Simplicial complexes: spectrum, homology and random walks

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Abstract

Random walks on a graph reflect many of its topological and spectral properties, such as connectedness, bipartiteness and spectral gap magnitude. In the first part of this paper we define a stochastic process on simplicial complexes of arbitrary dimension, which reflects in an analogue way the existence of higher dimensional homology, and the magnitude of the high-dimensional spectral gap originating in the works of Eckmann and Garland.

The second part of the paper is devoted to infinite complexes. We present a generalization of Kesten's result on the spectrum of regular trees, and of the connection between return probabilities and spectral radius. We study the analogue of the Alon-Boppana theorem on spectral gaps, and exhibit a counterexample for its high-dimensional counterpart. We show, however, that under some assumptions the theorem does hold - for example, if the codimension-one skeletons of the complexes in question form a family of expanders.

Our study suggests natural generalizations of many concepts from graph theory, such as amenability, recurrence/transience, and bipartiteness. We present some observations regarding these ideas, and several open questions.

Contents

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1 Introduction

There are well known connections between dynamical, topological and spectral properties of graphs: The random walk on a graph reflects both its topological and algebraic connectivity, which are reflected by the 0^{th}-homology and the spectral gap, respectively.

In this paper we present a stochastic process which takes place on simplicial complexes of arbitrary dimension and generalizes these connections. In particular, for a finite d-dimensional complex, the asymptotic behavior of the process reflects the existence of a nontrivial $(d-1)$-homology, and its rate of convergence is dictated by the $(d-1)$-dimensional spectral gap[†]. The study of the process on finite complexes occupies the first half of the paper. In the second half we turn to infinite complexes, studying the high-dimensional analogues of classic properties and theorems regarding infinite graphs. Both in the finite and the infinite cases, one encounters new phenomena along the familiar ones, which reveal that graphs present only a degenerated case of a broader theory.

In order to give a flavor of the results without plunging into the most general definitions, we present in §1.1, without proofs, the special case of regular triangle complexes. A summary of the paper and its main results both for finite and infinite complexes is presented in §1.2.

This manuscript is part of an ongoing research seeking to understand the notion of *high-dimensional expanders*. Namely, the analogue of expander graphs in the realm of simplicial complexes of general dimension. Here we discuss the dynamical aspect of expansion, i.e. asymptotic behavior of random walks, and its relation to spectral expansion and homology. In a previous paper we studied expansion from combinatorial and isoperimetric points of view [PRT12]. The study of high-dimensional expanders is currently a very active one, comprising the notions of geometric and topological expansion in [Gro10, FGL$^+$10, MW11], $\mathbb{F}_2$-coboundary expansion in [LM06, MW09, DK10, GW12, SKM12], and Ramanujan complexes in [CSŻ03, Li04, LSV05]. We refer the reader to [Lub13] for a survey of the current state of the field.

[†] The high-dimensional spectral gap originates in the classic works of Eckmann [Eck44] and Garland [Gar73], and appears in Definition 2.5 here.

1.1 Example - regular triangle complexes

First, let us observe the $\frac{1}{2}$-lazy random walk on a k-regular graph $G = (V, E)$: the walker starts at a vertex v_0, and at each step remains in place with probability $\frac{1}{2}$ or moves to each of its k neighbors with probability $\frac{1}{2k}$. Let $\mathbf{p}_n^{v_0}(v)$ denote the probability of finding the walker at the vertex v at time n. The following observations are classic:

(1) If G is finite, then $\mathbf{p}_\infty^{v_0} = \lim_{n\to\infty} \mathbf{p}_n^{v_0}$ exists, and it is constant if and only if G is connected.

(2) Furthermore, the rate of convergence is given by

$$\left\|\mathbf{p}_n^{v_0} - \mathbf{const}\right\| = O\left(\left(1 - \frac{1}{2}\lambda(G)\right)^n\right),$$

where $\lambda(G)$ is the spectral gap of G (the definition follows below).

(3) When G is infinite and connected, the spectral gap is related to the return probability of the walk by

$$\lim_{n\to\infty} \sqrt[n]{\mathbf{p}_n^{v_0}(v_0)} = 1 - \frac{1}{2}\lambda(G). \tag{1.1}$$

We recall the basic definitions: the *Laplacian* of G, which we denote by Δ^+, is the operator which acts on $\mathbb{R}^V$ by

$$\left(\Delta^+ f\right)(v) = f(v) - \frac{1}{k}\sum_{w\sim v} f(w)$$

(where $\sim$ denotes neighboring vertices). If G is finite, then its *spectral gap* $\lambda(G)$ is defined as the minimal Laplacian eigenvalue on a function whose sum on V vanishes. When G is infinite, its spectral gap is defined to be $\lambda(G) = \min \operatorname{Spec}\left(\Delta^+\big|_{L^2(V)}\right)$ (for more on this see §3.1).

Moving one dimension higher, let $X = (V, E, T)$ be a *k-regular triangle complex*. This means that $E \subseteq \binom{V}{2}$ (i.e. E consists of subsets of V of size 2, the *edges* of X), $T \subseteq \binom{V}{3}$ (the *triangles*), every edge is contained in exactly k triangles in T, and for every triangle $\{u, v, w\}$ in T, the edges forming its boundary, $\{u, v\}$, $\{u, w\}$ and $\{v, w\}$, are in E.

For $\{v, w\} \in E$ we denote the directed edge $\overset{v}{\bullet}\!\longrightarrow\!\overset{w}{\bullet}$ by $[v, w]$, and the set of all directed edges by $E_\pm$ (so that $|E_\pm| = 2|E|$). For $e \in E_\pm$, $\overline{e}$ denotes the edge with the same vertices and opposite direction, i.e. $\overline{[v, w]} = [w, v]$.

The following definition is the basis of the process which we shall study:

Definition 1.1. Two directed edges $e, e' \in E_\pm$ are called *neighbors* (indicated by $e \sim e'$) if they have the same origin or the same terminus, and the triangle they form is in the complex. Namely, if $e = [v, w]$ and $e' = [v', w']$, then $e \sim e'$ means that either $v = v'$ and $\{v, w, w'\} \in T$, or $w = w'$ and $\{v, v', w'\} \in T$.

We study the following lazy random walk on $E_\pm$: The walk starts at some directed edge $e_0 \in E_\pm$. At every step, the walker stays put with probability $\frac{1}{2}$, or else move to a uniformly chosen neighbor. Figure 1.1 illustrates one step of the process, in two cases (the right one is non-regular, but the walk is defined in the same manner).

As in the random walk on a graph, this process induces a sequence of distributions on $E_\pm$,

$$\mathbf{p}_n(e) = \mathbf{p}_n^{e_0}(e),$$

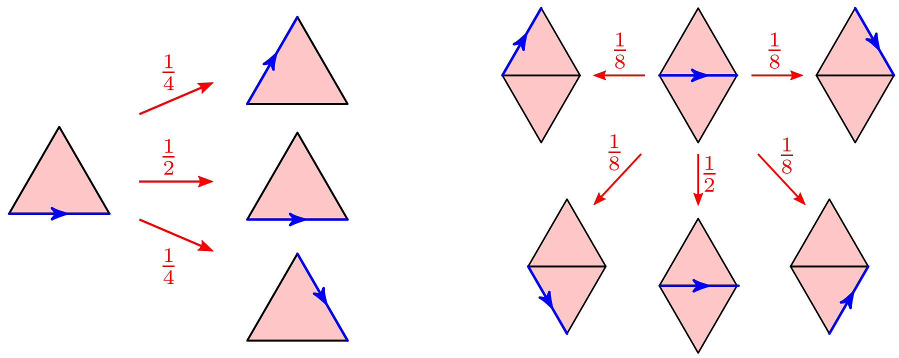

Figure 1.1: One step of the edge walk.

describing the probability of finding the walker at the directed edge e at time n (having started from e_0). However, studying the evolution of $\mathbf{p}_n$ amounts to studying the traditional random walk on the graph with vertices $E_\pm$ and edges defined by $\sim$. This will not take us very far, and in particular will not reveal the presence or absence of first homology in X. Instead, we study the evolution of what we call the "expectation process" on X:

$$\mathcal{E}_n(e) = \mathcal{E}_n^{e_0}(e) = \mathbf{p}_n^{e_0}(e) - \mathbf{p}_n^{e_0}(\overline{e}),$$

i.e. the probability of finding the walker at time n at e, minus the probability of finding it at the opposite edge $\overline{e}$ (for the reasons behind the name see Remark 2.4).

It is tempting to look at $\mathcal{E}_\infty^{e_0} = \lim_{n\to\infty} \mathcal{E}_n^{e_0}$ as is done in graphs, but a moment of reflection will show the reader that $\mathcal{E}_\infty^{e_0} \equiv 0$ for any finite triangle complex, and any starting point e_0. Namely, the probabilities of reaching e and $\overline{e}$ become arbitrarily close, for every e. While this might cause initial worry, it turns out that the rate of decay of $\mathcal{E}_n$ is always the same: for any finite triangle complex one has $\|\mathcal{E}_n^{e_0}\| = \Theta\left(\left(\frac{3}{4}\right)^n\right)$. It is therefore reasonable to turn our attention to the *normalized expectation process,*

$$\widetilde{\mathcal{E}}_n^{e_0}(e) = \left(\tfrac{4}{3}\right)^n \mathcal{E}_n^{e_0}(e) = \left(\tfrac{4}{3}\right)^n \left[\mathbf{p}_n^{e_0}(e) - \mathbf{p}_n^{e_0}(\overline{e})\right],$$

and observe its limit,

$$\widetilde{\mathcal{E}}_\infty^{e_0} = \lim_{n\to\infty} \widetilde{\mathcal{E}}_n^{e_0}.$$

For a finite triangle complex this limit always exists, and is nonzero. This is the object which reveals the first homology of the complex.

To see how, we need the following definition: We say that $f : E_\pm \to \mathbb{R}$ is *exact* if its sum along every closed path vanishes; namely, if

$$v_0 \sim v_1 \sim \ldots \sim v_n = v_0 \qquad \Longrightarrow \qquad \sum_{i=0}^{n-1} f([v_i, v_{i+1}]) = 0.$$

This is the one-dimensional analogue of constant functions (for reasons which will become clear in §2.2), and the following holds:

(1) For a finite X, $\widetilde{\mathcal{E}}_\infty^{e_0}$ is exact for every $e_0 \in E_\pm$ if and only if G has a trivial first homology.

(2) Furthermore, the rate of convergence is given by

$$\left\|\widetilde{\mathcal{E}}_n^{e_0} - \mathbf{exact}\right\| = O\left(\left(1 - \frac{1}{3}\lambda\left(X\right)\right)^n\right),$$

where $\lambda\left(X\right)$ is the spectral gap of X (see Definition 2.5).

(3) If X is infinite and every vertex in X is of infinite degree, then its spectral gap (which is defined in §3.2) is revealed by the "return expectation":

$$\sup_{e_0 \in E_\pm} \lim_{n\to\infty} \sqrt[n]{\widetilde{\mathcal{E}}_n^{e_0}\left(e_0\right)} = 1 - \frac{1}{3}\lambda\left(X\right).$$

What if one is interested not only in the existence of a first homology, but also in its dimension? The answer is manifested in the walk as well. In graphs the number of connected components is given by the dimension of $\operatorname{Span}\left\{\mathbf{p}_\infty^{v_0} \,|\, v_0 \in V\right\}$, and an analogue statement holds here (see Theorem 2.9).

Remark. If the non-lazy walk on a finite graph is observed, then apart from disconnectedness there is another obstruction for convergence to the uniform distribution: *bipartiteness.* We shall see that this is a special case of an obstruction in general dimension, which we call *disorientability* (see Definition 2.6). In our example we have avoided this problem by considering the lazy walk, both on graphs and on triangle complexes.

1.2 Summary of results

We give now a brief summary of the paper and its main results. The definitions of the terms which appear in this section are explained throughout the paper.

In §2.1 we define a p-lazy random walk on the oriented $(d-1)$-cells of a d-dimensional complex X, and associate with this walk the normalized expectation process $\widetilde{\mathcal{E}}_n^{\sigma_0}$. In §2.4 it is shown that the limit of this process $\widetilde{\mathcal{E}}_\infty^{\sigma_0} = \lim_n \widetilde{\mathcal{E}}_n^{\sigma_0}$ always exists and captures various properties of X, according to the amount of laziness p (this is an abridged version of Theorem 2.9):

Theorem. *When $\frac{d-1}{3d-1} < p < 1$, $\widetilde{\mathcal{E}}_\infty^{\sigma_0}$ is exact for every starting point σ_0 if and only if the $(d-1)$-homology of X is trivial. If furthermore $p \geq \frac{1}{2}$ then the rate of convergence is controlled by the spectral gap of X:*

$$\operatorname{dist}\left(\widetilde{\mathcal{E}}_n^{\sigma_0}, \widetilde{\mathcal{E}}_\infty^{\sigma_0}\right) = O\left(\left(1 - \frac{1-p}{p\left(d-1\right)+1}\lambda\left(X\right)\right)^n\right).$$

When $p = \frac{d-1}{3d-1}$, $\widetilde{\mathcal{E}}_\infty^{\sigma_0}$ is exact for every starting point σ_0 if and only if the $(d-1)$-homology of X is trivial, and in addition X has no disorientable $(d-1)$-components (see Definitions 2.2, 2.6).

Next, we move on to discuss infinite complexes, showing that they present new aspects which do not appear in infinite graphs. In §3.3 we define a family of simplicial complexes (which we call *arboreal complexes*) generalizing the notion of trees. In Theorem 3.3 we compute their spectra, generalizing Kesten's classic result on the spectrum of regular trees [Kes59]. The spectra of the regular arboreal complexes of high dimension and low regularity exhibit a surprising new phenomenon - an isolated eigenvalue.

Sections 3.4 and 3.5 are devoted to study the behavior of the spectrum with respect to a limit in the space of complexes. In particular we are interested in the high-dimensional analogue of the Alon-Bopanna theorem, which states that if a sequence of graphs G_n convergences to a graph G, then $\liminf_{n\to\infty} \lambda(G_n) \leq \lambda(G)$. We first show that in general this need not hold in higher dimension (Theorem 3.10). This uses the isolated eigenvalue of the 2-regular arboreal complex of dimension two, which is shown in Figure 3.1, as well as a study of the spectrum of balls in this complex (shown in Figure 3.2).

Even though the Alon-Bopanna theorem does not hold in general in high dimension, we show that under a variety of conditions it does hold (Theorem 3.11):

Theorem. *If* $X_n \overset{n\to\infty}{\longrightarrow} X$*, and one of the following holds:*

(1) The spectral gap of X *is nonzero,*

(2) zero is a non-isolated point in the spectrum of X*, or*

(3) the $(d-1)$*-skeletons of the complexes* X_n *form a family of* $(d-1)$*-expanders,*

then

$$\liminf_{n\to\infty} \lambda(X_n) \leq \lambda(X).$$

In §3.7 we show that the connection between the spectrum of a graph, and the return probability of the random walk on it (see e.g. [Kes59, Lemma 2.2]), generalizes to higher dimensions (Proposition 3.14).

The final section on infinite complexes addresses the high-dimensional analogues of the concepts of amenability, recurrence and transience, proving some properties of these (Proposition 3.17), and raising many open questions.

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2 Finite complexes

Throughout this section X is a finite d-dimensional simplicial complex on a finite vertex set V. This means that X is comprised of subsets of V, called *cells*, and the subset of every cell is also a cell. A cell of size $j+1$ (where $-1 \leq j$) is said to be of *dimension* j, and X^j denotes the set of j-cells - cells of dimension j. The dimension of X is the maximal dimension of a cell in it. The *degree* of a j-cell σ, denoted $\deg(\sigma)$, is the number of $(j+1)$-cells containing it. We shall assume that X is *uniform*, meaning that every cell is contained in some cell of dimension $d = \dim X$.

For $j \geq 1$, every j-cell $\sigma = \{\sigma_0, \ldots, \sigma_j\}$ has two possible orientations, corresponding to the possible orderings of its vertices, up to an even permutation (1-cells and the empty cell have only one orientation). We denote an oriented cell by square brackets, and a flip of orientation by an overbar. For example, one orientation of $\sigma = \{x,y,z\}$ is $[x,y,z]$, which is the same as $[y,z,x]$ and $[z,x,y]$. The other orientation of σ is $\overline{[x,y,z]} = [y,x,z] = [x,z,y] = [z,y,x]$. We denote by $X^j_\pm$ the set of oriented j-cells (so that $\left|X^j_\pm\right| = 2\left|X^j\right|$ for $j \geq 1$ and $X^j_\pm = X^j$ for $j = -1, 0$), and we shall occasionally denote by X^j_+ a choice of orientation for X^j, i.e. a subset of $X^j_\pm$ such that $X^j_\pm$ is the disjoint union of X^j_+ and $\left\{\overline{\sigma}\,\middle|\,\sigma \in X^j_+\right\}$.

The *faces* of a j-cell $\sigma = \{v_0, \ldots, v_j\}$ are the $(j-1)$-cells $\sigma \backslash v_0, \sigma \backslash v_1, \ldots, \sigma \backslash v_j$. An oriented j-cell $\sigma = [v_0, \ldots, v_j]$ $(2 \leq j \leq d)$ induces an orientation on its faces as follows: the face $\{v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_j\}$ is oriented as $(-1)^i [v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k]$, where $(-1)^i$ means taking the opposite orientation when $(-1)^i$ is -1.

