# Packing random graphs and hypergraphs 

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#### Abstract

We determine to within a constant factor the threshold for the property that two random $k$-uniform hypergraphs with edge probability $p$ have an edge-disjoint packing into the same vertex set. More generally, we allow the hypergraphs to have different densities. In the graph case, we prove a stronger result, on packing a random graph with a fixed graph.


## 1 Introduction

Let $G_{1}$ and $G_{2}$ be two $k$-uniform hypergraphs of order $n$. We say that $G_{1}$ and $G_{2}$ can be packed if they can be placed onto the same vertex set so that their edge sets are disjoint.

In the graph case, quite a lot is known. Bollobás and Eldridge [2] and Catlin [5] independently conjectured that if $\left(\Delta\left(G_{1}\right)+1\right)\left(\Delta\left(G_{2}\right)+1\right) \leq n+1$ then $G_{1}$ and $G_{2}$ can be packed. Sauer and Spencer [12] proved that graphs $G_{1}$ and $G_{2}$ of order $n$ can be packed if $\Delta\left(G_{1}\right) \Delta\left(G_{2}\right)<n / 2$. Let us note

[^0]that the conjectured bound would be tight: suppose that $n=a b-2$, and let $G_{1}=(b-1) K_{a} \cup K_{a-2}$ (the vertex-disjoint union of $b-1$ complete graphs of order $a$ and a complete graph of order $a-2)$ and $G_{2}=(a-1) K_{b} \cup K_{b-2}$. Then $\left(\Delta\left(G_{1}\right)+1\right)\left(\Delta\left(G_{2}\right)+1\right)=n+2$, but $G_{1}$ and $G_{2}$ cannot be packed.

For fixed $k \geq 3$, the graph example given above is easy to generalize: suppose that $n=(a-1)(b-1)(k-1)+a+b-3$. Let $G_{1}$ be the vertexdisjoint union of $b-1$ complete $k$-uniform graphs of order $(a-1)(k-1)+1$ and $a-2$ isolated vertices; let $G_{2}$ be the vertex-disjoint union of $a-1$ complete $k$-uniform graphs of order $(b-1)(k-1)+1$ and $b-2$ isolated vertices. Then $\Delta\left(G_{1}\right) \Delta\left(G_{2}\right)=\Theta\left(a^{k-1} b^{k-1}\right)=\Theta\left(n^{k-1}\right)$, but $G_{1}$ and $G_{2}$ cannot be packed. For another family of examples, choose $r<k$ and fix an $r$-set $A \subset[n]$. Let $G_{1}$ have all edges containing $A$, and $G_{2}$ be an $(n, k, r)$-design (these are now known to exist for suitable $n$ : see Keevash (9). $G_{1}$ and $G_{2}$ cannot be packed, and we have $\Delta\left(G_{1}\right)=\Theta\left(n^{k-r}\right)$ and $\Delta\left(G_{2}\right)=\Theta\left(n^{r-1}\right)$, and so again $\Delta\left(G_{1}\right) \Delta\left(G_{2}\right)=\Theta\left(n^{k-1}\right)$. On the positive side, much less is known. Teirlinck [13] (see Alon [1] for further results and discussion) showed that, for $n \geq 7$, any two Steiner triple systems $G_{1}, G_{2}$ can be packed: note that these satisfy $\Delta\left(G_{1}\right) \Delta\left(G_{2}\right)=\Theta\left(n^{2}\right)$. There are also some nice results when one of $G_{1}$ and $G_{2}$ has very small maximal degree: see Rödl, Ruciński and Taraz [11] and Conlon [6].

