

An application of Lemma 3.2 now shows that (21) holds w.l.p. ■

Assume then that (21) holds, and for convenience rename its bounds  $\vartheta np^{k-7/5} = r$  and  $np^{k-2}t = u$ ; we may of course assume  $r \geq 1$  if proving the first bound in (16) and  $u \geq 1$  if proving the second. We have (a bit crudely)

$$\begin{aligned} \Pr(|\nabla(D_1(i))| \geq \vartheta n^2 p^{k-1}) &< \Pr(\exists T \in \binom{V(i)}{r} : |\nabla(T)| \geq \vartheta n^2 p^{k-1}) \\ &< \binom{n}{r} \Pr(B((k-1)rn, p) \geq \vartheta n^2 p^{k-1}) \\ &< n^r (e(k-1)p^{3/5})^{\vartheta n^2 p^{k-1}} \\ &< p^{\Omega_\varepsilon(n^2 p^{k-1})} \end{aligned}$$

and

$$\begin{aligned} \Pr(|\nabla(D_2(i))| \geq kn^2 p^{k-1}t) &< \Pr(\exists T \in \binom{V(i)}{u} : |\nabla(T)| \geq kn^2 p^{k-1}t) \\ &< \binom{n}{u} \Pr(B((k-1)un, p) \geq kn^2 p^{k-1}t) \\ &< n^u \exp[-\Omega(n^2 p^{k-1}t)] \\ &< p^{\Omega(n^2 p^{k-1})}, \end{aligned}$$

with the third inequality in each case given by Lemma 3.1. ■

## 6 Proof of Lemma 4.2

(Here we are only interested in  $k \geq 5$ .) We bound the contribution of high-degree  $(A', C')$ -pairs, the argument for  $(B', C')$ -pairs being similar.

Let  $A''$  be the (random) set of vertices of  $A'$  involved in high-degree  $(A', C')$ -pairs—that is,  $A'' = \{a \in A' : \exists c \in C' d(a, c) > np^{3/2}\}$ —and define  $C''$  similarly. We will show that

$$\text{w.l.p. } |A''|, |C''| < np^{k-5/2} \quad (22)$$

and

$$\text{w.l.p. } w(a, c) < 2t\Psi(H_{\{1,3\}}, (1+\delta)np) \quad \forall (a, c) \in A' \times C'. \quad (23)$$

Combining these we find that the total weight of high degree  $(A', C')$ -pairs is w.l.p. at most

$$(np^{k-5/2})^2 2t\Psi(H_{\{1,3\}}, (1+\delta)np) < 4n^2 p^{3k-7} t\Psi(H_{\{1,3\}}, n) < \vartheta\Psi(H, n),$$

where the second inequality uses  $\Psi(H_{\{1,3\}}, n) \leq n^{-2} p^{-(2k-3)} \Psi(H, n)$  and  $4p^{k-4}t < \vartheta$  (see (10)). Since, as noted above, the same argument shows that the contribution of high-degree  $(B', C')$ -pairs is w.l.p. at most  $\vartheta\Psi(H, n)$ , the lemma follows.

*Proof of (22).* Given  $\nabla(C)$ , the events  $\{a \in A''\}$  are independent, with

$$\begin{aligned} \Pr(a \in A'') &< n(k-2) \Pr[B((1+\delta)np, p) > np^{3/2}] \\ &< n(k-2)(e(1+\delta)p^{1/2})^{np^{3/2}} < p^{0.4np^{3/2}} =: q, \end{aligned}$$

where we use (9), (10) and  $k \geq 5$  for the last inequality. Thus, since  $enq^{1/2} < 1$ , Lemma 3.2 gives (22) for  $A''$ , and of course the same argument applies to  $C''$ .  $\blacksquare$

*Proof of (23).* Here we have lots of room and just bound  $\max\{w_3(a) : a \in A'\}$ , a trivial upper bound on  $\max\{w(a, c) : a \in A', c \in C'\}$ . Since  $d(a) < (1+\delta)np$  (for  $a \in A'$ ) and  $v_1 \sim v_\ell \forall \ell \in [k] \setminus \{2, 3\}$ , Theorem 2.3(b) gives (inductively)

$$\begin{aligned} \Pr[\exists a \in A' w_3(a) \geq 2t\Psi(H_{\{1,3\}}, (1+\delta)np)] \\ < n \exp[-\Omega(f(k-2, (1+\delta)npt^{\frac{1}{k-2}}))] < p^{\Omega(n^2 p^{k-1})} \end{aligned}$$

(with verification of the second inequality, which does need (9) at one point, again left to the reader).  $\blacksquare$

## 7 Proof of Lemma 4.3

This requires special treatment when  $k = 4$ ; see the beginning of Section 7.2 for the reason for the split. In Sections 7.1 and 7.2 we set  $A'' = \{a : d_i(a) \leq (1 + \delta)np \ \forall i \geq 3\} \supseteq A'$  and define  $B''$  similarly.