Finally, we define the space of *j-forms* on X: these are functions on $X^j_\pm$ which are antisymmetric w.r.t. a flip of orientation:

$$\Omega^j = \Omega^j(X) = \left\{ f : X^j_\pm \to \mathbb{R} \,\middle|\, f(\overline{\sigma}) = -f(\sigma) \quad \forall \sigma \in X^{d-1}_\pm \right\}.$$

For $j = -1, 0$ there are no flips; Ω^0 is just the space of functions on the vertices, and $\Omega^{-1} = \{f : \{\varnothing\} \to \mathbb{R}\}$ can be identified in a natural way with $\mathbb{R}$. With every oriented j-cell $\sigma \in X^j$ we associate the Dirac j-form $\mathbb{1}_\sigma$ defined by

$$\mathbb{1}_\sigma(\sigma') = \begin{cases} 1 & \sigma' = \sigma \\ -1 & \sigma' = \overline{\sigma} \\ 0 & \text{otherwise} \end{cases}$$

(for $j = 0$ this is the standard Dirac function, and $\mathbb{1}_\varnothing$ is the constant 1).

2.1 The $(d-1)$-walk and expectation process

Let X be a uniform d-dimensional complex and $0 \leq p < 1$. The following process is the generalization of the edge walk from §1.1:

Definition 2.1. The p-lazy $(d-1)$-walk on a d-complex X is defined as follows:

- Two oriented $(d-1)$-cells $\sigma, \sigma' \in X^{d-1}_\pm$ are said to be *neighbors* (denoted $\sigma \sim \sigma'$) if there exists an oriented d-cell τ, such that both σ and $\overline{\sigma'}$ are faces of τ with the orientations induced by it (see Figure 1.1).
- The walk starts at an initial oriented $(d-1)$-cell σ_0, and at each step the walker stays in place with probability p, and with probability $(1-p)$ chooses uniformly one of its neighbors and moves to it.

Put differently, it is the Markov chain on $X^{d-1}_\pm$ with transition probabilities

$$\operatorname{Prob}(X_{n+1} = \sigma' | X_n = \sigma) = \begin{cases} p & \sigma' = \sigma \\ \frac{1-p}{d \deg(\sigma)} & \sigma' \sim \sigma \\ 0 & \text{otherwise} \end{cases},$$

(note that σ is contained in $\deg(\sigma)$ d-cells, and thus has $d \cdot \deg \sigma$ neighbors!)

We remark that neighboring cells can also be described in the following way: if $\sigma, \sigma' \in X^j_\pm$ and $j \geq 2$, then $\sigma \sim \sigma'$ iff the unoriented cell $\sigma \cup \sigma'$ is in X^d, and the intersection $\sigma \cap \sigma'$ inherits the same orientation from both σ and σ'. For $j = 1$, this can be interpreted as follows: two edges $e, e' \in X^1_\pm$ are neighbors if they bound a triangle in the complex, and the vertex at which they intersect "inherits the same orientation from both of them": it is either the origin of both e and e', or the terminus of both. Finally, for $j = 0$ Definition 2.1 gives the standard neighboring relation and p-lazy random walk on a graph.

Definition 2.2. We say that X is $(d-1)$-*connected* if the $(d-1)$-walk on it is irreducible, i.e., for every pair of oriented $(d-1)$-cells σ and σ' there exist a chain $\sigma = \sigma_0 \sim \sigma_1 \sim \ldots \sim \sigma_n = \sigma'$. Moreover, having such a chain defines an equivalence relation on the $(d-1)$-cells of X, whose classes we call the $(d-1)$-*components* of X.

Remark. Due to the assumption of uniformity, it is enough to observe unoriented cells - X is $(d-1)$-connected iff for every $\sigma, \sigma' \in X^{d-1}$ there exists a chain of unoriented $(d-1)$-cells $\sigma = \sigma_0, \sigma_1, \ldots, \sigma_n = \sigma'$ with $\sigma_i \cup \sigma_{i+1} \in X^d$ for all i. This is also equivalent to the assertion that for any $\tau, \tau' \in X^d$ there is a chain $\tau = \tau_0, \tau_1, \ldots, \tau_m = \tau'$ of d-cells with $\tau_i \cap \tau_{i-1} \in X^{d-1}$ for all i (this is sometimes referred to as a *chamber complex*). We note that it follows from uniformity that a $(d-1)$-connected complex is connected as a topological space. The other direction does not hold: the complex $\bowtie$ is not 1-connected, even though it is connected (and uniform).

Observing the $(d-1)$-walk on X, we denote by $\mathbf{p}_n^{\sigma_0}(\sigma)$ the probability that the random walk starting at σ_0 reaches σ at time n. We then have:

Definition 2.3. For $d \geq 2$, the *expectation process* on X starting at σ_0 is the sequence of $(d-1)$-forms $\{\mathcal{E}_n^{\sigma_0}\}_{n=0}^{\infty}$ defined by

$$\mathcal{E}_n^{\sigma_0}(\sigma) = \mathbf{p}_n^{\sigma_0}(\sigma) - \mathbf{p}_n^{\sigma_0}(\overline{\sigma}).$$

For $d=1$ (i.e. graphs) we simply define $\mathcal{E}_n^{v_0} = \mathbf{p}_n^{v_0}$.[†]

The *normalized expectation process* is defined to be

$$\widetilde{\mathcal{E}}_n^{\sigma_0}(\sigma) = \left(\frac{d}{p(d-1)+1}\right)^n \mathcal{E}_n^{\sigma_0}(\sigma) = \left(\frac{d}{p(d-1)+1}\right)^n \left[\mathbf{p}_n^{\sigma_0}(\sigma) - \mathbf{p}_n^{\sigma_0}(\overline{\sigma})\right],$$

where p is the laziness of the walk. In particular, for $d=1$ one has $\widetilde{\mathcal{E}}_n^{v_0} = \mathcal{E}_n^{v_0} = \mathbf{p}_n^{v_0}$ for all p.

The reason for this particular normalization is that for a lazy enough process (in particular for $p \geq \frac{1}{2}$) one has $\|\mathcal{E}_n^{\sigma_0}\| = \Theta\left(\left(\frac{p(d-1)+1}{d}\right)^n\right)$ (see (2.8)). Note that $\widetilde{\mathcal{E}}_0^{\sigma_0} = \mathcal{E}_0^{\sigma_0} = \mathbb{1}_{\sigma_0}$.

Remark 2.4. The name "expectation process" comes from the fact that for any $(d-1)$-form f

$$\mathbb{E}_n^{\sigma_0}[f] = \sum_{\sigma \in X_{\pm 1}^{d-1}} \mathbf{p}_n^{\sigma_0}(\sigma) f(\sigma) = \sum_{\sigma \in X^{d-1}} \mathcal{E}_n^{\sigma_0}(\sigma) f(\sigma)$$

where, as implied by the notation, $\mathcal{E}_n^{\sigma_0}(\sigma) f(\sigma)$ does not depend on the orientation of σ.

The evolution of the expectation process over time is given by $\mathcal{E}_{n+1}^{\sigma_0} = A\mathcal{E}_n^{\sigma_0}$, where $A = A(X,p)$ is the *transition operator* acting on Ω^{d-1} by

$$(Af)(\sigma) = pf(\sigma) + \frac{(1-p)}{d} \sum_{\sigma' \sim \sigma} \frac{f(\sigma')}{\deg(\sigma')} \qquad \left(f \in \Omega^{d-1}, \sigma \in X^{d-1}\right). \tag{2.1}$$

Note that the evolution of $\mathbf{p}_n^{\sigma_0}$ is given by the same A, acting on all functions from $X_{\pm}^d$ to $\mathbb{R}$, and not only on forms.

It is sometimes useful to observe the Markov operator $M = M(X,p)$ associated with this evolution, which is characterized by

$$\mathbb{E}_{n+1}^{\sigma_0}[f] = \mathbb{E}_n^{\sigma_0}[Mf],$$

[†] The results in the paper hold for graphs as well, using this definition of $\mathcal{E}_n^{v_0}$, but they are all familiar theorems. In some cases the proofs are slightly different, and we will not trouble to handle this special case.

and is given explicitly by

$$(Mf)(\sigma) = pf(\sigma) + \frac{1-p}{d\deg(\sigma)} \sum_{\sigma'\sim\sigma} f(\sigma') \qquad \left(f \in \Omega^{d-1}, \sigma \in X^{d-1}\right).$$

This is the transpose of A, w.r.t. to a natural choice of basis for $\Omega^{d-1}(X)$.

2.2 Simplicial complexes and Laplacians

For a cell σ and a vertex $v \notin \sigma$, we write $v \triangleleft \sigma$ if $v\sigma = \{v\} \cup \sigma$ is a cell in X. If σ is oriented, $\sigma = [\sigma_0, \ldots, \sigma_k]$, and $v \triangleleft \sigma$, then $v\sigma$ denotes the oriented cell $[v, \sigma_0, \ldots, \sigma_k]$.

For $0 \le k \le d$, the k^{th} *boundary operator* $\partial_k : \Omega^k \to \Omega^{k-1}$ is defined by

$$(\partial_k f)(\sigma) = \sum_{v \triangleleft \sigma} f(v\sigma).$$

In particular $\partial_0 : \Omega^0 \to \Omega^{-1}$ is defined by $(\partial_0 f)(\varnothing) = \sum_{v\in X^0} f(v)$.

The sequence $\left(\Omega^k, \partial_k\right)$ is the *simplicial chain complex* of X, meaning that $\partial_k\partial_{k+1} = 0$ for all k, giving rise to the *k-cycles* $Z_k = \ker\partial_k$, the *k-boundaries* $B_k = \operatorname{im}\partial_{k+1}$ and the (real) k^{th}-*homology* $H_k = {}^{Z_k}/_{B_k}$.

Given a weight function $w : X \to (0,\infty)$, Ω^k become inner product spaces (for $-1 \le k \le d$) with

$$\langle f, g\rangle = \sum_{\sigma\in X^k} w(\sigma) f(\sigma) g(\sigma) \qquad \forall f, g \in \Omega^k.$$

Note that the sum is over X^k and not $X^k_\pm$ - this is well defined since the value of $f(\sigma)g(\sigma)$ is independent of the orientation of σ.

Since X is finite the spaces Ω^k are finite dimensional, and there exist adjoint operators to the boundary operators ∂_k. These are the *co-boundary operators*, which are denoted by $\delta_k = \partial_k^* : \Omega^{k-1} \to \Omega^k$, and are given by

$$(\delta_k f)(\sigma) = (\partial_k^* f)(\sigma) = \frac{1}{w(\sigma)} \sum_{i=0}^{k} (-1)^i w(\sigma\backslash\sigma_i) f(\sigma\backslash\sigma_i) \quad 0 \le k \le d.$$

We will stick with the notation ∂_k^* until we get to infinite complexes, where sometimes δ_k is defined even though ∂_k is not. The *simplicial cochain complex* of X is (Ω_k, δ_k), and $Z^k = \ker\delta_{k+1}$, $B^k = \operatorname{im}\delta_k$, $H^k = {}^{Z^k}/_{B^k}$ are the *cocycles, coboundaries* and *cohomology*, respectively. Cocycles are also known as *closed forms*, and coboundaries as *exact forms*.

The following weight functions will be used throughout this paper†:

$$w(\sigma) = \begin{cases} \frac{1}{\deg\sigma} & \sigma \in X^{d-1} \\ 1 & \sigma \in X\backslash X^{d-1} \end{cases}.$$

Notice that for $\sigma \in X^{d-1}$

$$\frac{1}{w(\sigma)} = \deg(\sigma) = \left|\left\{\tau \in X^d \,\middle|\, \sigma \subset \tau\right\}\right| = \left|\{v \mid v \triangleleft \sigma\}\right| = \frac{1}{d}\left|\left\{\sigma' \in X^{d-1} \,\middle|\, \sigma' \sim \sigma\right\}\right|.$$

† Another natural weight function is the constant one. The obtained Laplacians are more convenient for isoperimetric analysis. For more details see [PRT12].

Due to our choice of weights, the inner product and coboundary operators are given by

$$\langle f,g\rangle=\begin{cases}\sum\limits_{\sigma\in X^k}f(\sigma)\,g(\sigma) & f,g\in\Omega^k,k\neq d-1\\ \sum\limits_{\sigma\in X^{d-1}}\frac{f(\sigma)g(\sigma)}{\deg\sigma} & f,g\in\Omega^{d-1}\end{cases}\tag{2.2}$$

$$(\delta_k f)(\sigma)=(\partial_k^* f)(\sigma)=\begin{cases}\sum\limits_{i=0}^{k}(-1)^i f(\sigma\backslash\sigma_i) & k\leq d-2\\ \deg(\sigma)\sum\limits_{i=0}^{d-1}(-1)^i f(\sigma\backslash\sigma_i) & k=d-1\\ \sum\limits_{i=0}^{d}\dfrac{(-1)^i f(\sigma\backslash\sigma_i)}{\deg(\sigma\backslash\sigma_i)} & k=d.\end{cases}\tag{2.3}$$

Finally, the *upper, lower, and full Laplacians* $\Delta^+,\Delta^-,\Delta:\Omega^{d-1}\to\Omega^{d-1}$ are defined by:

$$\begin{aligned}\Delta^+&=\partial_d\delta_d=\partial_d\partial_d^*\\ \Delta^-&=\delta_{d-1}\partial_{d-1}=\partial_{d-1}^*\partial_{d-1}\\ \Delta&=\Delta^++\Delta^-.\end{aligned}$$

An explicit calculation gives

$$\begin{aligned}\left(\Delta^+f\right)(\sigma)&=\sum_{v\lhd\sigma}(\partial_d^*f)(v\sigma)=\sum_{v\lhd\sigma}\sum_{i=0}^{d}\frac{(-1)^i f\left(v\sigma\backslash(v\sigma)_i\right)}{\deg\left(v\sigma\backslash(v\sigma)_i\right)}\\ &=f(\sigma)-\sum_{v\lhd\sigma}\sum_{i=0}^{d-1}\frac{(-1)^i f\left(v\left(\sigma\backslash\sigma_i\right)\right)}{\deg\left(v\left(\sigma\backslash\sigma_i\right)\right)}=f(\sigma)-\sum_{\sigma'\sim\sigma}\frac{f(\sigma')}{\deg(\sigma')}\end{aligned}\tag{2.4}$$

and

$$\left(\Delta^-f\right)(\sigma)=\deg\sigma\sum_{i=0}^{d-1}(-1)^i\sum_{v\lhd\sigma\backslash\sigma_i}f\left(v\sigma\backslash\sigma_i\right).\tag{2.5}$$

More generally, one can define the k-th upper, lower and full Laplacians $\Delta_k^+=\partial_{k+1}\delta_{k+1}$, $\Delta_k^-=\delta_k\partial_k$ and $\Delta_k=\Delta_k^++\Delta_k^-$. Apart from $k=d-1$, these will only make a brief appearance in §3.5. The kernel of Δ_k is the space of *harmonic k-forms*, denoted by $\mathcal{H}^k=\mathcal{H}^k(X)$.

The spaces defined so far are related by

$$\begin{aligned}Z_k&=\ker\partial_k=\ker\Delta_k^- & B_k&=\left(Z^k\right)^\perp=\operatorname{im}\partial_{k+1}=\operatorname{im}\Delta_k^+\\ Z^k&=\ker\delta_{k+1}=\ker\Delta_k^+ & B^k&=Z_k^\perp=\operatorname{im}\delta_k=\operatorname{im}\Delta_k^-\end{aligned}$$
$$\mathcal{H}^k=\ker\Delta_k=Z_k\cap Z^k=\left(B_k\oplus B^k\right)^\perp\cong H_k\cong H^k$$

The isomorphism between harmonic functions, homology and cohomology, which is sometimes called the discrete Hodge theorem, was first observed in [Eck44]. In a similar manner, there is a "discrete Hodge decomposition"

$$\Omega^k=\overbrace{B_k\oplus\underbrace{\mathcal{H}^k}_{}}^{Z_k}\!\!\underbrace{\oplus B^k}_{Z^k},\tag{2.6}$$

and all the Laplacians decompose with respect to it. All of these claims follow by linear algebra, using the fact that Ω^k is finite-dimensional (see [PRT12, §2] for the details). For infinite complexes the situation is more involved, and is addressed in §3.2.

2.3 The upper Laplacian spectrum

In this section we study the spectrum of the upper Laplacian Δ^+ of a finite complex X. First note that as $\Delta^+ = \partial_d \partial_d^*$, its spectrum is non-negative. Furthermore, zero is obtained precisely on closed forms, i.e. $\ker \Delta^+ = Z^{d-1}$. The space of closed forms always contains the exact forms, $B^{d-1} = \operatorname{im} \partial_{d-1}^*$, which are considered the *trivial zeros* in the spectrum of Δ^+. The existence of *nontrivial zeros* in the spectrum of Δ^+, i.e. closed forms which are not exact, indicates the existence of a nontrivial homology. This leads to the following definition:

Definition 2.5. The *spectral gap* of a finite d-dimensional complex X, denoted $\lambda(X)$, is

$$\lambda(X) = \min \operatorname{Spec}\left(\Delta^+|_{Z_{d-1}}\right) = \min \operatorname{Spec}\left(\Delta|_{Z_{d-1}}\right).$$

The *essential gap* of X, denoted $\widetilde{\lambda}(X)$, is

$$\widetilde{\lambda}(X) = \min \operatorname{Spec}\left(\Delta^+|_{B_{d-1}}\right) = \min \operatorname{Spec}\left(\Delta|_{B_{d-1}}\right)$$

(the transition from Δ to Δ^+ follows from the fact that Δ^- vanishes on Z_{d-1}.)

Since λ vanishes precisely when the $(d-1)$-homology of X is nontrivial, it should be thought of as giving a quantitative measure for the "triviality of the homology". For example, in graphs, having $\lambda(X)$ far away from zero is a measure of high-connectedness, or "high triviality of the 0^{th}-homology".