In this paper, we consider what happens when $G_{1}$ and $G_{2}$ are random hypergraphs. For integers $k, n$ and $p \in[0,1]$, we write $\mathcal{G}(n, k, p)$ for the random $k$-uniform hypergraph on $n$ vertices in which each possible edge is present indepedently with probability $p$; when $k=2$, we write $\mathcal{G}(n, p)=$ $\mathcal{G}(n, 2, p)$. In the graph case, with $G_{1} \in \mathcal{G}(n, p)$ and $G_{2} \in \mathcal{G}(n, q)$, the extremal results mentioned above suggest that we should expect a condition of form $p q n \leq c$ (for suitable $c$ ) to be able to pack $G_{1}$ and $G_{2}$. More generally, for $k$-uniform hypergraphs, we might hope for a condition of form $p q n^{k-1} \leq c$, as this would give $\Delta\left(G_{1}\right) \Delta\left(G_{2}\right)=O\left(n^{k-1}\right)$ with high probability (i.e. with probability $1-o(1)$ as $n \rightarrow \infty)$ provided $p, q$ are not extremely small (for instance $\min \{p, q\} \gg \log n / n^{k-1}$ is enough). In fact, we shall show here that it is possible to pack rather denser graphs: if $G_{1}$ and $G_{2}$ are both random then we can allow an additional factor $\log n$ in the product pqn $n^{k-1}$, but not more. (We note that a similar phenomenon occurs when we try to minimize the overlap of two random hypergraphs: see Bollobás and Scott [3] and Ma, Naves and Sudakov [10].)

We will prove the following theorem.

Theorem 1. Let $\delta \in(0,1)$. For every $k \geq 2$, there exists $\varepsilon>0$ such that the following holds. Let $p=p(n)$ and $q=q(n)$ be positive reals such that

- $\max \{p, q\} \leq 1-\delta$
- $p q \leq \varepsilon \log n / n^{k-1}$.

Let $G_{1} \in \mathcal{G}(n, k, p)$ and $G_{2} \in \mathcal{G}(n, k, q)$ be random $k$-uniform hypergraphs of order $n$. Then, with high probability, there is a packing of $G_{1}$ and $G_{2}$.

Note that if $p q=\varepsilon \log n / n^{k-1}$ then with high probability $G_{1}$ and $G_{2}$ satisfy $\Delta\left(G_{1}\right) \Delta\left(G_{2}\right)=\Theta\left(n^{k-1} \log n\right)$.

The bound on $p q$ in Theorem 1 is easily seen to be sharp up to the constant. Indeed, if $G_{1} \in \mathcal{G}(n, k, p)$ and $G_{2} \in \mathcal{G}(n, k, q)$ then the probability that $G_{1}$ and $G_{2}$ can be packed is at most the expected number of packings

$$
n!(1-p q)^{\binom{n}{k}} \leq \exp \left(n \log n-(1+o(1)) p q n^{k} / k!\right)
$$

which is $o(1)$ if $p q \geq \alpha \log n / n^{k-1}$ for any constant $\alpha>k$ !. In particular, if we take $p=q$, then combining this bound with Theorem 1 shows that the threshold density for two random $k$-uniform hypergraphs to be unpackable is $\Theta\left(\sqrt{\log n / n^{k-1}}\right)$.

In the case of graphs, we will in fact prove a much stronger result: it turns out that we can take just one of the two graphs to be random. Indeed, we prove the following.

Theorem 2. For all $\gamma, K>0$ and $\delta \in(0,1)$ there exists $\varepsilon>0$ such that the following holds. Let $p=p(n)$ and $q=q(n)$ be positive reals such that

- $p \leq 1-\delta$
- $q \leq n^{-\gamma}$
- $p q n \leq \varepsilon \log n$.

Let $G_{1}$ be a graph of order $n$ with maximal degree at most $q n$ and let $G_{2} \in$ $\mathcal{G}(n, p)$. Then with failure probability $O\left(n^{-K}\right)$ there is a packing of $G_{1}$ and $G_{2}$.

The rest of the paper is organized as follows. In Section 2 we prove Theorem 2, and in Section 3 we prove the extension to hypergraphs. We conclude in Section 4 with some open problems.

## 2 Packing random graphs

The aim of this section is to prove Theorem 2, We begin by noting a couple of standard facts.