### 7.1 Proof for $k \geq 5$

For reasons that will be explained as we proceed, we need somewhat different arguments for large and small values of  $p$ .

*Case 1:*  $np^{(k-1)/2} \geq \log^4 n$ . Let  $C_b = \{c \in C \cap N(b) : d(b, c) \leq np^{3/2}\}$  and

$$W(A) = \{a : \exists b \in B'', \sum_{c \in C_b \cap N(a)} w_1(b, c) > \zeta\} \supseteq \{a : \exists b, w^*(a, b) > \zeta\}$$

(see (15) for  $\zeta$ ), and define  $W(B)$  similarly.

*Remark.* While it may seem more natural to define  $W(A)$ ,  $W(B)$  in terms of  $w(a, b)$  or  $w^*(a, b)$ , the present definition has the advantage of not depending on  $\nabla(A, B)$ . We will see something similar in Case 2.

The point requiring most work here is

$$\text{w.l.p.} \quad |W(A)|, |W(B)| < \vartheta np^{(k-1)/2} t^3. \quad (24)$$

Given this, the rest of the argument goes as follows. According to Lemma 3.1, (24) implies

$$\text{w.l.p.} \quad |\nabla(W(A), W(B))| < \vartheta n^2 p^{k-1} \quad (25)$$

(since, given the inequality in (24),  $|\nabla(W(A), W(B))| \sim B(m, p)$  for some  $m < \vartheta^2 n^2 p^{k-1} t^6$ ; note the inequalities in (24) and (25) depend on separate sets of random edges). On the other hand, an inductive application of Theorem 2.3(b) gives

$$\text{w.l.p.} \quad w^*(a, b) < 2\Psi(H_{\{1,2\}}, (1 + \delta)np) \quad \forall a, b \quad (26)$$

(using the fact that we are in Case 1 and noting that  $d(a) > (1 + \delta)np$  implies  $w^*(a, b) = 0$ ).

Finally, the combination of (25) and (26) bounds the number of type three copies of  $H$  by  $\vartheta n^2 p^{k-1} \cdot 2\Psi(H_{\{1,2\}}, (1 + \delta)np) < 4\vartheta\Psi(H, n)$ .  $\blacksquare$

Proofs of the two assertions in (24) being similar, we just deal with  $W(A)$ . We first show

$$\text{w.l.p.} \quad w_1(b, c) < 2tn^{k-3}p^{e_H-(3k-3)/2} =: \gamma \quad \forall b \in B'' \text{ and } c \in C_b \quad (27)$$

and

$$\text{w.l.p.} \quad w_1(b) < 4n^{k-2}p^{e_H-(k-1)} \quad \forall b \in B''. \quad (28)$$

These will imply, via Lemma 3.3, that the events  $\{a \in W(A)\}$  are unlikely, and then (24) will be an application of Lemma 3.2.

Each of (27) and (28) is given (inductively) by Theorem 2.3(b), with small differences in arithmetic depending on  $d(v_2)$  and  $d(v_3)$ : say we are *in* (a), (b) or (c) according to whether  $(d(v_2), d(v_3))$  is  $(k-1, k-1)$ ,  $(k-1, k-2)$  or  $(k-2, k-2)$ .

For (27) we first observe that, given  $\nabla(B \cup C)$  and  $c \in C_b$ ,  $w_1(b, c)$  is stochastically dominated by  $X := X(H_{\{1,2,3\}}, np^{3/2})$  in (a) and (c), and by the sum of  $\lfloor 1/p \rfloor$  copies of  $X$  in (b). (For the latter assertion, let  $\ell$  be the index for which  $v_3 \not\sim v_\ell$  and, recalling that  $b \in B''$ , partition  $N(b) \cap V_\ell = V_1 \cup \dots \cup V_{\lfloor 1/p \rfloor}$  with each block of size at most  $np^{3/2}$ .) Theorem 2.3(b) thus gives the upper bound

$$n^2 \lfloor 1/p \rfloor \exp[-\Omega(f(k-3, np^{3/2}t^{1/(k-3)}))] < p^{\Omega(n^2p^{k-1})} \quad (29)$$

on either

$$\Pr(\exists b \in B'', c \in C_b : w_1(b, c) > 2t\Psi(H_{\{1,2,3\}}, np^{3/2}))$$

(if we are in (a) or (c)) or

$$\Pr(\exists b \in B'', c \in C_b : w_1(b, c) > 2t \lfloor 1/p \rfloor \Psi(H_{\{1,2,3\}}, np^{3/2}))$$

(if we are in (b)), the inequality in (29) holding because we are in Case 1. (Note that in (29) the  $\lfloor 1/p \rfloor$  is needed only when we are “in (b),” and the term involving  $t$  only when  $k = 5$ .)