In contrast, $\widetilde{\lambda}$ never vanishes, as $B_{d-1} = \left(Z^{d-1}\right)^{\perp} = (\ker \Delta^+)^{\perp}$. If the $(d-1)$-homology is nontrivial then $\lambda = \widetilde{\lambda}$, so that $\widetilde{\lambda}$ is only of additional interest when the homology vanishes. In a disconnected graph $\widetilde{\lambda}$ controls the mixing rate as λ does for a connected graph, and we will see that the same happens in higher dimension (see (2.10)).

Until now we have studied one extremity of $\operatorname{Spec} \Delta^+$. The other side relates to the following definition:

Definition 2.6. A *disorientation* of a d-complex X is a choice of orientation X_+^d of its d-cells, so that whenever $\sigma, \sigma' \in X_+^d$ intersect in a $(d-1)$-cell they induce the same orientation on it. If X has a disorientation it is said to be *disorientable*.

Remarks.

(1) A disorientable 1-complex is precisely a bipartite graph, and thus disorientability should be thought of as a high-dimensional analogue of bipartiteness. Another natural analogue is "$(d+1)$-partiteness": having some partition $A_0, \ldots, A_d$ of V so that every d-cell contains one vertex from each A_i. A $(d+1)$-partite complex is easily seen to be disorientable, but the opposite does not necessarily hold for $d \geq 2$.

(2) Notice the similarity to the notion of orientability: a d-complex is orientable if there is a choice of orientations of its d-cells, so that cells intersecting in a codimension one cell induce *opposite* orientations on it. However, orientability implies that $(d-1)$-cells have degrees at most two, where as disorientability impose no such restrictions. Note that a complex can certainly be both orientable and disorientable (e.g. Figure 2.1(a)).

Proposition 2.7. *Let X be a finite complex of dimension d.*

(1) $\operatorname{Spec} \Delta^+(X)$ *is the disjoint union of* $\operatorname{Spec} \Delta^+(X_i)$ *where X_i are the $(d-1)$-components of X.*

(2) The spectrum of $\Delta^+ = \Delta^+(X)$ *is contained in* $[0, d+1]$.

(3) Zero is achieved on the closed $(d-1)$*-forms,* Z^{d-1}.

(4) If X *is* $(d-1)$*-connected, then* $d+1$ *is in the spectrum iff* X *is disorientable, and is achieved on the boundaries of disorientations (see (2.7)).*

Proof. (1) follows from the observation that Δ^+ decomposes w.r.t. the decomposition $\Omega^{d-1}(X) = \bigoplus_i \Omega^{d-1}(X_i)$. We have already seen *(3)*, and the fact that the spectrum is nonnegative. Now, assume that $\Delta^+ f = \lambda f$. Choose $\sigma \in X^{d-1}$ which maximize $\frac{|f(\sigma)|}{\deg(\sigma)}$. By (2.4),

$$\lambda f(\sigma) = \left(\Delta^+ f\right)(\sigma) = f(\sigma) - \sum_{\sigma' \sim \sigma} \frac{f(\sigma')}{\deg(\sigma')}$$

and therefore

$$|\lambda f(\sigma)| \leq |f(\sigma)| + \sum_{\sigma' \sim \sigma} \frac{|f(\sigma')|}{\deg(\sigma')} \leq (d+1)|f(\sigma)|,$$

(since $\#\{\sigma' \,|\, \sigma' \sim \sigma\} = d \deg \sigma$), hence $\lambda \leq d+1$ and *(2)* is obtained.

Next, assume that X is $(d-1)$-connected and that X_+^d is a disorientation. Define

$$F(\tau) = \begin{cases} 1 & \tau \in X_+^d \\ -1 & \tau \in X_\pm^d \backslash X_+^d \end{cases}, \tag{2.7}$$

and $f = \partial_d F$. For any $\sigma \in X_\pm^{d-1}$, there exists some vertex v with $v \lhd \sigma$ (since X is uniform). Furthermore, by the assumption on X_+^d, if $v \lhd \sigma$ and $v' \lhd \sigma$ for vertices v, v' then $v\sigma \in X_+^d$ if and only if $v'\sigma \in X_+^d$, and thus

$$f(\sigma) = (\partial_d F)(\sigma) = \sum_{v \lhd \sigma} F(v\sigma) = \deg(\sigma) F(\tau)$$

where τ is any d-cell containing σ. If σ and σ' are neighboring $(d-1)$-faces in $X_\pm^{d-1}$, then by definition, for some $\tau \in X_\pm^d$, σ is a face of τ and σ' is a face of $\overline{\tau}$, so that

$$\frac{f(\sigma)}{\deg \sigma} + \frac{f(\sigma')}{\deg \sigma'} = F(\tau) + F(\overline{\tau}) = 0,$$

and consequently for any $\sigma \in X_\pm^{d-1}$

$$\left(\Delta^+ f\right)(\sigma) = f(\sigma) - \sum_{\sigma' \sim \sigma} \frac{f(\sigma')}{\deg(\sigma')} = f(\sigma) - \sum_{\sigma' \sim \sigma} \frac{-f(\sigma)}{\deg(\sigma)} = (d+1) f(\sigma),$$

so that f is a Δ^+-eigenform with eigenvalue $d+1$.

In the other direction, assume that X is $(d-1)$-connected and that $\Delta^+ f = (d+1) f$ for some $f \in \Omega^{d-1}(X) \setminus \{0\}$. Fix some $\widetilde{\sigma} \in X_\pm^{d-1}$ which maximize $\frac{|f(\sigma)|}{\deg \sigma}$, normalize f so that $\frac{|f(\widetilde{\sigma})|}{\deg \widetilde{\sigma}} = 1$, and define

$$F = \frac{\partial_d^* f}{d+1}, \quad X_+^d = \left\{\tau \in X_\pm^d \,\middle|\, F(\tau) > 0\right\}.$$

We have $f = \frac{\Delta^+ f}{d+1} = \frac{\partial_d \partial_d^* f}{d+1} = \partial_d F$ by assumption, and we proceed to show that X_+^d is a disorientation with F the corresponding form as in (2.7). By the definition of Δ^+

$$\deg \widetilde{\sigma} = |f(\widetilde{\sigma})| = \frac{1}{d} \left| \sum_{\sigma \sim \widetilde{\sigma}} \frac{f(\sigma)}{\deg(\sigma)} \right| \leq \frac{1}{d} \sum_{\sigma \sim \widetilde{\sigma}} \frac{|f(\sigma)|}{\deg(\sigma)} \leq \frac{1}{d} \sum_{\sigma \sim \widetilde{\sigma}} 1 = \deg \widetilde{\sigma},$$

so that $\frac{|f(\sigma)|}{\deg \sigma} = 1$ for every $\sigma \sim \widetilde{\sigma}$. Continuing in this manner, $(d-1)$-connectedness implies that $\frac{|f(\sigma)|}{\deg \sigma} \equiv 1$ on all $X_\pm^d$. Using again the definition of Δ^+, for any σ in $X_\pm^d$

$$\frac{f(\sigma)}{\deg \sigma} = -\frac{1}{\deg \sigma \cdot d} \sum_{\sigma' \sim \sigma} \frac{f(\sigma')}{\deg(\sigma')}.$$

Since the r.h.s is an average over terms whose absolute value is that of the l.h.s this gives $\frac{f(\sigma')}{\deg \sigma'} = -\frac{f(\sigma)}{\deg \sigma}$ whenever $\sigma \sim \sigma'$, hence

$$F(\tau) = \frac{1}{d+1} \sum_{i=0}^{d} \frac{(-1)^i f(\tau \backslash \tau_i)}{\deg(\tau \backslash \tau_i)} = \frac{f(\tau \backslash \tau_0)}{\deg(\tau \backslash \tau_0)}$$

is always of absolute value one. Furthermore, if $\tau, \tau' \in X_\pm^d$ intersect in a face σ and induce opposite orientations on it, then $\tau = v\sigma$ and $\tau' = \overline{v'\sigma}$ for some vertices v, v', hence

$$F(\tau) = F(v\sigma) = \frac{f(\sigma)}{\deg \sigma} = F(v'\sigma) = -F\left(\overline{v'\sigma}\right) = -F(\tau')$$

which concludes the proof. □

2.4 Walk and spectrum

The $(d-1)$-walk defined in §2.1 is related to the Laplacians from §2.2 as follows:

Proposition 2.8. *Observe the p-lazy $(d-1)$-walk on X starting at $\sigma_0 \in X_\pm^{d-1}$. Then*

1. *The transition operator $A = A(X,p)$ is given by*

$$A = \frac{p(d-1)+1}{d} \cdot I - \frac{1-p}{d} \cdot \Delta^+,$$

so that

$$\mathcal{E}_n^{\sigma_0} = A^n \mathcal{E}_0^{\sigma_0} = \left(\frac{p(d-1)+1}{d} \cdot I - \frac{1-p}{d} \cdot \Delta^+ \right)^n \mathcal{E}_0^{\sigma_0}.$$

2. *The spectrum of A is contained in $\left[2p-1, \frac{p(d-1)+1}{d}\right]$, with $2p-1$ achieved by disorientations, and $\frac{p(d-1)+1}{d}$ by closed forms (Z^{d-1}).*

3. *The expectation process satisfies*

$$\frac{1}{\sqrt{K_{d-2}K_{d-1}}} \left(\frac{p(d-1)+1}{d} \right)^n \leq \| \mathcal{E}_n^{\sigma_0} \| \leq \max \left(|2p-1|, \frac{p(d-1)+1}{d} \right)^n \tag{2.8}$$

where K_j is the maximal degree of a j-cell in X.

Proof. *(1)* follows trivially from (2.1) and (2.4), and Proposition 2.7 then implies *(2)*. The upper bound in *(3)* follows from *(2)* by $\mathcal{E}_n^{\sigma_0} = A^n \mathcal{E}_0^{\sigma_0}$ and $\|\mathcal{E}_0^{\sigma_0}\| = \|\mathbb{1}_{\sigma_0}\| = \frac{1}{\sqrt{\deg \sigma_0}} \leq 1$. For the lower bound, let v be a vertex in σ_0, and $\sigma_0, \ldots, \sigma_k$ the $(d-1)$-cells containing $\sigma_0 \backslash v$. Define $f = \partial_d^* \mathbb{1}_{\sigma_0 \backslash v} = \sum_{i=0}^k \deg \sigma_i \cdot \mathbb{1}_{\sigma_i}$, so that $f \in Z^{d-1}$ and $\|f\|^2 = \sum_{i=0}^k \deg \sigma_i \leq K_{d-2} K_{d-1}$. Since Δ^+ decomposes w.r.t. the orthogonal sum $\Omega^{d-1} = Z^{d-1} \oplus B_{d-1}$ so does $A = \frac{p(d-1)+1}{d} \cdot I - \frac{1-p}{d} \cdot \Delta^+$, hence by *(2)*

$$\begin{aligned} \|\mathcal{E}_n^{\sigma_0}\| = \|A^n \mathbb{1}_{\sigma_0}\| &\geq \left(\tfrac{p(d-1)+1}{d}\right)^n \|\mathbb{P}_{Z^{d-1}}\left(\mathbb{1}_{\sigma_0}\right)\| \geq \left(\tfrac{p(d-1)+1}{d}\right)^n \left|\left\langle \tfrac{f}{\|f\|}, \mathbb{1}_{\sigma_0} \right\rangle\right| \\ &= \left(\tfrac{p(d-1)+1}{d}\right)^n \frac{|f(\sigma_0)|}{\|f\| \deg \sigma_0} \geq \frac{1}{\sqrt{K_{d-2} K_{d-1}}} \left(\tfrac{p(d-1)+1}{d}\right)^n. \end{aligned}$$

□

This proposition leads to the connection between the asymptotic behavior of the $(d-1)$-walk and the homology and spectrum of the complex:

Theorem 2.9. *Let $\widetilde{\mathcal{E}}_n^\sigma$ be the normalized expectation process associated with the p-lazy $(d-1)$-walk on X starting from σ (see Definitions 2.1, 2.3). Then $\widetilde{\mathcal{E}}_\infty^\sigma = \lim_{n\to\infty} \widetilde{\mathcal{E}}_n^\sigma$ exists and satisfies the following:*

(1) If $\frac{d-1}{3d-1} < p < 1$, then $\widetilde{\mathcal{E}}_\infty^\sigma$ is exact for every starting point σ if and only if $H_{d-1}(X) = 0$.[†] If furthermore $p \geq \frac{1}{2}$ then

$$\operatorname{dist}\left(\widetilde{\mathcal{E}}_n^\sigma, B^{d-1}\right) = O\left(\left(1 - \frac{1-p}{p(d-1)+1}\lambda(X)\right)^n\right). \tag{2.9}$$

(2) More generally, the dimension of $H_{d-1}(X)$ equals the dimension of $\operatorname{Span}\left\{\mathbb{P}_{Z_{d-1}}\left(\widetilde{\mathcal{E}}_\infty^\sigma\right) \middle| \sigma \in X^{d-1}\right\}$.

(3) If $p = \frac{d-1}{3d-1}$ then $\widetilde{\mathcal{E}}_\infty^\sigma$ is exact for all σ if and only if X has a trivial $(d-1)$-homology and no disorientable $(d-1)$-components.

(4) More generally, if $\frac{d-1}{3d-1} < p < 1$ then $\widetilde{\mathcal{E}}_\infty^\sigma$ is closed, and likewise for $p = \frac{d-1}{3d-1}$, unless X has a disorientable $(d-1)$-component. If $p \geq \frac{1}{2}$ then

$$\operatorname{dist}\left(\widetilde{\mathcal{E}}_n^\sigma, Z^{d-1}\right) = O\left(\left(1 - \frac{1-p}{p(d-1)+1}\widetilde{\lambda}(X)\right)^n\right). \tag{2.10}$$

Proof.

Case (*i*) – $\boldsymbol{\frac{d-1}{3d-1} < p < 1}$: We have $|2p-1| < \frac{p(d-1)+1}{d}$, so that $\|A\| = \max \operatorname{Spec} A = \frac{p(d-1)+1}{d}$. Thus,

$$\operatorname{Spec} A\big|_{B_{d-1}} \subseteq \left[2p-1, \frac{p(d-1)+1}{d}\right) \subseteq (-\|A\|, \|A\|).$$

[†] Note that the first value of p for which the homology can be studied via the walk in every dimension is $p = \frac{1}{3}$.

Since A decomposes w.r.t. $\Omega^{d-1} = B_{d-1} \oplus Z^{d-1}$, and $A\big|_{Z^{d-1}} = \|A\| \cdot I\big|_{Z^{d-1}}$, this means that $\left(\frac{A}{\|A\|}\right)^n$ converges to the orthogonal projection $\mathbb{P}_{Z^{d-1}}$. Now $\widetilde{\mathcal{E}}_n^\sigma = \left(\frac{d}{p(d-1)+1}\right)^n \mathcal{E}_n^\sigma = \left(\frac{A}{\|A\|}\right)^n \mathcal{E}_0^\sigma$, which shows that

$$\widetilde{\mathcal{E}}_\infty^\sigma = \mathbb{P}_{Z^{d-1}}\left(\widetilde{\mathcal{E}}_0^\sigma\right) = \mathbb{P}_{Z^{d-1}}\left(\mathcal{E}_0^\sigma\right) = \mathbb{P}_{Z^{d-1}}\left(\mathbb{1}_\sigma\right). \tag{2.11}$$

In particular $\widetilde{\mathcal{E}}_\infty^\sigma$ is closed, so that if the homology of X is trivial then it is exact. On the other hand, assume that $\widetilde{\mathcal{E}}_\infty^\sigma$ is exact for all σ: then

$$\widetilde{\mathcal{E}}_\infty^\sigma = \mathbb{P}_{Z^{d-1}}\left(\mathcal{E}_0^\sigma\right) = \mathbb{P}_{Z^{d-1}}\left(\mathbb{1}_\sigma\right) = \mathbb{P}_{B^{d-1}}\left(\mathbb{1}_\sigma\right) + \mathbb{P}_{\mathcal{H}^{d-1}}\left(\mathbb{1}_\sigma\right)$$

so that $\mathbb{P}_{\mathcal{H}^{d-1}}\left(\mathbb{1}_\sigma\right) = 0$ by (2.6). As $\{\mathbb{1}_\sigma\}$ span Ω^{d-1}, this shows that $H_{d-1} \cong \mathcal{H}^{d-1} = 0$. To further understand the dimension of the homology, observe that

$$\operatorname{Span}\left\{\mathbb{P}_{Z_{d-1}}\left(\widetilde{\mathcal{E}}_\infty^\sigma\right) \,\middle|\, \sigma \in X^{d-1}\right\} = \mathcal{H}^{d-1}(X),$$

which follows from

$$\mathbb{P}_{Z_{d-1}}\left(\widetilde{\mathcal{E}}_\infty^\sigma\right) = \mathbb{P}_{Z_{d-1}}\left(\mathbb{P}_{Z^{d-1}}\left(\mathbb{1}_\sigma\right)\right) = \mathbb{P}_{\mathcal{H}^{d-1}}\left(\mathbb{1}_\sigma\right).$$

If $p \geq \frac{1}{2}$ then we know not only that $\|A\| = \max \operatorname{Spec} A$ but also that $\left\|A\big|_{Z_{d-1}}\right\| = \max \operatorname{Spec}\left(A\big|_{Z_{d-1}}\right)$, which allows us to say more. In this case A is positive semidefinite, so that (2.10) follows by

$$\begin{aligned}\left\|\left(\frac{d}{p(d-1)+1}A\right)^n - \mathbb{P}_{Z^{d-1}}\right\| &= \left\|\left(\frac{d}{p(d-1)+1}A\big|_{B_{d-1}}\right)^n\right\| \\ &= \left\|\left(I - \frac{1-p}{p(d-1)+1}\cdot\Delta^+\right)^n\big|_{B_{d-1}}\right\| = \left(1 - \frac{1-p}{p(d-1)+1}\widetilde{\lambda}(X)\right)^n,\end{aligned}$$

which gives (2.9) as well when the homology is trivial.