We will use the following Chernoff-type inequalities. Let $X$ be a sum of Bernoulli random variables, and let $\mu=\mathbb{E} X$. Then for $t>0$, we have

$$
\begin{equation*}
\mathbb{P}[X \leq \mathbb{E} X-t] \leq \exp \left(-t^{2} / 2 \mu\right) \tag{1}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathbb{P}[X \geq \mathbb{E} X+t] \leq \exp \left(-t^{2} /(2 \mu+2 t / 3)\right) \tag{2}
\end{equation*}
$$

(see, e.g., [8, Theorems 2.1 and 2.8] or [4, Chapter 2]). Inequality (2) is often called Bernstein's inequality.

It will also be useful to note a simple (and standard) fact about the binomial distribution, see e.g., [8, Corollary 2.4].

Proposition 3. For every $K>0$ there is $\delta>0$ such that if $x>0$ and $X \sim$ $\operatorname{Bi}(n, p)$ is a binomial random variable with $n p \leq \delta x$ then $\mathbb{P}[X \geq x] \leq e^{-K x}$.

Proof. This is standard; we include a proof for completeness. We have, assuming as we may that $x$ is an integer,

$$
\mathbb{P}[X \geq x] \leq\binom{ n}{x} p^{x} \leq\left(\frac{e n p}{x}\right)^{x} \leq(e \delta)^{x}
$$

where we have used the standard bound $\binom{n}{k} \leq(e n / k)^{k}$ in the second line. The result follows by choosing $\delta=e^{-K-1}$.

Our first lemma is the following, which shows that, if $\mathcal{A}$ is a large, sparse set system then a random set (of suitable size) is quite likely to be disjoint from some member of $\mathcal{A}$.

Lemma 4. For all $\delta, \gamma \in(0,1)$ there is $\varepsilon>0$ such that the following holds for all sufficiently large $n$. Let $d=n^{1-\gamma}$, let $X$ be any set, and let $\mathcal{A}$ be a set sequence in $\mathcal{P}(X)$ such that:

- $|\mathcal{A}| \geq n$
- every element of $X$ belongs to at most $d$ sets from $\mathcal{A}$
- all sets in $\mathcal{A}$ have size at most $\varepsilon \log n$.

Let $B \subset X$ be a random set where each element of $X$ independently belongs to $B$ with probability $1-\delta$. Then $B$ is disjoint from at least $n^{1-\gamma / 4}$ sets of $\mathcal{A}$, with failure probability $O\left(\exp \left(-n^{\gamma / 3}\right)\right)$.

Proof. This can be proved in more than one way (an alternative proof pointed out by a referee runs an element exposure martingale on X and then applies the Hoeffding-Azuma inequality).

We may assume that $|\mathcal{A}|=n$. We choose a small $\varepsilon>0$, and assume that $n$ is large. We ignore below insignificant roundings to integers.

We begin by partitioning $\mathcal{A}$ into sets of pairwise disjoint elements. Let $G$ be the intersection graph of $\mathcal{A}$ : so the vertices of $G$ are the elements of $\mathcal{A}$, and $G$ has edges $A A^{\prime}$ whenever $A \cap A^{\prime}$ is nonempty. Since every vertex belongs to at most $d$ sets from $\mathcal{A}$, and every set has size at most $\varepsilon \log n$, each set in $\mathcal{A}$ meets at most $\varepsilon d \log n$ other sets. Thus $G$ has maximal degree at most $\varepsilon d \log n$. It follows by a theorem of Hajnal and Szemerédi [7] that $G$ has a colouring with at most $\varepsilon d \log n+1$ colours in which the sizes of distinct colour classes differ by at most 1 . Thus we may partition $G$ into independent sets (and so $\mathcal{A}$ into collections of pairwise disjoint sets) of size at least $n /(\varepsilon d \log n+1) \geq n^{\gamma / 2}$.