To complete the proof of (27) it just remains to check that  $\gamma$  (recall this is the right hand side of (27)) is an upper bound on  $2t\Psi(H_{\{1,2,3\}}, np^{3/2})$  if we are in (a) or (c), and on  $2t \lfloor 1/p \rfloor \Psi(H_{\{1,2,3\}}, np^{3/2})$  if we are in (b). ■

The proof of (28) is similar. Here, because we are in Case 1, Theorem 2.3(b) gives the bound

$$n \lfloor 1/p \rfloor \exp[-\Omega(f(k-2, (1+\delta)np))] < p^{\Omega(n^2p^{k-1})}$$

on  $\Pr(\exists b \in B'' \ w_1(b) > 2\Psi(H_{\{1,2\}}, (1+\delta)np))$  if we are in (a) or (b), and on  $\Pr(\exists b \in B'' \ w_1(b) > 2\lfloor 1/p \rfloor \Psi(H_{\{1,2\}}, (1+\delta)np))$  if we are in (c); and it's easy to check that  $2\Psi(H_{\{1,2\}}, (1+\delta)np)$  or  $2\lfloor 1/p \rfloor \Psi(H_{\{1,2\}}, (1+\delta)np)$  (as appropriate) is less than  $4n^{k-2}p^{e_H-(k-1)}$ . ■

Finally we return to (24). Fix (and condition on) any value of  $E(G) \setminus \nabla(A, C)$  satisfying the inequalities in (27) and (28). It is enough to show that, under this conditioning and for any  $a$ ,

$$\Pr(a \in W(A)) < \exp[-\Omega(np^{(k-1)/2}/t^2)] =: q, \quad (30)$$

since then Lemma 3.2 implies, using  $enq^{1/2} < 1$  and the fact that the events  $\{a \in W(A)\}$  are independent,

$$|W(A)| <^* \vartheta np^{(k-1)/2} t^3.$$

(The assertion  $enq^{1/2} < 1$  (or  $enq^c < 1$ ) imposes the most stringent requirement on  $p$  for Case 1.)

For (30) we observe that (28) gives (for any  $a$  and  $b \in B''$ )

$$\mathbf{E} \sum_{c \in C_b \cap N(a)} w_1(b, c) = p \sum_{c \in C_b} w_1(b, c) \leq p w_1(b) < 4n^{k-2}p^{e_H-k+2} < \zeta/2,$$

whence, using Lemma 3.3 with (27), we have

$$\begin{aligned} \Pr(a \in W(A)) &< \Pr\left(\exists b \in B'' \sum \{w_1(b, c) : c \in C_b \cap N(a)\} > \zeta\right) \\ &< n \exp[-\Omega(\zeta/\gamma)] < n \exp[-\Omega(np^{(k-1)/2}/t^2)] \\ &< \exp[-\Omega(np^{(k-1)/2}/t^2)]. \end{aligned}$$

■

*Case 2:*  $np^{(k-1)/2} < \log^4 n$ . Recall that for very small  $p$ —in particular for  $p$  in the present range—and  $H \neq K_k$ , Theorem 2.3 is contained in Lemma 2.5; we may thus assume  $H = K_k$ . Let  $H' = H - v_1v_2$  and, writing  $w'$  for  $w_{H'}$ , set

$$W(A) = \{a : \exists b \in B'', w'(a, b) > \zeta\} \supseteq \{a : \exists b \ w^*(a, b) > \zeta\}, \quad (31)$$

and define  $W(B)$  similarly. (We could also work directly with  $w(a, b)$  and avoid the extra definitions; but the present treatment, which we will see



again below, is more natural in that it allows us to ignore the essentially irrelevant  $\nabla(A, B)$ .)

The argument here is similar to that for Case 1. We again show that membership in  $W(A)$ ,  $W(B)$  is unlikely, leading to

$$\text{w.l.p.} \quad |W(A)|, |W(B)| < \log^8 n, \quad (32)$$

which, in view of Lemma 3.1, again gives

$$\text{w.l.p.} \quad |\nabla(W(A), W(B))| < \vartheta n^2 p^{k-1}. \quad (33)$$

On the other hand we will show, by an argument somewhat different from others seen here,

$$\text{w.l.p.} \quad \mathbf{w}^*(a, b) < n^{k-2} p^{\binom{k-1}{2}} \quad \forall a, b. \quad (34)$$

Combining this with (33) gives Lemma 4.3 (for the present case).