Case *(ii)* – $p = \frac{d-1}{3d-1}$: Now, $|2p-1| = \frac{p(d-1)+1}{d} = \|A\|$. If X has no disorientable $(d-1)$-components then again $\operatorname{Spec} A\big|_{B_{d-1}} \subseteq (-\|A\|, \|A\|)$, which gives (2.11), and everything is as before. On the other hand, let us assume that $\widetilde{\mathcal{E}}_\infty^\sigma$ is closed for all σ. Denoting by Ω_λ^{d-1} the λ-eigenspace of A, now $\left(\frac{d}{p(d-1)+1}A\right)^{2n}$ converges to $\mathbb{P}_{Z^{d-1}} + \mathbb{P}_{\Omega_{2p-1}^{d-1}}$ (Δ^+ is diagonalizable and consequently so is A). Since $\widetilde{\mathcal{E}}_\infty^\sigma$ is closed this shows that $\mathbb{P}_{\Omega_{2p-1}^{d-1}}\left(\mathbb{1}_\sigma\right) = 0$, and consequently that $\Omega_{2p-1}^{d-1} = 0$, i.e. X has no disorientable $(d-1)$-components. □

Remarks.

(1) The study of complexes via $(d-1)$-walk gives a conceptual reason to the fact that the high-dimensional case is harder than that of graphs: while graphs are studied by the evolution of probabilities, analogue properties of high-dimensional complexes are reflected in the expectation process. As this is given by the difference of two probability vectors, it is much harder to analyze. Several examples of this appear in the open questions in §4.

(2) In order to study the connectedness of a graph it is enough to observe the walk starting at one vertex. If $\mathbf{p}_\infty^{v_0}$ is not exact (i.e. not proportional to the degree function) for even one v_0, then the graph is necessarily disconnected. In general dimension, however, this is not enough: there are complexes (even $(d-1)$-connected ones!) with nontrivial $(d-1)$-homology, such that $\widetilde{\mathcal{E}}_\infty^\sigma$ is exact for a carefully chosen σ.

(3) If one starts the process with a general initial distribution $\mathbf{p}_0$ instead of the Dirac probability $\mathbb{1}_\sigma$, then Theorem 2.9 holds for the corresponding expectation process (i.e. $\mathcal{E}_0(\sigma) = \mathbf{p}_0(\sigma) - \mathbf{p}_0(\overline{\sigma})$, $\mathcal{E}_{n+1} = A\mathcal{E}_n$). Furthermore, in these settings a disorientable component corresponds to a distribution for which $\widetilde{\mathcal{E}}_n$ is 2-periodic for $p = \frac{d-1}{3d-1}$ (see Figure 2.1(a)); a nontrivial homology corresponds to a distribution which induces a stationery non-exact $\widetilde{\mathcal{E}}_n$ for $p \geq \frac{d-1}{3d-1}$ (see Figure 2.1(b)).

(a) (b)

Figure 2.1: Two distributions on the edges of 2-complexes (the orientations drawn have uniform probability, and their inverses probability zero). (a) is a distribution for which $\widetilde{\mathcal{E}}_n = \left(\frac{5}{3}\right)^n \mathcal{E}_n$ is 2-periodic under the $\frac{1}{5}$-lazy walk; (b) is a distribution for which $\widetilde{\mathcal{E}}_n$ is stable and non exact (under the p-lazy walk, $p > \frac{1}{5}$).

3 Infinite complexes

3.1 Infinite graphs

We move to the case of infinite complexes, starting with infinite graphs. Recall that for a finite graph $G = (V, E)$, we observed $\Delta^+ = \Delta^+(G)$, and defined

$$\lambda(G) = \min \operatorname{Spec} \Delta^+\big|_{(B^0)^\perp} = \min \operatorname{Spec} \Delta^+\big|_{Z_0}.$$

In contrast, when G is an infinite graph (i.e. $|V| = \infty$) one usually restrict his attention to $L^2(V)$ and define

$$\lambda(G) = \min \operatorname{Spec} \Delta^+\big|_{L^2(V)}. \tag{3.1}$$

Here there is no restriction to Z_0, nor to $\left(B^0\right)^\perp$. These two spaces, which coincide in the finite dimensional case, since

$$Z_0 = \ker \partial_0 = (\operatorname{im} \partial_0^*)^\perp = \left(B^0\right)^\perp, \tag{3.2}$$

fail to do so in the infinite settings. First, Z_0 is not even defined, as $(\partial_0 f)(\varnothing) = \sum_{v\in V} f(v)$ has no meaning for general $f \in L^2(V)$. One can observe $B^0 = \operatorname{im} \delta_0$, taking (2.3) as the definition of δ_0 (as ∂_0 is not defined). With this definition, B^0 consists of the scalar multiples of the degree function. Since these are never in $L^2(V)$ (assuming as always that there are no isolated vertices), we have $B^0 = 0$ and $\left(B^0\right)^\perp = L^2(V)$, justifying (3.1). Another thing which fails here is the chain complex property $\partial_0\partial_1 = 0$: there may exist $f \in \Omega^1(G)$ such that $\partial_0\partial_1 f$ is defined and nonzero. For example, take $V = \mathbb{Z}$, $E = \{\{i, i+1\} \mid i \in \mathbb{Z}\}$, and $f([i, i+1]) = \begin{cases} 0 & i < 0 \\ 1 & 0 \leq i \end{cases}$. Here $\partial_1 f = \mathbb{1}_0$, and thus $(\partial_0\partial_1 f)(\varnothing) = 1$. If G is transient, e.g. the $\mathbb{Z}^3$ graph, or a k-regular tree with $k \geq 3$, there are even such f in L^2 - see §3.8.

3.2 Infinite complexes of general dimension

For a complex X of dimension d, and $-1 \leq k \leq d$, we denote

$$\Omega^k_{L^2} = \Omega^k_{L^2}(X) = \left\{ f \in \Omega^k(X) \,\middle|\, \|f\|^2 < \infty \right\} \subseteq \Omega^k(X),$$

where we recall that

$$\|f\|^2 = \sum_{\sigma \in X^k} w(\sigma) f(\sigma)^2 = \begin{cases} \sum_{\sigma \in X^k} f(\sigma)^2 & k \neq d-1 \\ \sum_{\sigma \in X^k} \frac{f(\sigma)^2}{\deg \sigma} & k = d-1 \end{cases}.$$

Whenever referring to infinite complexes, the domain of all operators (i.e. $\partial, \delta, \Delta^+, \Delta^-, \Delta$) is assumed to be $\Omega^k_{L^2}$, unless explicitly stated that we are interested in Ω^k.

Let us examine these operators. We shall always assume that the $(d-1)$-cells in X have globally bounded degrees, which ensures that the boundary and coboundary operators $\partial_d : \Omega^d \to \Omega^{d-1}$, $\delta_d : \Omega^{d-1} \to \Omega^d$ are defined, bounded, and adjoint to one another, so that $\Delta^+ = \partial_d \delta_d = \partial_d \partial_d^*$ is bounded and self-adjoint. We do not assume that the degrees in other dimensions are bounded, as this would rule out infinite graphs, for example. This means that in general δ_k does not take $\Omega^{k-1}_{L^2}$ into $\Omega^k_{L^2}$ but only to Ω^k, and ∂_k need not even be defined. In particular, one cannot always define Δ^-.

The cochain property $\delta_k \delta_{k-1} = 0$ always holds, whereas in general $\partial_{k-1}\partial_k(f)$ can be defined and nonzero for some $f \in \Omega^k_{L^2}$. If the degrees of $(k-1)$-cells are bounded, then δ_k and ∂_k are bounded and $\delta_k = \partial_k^*$. Thus, if the degrees of $(k-1)$-cells and $(k-2)$-cells are globally bounded one has $\partial_{k-1}\partial_k = (\delta_k \delta_{k-1})^* = 0^* = 0$ as well.

In contrast with infinite graphs, an infinite d-complex may have $(d-2)$-cells of finite degree, so that the image of δ_{d-1} may contain L^2-coboundaries. For example, if v is a vertex of finite degree in an infinite triangle complex, then the "star" $\delta_1 \mathbb{1}_v$ is an L^2-coboundary. We denote by B^{d-1} the L^2-coboundaries, i.e. $B^{d-1} = \operatorname{im} \delta_{d-1} \cap \Omega^{d-1}_{L^2}$. In order to avoid trivial zeros in the spectrum of Δ^+, we define $Z_{d-1} = \left(B^{d-1}\right)^\perp$ (the orthogonal complement in $\Omega^{d-1}_{L^2}$), and

$$\lambda(X) = \min \operatorname{Spec} \Delta^+\big|_{Z_{d-1}}.$$

We stress out that Z_{d-1} is not necessarily the kernel of ∂_{d-1} (which is not even defined in general). If the $(d-2)$-degrees are globally bounded then ∂_{d-1} is defined and dual to δ_{d-1}, and this gives inclusion in one direction:

$$Z_{d-1} = \left(B^{d-1}\right)^\perp = \left(\operatorname{im} \delta_{d-1}\right)^\perp \subseteq \ker \partial_{d-1}. \tag{3.3}$$

For finite complexes there is an equality here (as in (3.2)) due to dimension considerations.

In infinite graphs we had $B^0 = 0$, $Z_0 = \Omega^0_{L^2} = L^2(V)$ and $\lambda = \min \operatorname{Spec} \Delta^+\big|_{L^2(V)}$. The following lemma shows that this happens whenever all $(d-2)$-cells are of infinite degree:

Lemma 3.1. *If X is a d-complex whose $(d-2)$-cells are all of infinite degree, then $B^{d-1} = 0$ and thus $\lambda(X) = \min \operatorname{Spec} \Delta^+$.*

Proof. Let $f \in \Omega^{d-2}$ be such that $\delta_{d-1} f \in \Omega^{d-1}_{L^2} \setminus \{0\}$. Choose $\tau \in X^{d-2}_\pm$ for which $f(\tau) > 0$, and let $\{\sigma_i\}_{i=1}^\infty$ be a sequence of $(d-1)$-cells containing τ. Since $\sum_{i=1}^\infty \left(\delta_{d-1} f\right)^2(\sigma_i) \leq \|\delta_{d-1} f\|^2 <$

∞, for infinitely many i we have $|(\delta_{d-1}f)(\sigma_i)| \leq \frac{f(\tau)}{2}$. Since τ contributes $f(\tau)$ to $(\delta_{d-1}f)(\sigma_i)$, one of the other faces of σ_i must be of absolute value at least $\frac{f(\tau)}{2(d-1)}$. Since these faces are all different $(d-2)$-cells (if $\sigma_i \cap \sigma_j$ contains τ and another $(d-2)$-cell, then $\sigma_i = \sigma_j$), we have $\|f\| = \infty$. $\square$

3.3 Example - arboreal complexes

Definition 3.2. We say that a d-complex is *arboreal* if it is $(d-1)$-connected, and has no simple d-loops. That is, there are no non-backtracking closed chains of d-cells, $\sigma_0, \sigma_1, \ldots, \sigma_n = \sigma_0$ s.t. $\dim(\sigma_i \cap \sigma_{i+1}) = d-1$ (σ_i and σ_{i+1} are adjacent) and $\sigma_i \neq \sigma_{i+2}$ (the chain is non-backtracking).

For $d = 1$, these are simply trees. As in trees, there is a unique k-regular arboreal d-complex for every $k \in \mathbb{N}$, and we denote it by T_k^d. It can be constructed as follows: start with a d-cell, and attach to each of its faces $k-1$ new d-cells. Continue by induction, adding to each face of a d-cell in the boundary $k-1$ new d-cells at every step. For example, the 2-regular arboreal triangle complex T_2^2 can be thought of as an ideal triangulation of the hyperbolic plane, depicted in Figure 3.1.

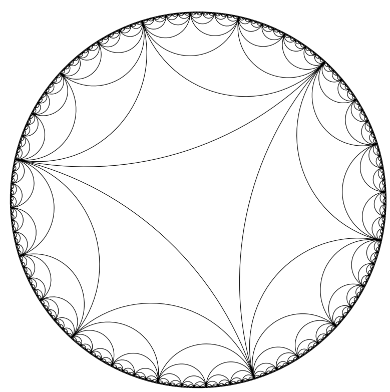


Figure 3.1: The 2-regular arboreal triangle complex T_2^2.

The following theorem describes the spectrum of regular arboreal complexes:

Theorem 3.3. *The spectrum of the non-lazy transition operator on* the *k-regular arboreal d-complex* is

$$\operatorname{Spec} A\left(T_k^d, 0\right) = \begin{cases} \left[\frac{1-d-2\sqrt{d(k-1)}}{kd}, \frac{1-d+2\sqrt{d(k-1)}}{kd}\right] \cup \left\{\frac{1}{d}\right\} & 2 \leq k \leq d \\ \left[\frac{1-d-2\sqrt{d(k-1)}}{kd}, \frac{1-d+2\sqrt{d(k-1)}}{kd}\right] & d < k. \end{cases} \tag{3.4}$$

Remarks.

(1) For $d = 1$ this gives the spectrum of the k-regular tree, which is a famous result of Kesten [Kes59]:
$$\operatorname{Spec} A\left(T_k^1, 0\right) = \left[-\frac{2\sqrt{k-1}}{k}, \frac{2\sqrt{k-1}}{k}\right].$$

(2) Since for $2 \leq k \leq d$ the value $\frac{1}{d}$ is an isolated value of the spectrum of T_k^d, it follows that it is in fact an eigenvalue. This is a major difference from the case of graphs, where the value $\frac{1}{d} = 1$ cannot be an eigenvalue for infinite graphs. This phenomena will play a crucial role in the counterexample for the Alon-Boppana theorem in general dimension (see §3.5-3.6).

(3) Another phenomena which does not occur in the case of graphs, is that in the region $2 \leq k \leq d$ the spectrum expands as k becomes larger. The spectrum is maximal (as a set) for $k = d+1$, where $\operatorname{Spec} A\left(T_{d+1}^d, 0\right) = \left[-\frac{3d-1}{d(d+1)}, \frac{1}{d}\right]$, merging with the isolated eigenvalue which appear for smaller k.

(4) The spectra of the Laplacian $\Delta^+ = \Delta^+\left(T_k^d\right)$, and of the p-lazy transition operator $A_p = A\left(T_k^d, p\right)$, are obtained from (3.4) using $\Delta^+ = I - d \cdot A$ and $A_p = p \cdot I + (1-p) \cdot A$.

In order to prove Theorem 3.3 we will need the following lemma, for the idea of which we are indebted to Jonathan Breuer:

Lemma 3.4. *Let X be any set, and $L^2(X)$ the Hilbert space of complex functions of finite L^2-norm on X (with respect to the counting measure). Let A be a bounded self adjoint operator on $L^2(X)$, and $a < b \in \mathbb{R}$, such that the following hold:*

(1) For every $x \in X$ and $a \leq \lambda \leq b$, there exists $\psi_x^\lambda \in L^2(X)$ such that $(A - \lambda I)\psi_x^\lambda = \mathbb{1}_x$.

(2) The integral $\int_a^b c(\lambda)^2 \, d\lambda$ is finite, where $c(\lambda) = \sup_{x \in X} \left\|\psi_x^\lambda\right\|$.

Then $(a,b) \cap \operatorname{Spec}(A) = \varnothing$.

Proof. We show that $\mathbb{P}_{[a,b]}$, the spectral projection of A on the interval $[a,b]$, is zero, and the conclusion $(a,b) \cap \operatorname{Spec}(A) = \varnothing$ follows by the spectral theorem. Stone's formula states that

$$(s)\lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_a^b \left[(A - \lambda - i\varepsilon)^{-1} - (A - \lambda + i\varepsilon)^{-1}\right] d\lambda = \mathbb{P}_{(a,b)} + \frac{1}{2}\mathbb{P}_{\{a,b\}}$$

where $\mathbb{P}_{(a,b)}$ and $\mathbb{P}_{\{a,b\}}$ the spectral projections of A on (a,b) and $\{a,b\}$ respectively, and $(s)\lim$ denotes a limit in the strong sense. Denoting $\mathbb{P} = \mathbb{P}_{(a,b)} + \frac{1}{2}\mathbb{P}_{\{a,b\}}$, this gives for every $x \in X$

$$\lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_a^b \left\langle \left[(A - \lambda - i\varepsilon)^{-1} - (A - \lambda + i\varepsilon)^{-1}\right] \mathbb{1}_x, \mathbb{1}_x \right\rangle d\lambda = \langle \mathbb{P}\mathbb{1}_x, \mathbb{1}_x \rangle$$

Evaluating the right hand side we get

$$\begin{aligned}
\langle \mathbb{P}\mathbb{1}_x, \mathbb{1}_x \rangle &= \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_a^b \left\langle \left[(A - \lambda - i\varepsilon)^{-1} - (A - \lambda + i\varepsilon)^{-1}\right] \mathbb{1}_x, \mathbb{1}_x \right\rangle d\lambda \\
&= \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_a^b \left\langle \left[(A - \lambda - i\varepsilon)^{-1} - (A - \lambda + i\varepsilon)^{-1}\right] (A - \lambda)\psi_x^\lambda, (A - \lambda)\psi_x^\lambda \right\rangle d\lambda \\
&= \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_a^b \left\langle (A - \lambda + i\varepsilon)^{-1} \left[A - \lambda + i\varepsilon - A + \lambda + i\varepsilon\right] (A - \lambda - i\varepsilon)^{-1} (A - \lambda)^2 \psi_x^\lambda, \psi_x^\lambda \right\rangle d\lambda \\
&= \lim_{\varepsilon \downarrow 0} \frac{\varepsilon}{\pi} \int_a^b \left\langle \left((A - \lambda)^2 + \varepsilon^2\right)^{-1} (A - \lambda)^2 \psi_x^\lambda, \psi_x^\lambda \right\rangle d\lambda \\
&\leq \lim_{\varepsilon \downarrow 0} \frac{\varepsilon}{\pi} \int_a^b \left\| \left((A - \lambda)^2 + \varepsilon^2\right)^{-1} (A - \lambda)^2 \right\| c(\lambda)^2 \, d\lambda.
\end{aligned}$$

Defining $f_{\varepsilon,\lambda}(t) = \frac{(t-\lambda)^2}{(t-\lambda)^2+\varepsilon^2}$, we have $|f_{\varepsilon,\lambda}(t)| \le 1$ for every $t,\lambda \in \mathbb{R}$ and $\varepsilon > 0$, and thus $\|f_{\varepsilon,\lambda}(A)\| \le 1$. Therefore, using *(2)*, the last limit above is zero. Consequently, for any $x, y \in X$

$$|\langle \mathbb{P}\mathbb{1}_x, \mathbb{1}_y\rangle| = |\langle \mathbb{P}\mathbb{1}_x, \mathbb{P}\mathbb{1}_y\rangle| \le \langle \mathbb{P}\mathbb{1}_x, \mathbb{P}\mathbb{1}_x\rangle^{\frac{1}{2}} \cdot \langle \mathbb{P}\mathbb{1}_y, \mathbb{P}\mathbb{1}_y\rangle^{\frac{1}{2}} = 0.$$

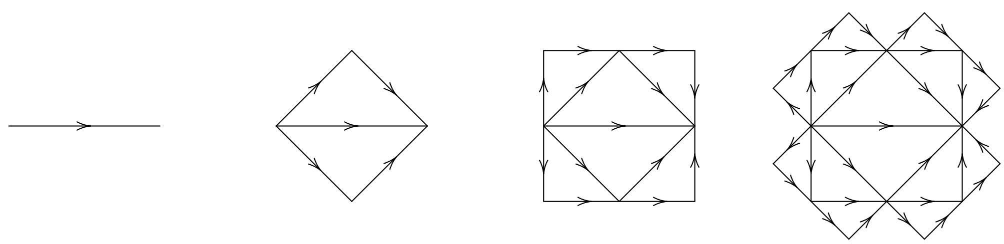
It follows that for general $f \in L^2(X)$

$$\langle \mathbb{P}f, f\rangle = \left\langle \mathbb{P}\left(\sum_{x\in X} f(x)\mathbb{1}_x\right), \sum_{y\in X} f(y)\mathbb{1}_y \right\rangle = \sum_{x,y\in X} f(v)f(w)\langle \mathbb{P}\mathbb{1}_x, \mathbb{1}_y\rangle = 0,$$

which implies that $\mathbb{P} = 0$, hence also $\mathbb{P}_{(a,b)}$ and $\mathbb{P}_{\{a,b\}}$, and therefore also $\mathbb{P}_{[a,b]}$. □

Proof of Theorem 3.3. Let $X = T_k^d$, and $\Lambda_\pm = \frac{1-d\pm 2\sqrt{d(k-1)}}{kd}$. The proof is separated into two parts. First we prove that every $\Lambda_- \le \lambda \le \Lambda_+$, and also $\lambda = \frac{1}{d}$ when $k \le d$, is in the spectrum, by exhibiting an appropriate eigenform or an approximate one. In the second part we use Lemma 3.4 to prove that there are no other points in the spectrum.

Define an orientation X_+^{d-1} as follows: choose an arbitrary $(d-1)$-cell $\sigma_0 \in X_\pm^{d-1}$ and place it in X_+^{d-1}. Then add to X_+^{d-1} all the $k \cdot d$ neighbors of σ_0. Next, for every neighbor τ of the recently added $k \cdot d$ cells, add τ to X_+^{d-1}, unless τ or $\overline{\tau}$ is already there. Continue expanding in this manner, adding at each stage the neighbors of the last "layer" which are further away from the starting cell σ_0. Apart from orientation, this process gives X_+^{d-1} a layer structure: $\{\sigma_0\}$ is the 0^{th} layer, its neighbors the 1^{st} layer, and so on. We denote by $S_n(X,\sigma_0)$ the n^{th} layer, and also write $B_n(X,\sigma_0) = \bigcup_{k\le n} S_k(X,\sigma_0)$ for the "n^{th} ball" around σ_0. Figure 3.2 demonstrates this for the first four layers of T_2^2.



$B_0\left(T_2^2,\sigma_0\right)$ $B_1\left(T_2^2,\sigma_0\right)$ $B_2\left(T_2^2,\sigma_0\right)$ $B_3\left(T_2^2,\sigma_0\right)$

Figure 3.2: The orientation at the zeroth, first, second, and third layers of $X = T_2^2$.

We shall study X_+^{d-1}-spherical forms, i.e. forms in $\Omega^{d-1}(X)$ which are constant on each layer of X_+^{d-1}. For such a form f we will make some abuse of notation and write $f(n)$ for the value of f on the cells in the n^{th} layer of X_+^{d-1}. As in regular trees, if one allows forms which are not in L^2, then for every $\lambda \in \mathbb{R}$ there is a unique (up to a constant) X_+^{d-1}-spherical eigenform f with eigenvalue λ. This form is given explicitly by

$$f(n) = \left(\frac{\lambda - \alpha_-}{\alpha_+ - \alpha_-}\right)\cdot \alpha_+^n + \left(\frac{\alpha_+ - \lambda}{\alpha_+ - \alpha_-}\right)\cdot \alpha_-^n,$$

where

$$\alpha_\pm = \frac{d-1+dk\lambda \pm \sqrt{(d-1+dk\lambda)^2 - 4d(k-1)}}{2d(k-1)}, \tag{3.5}$$

except for the case $\alpha_+ = \alpha_-$, which happens when $\lambda \in \{\Lambda_-, \Lambda_+\}$. In this case f is given by

$$f(n) = (1-n)\left(\frac{(d-1)+dk\lambda}{2d\,(k-1)}\right)^n + \lambda n\left(\frac{(d-1)+dk\lambda}{2d\,(k-1)}\right)^{n-1},$$

but this will not concern us as the spectrum is closed, and it is therefore enough to show that (Λ_-, Λ_+) is contained in it to deduce this for $[\Lambda_-, \Lambda_+]$.

The term inside the root in (3.5) is negative for $\Lambda_- < \lambda < \Lambda_+$, hence in this case $|\alpha_+| = |\alpha_-| = \frac{1}{\sqrt{d(k-1)}}$. We claim the following: for any $\Lambda_- < \lambda < \Lambda_+$ there exist $0 < c_1 < c_2 < \infty$ (which depend on λ) such that

(1) For all $n \in \mathbb{N}$,

$$|f\,(n)| \le c_2 \left(\frac{1}{\sqrt{d(k-1)}}\right)^n. \tag{3.6}$$

(2) For infinitely many $n \in \mathbb{N}$,

$$c_1 \left(\frac{1}{\sqrt{d(k-1)}}\right)^n \le |f\,(n)|. \tag{3.7}$$

Indeed, (1) follows from $|f\,(n)| \le \left[\left|\frac{\lambda-\alpha_-}{\alpha_+-\alpha_-}\right| + \left|\frac{\alpha_+-\lambda}{\alpha_+-\alpha_-}\right|\right]\left(\frac{1}{\sqrt{d(k-1)}}\right)^n$ (as $\alpha_+ \neq \alpha_-$ for $\Lambda_- < \lambda < \Lambda_+$). Next, denote $\gamma = \frac{\lambda-\alpha_-}{\alpha_+-\alpha_-}$ and observe that

$$|f\,(n)|\,[d(k-1)]^{\frac{n}{2}} = \left|\gamma\alpha_+^n + \overline{\gamma}\alpha_-^n\right|[d(k-1)]^{\frac{n}{2}} = 2\Re\left(\gamma\left(\alpha_+\sqrt{d\,(k-1)}\right)^n\right).$$

If (2) fails, then $|f\,(n)|\,[d(k-1)]^{\frac{n}{2}} \overset{n\to\infty}{\longrightarrow} 0$. Since $\left|\alpha_+\sqrt{d\,(k-1)}\right| = 1$, this means that $n\arg\alpha_+ \overset{n\to\infty}{\longrightarrow} \frac{\pi}{2} - \arg\gamma \pmod{\pi}$, hence $\alpha_+ \in \mathbb{R}$, which is false.

Even though f is not in $\Omega_{L^2}^{d-1}(X)$ it induces a natural sequence of approximate eigenforms:

$$f_n(\sigma) = \begin{cases} f\,(k) & \sigma \in S_k\,(X,\sigma_0) \text{ and } k \le n \\ -f\,(k) & \overline{\sigma} \in S_k\,(X,\sigma_0) \text{ and } k \le n \\ 0 & \text{otherwise.} \end{cases}$$

To see this, observe that $(A_0 - \lambda)\,f = 0$, and that f_n coincides with f on $B_n\,(X,\sigma_0)$ for $k \le n$ and vanishes on $\left(T_d^k\right)^{d-1} \backslash B_n\,(X,\sigma_0)$. It follows that $(A_0-\lambda)\,f_n$ is supported on $S_n\,(X,\sigma_0) \cup S_{n+1}\,(X,\sigma_0)$, and by $|S_n\,(X,\sigma_0)| = d^n k\,(k-1)^{n-1}$, the definition of A_0, and (3.6)

$$\begin{aligned}
\frac{\|(A_0-\lambda)\,f_n\|^2}{\|f_n\|^2} &= \frac{|S_n\,(X,\sigma_0)|\left(\frac{1}{dk}\left[f\,(n-1)-(d-1)\,f\,(n)\right]-\lambda f(n)\right)^2 + |S_{n+1}\,(X,\sigma_0)|\left(\frac{1}{dk}f\,(n)\right)^2}{\sum_{j=0}^n |S_j\,(X,\sigma_0)|\,f^2(j)} \\
&= \frac{d^n k\,(k-1)^{\,n-1}\cdot\left(-\frac{k-1}{k}f\,(n+1)\right)^2 + d^{n+1}k\,(k-1)^{\,n}\left(\frac{1}{dk}f\,(n)\right)^2}{f^2\,(0) + \sum_{j=1}^n d^j k\,(k-1)^{j-1}\,f^2\,(j)} \\
&= \frac{d^n k^{-1}\,(k-1)^{\,n+1} f\,(n+1)^2 + d^{n-1}k^{-1}\,(k-1)^{\,n} f\,(n)^2}{f^2\,(0) + \sum_{j=1}^n d^j k\,(k-1)^{j-1}\,f^2\,(j)} \\
&\le \frac{\frac{2c_2^2}{dk}}{f^2\,(0) + \frac{k}{k-1}\sum_{j=1}^n \left[d\,(k-1)\right]^j f\,(j)^2}.
\end{aligned}$$

By (3.7), the denominator becomes arbitrarily large as n grows, and therefore $\frac{\|(A_0-\lambda)f_n\|^2}{\|f_n\|^2} \to 0$ and $\lambda \in \operatorname{Spec} A_0$.

Turning to the isolated eigenvalues in (3.4), one can easily check that $f(n) = \frac{1}{d^n}$ is an eigenform with eigenvalue $\frac{1}{d}$, and for $2 \le k \le d$ it is in L^2. This concludes the first part of the proof.

Next assume that $\lambda \in \left(-1, \frac{1}{d}\right) \setminus [\Lambda_-, \Lambda_+]$. We show that in this case Lemma 3.4 can be applied. Let σ_0 and X_+^{d-1} be as before, including the layer structure. Define the following X_+^{d-1}-spherical forms:

$$\psi_{\sigma_0}^{\lambda}(n) = \frac{\alpha_+^n}{\alpha_+ - \lambda}, \qquad \varphi_{\sigma_0}^{\lambda}(n) = \frac{\alpha_-^n}{\alpha_- - \lambda}. \tag{3.8}$$

The functions $\psi_{\sigma_0}^{\lambda}$ is defined whenever $\lambda \neq \alpha_+$, which holds unless $\lambda = \frac{1}{d}$ and $k \le d+1$ (see (3.5)). Similarly, $\varphi_{\sigma_0}^{\lambda}$ is defined unless $\lambda = -1$, or $\lambda = \frac{1}{d}$ and $k \le d+1$. It is straightforward to verify that

$$(A_0 - \lambda I)\,\psi_{\sigma_0}^{\lambda} = (A_0 - \lambda I)\,\varphi_{\sigma_0}^{\lambda} = \mathbb{1}_{\sigma_0}$$

whenever the functions are defined. For every X_+^{d-1}-spherical form f one has

$$\|f\|^2 = \sum_{n=0}^{\infty} |S_n(X,\sigma_0)|\, f^2(n) = f^2(0) + \frac{k}{k-1}\sum_{n=1}^{\infty} [(k-1)\,d]^n f^2(n)\,. \tag{3.9}$$

One can verify that $0 < d\,(k-1)\,\alpha_+^2 < 1$ holds for all $\lambda < \Lambda_-$, and thus by (3.8) and (3.9) $\left\|\psi_{\sigma_0}^{\lambda}\right\|$ is finite. In fact, $\left\|\psi_{\sigma_0}^{\lambda}\right\|$ is continuous w.r.t. λ in this region, so that it is bounded on every interval $[a,b] \subseteq (-\infty, \Lambda_-)$. Furthermore, for any $\sigma \in X^{d-1}$ there is an isometry of T_k^d which takes σ_0 to σ, and thus $\psi_{\sigma_0}^{\lambda}$ to a form ψ_{σ}^{λ} with the same L^2-norm as $\psi_{\sigma_0}^{\lambda}$, and which satisfies $(A_0 - \lambda I)\,\psi_{\sigma}^{\lambda} = \mathbb{1}_{\sigma}$. We can now invoke Lemma 3.4 for $[a,b] \subseteq (-\infty, \Lambda_-)$, using $\psi_{\sigma_0}^{\lambda}$ and its translations by isometries, and obtain that $(a,b) \cap \operatorname{Spec} A_0 = \varnothing$. Thus, $\operatorname{Spec} A_0$ does not intersect $(-\infty, \Lambda_-)$.

Similarly, $0 < d\,(k-1)\,\alpha_-^2 < 1$ holds for all $\lambda > \Lambda_+$, so that the same argumentation for $\varphi_{\sigma_0}^{\lambda}$ shows that $\operatorname{Spec} A_0$ does not intersect (Λ_+, ∞), provided that $d+1 < k$. When $k \le d+1$ we know that $\frac{1}{d} \in \operatorname{Spec} A_0$, and we need to show that $\operatorname{Spec} A_0$ does not intersect $(\Lambda_+, \infty) \setminus \left\{\frac{1}{d}\right\}$. This is done in the same manner, observing intervals $[a,b] \subseteq \left(\Lambda_+, \frac{1}{d}\right)$ and $[a,b] \subseteq \left(\frac{1}{d}, \infty\right)$ separately. $\square$

3.4 Continuity of the spectral measure

In this section we generalize parts of Grigorchuk and Żuk's work on graphs [GŻ99] to general simplicial complexes. We assume throughout the section that all d-complexes referred to are $(d-1)$-connected, and that families and sequences of d-complexes we encounter have globally bounded $(d-1)$-degrees.

For a uniform d-complex X we define the distance between two $(d-1)$-cells to be the minimal length of a $(d-1)$-chain connecting them:

$$\operatorname{dist}(\sigma, \sigma') = \min\left\{ n \;\middle|\; \begin{array}{c} \exists \sigma_0, \sigma_1, \ldots, \sigma_n = \sigma_0 \in X^{d-1} \text{ s.t.} \\ \sigma_i \cup \sigma_{i+1} \in X^d \quad \forall i \end{array} \right\}.$$

We denote by $B_n(X,\sigma)$ the ball of radius n around σ in X, which is the maximal subcomplex of X all of whose $(d-1)$-cells are of distance at most n from $\sigma^{\dagger}$. A *marked d-complex* (X,σ) is

† this is similar to $B_n(X,\sigma)$ defined in the proof of theorem 3.3, but there $B_n(X,\sigma)$ referred only to the $(d-1)$-cells, and here to the entire subcomplex a d-complex with a choice of a $(d-1)$-cell σ. On the space of marked d-complexes with finite $(d-1)$-degrees one can define a metric by

$$\operatorname{dist}\left(\left(X_1,\sigma_1\right),\left(X_2,\sigma_2\right)\right)=\inf\left\{\frac{1}{n+1}\ :\ B_n\left(X_1,\sigma_1\right)\text{ is isometric to }B_n\left(X_2,\sigma_2\right)\right\}$$

Remarks.

(1) A limit (X,σ) of a sequence (X_n,σ_n) in this space is unique up to isometry.

(2) For every $K\in\mathbb{N}$, the subspace of d-complexes with $(d-1)$-degrees bounded by K is compact. This is due to the fact that there is only a finite number of possibilities for a ball of radius n, so that every sequence has a converging subsequence by a diagonal argument (see [GŻ99] for details).