*Proof of (32).* Of course it's enough to prove the assertion for  $W(A)$ . We first observe that

$$\text{w.l.p.} \quad \mathbf{w}_1(b) < 2t\Psi(H_{\{1,2\}}, (1+\delta)np) < 4t \log^{4k-8} n =: m \quad \forall b \in B''; \quad (35)$$

as elsewhere, this is given by an inductive application of Theorem 2.3(b), which says that, for any  $b \in B''$ ,

$$\begin{aligned} \Pr(\mathbf{w}_1(b) > 2t\Psi(H_{\{1,2\}}, (1+\delta)np)) &< \exp[-\Omega(f(k-2, (1+\delta)np t^{1/(k-2)}))] \\ &< p^{\Omega(n^2 p^{k-1})}. \end{aligned}$$

(Note that for very small  $p$  the extra factor  $t$  in (35)—which did not appear in (28)—is needed for the final inequality here.)

We now condition on  $E(G) \setminus \nabla(A)$  and assume that, as in (35),  $\mathbf{w}_1(b) < m \quad \forall b \in B''$ . Note that  $a \in W(A)$  means (at least) that there is some  $b \in B''$  with

$$\mathbf{w}'(a, b) \geq 3^{k-2}. \quad (36)$$

For  $i \in \{3, \dots, k\}$  (and any  $b$ ), let  $V_i^*(b)$  be the set of vertices of  $V_i$  lying on copies of  $H_1$  that contain  $b$ . Since

$$\mathbf{w}'(a, b) \leq \prod_{i=3}^k |N(a) \cap V_i^*(b)|,$$

(36) at least requires  $|N(a) \cap (\cup_{i=3}^k V_i^*(b))| \geq 3(k-2)$ ; so the probability (for a given  $a$ ) that there is some  $b$  for which (36) holds is at most

$$n \Pr(B((k-2)m, p) \geq 3(k-2)) < p^{-(k-1)/2 + (1-o(1))3(k-2)} < p^{k-1} =: q.$$

But then, since (say)  $enq^{3/4} < 1$ , Lemma 3.2 gives (32). ■

*Remark.* Of course (34) is the counterpart of (26) of Case 1 (since  $H$  is now  $K_k$  the two bounds differ only by small constant factors); but for very small  $p$  the simple inductive derivation of (26) using Theorem 2.3(b) no longer applies, since  $f(k-2, (1+\delta)np)$  may be much smaller than  $f(k, n)$ .

*Proof of (34).* We may assume  $b \in B'$  as otherwise  $w^*(a, b) = 0$ . For  $i \in \{3, \dots, k\}$  let

$$V_i^*(a, b) = \{v \in V_i : \text{some copy of } H \text{ on } a, b \text{ contains } v\}.$$

We will show that

$$\text{w.l.p.} \quad |\nabla(V_i^*(a, b), V_j^*(a, b))| < n^2 p^{k-1} \quad \forall i, j, a \text{ and } b \in B'. \quad (37)$$

That this gives (34) is essentially a special case of a theorem of N. Alon [1], the precise statement used here (see the proof of Theorem 1.1 in [7]) being: an  $r$ -partite graph with at most  $\ell$  edges between any two of its parts contains at most  $\ell^{r/2}$  copies of  $K_r$ .

For the proof of (37) we fix  $a, b$  and  $i < j$ , and think of choosing edges of  $G$  in the order: (i)  $\nabla(b, V_3 \cup \dots \cup V_k)$ ; (ii)  $\nabla(V_\alpha, V_\beta)$  for all  $3 \leq \alpha < \beta \leq k$  except  $(\alpha, \beta) = (i, j)$ ; (iii)  $\nabla(a, V_i \cup V_j)$ ; (iv)  $\nabla(V_i, V_j)$ . (The remaining edges are irrelevant here.)

Let  $H'' = H_1 - v_i v_j$ . Since  $b \in B'$ , Lemma 2.5 gives (since we are in Case 2)

$$w_{H''}(b) <^* 2\Psi(H_{1,2} - v_i v_j, (1+\delta)np) =: m. \quad (38)$$

Let  $V_i^*$  be the set of vertices of  $V_i$  contained in copies of  $H''$  that contain  $b$ , and define  $V_j^*$  similarly.

If the bound in (38) holds, then each of  $V_i^*, V_j^*$  has size at most  $m < p^{-1} \log^{O(1)} n$ ; an application of Lemma 3.1 thus shows that w.l.p. each of  $N(a) \cap V_i^*$ ,  $N(a) \cap V_j^*$  (and thus also  $V_i^*(a, b)$ ,  $V_j^*(a, b)$ ) has size at most (say)  $p^{-1/4}$ , and a second application gives (37). ■

## 7.2 Proof for $k = 4$

For  $k = 4$ , as in Case 2 above, we can't simply invoke induction to obtain (26), since  $f(2, (1+\delta)np)$  ( $\approx n^2 p^3$ ) is smaller than  $f(4, n)$ . This is the main reason a separate argument is needed for  $k = 4$ .

*Proof.* In this section, for  $x, y \in G$  let  $d(x, y) = \max_{j \geq 3} d_j(x, y)$ . We consider the possibilities  $H = K_4$  and  $H = K_4^-$  ( $K_4$  with an edge removed) separately.

*Case 1.*  $H = K_4$ . Now  $ab$  is heavy if  $w^*(a, b) > 225n^2p^3/s$ . Here it will be helpful to work with  $w$  rather than  $w^*$ . We treat (heavy)  $ab$ 's with  $w(a, b) > n^2p^3$  and those with  $w(a, b) \in (225n^2p^3/s, n^2p^3]$  separately.

To bound the contribution of edges of the first type, set

$$A^* = \{a : \exists b \in B'', w'(a, b) > n^2p^3\} \supseteq \{a : \exists b \in B', w(a, b) > n^2p^3\}$$

(where  $w'$  is as in the paragraph containing (31)), and define  $B^*$  similarly. We first show

$$\text{w.l.p.} \quad |A^*|, |B^*| < np^{7/4}. \quad (39)$$

To see this (for  $A^*$ , say) we condition on the value of  $\nabla(B, C \cup V_4)$  and consider  $\Pr(a \in A^*)$ . Noting that for any  $a$  and  $b \in B''$ ,

$$\Pr(w'(a, b) \geq n^2p^3) \leq \Pr(d(a, b) > np^{5/4}) + \Pr(w'(a, b) \geq n^2p^3 | d(a, b) \leq np^{5/4})$$

(where  $5/4$  is just a convenient value between 1 and  $3/2$ ), we have

$$\begin{aligned} \Pr(a \in A^*) &< n[2\Pr(B((1+\delta)np, p) > np^{5/4}) + \Pr(B(n^2p^{5/2}, p) > n^2p^3)] \\ &\leq p^{\Omega(np^{5/4})} + p^{\Omega(n^2p^3)}. \end{aligned} \quad (40)$$

Since (given  $\nabla(B, C \cup V_4)$ ) the events  $\{a \in A^*\}$  are independent, Lemma 3.2 now gives (39). (Note that when the second term dominates (40), Lemma 3.2 gives  $A^* = \emptyset$  w.l.p.)

On the other hand, again using Lemma 3.1, we have

$$\begin{aligned} \Pr(\exists a \in A, b \in B' : w(a, b) > n^2p^3t) &< n^2\Pr(B((1+\delta)^2n^2p^2, p) > n^2p^3t) \\ &< p^{\Omega(n^2p^3)}, \end{aligned}$$

and combining this with (39) gives

$$\sum \{w^*(a, b) : w(a, b) > n^2p^3\} <^* |A^*||B^*|n^2p^3t <^* n^4p^{6.5}t \quad (< \vartheta n^4p^6).$$

For  $ab$  of the second type (i.e. with  $w(a, b) \in (225n^2p^3/s, n^2p^3]$ ), we take  $J = 15np^{3/2}/\sqrt{s}$ , set  $A_J = \{a : \exists b \in B'', d(a, b) > J\}$ , and define  $B_J$  similarly. Given  $\nabla(B, C \cup V_4)$  the events  $\{a \in A_J\}$  are independent with, for each  $a$ ,

$$\Pr(a \in A_J) < 2n\Pr(B((1+\delta)np, p) > J) < 2np^{(1-o(1))J/2} =: q.$$

(using  $e(1 + \delta)np^{3/2+o(1)} < J$  for the second inequality). Since  $enq^{1/2} < 1$  (to see this, note  $J$  is always at least 15, and is  $n^{\Omega(1)}$  if  $p > n^{-2/3+\Omega(1)}$ ), Lemma 3.2 gives

$$|A_J| <^* \sqrt{\vartheta} n^2 p^3 / J.$$

Of course an identical discussion applies to  $|B_J|$ , so we have  $|A_J||B_J| <^* \vartheta sn^2 p^3$  and, by Lemma 3.1,

$$|\nabla(A_J, B_J)| <^* \vartheta n^2 p^3.$$

Thus, finally,

$$\begin{aligned} \sum \{w^*(a, b) : ab \text{ heavy}, w(a, b) \in (n^2 p^3 / s, n^2 p^3]\} \\ <^* |\nabla(A_J, B_J)| n^2 p^3 = \vartheta n^4 p^6 \end{aligned}$$