Our next goal is to study the relation of this metric to the spectra of complexes. We use some standard spectral theoretical results which we summarize as follows: Let X be a countable set with a weighted counting measure w, i.e., $\int_X f=\sum_{x\in X}w(x)f(x)$, and A a self-adjoint operator on $L^2(X,w)$. For every $x\in X$, the spectral measure μ_x is the unique regular Borel measure on $\mathbb{C}$ such that for every polynomial $P(t)\in\mathbb{C}[t]$

$$\langle P(A)\mathbb{1}_x,\mathbb{1}_x\rangle=\int_{\mathbb{C}}P(z)d\mu_x(z),$$

where $\mathbb{1}_x$ is the Dirac function of the point x. For $x,y\in X$ the spectral measure $\mu_{x,y}$ is the unique regular Borel measure on $\mathbb{C}$ such that for every polynomial P

$$\langle P(A)\mathbb{1}_x,\mathbb{1}_y\rangle=\int_{\mathbb{C}}P(z)d\mu_{x,y}(z).$$

The spectrum of A can be inferred from the spectral measures by

$$\operatorname{Spec}A=\bigcup_{x,y\in X}\operatorname{supp}\mu_{x,y}=\bigcup_{x\in X}\operatorname{supp}\mu_x. \tag{3.10}$$

We wish to apply this mechanism to the analysis of the action of $A=A(X,0)=\frac{I-\Delta^+}{d}$ on $\Omega_{L^2}^{d-1}$ (with the inner product as in (2.2)), and this is justified by observing that for any choice of orientation X_+^{d-1} of X^{d-1}, we have an isometry $\Omega_{L^2}^{d-1}\cong L^2\left(X_+^{d-1},w\right)$, where $w(\sigma)=\frac{1}{\deg\sigma}$. For any $\sigma\in X^{d-1}$ we denote by μ_σ^X the spectral measure of A w.r.t. $\mathbb{1}_\sigma$. Similarly, $\mu_{\sigma,\sigma'}^X$ denotes the spectral measure of A w.r.t. $\mathbb{1}_\sigma$ and $\mathbb{1}_{\sigma'}$.

Lemma 3.5. *If* $\lim\limits_{n\to\infty}(X_n,\sigma_n)=(X,\sigma)$ *then* $\mu_{\sigma_n}^{X_n}$ *converges weakly to* μ_σ^X.

Proof. For regular finite Borel measures on $\mathbb{R}$ with compact support, weak convergence follows from convergence of the moments of the measures (see e.g. [Fel66, §VIII.1]). For $m\geq 0$ the m^{th} moment of μ_σ^X, denoted $\left(\mu_\sigma^X\right)^{(m)}$, is given by

$$\left(\mu_\sigma^X\right)^{(m)}=\int_{\mathbb{C}}z^m d\mu_\sigma^X(z)=\langle A^m\mathbb{1}_\sigma,\mathbb{1}_\sigma\rangle=\langle A^m\mathcal{E}_0^\sigma,\mathbb{1}_\sigma\rangle=\langle\mathcal{E}_m^\sigma,\mathbb{1}_\sigma\rangle=\frac{\mathcal{E}_m^\sigma(\sigma)}{\deg\sigma},$$

where $\mathcal{E}_m^\sigma$ is the 0-lazy expectation process starting at σ, at time m. However,

$$\mathcal{E}_m^\sigma(\sigma)=\mathbf{p}_m^\sigma(\sigma)-\mathbf{p}_m^\sigma(\overline{\sigma})$$

is determined by the structure of the complex in the ball $B_m(X,\sigma)$. For large enough n, $B_m(X,\sigma)$ is isometric to $B_m(X_n,\sigma_n)$, which implies that $\left(\mu_{\sigma_n}^{X_n}\right)^{(m)}=\left(\mu_\sigma^X\right)^{(m)}$. ☐

3.5 Alon-Boppana type theorems

Definition 3.6. A sequence of d-complexes X_n, whose $(d-1)$-degrees are bounded globally, is said to converge to the complex X (written $X_n \overset{n\to\infty}{\longrightarrow} X$) if (X_n, σ_n) converges to (X, σ) for some choice of $\sigma_n \in X_n^{d-1}$ and $\sigma \in X^{d-1}$.

In particular, if X is an infinite d-complex with bounded $(d-1)$-degrees, and $\{X_n\}$ is a sequence of quotients of X whose injectivity radii approach infinity, then $X_n \overset{n\to\infty}{\longrightarrow} X$.

The following is (one form of) the classic Alon-Boppana theorem:

Theorem 3.7 (Alon-Boppana)**.** *Let G_n be a sequence of graphs whose degrees are globally bounded, and G a graph s.t. $G_n \overset{n\to\infty}{\longrightarrow} G$. Then*

$$\liminf_{n\to\infty} \lambda(G_n) \leq \lambda(G).$$

In the literature one encounters many variations on this formulation: some refer only to quotients of G, some only to regular graphs, and some are quantitative (e.g. [Nil91]).

In this section we study the analogue question for complexes of general dimension. We start with the following:

Theorem 3.8. *If $X_n \overset{n\to\infty}{\longrightarrow} X$ and $\lambda \in \operatorname{Spec} A(X,0)$, there exist $\lambda_n \in \operatorname{Spec} A(X_n,0)$ with $\lim\limits_{n\to\infty} \lambda_n = \lambda$. The same holds for the corresponding Laplacians Δ_X^+ and $\Delta_{X_n}^+$.*

Proof. Let σ_n, σ be as in Definition 3.6. Since $\lambda \in \operatorname{Spec} A(X,0)$, for every $\varepsilon > 0$ there exists $\sigma' \in X^{d-1}$ such that $\mu_{\sigma'}^X((\lambda-\varepsilon, \lambda+\varepsilon)) > 0$. We denote $r = \operatorname{dist}(\sigma, \sigma')$, and restrict our attention to the tail of $\{(X_n, \sigma_n)\}$ in which $B_r(X_n, \sigma_n)$ is isometric to $B_r(X, \sigma)$. If σ_n' is the image of σ' under such an isometry, and $d_n = \max\{k \mid B_k(X_n, \sigma_n) \cong B_k(X, \sigma)\}$, then $B_{d_n - r}(X_n, \sigma_n') \cong B_{d_n-r}(X, \sigma')$, and since $d_n - r \to \infty$ we have $(X_n, \sigma_n') \to (X, \sigma')$. By Lemma 3.5, $\mu_{\sigma_n'}^{X_n}((\lambda-\varepsilon, \lambda+\varepsilon)) > 0$ for large enough n and therefore $\operatorname{Spec} A(X_n, 0)$ intersects $(\lambda-\varepsilon, \lambda+\varepsilon)$. The result for the Laplacians follows from the fact that $\Delta^+ = I - d\cdot A$. $\square$

In particular this gives:

Corollary 3.9. *If $X_n \overset{n\to\infty}{\longrightarrow} X$ then* $\operatorname{Spec} A_X \subseteq \overline{\bigcup_n \operatorname{Spec} A_{X_n}}$.

This is an analogue of [Li04, Thm. 4.3], which is also regarded sometimes as an Alon-Boppana theorem. In [Li04] the same statement is proved for the Hecke operators acting on $X = \mathcal{B}_{n,F}$, the Bruhat-Tits building of type $\widetilde{A}_n$, and on a sequence of quotients of X whose injectivity radii approach infinity.

Returning to the formulation of Alon-Boppana with spectral gaps, Theorem 3.8 yields as an immediate result that if $X_n \overset{n\to\infty}{\longrightarrow} X$ then

$$\liminf_{n\to\infty} \min \operatorname{Spec} \Delta_{X_n}^+ \leq \min \operatorname{Spec} \Delta_X^+ \leq \lambda(X). \tag{3.11}$$

In order to obtain the higher dimensional analogue of the Alon-Boppana theorem one would like to verify that this holds also when the spectrum of $\Delta_{X_n}^+$ is restricted to $Z_{d-1} = \left(B^{d-1}\right)^\perp$. But while this holds for graphs, the situation is more involved in general dimension. First of all, it does not hold in general:

Theorem 3.10. *Let T_2^2 be the arboreal 2-regular triangle complex (Figure 3.1), and $X_r = B_r\left(T_2^2, e_0\right)$ be the ball of radius r around an edge in it (as in Figure 3.2). Then $\lim_{r\to\infty} \lambda\left(X_r\right) = \frac{3}{2} - \sqrt{2}$, while $\lambda\left(T_2^2\right) = 0$.*

The proof follows in the next section. Before we delve into this counterexample, let us exhibit first several cases in which the Alon-Boppana analogue does hold:

Theorem 3.11. *If $X_n \overset{n\to\infty}{\longrightarrow} X$, and one of the following holds:*

(1) Zero is not in $\operatorname{Spec} \Delta_X^+\big|_{Z_{d-1}}$ (i.e. $\lambda(X) \neq 0$),

(2) zero is a non-isolated point in $\operatorname{Spec} \Delta_X^+\big|_{Z_{d-1}}$, or

(3) the $(d-1)$-skeletons of the complexes X_n form a family of $(d-1)$-expanders,

then

$$\liminf_{n\to\infty} \lambda\left(X_n\right) \leq \lambda\left(X\right).$$

Proof. By Theorem 3.8 there exist $\lambda_n \in \operatorname{Spec} \Delta_{X_n}^+$ with $\lambda_n \to \lambda(X)$. If *(1)* holds, then $\lambda_n > 0$ for large enough n, which implies that $\lambda_n \in \operatorname{Spec} \Delta_{X_n}^+\big|_{Z_{d-1}}$, hence $\lambda(X_n) = \min \operatorname{Spec} \Delta_{X_n}^+\big|_{Z_{d-1}} \leq \lambda_n$. Thus, $\liminf_{n\to\infty} \lambda(X_n) \leq \liminf_{n\to\infty} \lambda_n = \lambda(X)$. If *(2)* holds then there are $\mu_n \in \operatorname{Spec} \Delta_X^+ \setminus \{0\}$ with $\mu_n \to \lambda(X)$. For every μ_n there is a sequence $\lambda_{n,m} \in \operatorname{Spec} \Delta_{X_m}^+\big|_{Z_{d-1}}$ with $\lambda_{n,m} \overset{m\to\infty}{\longrightarrow} \mu_n$, and $\lambda_{n,n} \to \lambda(X)$.

In *(3)* we mean that the $(d-2)$-cells in X_n have globally bounded degrees, and the $(d-2)$-dimensional spectral gaps

$$\lambda_{d-2}\left(X_n\right) = \min \operatorname{Spec} \Delta_{d-2}^+\big|_{Z_{d-2}(X_n)}$$

are bounded away from zero (see Remark (1) after the proof). For example, if X_n are triangle complexes, this means that their underlying graphs form a family of expander graphs in the classical sense. By the previous cases, we can assume that $\lambda(X) = 0$, and furthermore that zero is an isolated point in $\operatorname{Spec} \Delta_X^+\big|_{Z_{d-1}}$. This implies that it is an eigenvalue, so that there exists $0 \neq f \in Z_{d-1}(X) = B^{d-1}(X)^{\perp}$ with $\Delta_X^+ f = 0$.

Since $X_n \overset{n\to\infty}{\longrightarrow} X$ there exist $\sigma_n \in X_n$, $\sigma_\infty \in X$, a sequence $r(n) \to \infty$, and isometries $\psi_n : B_{r(n)}\left(X_n, \sigma_n\right) \overset{\cong}{\longrightarrow} B_{r(n)}\left(X, \sigma_\infty\right)$. Define $f_n \in \Omega_{L^2}^{d-1}\left(X_n\right)$ by

$$f_n(\tau) = \begin{cases} f\left(\psi_n(\tau)\right) & \operatorname{dist}\left(\tau, \sigma_n\right) \leq r(n) \\ 0 & r(n) < \operatorname{dist}\left(\tau, \sigma_n\right). \end{cases}$$

We first claim that $\|\Delta^+ f_n\|$ and $\|\Delta^- f_n\|$ converge to zero ($\Delta^- = \Delta^-(X_n)$ are defined since the $(d-2)$-degrees are bounded). Since f_n is zero outside $B_{r(n)}\left(X_n, \sigma_n\right)$ and coincide with f on it, by $\Delta^+ f = 0$ we have

$$\begin{aligned} \left\|\Delta^+ f_n\right\|^2 &= \sum_{\sigma \in X_n^{d-1}} \left|\Delta^+ f_n(\sigma)\right|^2 = \sum_{\sigma : r(n) \leq \operatorname{dist}(\sigma, \sigma_n) \leq r(n)+1} \left|\Delta^+ f_n(\sigma)\right|^2 \\ &= \sum_{\sigma : r(n) \leq \operatorname{dist}(\sigma, \sigma_n) \leq r(n)+1} \left| f_n(\sigma) - \sum_{\sigma' \sim \sigma} \frac{f_n(\sigma')}{\deg \sigma'} \right|^2. \end{aligned}$$

Using $\left(\sum_{i=1}^{k} a_i\right)^2 \le k \sum_{i=1}^{k} a_i^2$ this gives

$$\left\|\Delta^+ f_n\right\|^2 \le (dK+1) \sum_{\sigma : r(n) \le \operatorname{dist}(\sigma,\sigma_n) \le r(n)+1} \left[\left|f_n(\sigma)\right|^2 + \sum_{\sigma' \sim \sigma} \left|f_n(\sigma')\right|^2 \right],$$

where K is a bound on the degree of $(d-1)$-cells in X and X_n. Since every $(d-1)$-cell has at most dK neighbors, we have

$$\begin{aligned} \left\|\Delta^+ f_n\right\|^2 &\le dK(dK+1) \sum_{\sigma : r(n)-1 \le \operatorname{dist}(\sigma,\sigma_n) \le r(n)+2} \left|f_n(\sigma)\right|^2 \\ &\le dK(dK+1) \left\| f\big|_{X \setminus B_X(\sigma, r(n)-2)} \right\|^2 \overset{n\to\infty}{\longrightarrow} 0. \end{aligned}$$

The reasoning for $\|\Delta^- f_n\| \to 0$ (see (2.5)) is analogous: (3.3) gives $\Delta^- f = 0$, and the assumptions that $(d-2)$-degrees are globally bounded yields similar bounds as done for Δ^+.

For every n write $f_n = z_n + b_n$, with $z_n \in Z_{d-1}(X_n)$ and $b_n \in B^{d-1}(X_n)$. It is enough to show that $\|z_n\|$ are bounded away from zero, since then $\frac{\|\Delta^+ z_n\|}{\|z_n\|} = \frac{\|\Delta^+ f_n\|}{\|z_n\|} \to 0$, showing that $\lambda(X_n) = \min \operatorname{Spec}\left(\Delta^+\big|_{Z_{d-1}(X_n)}\right)$ converge to zero.

Assume therefore that there are arbitrarily small $\|z_n\|$, and by passing to a subsequence that $\|z_n\| \to 0$. Then $\|b_n\| \to \|f\| > 0$, giving $\frac{\|\Delta^- b_n\|}{\|b_n\|} = \frac{\|\Delta^- f_n\|}{\|b_n\|} \to 0$. This implies that $\lambda'_n = \min \operatorname{Spec}\left(\Delta^-\big|_{B^{d-1}(X_n)}\right)$ converge to zero. However,

$$\begin{aligned} \lambda'_n &= \min \operatorname{Spec}\left(\Delta^-\big|_{B^{d-1}(X_n)}\right) = \min \operatorname{Spec}\left(\partial^*_{d-1}\partial_{d-1}\big|_{B^{d-1}(X_n)}\right) \\ &\overset{\star}{=} \min \operatorname{Spec}\left(\partial_{d-1}\partial^*_{d-1}\big|_{B_{d-2}(X_n)}\right) = \min \operatorname{Spec}\left(\Delta^+_{d-2}\big|_{B_{d-2}(X_n)}\right) \\ &\ge \min \operatorname{Spec}\left(\Delta^+_{d-2}\big|_{Z_{d-2}(X_n)}\right) = \lambda_{d-2}(X_n) \end{aligned}$$

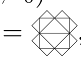
where $\star$ is due to the fact that B^{d-1} and B_{d-1} are the orthogonal complements of $\ker \partial_{d-1}$ and $\ker \partial^*_{d-1}$ respectively. This is a contradiction, since $\lambda_{d-2}(X_n)$ are bounded away from zero. $\square$

Remarks.

(1) If $X^{(j)}$ denote the j-skeleton of a complex X, i.e. the subcomplex consisting of all cells of dimension $\le j$, then one can look at $\lambda\left(X_n^{(d-1)}\right)$ instead of at $\lambda_{d-2}(X_n)$. Since we have different weight functions in codimension one, these are not equal. However, since we assumed that all $(d-1)$ and $(d-2)$ degrees are globally bounded (and nonzero), the norms induced by these choices of weight functions are equivalent, and thus $\lambda\left(X_n^{(d-1)}\right)$ are bounded away from zero iff $\lambda_{d-2}(X_n)$ are.

(2) The Alon-Boppana theorem for graphs follows from condition *(2)* in this Proposition (as done in [GŻ99]), since zero is never an isolated point in the spectrum of the Laplacian of an infinite connected graph. Otherwise, it would correspond to an eigenfunction, which is some multiple of the degree function, hence not in L^2.

3.6 Analysis of balls in T_2^2

In this section we analyze the spectrum of balls in the 2-regular triangle complex T_2^2, proving in particular that they constitute a counterexample for the higher-dimensional analogue of Alon-Boppana (Theorem 3.10). We denote here $X_r = B_r\left(T_2^2, e_0\right)$ the ball of radius r around an edge e_0 in T_2^2: X_0 is a single edge, $X_1 =$ , $X_2 =$ , $X_3 =$ , and so on. For $r \geq 1$ we define three $r \times r$ matrices denoted $M_{++}^{(r)}, M_{+-}^{(r)}, M_{--}^{(r)}$, and for $r \geq 0$ a $(r+1) \times (r+1)$ matrix $M_{-+}^{(r)}$, as follows:

$$M_{-+}^{(0)} = \left(1\right), \qquad M_{++}^{(1)} = M_{+-}^{(1)} = M_{--}^{(1)} = \left(0\right)$$
$$M_{-+}^{(1)} = \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix}, \qquad M_{++}^{(2)} = M_{+-}^{(2)} = \begin{pmatrix} \frac{1}{2} & -1 \\ -1 & 2 \end{pmatrix}, \qquad M_{--}^{(2)} = \begin{pmatrix} \frac{3}{2} & -1 \\ -1 & 2 \end{pmatrix}$$

$$M_{++}^{(r)} = M_{+-}^{(r)} = \left.\begin{pmatrix} \frac{1}{2} & -1 & & & \\ -\frac{1}{2} & \frac{3}{2} & -1 & & \\ & & \ddots & & \\ & & -\frac{1}{2} & \frac{3}{2} & -1 \\ & & & -1 & 2 \end{pmatrix}\right\} r$$

$$M_{--}^{(r)} = \left.\begin{pmatrix} \frac{3}{2} & -1 & & & \\ -\frac{1}{2} & \frac{3}{2} & -1 & & \\ & & \ddots & & \\ & & -\frac{1}{2} & \frac{3}{2} & -1 \\ & & & -1 & 2 \end{pmatrix}\right\} r$$

$$M_{-+}^{(r)} = \left.\begin{pmatrix} 1 & -2 & & & \\ -\frac{1}{2} & \frac{3}{2} & -1 & & \\ & & \ddots & & \\ & & -\frac{1}{2} & \frac{3}{2} & -1 \\ & & & -1 & 2 \end{pmatrix}\right\} r+1$$

Theorem 3.12. *The spectrum of $X_r = B_r\left(T_2^2\right)$ is given (including multiplicities) by*

$$\operatorname{Spec} \Delta^+ \left(X_r\right) = \operatorname{Spec} M_{++}^{(r)} \cup \operatorname{Spec} M_{+-}^{(r)} \cup \operatorname{Spec} M_{--}^{(r)} \cup \operatorname{Spec} M_{-+}^{(r)} \cup \bigcup_{j=1}^{r-1} \left[\operatorname{Spec} M_{++}^{(j)}\right]^{2^{r-j+1}}$$

where $[X]^i$ means that X is repeated i times.

To make this clear, this gives

$$\left|\operatorname{Spec} \Delta^+ \left(X_r\right)\right| = 4r + 1 + \sum_{j=1}^{r-1} 2^{r-j+1} \cdot j = 2^{r+2} - 3 = \left|X_r^1\right| = \dim \Omega^1 \left(X_r\right),$$

as ought to be.

Proof. The symmetry group of X_r (for $r \geq 1$) is $G = \{id, \tau_h, \tau_v, \sigma\}$, where τ_h is the horizontal reflection, τ_v is the vertical reflection (around the middle edge e_0), and $\sigma = \tau_h \circ \tau_v = \tau_v \circ \tau_h$ is a rotation by π. The irreducible representations of G are given in Table 3.1.

We define four orientations for X_r, denoted $X_r^{\pm\pm}$, demonstrated in Figure 3.3. In all of them e_0 is oriented from left to right, and the first (top right) quadrant is oriented clockwise. Each of the other quadrants is then oriented according to the corresponding representation, e.g. X_r^{+-} satisfies

	e	τ_h	τ_v	σ
V_{++}	1	1	1	1
V_{+-}	1	1	-1	-1
V_{-+}	1	-1	1	-1
V_{--}	1	-1	-1	1

Table 3.1: The irreducible representations of $G = \mathrm{Sym}\,(X_r)$.

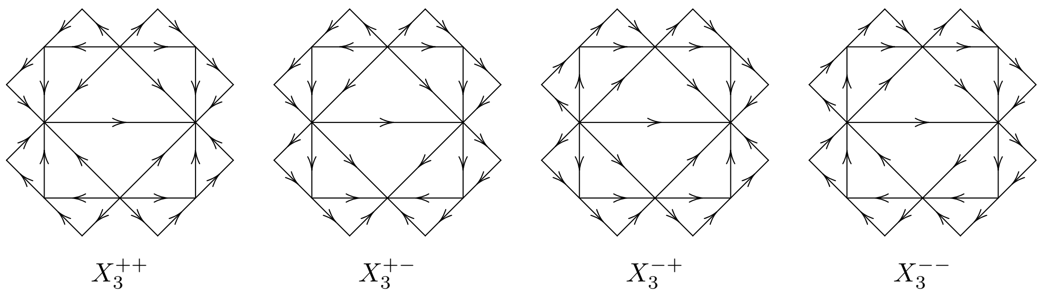


Figure 3.3: The four choices of orientations for X_r, depicted for $r = 3$.

the following: for every oriented edge e, if $e \in X_r^{+-}$ then $\tau_h e \in X_r^{+-}$, while $\tau_v e, \sigma e \notin X_r^{+-}$ (so that $\overline{\tau_v e}, \overline{\sigma e} \in X_r^{+-}$).

The space of 1-forms $\Omega^1(X_r)$ is naturally a representation of $G = \mathrm{Sym}\,(X_r)$, by $(\gamma f)(e) = f\left(\gamma^{-1}e\right)$ (where $\gamma \in G$, $f \in \Omega^1(X_r)$, $e \in X_r^1$). We denote by $\Omega_{\pm\pm}^{(r)} = \Omega_{\pm\pm}^1(X^r)$ its $V_{\pm\pm}$-isotypic components. For example, $f \in \Omega_{+-}^{(r)}$ if and only if it satisfies $\tau_h f = f$ and $\tau_v f = -f$ (which implies that $\sigma f = \tau_v \tau_h f = -f$).

We say that a 1-form on X_r is $++$-*spherical*, denoted $f \in \mathcal{S}_{++}^{(r)}$, if it is

(1) spherical in absolute value (i.e. $|f(e)| = |f(e')|$ whenever $\mathrm{dist}\,(e_0, e) = \mathrm{dist}\,(e_0, e')$), and

(2) V_{++}-isotypic (namely $f \in \Omega_{++}^{(r)}$, or equivalently, f is of constant sign on X_r^{++}).

The definition of $\mathcal{S}_{+-}^{(r)}, \mathcal{S}_{-+}^{(r)}, \mathcal{S}_{--}^{(r)}$ are analogue.

Let $e_1, \ldots, e_r$ be edges in the first quadrant of X_r oriented as in $X_r^{\pm\pm}$, and with $\mathrm{dist}\,(e_i, e_0) = i$. Let f be an eigenform of Δ^+ with eigenvalue λ, which is in one of the $\mathcal{S}_{\pm\pm}^{(r)}$. Then for $2 \le i \le r-1$

$$\lambda f(e_i) = \left(\Delta^+ f\right)(e_i) = f(e_i) - \frac{1}{2}\left[f(e_{i-1}) - f(e_i) + 2f(e_{i+1})\right]$$

and

$$\lambda f(e_r) = \left(\Delta^+ f\right)(e_r) = f(e_r) - \left[f(e_{r-1}) - f(e_r)\right].$$

The behavior of f around e_0, e_1 depends on the isotypic component. We assume $r \ge 2$, and leave it to the reader to verify the cases $r = 0, 1$. Every form in $\Omega_{++}^{(r)}, \Omega_{+-}^{(r)}, \Omega_{--}^{(r)}$ must vanish on the middle edge e_0: for the first two, since

$$f(e_0) = (\tau_h f)(e_0) = f(\tau_h e_0) = f(\overline{e_0}) = -f(e_0),$$

and for the last one since $f(e_0) = (-\tau_v f)(e_0) = -f(\tau_v e_0) = -f(e_0)$. For a spherical $(-+)$-functions we have

$$\lambda f(e_0) = \left(\Delta^+ f\right)(e_0) = f(e_0) - \frac{1}{2}\left[4 \cdot f(e_1)\right],$$

and at e_1 we have (using the fact that $f(e_0) = 0$ for $f \in \Omega_{++}^{(r)}, \Omega_{+-}^{(r)}, \Omega_{--}^{(r)}$)

$$\lambda f(e_1) = \left(\Delta^+ f\right)(e_1) = \begin{cases} f(e_1) - \frac{1}{2}\left[f(e_1) + 2f(e_2)\right] & f \in \Omega_{++}^{(r)}, \Omega_{+-}^{(r)} \\ f(e_1) - \frac{1}{2}\left[f(e_0) - f(e_1) + 2f(e_2)\right] & f \in \Omega_{-+}^{(r)} \\ f(e_1) - \frac{1}{2}\left[-f(e_1) + 2f(e_2)\right] & f \in \Omega_{--}^{(r)}. \end{cases}$$

The matrices $M_{\pm\pm}^{(r)}$ represent these equations, and thus the $++$-spherical spectrum of X^r is $\operatorname{Spec} \Delta^+\big|_{\mathcal{S}_{++}^{(r)}} = \operatorname{Spec} M_{++}^{(r)}$, and likewise for the other $\mathcal{S}_{\pm\pm}^{(r)}$.

Until now we have only accounted for the spherical part of $\Omega^1(X)$, finding in total $4r + 1$ eigenvalues. The other eigenvalues are obtained by using spherical eigenforms of X^i with $i < r$.

Denote by $X_r^{\mathfrak{h}}$ the upper half of X_r, including e_0, which is a fundamental domain for $\{id, \tau_v\}$. Observe that $X_r \backslash \overset{\circ}{X}_1$ (by which we mean X_r after deleting e_0 and the two triangles adjacent to it, but not the other four edges), is comprised of four copies of $X_{r-1}^{\mathfrak{h}}$, which intersect only in vertices. Denote these four copies of $X_{r-1}^{\mathfrak{h}}$ by $Y_1, \ldots, Y_4$. Let $f \in \mathcal{S}_{++}^{(r-1)}$ be a $(++)$-spherical λ-eigenform on X_{r-1}, and define $g \in \Omega^1(X_r)$ by $g\big|_{Y_1} = f\big|_{X_{r-1}^{\mathfrak{h}}}$ and $g\big|_{Y_2} = g\big|_{Y_3} = g\big|_{Y_4} = 0$. We show now that g is a λ-eigenform of X^r. Since $f \in \Omega_{++}^{(r-1)}$, $g(e_1) = f(e_0) = 0$, where e_1 is the edge incident to e_0 in Y_1. Therefore, $\Delta^+ g = \lambda g$ holds everywhere outside Y_1. It also holds at e_1, since if e_2, e_2' are the two edges incident to e_1 in Y_1, then $g(e_2) = -g(e_2')$ since f is symmetric with respect to τ_h. Obviously, $\Delta^+ g = \lambda g$ holds in $Y_1 \backslash \{e_1\}$, and we are done. We could have taken $g\big|_{Y_i} = f\big|_{X_{r-1}^{\mathfrak{h}}}$ for any $i \in \{1, 2, 3, 4\}$, and the resulting eigenforms are independent. We remark that taking $f \in \Omega_{+-}^{(r-1)}$ would also work, but would give again the same eigenforms, while $f \in \Omega_{-+}^{(r)}, \Omega_{--}^{(r)}$ would not define an eigenform on X_r.

More generally, $X_r \backslash \overset{\circ}{X}_j$ is comprised of 2^{j+1} copies of $X_{r-j}^{\mathfrak{h}}$, and in a similar way every eigenform of $\Delta^+\big|_{\mathcal{S}_{++}^{(r-j)}}$ contributes 2^{j+1} eigenforms to X^r. We recall that for $f \in \mathcal{S}_{++}^{(r-j)}$ we always have $f(e_0) = 0$, and observe that due to the recursion relations if $f \neq 0$ then $f(e_1) \neq 0$. Therefore, the eigenforms obtained from copies of $X_{r-j}^{\mathfrak{h}}$ for various j are all linearly independent, as they are supported outside different balls in X^r. Together with the $4r + 1$ spherical eigenforms, this accounts for

$$4r + 1 + \sum_{j=1}^{r} 2^{j+1} \cdot \left|\operatorname{Spec} \Delta^+\big|_{\mathcal{S}_{++}^{(r-j)}}\right| = 4r + 1 + \sum_{j=1}^{r-1} 2^{j+1}(r - j) = 2^{r+2} - 3$$

independent eigenforms, and since this is the dimension of $\Omega^1(X_r)$ we are done. $\square$

Proposition 3.13. *For every $r \in \mathbb{N}$ and $\lambda \in \operatorname{Spec} M_{\pm\pm}^{(r)}$, either $\lambda = 0$ or $\frac{3}{2} - \sqrt{2} < \lambda$.*

Proof. Let $p_{++}^{[r]}(\lambda) = \det\left(M_{++}^{(r)} - \lambda I\right)$, and similarly for the other $\pm\pm$. Expanding $M_{--}^{(r)} - \lambda I$

by minor gives

$$p_{--}^{[1]}(\lambda) = 1-\lambda, \quad p_{--}^{[2]}(\lambda) = \lambda^2 - \frac{7}{2}\lambda + 2, \quad p_{--}^{[3]} = -\lambda^3 + 5\lambda^2 - \frac{27}{4}\lambda + 2$$
$$p_{--}^{[r]}(\lambda) = \left(\frac{3}{2} - \lambda\right) p_{--}^{[r-1]}(\lambda) - \frac{1}{2} p_{--}^{[r-2]}(\lambda) \qquad (r \geq 4).$$

This yields a quadratic recurrence formula in $\mathbb{Q}[\lambda]$ whose solution (for $r \geq 2$) is $p_{--}^{[r]}(\lambda) = \alpha(\lambda)\mu_+(\lambda)^r + \beta(\lambda)\mu_-(\lambda)^r$, where

$$\alpha(\lambda) = 2 - \beta(\lambda) = \frac{(2\lambda-2)\sqrt{4\lambda^2 - 12\lambda + 1} + 4\lambda^2 - 10\lambda - 2}{(2\lambda-3)\sqrt{4\lambda^2 - 12\lambda + 1} + 4\lambda^2 - 12\lambda + 1},$$
$$\mu_\pm(\lambda) = \frac{3}{4} - \frac{\lambda}{2} \pm \frac{1}{4}\sqrt{4\lambda^2 - 12\lambda + 1}.$$

For $0 < \lambda < \frac{3}{2} - \sqrt{2}$ one can verify that $\beta(\lambda) < 0 < \alpha(\lambda)$ and $0 < \mu_-(\lambda) < \mu_+(\lambda)$, and for $r \geq 2$

$$p_{--}^{[r]}(\lambda) = \mu_+(\lambda)^r \left(\alpha(\lambda) + \beta(\lambda)\left(\frac{\mu_-(\lambda)}{\mu_+(\lambda)}\right)^r\right) \geq \mu_+(\lambda)^r \left(\alpha(\lambda) + \beta(\lambda)\left(\frac{\mu_-(\lambda)}{\mu_+(\lambda)}\right)^2\right)$$
$$= \mu_+(\lambda)^{r-2}\left(\alpha(\lambda)\mu_+(\lambda)^2 + \beta(\lambda)\mu_-(\lambda)^2\right) = \mu_+(\lambda)^{r-2} p_{--}^{[2]}(\lambda) > 0.$$

Using the solution for $p_{--}^{[r]}$ one can write $p_{+-}^{[r]}$, for $r \geq 4$, as

$$p_{+-}^{[r]}(\lambda) = \left(\frac{1}{2} - \lambda\right) p_{--}^{[r-1]}(\lambda) - \frac{1}{2} p_{--}^{[r-2]}(\lambda)$$
$$= \alpha(\lambda)\left(\left(\frac{1}{2} - \lambda\right)\mu_+(\lambda) - \frac{1}{2}\right)\mu_+(\lambda)^{r-2} + \beta(\lambda)\left(\left(\frac{1}{2} - \lambda\right)\mu_-(\lambda) - \frac{1}{2}\right)\mu_-(\lambda)^{r-2}$$

Now $\alpha(\lambda)\left(\left(\frac{1}{2} - \lambda\right)\mu_+(\lambda) - \frac{1}{2}\right) < 0 < \beta(\lambda)\left(\left(\frac{1}{2} - \lambda\right)\mu_-(\lambda) - \frac{1}{2}\right)$ for $0 < \lambda < \frac{3}{2} - \sqrt{2}$, and it follows that $p_{+-}^{[r]}(\lambda)$ does not vanish in this interval. This takes care of $p_{++}^{[r]}(\lambda)$ as well, since $M_{++}^{[r]} = M_{+-}^{[r]}$. The considerations for $p_{-+}^{[r]}(\lambda)$ are similar, and we leave them to the reader. □

We can conclude now that $\{X_r\}_{r\in\mathbb{N}}$ constitute a counterexample for high-dimensional Alon-Boppana:

Proof of Theorem 3.10. By the results in this section, the spectrum of $\Delta_{X_r}^+$ is contained in $\{0\} \cup \left(\frac{3}{2} - \sqrt{2}, 3\right]$. Since X_r is contractible, its first homology is trivial and thus the zeros in the spectrum all belong to coboundaries, i.e., $\operatorname{Spec} \Delta_{X_r}^+\big|_{Z_1} \subseteq \left(\frac{3}{2} - \sqrt{2}, 3\right]$. Therefore, $\liminf_{r\to\infty} \lambda(X_r) \geq \frac{3}{2} - \sqrt{2}$. In fact, $\liminf_{r\to\infty} \lambda(X_r) = \frac{3}{2} - \sqrt{2}$. This follows from $\frac{3}{2} - \sqrt{2} \in \operatorname{Spec} T_2^2$ (by Theorem 3.3), together with Theorem 3.8, which asserts that there exist $\lambda_r \in \operatorname{Spec} \Delta_{X_r}^+$ such that $\lambda_r \to \frac{3}{2} - \sqrt{2}$. As λ_r can be assumed to be nonzero, they are in fact in $\operatorname{Spec} \Delta_{X_r}^+\big|_{Z_1}$, so that $\liminf_{r\to\infty} \lambda(X_r) \leq \lim_{r\to\infty} \lambda_r = \frac{3}{2} - \sqrt{2}$. Finally, by Lemma 3.1 and Theorem 3.3 we have $\lambda\left(T_2^2\right) = 0$. □

3.7 Spectral radius and random walk

The spectral radius of an operator T is $\rho(T) = \max\{|\lambda| \mid \lambda \in \operatorname{Spec} T\}$. If T is a self-adjoint operator on a Hilbert space then $\rho(T) = \|T\|$. In this section we observe the transition operator $A = A(X,p)$ acting on $\Omega_{L^2}^{d-1}$, and relate it to the asymptotic behavior of the expectation process on X. Under additional conditions, this can be translated to a result on the spectral gap of the complex.