■

*Case 2:*  $H = K_4^-$ . Recall that  $v_3 v_4$  is the missing edge and an edge  $ab$  is heavy if  $w^*(a, b) > 225\Psi(H, n, p)/(n^2 p^3 s) = 225n^2 p^2 / s$ . We proceed more or less as in the second part of Case 1.

Set  $J = 15np/\sqrt{s}$ ,  $A_J = \{a : \exists b \in B'', d(a, b) > J\}$  and  $B_J = \{b : \exists a \in A'', d(a, b) > J\}$ . Given  $\nabla(B, C \cup V_4)$  the events  $\{a \in A_J\}$  are independent with, for each  $a$ ,

$$\Pr(a \in A_J) \leq 2n \Pr(B((1 + \delta)np, p) > J) < 2np^{J/2} < p^{J/3} =: q$$

(using Lemma 3.1 and  $J > ep^{-1/2}(1 + \delta)np^2$  for the second inequality). Since (say)  $enq^{1/2} < 1$ , Lemma 3.2 gives

$$|A_J| <^* n^2 p^3 / J,$$

and similarly for  $B_J$ . Since  $ab$  heavy at least requires  $a \in A_J, b \in B_J$  and  $a \in A'$  (and since  $a \in A'$  implies  $w(a, b) < ((1 + \delta)np)^2$ ), this says that the number of type three copies of  $H$  is at most

$$|A_J||B_J|((1 + \delta)np)^2 <^* (n^2 p^3 / J)^2 ((1 + \delta)np)^2 < \vartheta n^4 p^5$$

■

## 8 Proof of Lemma 4.4

As earlier, set  $H' = H - v_1 v_2$  and  $\mathbf{w}' = \mathbf{w}_{H'}$ . Let  $X' = \sum_{a \in A, b \in B} \mathbf{w}'(a, b)$ . Then  $X' = X_{H'}$  depends only on  $E(G) \setminus \nabla(A, B)$ . Thus

$$X' <^* (1 + \vartheta) \Psi(H', n) = (1 + \vartheta) \Psi(H, n)/p, \quad (41)$$

where the inequality is given by induction if  $d(v_2) = k - 1$  and by Lemma 2.4 if  $d(v_2) = k - 2$ .

Then

$$Y := \sum_{a \in A, b \in B} \min\{\mathbf{w}'(a, b), \zeta\} \mathbf{1}_{\{ab \in E(G)\}} \geq \sum_{a \in A, b \in B} \mathbf{w}^*(a, b) \mathbf{1}_{\{\mathbf{w}^*(a, b) \leq \zeta\}}.$$

In view of (41) it's enough to show that under any conditioning on  $E(G) \setminus \nabla(A, B)$  for which  $X' < (1 + \vartheta) \Psi(H, n)/p$ ,

$$\Pr(Y > (1 + 2\vartheta) \Psi(H, n)) < \exp[-\Omega_\vartheta(n^2 p^{k-1} s)] \quad (= \exp[-\Omega_\vartheta(f(k, n, p))]).$$

But under any such conditioning (or any conditioning on  $E(G) \setminus \nabla(A, B)$ ), the r.v.'s  $\mathbf{1}_{\{ab \in E(G)\}}$  are independent; so, noting  $\mathbf{E}Y \leq pX' < (1 + \vartheta) \Psi(H, n)$  and using Lemma 3.3, we have

$$\Pr(Y > (1 + 2\vartheta) \Psi(H, n)) < \exp[-\Omega_\vartheta(\Psi(H, n)/\zeta)] = \exp[-\Omega_\vartheta(n^2 p^{k-1} s)].$$

■

## 9 Proof of Theorem 1.2

Recall here  $H = K_k$ . Set  $r = \lceil 2\mathbf{E}\xi_H \rceil = \lceil 2\binom{n}{k} p^{\binom{k}{2}} \rceil$ . Note that we only need to prove Theorem 1.2 for small  $p$ , for simplicity say  $p < n^{-2/(k-1)} \log n$ , since above this  $f(k, n, p) = n^2 p^{k-1} \log(1/p)$  and the theorem is given by the lower bound in (5). It will thus be enough to show

**Proposition 9.1.** *For  $n^{-2/(k-1)} \leq p < n^{-2/(k-1)} \log n$ ,*

$$\Pr(\xi_H = r) > \exp[-O(r)]$$

*Proof.* (This is an easy generalization of the argument for  $k = 3$  given in [6].) The number of sets  $S$  of  $r$  vertex-disjoint copies of  $H$  in  $K_n$  is

$$s := \frac{(n)_{rk}}{r!(k!)^r} > \left( \frac{n^k}{rk^k} \right)^r. \quad (42)$$

For such an  $S$ , let  $Q_S$  and  $R_S$  be the events  $\{G \text{ contains all members of } S\}$  and  $\{S \text{ is the set of } H\text{'s of } G\}$ . We have  $\Pr(Q_S) = p^{r\binom{k}{2}}$  and will show (for any  $S$ )

$$\Pr(R_S|Q_S) = \exp[-O(r)], \quad (43)$$

whence (using (42))

$$\begin{aligned} \Pr(\xi_H = r) &> \sum_S \Pr(Q_S) \Pr(R_S|Q_S) = sp^{r\binom{k}{2}} \exp[-O(r)] \\ &> \left( \frac{n^k p^{\binom{k}{2}}}{rk^k} \right)^r \exp[-O(r)] = \exp[-O(r)]. \end{aligned}$$

For the proof of (43), fix  $S$ ; let  $W$  be the union of the vertex sets of the copies of  $H$  in  $S$ ; and for  $i = 0, \dots, k$ , let  $T(i)$  be the set of  $H$ 's (in  $K_n$ ) having exactly  $i$  vertices outside  $W$ . We have

$$\begin{aligned} \Pr(R_S|Q_S) &\geq (1-p)^{|T(0)|} \prod_{i=1}^k \left( 1 - p^{\binom{i}{2} + (k-i)i} \right)^{|T(i)|} \\ &= \exp[-O(r)]. \end{aligned} \quad (44)$$

Here the first inequality is given by Harris' Inequality [8] (which for our purposes says that for a product probability measure  $\mu$  on  $\{0, 1\}^E$  (with  $E$  a finite set) and events  $\mathcal{A}_i \subseteq \{0, 1\}^E$  that are either all increasing or all decreasing,  $\mu(\cap \mathcal{A}_i) \geq \prod \mu(\mathcal{A}_i)$ ), and for the second we can use, say,  $|T(i)| < n^i (rk)^{k-i}$  for  $0 \leq i \leq k$ . (We omit the easy arithmetic, just noting that all factors but the last (that is,  $i = k$ ) in (44) are actually much larger than  $\exp[-O(r)]$ .)

■

## 10 Concluding Remarks

Of course the big question is, what is the true behavior of the probability (1) for general  $H$ ? We continue to use  $\xi_H$  for  $\xi_H^{n,p}$ , and here confine ourselves to  $\eta = 1$ ; that is, we're interested in  $\Pr(\xi_H > 2\mathbf{E}\xi_H)$ . As usual we don't ask for more than the order of magnitude of the exponent.

One can show, mainly following the argument of Section 9, that for any  $K \subseteq H$

$$\Pr(\xi_H \geq 2\mathbf{E}\xi_H) > \exp[-O_H(\Psi(K, n, p))] \quad (45)$$

(where, recall,  $\Psi(K, n, p) = n^{v_K} p^{e_K}$ ). As far as we can see, it could be that the truth in (1) is always given by the largest of the lower bounds in (45) and (5). For the latter we (finally) define

$$M_H(n, p) = \begin{cases} n^2 p^{\Delta_H} & \text{if } p \geq n^{-1/\Delta_H} \\ \min_{K \subseteq H} (\Psi(K, n, p))^{1/\alpha_K^*} & \text{if } n^{-1/m_H} \leq p \leq n^{-1/\Delta_H} \end{cases} \quad (46)$$

(where, as usual,  $\alpha^*$  is fractional independence number; see e.g. [11] or [4]). This is not quite the same as the quantity  $M_H^*(n, p)$  used in [11], but, as shown in their Theorem 1.5, the two agree up to a constant factor; so the difference is irrelevant here.

**Conjecture 10.1.** *For any  $H$  and  $p > n^{-1/m_H}$ ,*

$$\Pr(\xi_H \geq 2\mathbf{E}\xi_H) = \exp[-\Theta_H(\min\{\min_{K \subseteq H, e_K > 0} \Psi(K, n, p), M_H(n, p)t\})]. \quad (47)$$

(Recall  $t = \log(1/p)$ .) We remark without proof (it is not quite obvious as far as we know) that, for a given  $H$ , the set of  $p$  for which the (outer) minimum in (47) is  $M_H(n, p)t$  is the interval  $[p_K, 1]$ , where  $K$  is a smallest subgraph of  $H$  with  $m_K = m_H$  and  $p_K$  is the unique  $p$  for which  $\Psi(K, n, p) = M_H(n, p) \log(1/p)$ .