Proposition 3.14. *Let $\mathcal{E}_n^\sigma$ be the expectation process associated with the p-lazy $(d-1)$-walk on a finite or countable d-complex X with bounded $(d-1)$-degrees.*

(1) For all values of p

$$\sup_{\sigma\in X_\pm^{d-1}} \limsup_{n\to\infty} \sqrt[n]{|\mathcal{E}_n^\sigma(\sigma)|} = \|A\| = \rho(A).$$

(2) If $\frac{1}{2} \le p \le 1$ then

$$\sup_{\sigma\in X_\pm^{d-1}} \limsup_{n\to\infty} \sqrt[n]{\mathcal{E}_n^\sigma(\sigma)} = \|A\| = \max\operatorname{Spec} A = \frac{p(d-1)+1}{d} - \frac{1-p}{d}\cdot\min\operatorname{Spec}\Delta^+.$$

(3) If $\frac{1}{2} \le p \le 1$ and all $(d-2)$-cells in X are of infinite degree, then

$$\sup_{\sigma\in X_\pm^{d-1}} \limsup_{n\to\infty} \sqrt[n]{\mathcal{E}_n^\sigma(\sigma)} = \frac{p(d-1)+1}{d} - \frac{1-p}{d}\cdot\lambda(X).$$

Proof. For an oriented $(d-1)$-cell σ,

$$\mathcal{E}_n^\sigma(\sigma) = A^n\mathcal{E}_0^\sigma(\sigma) = \deg\sigma\,\langle A^n \mathbb{1}_\sigma, \mathbb{1}_\sigma\rangle = \deg\sigma\int_{\mathbb{C}} z^n d\mu_\sigma(z) = \deg\sigma\int_{\operatorname{Spec} A} z^n d\mu_\sigma(z),$$

where μ_σ is the spectral measure of A with respect to $\mathbb{1}_\sigma$. It follows that

$$\limsup_{n\to\infty}\sqrt[n]{|\mathcal{E}_n^\sigma(\sigma)|} = \limsup_{n\to\infty}\sqrt[n]{\deg\sigma\left|\int_{\operatorname{supp}\mu_\sigma} z^n d\mu_\sigma(z)\right|} = \max\{|\lambda| \mid \lambda\in\operatorname{supp}\mu_\sigma\},$$

and by $\operatorname{Spec}(A) = \bigcup_{\sigma\in X_\pm^{d-1}} \operatorname{supp}(\mu_\sigma)$ (see (3.10))

$$\sup_{\sigma\in X_\pm^{d-1}} \limsup_{n\to\infty}\sqrt[n]{|\mathcal{E}_n^\sigma(\sigma)|} = \sup_{\sigma\in X_\pm^{d-1}} \max\{|\lambda| \mid \lambda\in\operatorname{supp}\mu_\sigma\} = \rho(A),$$

settling *(1)*. Since $\operatorname{Spec}(A) \subseteq \left[2p-1, \frac{p(d-1)+1}{d}\right]$, in the case $p \ge \frac{1}{2}$ the spectrum of A is nonnegative. Therefore,

$$\mathcal{E}_n^\sigma(\sigma) = A^n\mathcal{E}_0^\sigma(\sigma) = \deg\sigma\,\langle A^n\mathbb{1}_\sigma, \mathbb{1}_\sigma\rangle \ge 0$$

so that $|\mathcal{E}_n^\sigma(\sigma)| = \mathcal{E}_n^\sigma(\sigma)$, and in addition $\rho(A) = \max\operatorname{Spec} A$. This accounts for *(2)*, and combining this with Lemma 3.1 gives *(3)*. $\square$

This proposition is a generalization of the classic connection between return probability and spectral radius in an infinite connected graph. Namely, for any vertex v the non-lazy walk on this graph satisfies

$$\lim_{n\to\infty} \sqrt[n]{\mathbf{p}_n^v(v)} = 1 - \lambda(G) = \max \operatorname{Spec} A = \|A\| = \rho(A),$$

where A is the transition operator of the walk. There are slight differences, though: in general dimension $p \geq \frac{1}{2}$ is needed for some of these equalities, and in addition one must take the supremum over all possible starting points for the process. For graphs this is not necessary (provided the graph is connected), and we do not know whether the same is true in general dimension. One case in which this is not necessary is when the complex is $(d-1)$-transitive, in the sense that its symmetry group acts transitively on X^{d-1}. This (together with Theorem 3.3) leads to the following corollary:

Corollary 3.15. *For the k-regular arboreal d-complex T_k^d, the non-lazy random walk starting at any $(d-1)$-cell σ satisfies*

$$\limsup_{n\to\infty} \sqrt[n]{\left|\mathbf{p}_n^\sigma(\sigma) - \mathbf{p}_n^\sigma(\overline{\sigma})\right|} = \frac{d-1+2\sqrt{d(k-1)}}{kd}.$$

For $p \geq \frac{1}{2}$, the p-lazy walk satisfies

$$\limsup_{n\to\infty} \sqrt[n]{\mathbf{p}_n^\sigma(\sigma) - \mathbf{p}_n^\sigma(\overline{\sigma})} = \begin{cases} p + \frac{1-p}{d} & 2 \leq k \leq d+1 \\ p + (1-p)\frac{1-d+2\sqrt{d(k-1)}}{kd} & d+1 \leq k \end{cases}.$$

Another corollary of Proposition 3.14 is the following:

Corollary 3.16. *If $\dim X = d$ and there exists some $\tau \in X^{d-2}$ of finite degree (in particular, if X is finite), then the $p \geq \frac{1}{2}$ lazy random walk satisfies*

$$\sup_{\sigma\in X_\pm^{d-1}} \limsup_{n\to\infty} \sqrt[n]{\mathbf{p}_n^\sigma(\sigma) - \mathbf{p}_n^\sigma(\overline{\sigma})} = p + \frac{1-p}{d}.$$

Proof. The form $\delta_{d-1}\mathbb{1}_\tau$ is in $\Omega_{L^2}^{d-1}$ and in $\ker \delta_d$, so that $0 \in \operatorname{Spec}\Delta^+$. □

3.8 Amenability, transience and recurrence

An infinite connected graph with finite degrees is said to be *amenable* if its Cheeger constant

$$h(X) = \min_{\substack{A\subseteq V \\ 0<|A|<\infty}} \frac{|E(A, V\backslash A)|}{|A|}$$

is zero. It is called *recurrent* if with probability one the random walk on it returns to its starting point, and *transient* otherwise. A nonamenable graph is always transient.

All three notions have many equivalent characterizations. Among these are the following, which relate to the Laplacian of the graph:

(1) If X has bounded degrees, then it is amenable if and only if $\lambda(X) = \min \operatorname{Spec}\Delta^+ = 0$. This follows from the so-called discrete Cheeger inequalities due to Tanner, Dodziuk, and Alon-Milman [Dod84, Tan84, AM85, Alo86].

(2) X is transient if and only if $\mathbb{E}^v \left[\substack{\text{number of} \\ \text{visits to } v}\right] = \sum_{n=0}^{\infty} \mathbf{p}_n^v(v) < \infty$ for some v, or equivalently for all v.

(3) X is transient if and only if there exists $f \in \Omega^1_{L^2}(X)$ such that $\partial f = \mathbb{1}_v$ for some v, or equivalently for all v [Lyo83].

This suggests observing the following generalizations of these notions for a simplicial complex of dimension d:

(**A**) $\lambda(X) = 0$.

(**A**$'$) $\min \operatorname{Spec} \Delta^+ = 0$.

(**T**) $\sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_n^\sigma(\sigma) < \infty$ for every $\sigma \in X^{d-1}$, where $\widetilde{\mathcal{E}}$ is the normalized expectation process of laziness p on X, for some $\frac{1}{2} \le p < 1$ (see Proposition 3.17*(5)*).

(**T**$'$) For every $\sigma \in X^{d-1}$ there exists $f \in \Omega^d_{L^2}(X)$ such that $\partial_d f = \mathbb{1}_\sigma$.

For infinite graphs, (**A**) and (**A**$'$) are the same and are equivalent to amenability, and (**T**) (for any p) and (**T**$'$) are equivalent to transience. These definitions suggests many questions, some of which are presented in the next section. The next proposition points out some observations regarding them. Let us also define the property:

(**S**) All $(d-2)$-cells in X have infinite degrees,

which holds in any infinite graph.

Proposition 3.17. *Let X be a complex of dimension d with bounded $(d-1)$-degrees. Then*

(1) $(\mathbf{A}) \Rightarrow (\mathbf{A}')$.

(2) $(\mathbf{A}') + (\mathbf{S}) \Rightarrow (\mathbf{A})$.

(3) $\neg(\mathbf{A}') \Rightarrow (\mathbf{T}') \Rightarrow (\mathbf{S})$.

(4) $\neg(\mathbf{A}') \Rightarrow (\mathbf{T})$.

(5) If $(\mathbf{T})$ *holds for some $\frac{1}{2} \le p < 1$, then it holds for every such p.*

(6) If zero is an isolated point in $\operatorname{Spec} \Delta^+$ *then* $\neg(\mathbf{T})$.

Proof. *(1)* is trivial and *(2)* follows from Lemma 3.1.

(3) If $(\mathbf{A}')$ fails then $0 \notin \operatorname{Spec} \Delta^+$, which means that Δ^+ is invertible on $\Omega^{d-1}_{L^2}(X)$. Thus, for every $\sigma \in X^{d-1}$ there exists $\psi \in \Omega^{d-1}_{L^2}$ s.t. $\Delta^+ \psi = \mathbb{1}_\sigma$, and taking $f = \delta_d \psi$ gives $(\mathbf{T}')$. If $(\mathbf{S})$ fails, then some $\tau \in X^{d-2}$ has finite degree. In this case for any $f \in \Omega^d$ one has

$$(\partial_{d-1}\partial_d f)(\tau) = \sum_{v \lhd \tau} (\partial_d f)(v\tau) = \sum_{v \lhd \tau} \sum_{w \lhd v\tau} f(wv\tau) = 0,$$

since this sums over every d-cell containing τ exactly twice, with opposite orientations. If σ is any $(d-1)$-cell containing τ, then $\partial_d f = \mathbb{1}_\sigma$ would give $0 = (\partial_{d-1}\partial_d f)(\tau) = (\partial_{d-1}\mathbb{1}_\sigma)(\tau) = 1$, so that $(\mathbf{T}')$ fails.

(4) If $\min \operatorname{Spec} \Delta^{+} > 0$ then by Proposition 3.14*(2)*

$$\sup_{\sigma \in X_{\pm}^{d-1}} \limsup_{n \to \infty} \sqrt[n]{\widetilde{\mathcal{E}}_{n}^{\sigma}(\sigma)} = 1 - \frac{1-p}{p(d-1)+1} \cdot \min \operatorname{Spec} \Delta^{+} < 1$$

which gives $\sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n}^{\sigma}(\sigma) < \infty$ for every σ.

(5) Let $\frac{1}{2} \leq p$, and denote by $\left\{\widetilde{\mathcal{E}}_{n}^{p,\sigma}\right\}_{n=0}^{\infty}$ the p-lazy normalized expectation process starting from σ and $\widetilde{A}_{p} = \frac{p(d-1)+1}{d} \cdot A_{p}$. Recall that $\widetilde{\mathcal{E}}_{n}^{p,\sigma} = \widetilde{A}_{p}^{n} \widetilde{\mathcal{E}}_{0}^{p,\sigma} = \widetilde{A}_{p}^{n} \mathbb{1}_{\sigma}$, and let $\mu_{p} = \mu_{\sigma}^{\widetilde{A}_{p}}$ be the spectral measure of $\widetilde{A}_{p}$ w.r.t. σ. Then

$$\sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n}^{p,\sigma}(\sigma) = \sum_{n=0}^{\infty} \widetilde{A}_{p}^{n} \mathbb{1}_{\sigma}(\sigma) = \deg \sigma \sum_{n=0}^{\infty} \left\langle \widetilde{A}_{p}^{n} \mathbb{1}_{\sigma}, \mathbb{1}_{\sigma} \right\rangle = \deg \sigma \sum_{n=0}^{\infty} \int_{\operatorname{Spec} \widetilde{A}_{p}} \lambda^{n} d\mu_{p}(\lambda).$$

Since $\operatorname{Spec} \widetilde{A}_{p} \subseteq \left[\frac{d(2p-1)}{p(d-1)+1}, 1\right] \subseteq [0,1]$, by monotone convergence

$$\sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n}^{p,\sigma}(\sigma) = \deg \sigma \sum_{n=0}^{\infty} \int_{\operatorname{Spec} \widetilde{A}_{p}} \lambda^{n} d\mu_{p}(\lambda) = \deg \sigma \int_{\operatorname{Spec} \widetilde{A}_{p}} \frac{d\mu_{p}(\lambda)}{1-\lambda} \tag{3.12}$$

which is to be understood as ∞ if μ_{p} has an atom at $\lambda = 1$. Given $p < q < 1$ one has $\widetilde{A}_{q} = \pi \widetilde{A}_{p} + (1-\pi) I$, where $\pi = \pi(p,q,d) = \frac{1-q}{1-p} \cdot \frac{p(d-1)+1}{q(d-1)+1} \in (0,1)$. The spectral measure of $\widetilde{A}_{q}$ w.r.t. σ is thus given by $\mu_{q} = \mu_{\sigma}^{\widetilde{A}_{q}} = \mu_{\sigma}^{\widetilde{A}_{p}} \circ g^{-1}$ where $g(\lambda) = \pi\lambda + 1 - \pi$, so that

$$\sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n}^{q,\sigma}(\sigma) = \deg \sigma \int_{\operatorname{Spec}(\widetilde{A}_{q})} \frac{d\mu_{q}(\lambda)}{1-\lambda} = \deg \sigma \int_{\operatorname{Spec} \widetilde{A}_{p}} \frac{d\mu_{p}(\lambda)}{1-(\pi\lambda+1-\pi)} = \frac{1}{\pi} \sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n}^{p,\sigma}(\sigma)$$

which completes the proof. Finally, *(6)* follows from (3.12) as an isolated point in the spectrum implies an atom at 1.

□

4 Open Questions

(1) In an infinite connected graph, the limit of $\sqrt[n]{\mathbf{p}_{n}^{v}(v)}$ (which describes the spectral radius of the transition operator, see §3.7) is independent of the starting point v. Is the same true in higher dimension? Namely, is $\limsup_{n \to \infty} \sqrt[n]{\mathcal{E}_{n}^{\sigma}(\sigma)}$ independent of σ for a $(d-1)$-connected complex?

(2) If $X \twoheadrightarrow Y$ is a covering map of graphs, then $\lambda(X) \geq \lambda(Y)$ (see e.g. [Kes59, Lemma 3.1], but beware - Kesten uses $\lambda(X)$ for what we denote by $1 - \lambda(X)$). Does the same holds in higher dimension? If $\pi : X \twoheadrightarrow Y$ is a covering map of d-complexes, then the same argumentation as in graphs shows that for any $\widetilde{\sigma} \in X^{d-1}$ and $\sigma = \pi(\widetilde{\sigma}) \in Y^{d-1}$ one has $\mathbf{p}_{n}^{\widetilde{\sigma}}(\widetilde{\sigma}) \leq \mathbf{p}_{n}^{\sigma}(\sigma)$ and also $\mathbf{p}_{n}^{\widetilde{\sigma}}\left(\overline{\widetilde{\sigma}}\right) \leq \mathbf{p}_{n}^{\sigma}(\overline{\sigma})$. This, however, does not suffice to show that $\mathcal{E}_{n}^{\widetilde{\sigma}}(\widetilde{\sigma}) \leq \mathcal{E}_{n}^{\sigma}(\sigma)$. Showing that this hold (or even that it holds after taking n^{th}-roots and letting $n \to \infty$) would give the desired result.

(3) It is not hard to see that a $(d+1)$-partite d-complex is disorientable, but for $d \geq 2$ one can also construct examples of disorientable complexes which are not $(d+1)$-partite. It seems reasonable to conjecture that for simply connected complexes these properties coincide. Is this indeed the case?

(4) The suggestions for higher-dimensional analogues of amenability and transience raise several questions:

   (a) Can high amenability and transience be characterized in non-spectral terms (i.e. combinatorial expansion, or some $1-0$ event in the $(d-1)$-walk model)?

   (b) Are the transience properties $(\mathbf{T})$ and $(\mathbf{T}')$ equivalent under some conditions?

   (c) Are all d-complexes with degrees bounded by $d+1$ d-amenable?

(5) In classical settings, the Brownian motion on a Riemannian manifold constitute a continuous limit of the discrete random walk. Can one define a continuous process, say, on the $(d-1)$-sphere bundle of a Riemannian manifold, which relates to its $(d-1)$-homology and to the spectrum of the Laplace-Beltrami operator on $(d-1)$-forms?

(6) There are surprising and useful connections between random walks on graphs and electrical networks (see e.g. [DS84, LP05]). Can a parallel theory be devised for the random $(d-1)$-walk on d-complexes?

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