Conjecture 10.1 gives a different perspective on the observation from [11, Section 8.1] that  $H = K_2$  shows that the lower bound in (5) is not always tight. In this case  $M_H(n, p) = n^2 p$  for the full range of  $p$  above and, of course,  $\xi_H$  is just  $\text{Bin}(\binom{n}{2}, p)$ ; so the upper bound in (5) is the truth. But in fact (45) shows (with a little thought) that the lower bound in (5) is not tight for *any*  $H$  and sufficiently small  $p$  ( $> n^{-1/m_H}$ ), since for small enough  $p$  one of the terms  $\Psi(K, n, p)$  in (47) is  $o(M_H(n, p)t)$ . What's special about  $K_2$  is that it is the only (connected)  $H$  for which the best lower bound is *never* given by (5); that is, the minimum in (47) is never  $M_H(n, p)t$ .

It also seems interesting to estimate

$$\Pr(\xi_H \geq \gamma \mathbf{E}\xi_H) \quad (48)$$

when  $\gamma = \gamma(n) = \omega(1)$ . The present results essentially do this for  $H = K_k$  and “generic”  $p$ ; precisely, Theorem 2.3(b) implies (using a mild variant of

Proposition 2.1)

$$\Pr(\xi_H > 2\tau\Psi(H, n, p)) < \exp[-\Omega(f(k, n\tau^{1/k}, p))], \quad (49)$$

which, for  $p$  in the range where  $f(k, n\tau^{1/k}, p) = n^2\tau^{2/k}p^{k-1}t$ , is (up to the constant in the exponent) the probability of containing a clique of size  $np^{(k-1)/2}(2\tau)^{1/k}$  (provided this is not more than  $\binom{n}{k}$ ). Of course the trick that gets Theorem 2.3(b) from Theorem 2.3(a) is general, so results on Conjecture 10.1 give corresponding upper bounds for (48); but these bounds will not be tight in general, and at this writing we don't have a good guess as to the general truth in (48).

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## References

- [1] N. Alon, On the number of subgraphs of prescribed type of graphs with a given number of edges. *Israel J. Math.* **38** (1981), no. 1-2, 116-130.
- [2] N. Alon and J.H. Spencer, *The Probabilistic Method*, Wiley, New York, 2000.
- [3] J. Beck and W. Chen, *Irregularities of Distribution*, Cambridge Univ. Pr., Cambridge, 1987.
- [4] B. Bollobás, *Modern Graph Theory*, Springer, New York, 1998.
- [5] S. Chatterjee, The missing log in large deviations for subgraph counts (2010), <http://arxiv.org/abs/1003.3498>.
- [6] B. DeMarco and J. Kahn, Upper Tails for Triangles (2010), <http://arxiv.org/abs/1005.4471>.
- [7] E. Friedgut and J. Kahn, On the number of copies of one hypergraph in another *Israel J. Math.* **105** (1998), 251-256.
- [8] T.E. Harris, A lower bound on the critical probability in a certain percolation process, *Proc. Cam. Phil. Soc.* **56** (1960), 13-20.
- [9] S. Janson, Poisson approximation for large deviations. *Random Struct. Alg.* **1** (1990), 221-230.



- [10] S. Janson, T. Łuczak and A. Ruciński, *Random Graphs*, Wiley, New York, 2000.
- [11] S. Janson, K. Oleszkiewicz and A. Ruciński, Upper tails for subgraph counts in random graphs, *Israel J. Math.* **142** (2004), 61-92.
- [12] S. Janson and A. Ruciński, The infamous upper tail, *Random Structures & Algorithms* **20** (2002), 317-342.
- [13] S. Janson and A. Ruciński, The deletion method for upper tail estimates, *Combinatorica* **24** (2004), 615-640.
- [14] J. H. Kim and V. H. Vu, Concentration of multivariate polynomials and its applications, *Combinatorica* **20** (2000), 417-434.
- [15] J. H. Kim and V. H. Vu, Divide and conquer martingales and the number of triangles in a random graph, *Random Structures & Algorithms* **24** (2004), 166-174.
- [16] V.H Vu, On the concentration of multivariate polynomials with small expectation, *Random Structures & Algorithms*, **16(4)** (2000), 344-363.
- [17] V. H. Vu, A large deviation result on the number of small subgraphs of a random graph, *Combin. Probab. Comput.* **10** (2001), 79-94.

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