Let $\mathcal{A}^{\prime}$ be one of these collections of pairwise disjoint sets, and set $m=$ $\left|\mathcal{A}^{\prime}\right| \geq n^{\gamma / 2}$. The random set $B$ is disjoint from each member of $\mathcal{A}^{\prime}$ independently with probability at least $\delta^{\varepsilon \log n}=n^{-\varepsilon \log (1 / \delta)}>n^{-0.01 \gamma}$ provided we have chosen a sufficiently small $\varepsilon$; it follows that the probability that $B$ is disjoint from fewer than $m / n^{\gamma / 4}$ sets in $\mathcal{A}^{\prime}$ is at most

$$
\begin{aligned}
\binom{m}{m / n^{\gamma / 4}}\left(1-n^{-0.01 \gamma}\right)^{m-m / n^{\gamma / 4}} & \leq\left(\frac{e m}{m / n^{\gamma / 4}}\right)^{m / n^{\gamma / 4}} \exp \left(-n^{-0.01 \gamma} m / 2\right) \\
& <e^{m \log n / n^{\gamma / 4}} e^{-n^{-0.01 \gamma} m / 2} \\
& <e^{-n^{-0.01 \gamma} m / 4}
\end{aligned}
$$

provided $n$ is sufficiently large. There are $\varepsilon d \log n+1=o(n)$ colour classes, so with failure probability $o\left(n e^{-n^{-0.01 \gamma} n^{\gamma / 2} / 4}\right)=O\left(e^{-n^{\gamma / 3}}\right), B$ is disjoint from at least a fraction $n^{-\gamma / 4}$ of the sets in each colour class, and hence is disjoint from at least $n^{1-\gamma / 4}$ sets in $\mathcal{A}$.

For positive integers $m, n$, and $p \in[0,1]$ we write $\mathcal{S}(n, m, p)$ for a random sequence $\left(S_{i}\right)_{i=1}^{m}$ of $m$ subsets of $[n]$, where the subsets are independent and each set independently contains each element of $[n]$ with probability $p$.

Equivalently, we could consider a random $m \times n$ matrix with entries 0 and 1 , where each element independently takes value 1 with probability $p$. We shall refer to $S \in \mathcal{S}(n, m, p)$ as a random set sequence.

Given two random set sequences $\mathcal{A} \in \mathcal{S}(m, n, p)$ and $\mathcal{A}^{\prime} \in \mathcal{S}(m, n, q)$, where $m \leq n$, it will be useful to pair up the sets from $\mathcal{A}$ and $\mathcal{A}^{\prime}$ so that each pair is disjoint. For $A \in \mathcal{A}$ and $A^{\prime} \in \mathcal{A}^{\prime}$, the probability that $A$ and $A^{\prime}$ are disjoint is $(1-p q)^{n} \leq \exp (-n p q)$, so if $p q>2 \log n / n$ it is likely that we do not have any disjoint pairs at all. However, if $p q<c \log n / n$, for small enough $c$, we will show that such a pairing is possible. In fact we will prove a much stronger result: we can take just one of the set systems to be random, provided the other satisfies certain sparsity conditions.
Lemma 5. For all $K>0$ and $\eta, \gamma, \delta \in(0,1)$ there is $\varepsilon>0$ such that the following holds for all sufficiently large $n$. Suppose that $p=p(n), q=q(n) \in$ $[0,1]$ satisfy $0 \leq p<1-\delta$ and $p q<\varepsilon \log n / n$. Let $m \in\left[n^{\eta}, n\right]$ be an integer and set $d=m^{1-\gamma}$, and suppose that $\mathcal{A}=\left(A_{i}\right)_{i=1}^{m}$ is a sequence of subsets of $[n]$ such that

- every $i \in[n]$ belongs to at most $d$ sets from $\mathcal{A}$
- $\max _{A \in \mathcal{A}}|A| \leq q n$.

Let $\mathcal{B}=\left(B_{i}\right)_{i=1}^{m} \in S(n, m, p)$ be a random set sequence, and let $H$ be the bipartite graph with vertex classes $\mathcal{A}$ and $\mathcal{B}$, where we join $A_{i}$ to $B_{j}$ if $A_{i} \cap$ $B_{j}=\emptyset$. Then, with failure probability $O\left(n^{-K}\right), H$ has minimal degree at least $m^{1-\gamma / 4}$; furthermore, $H$ has a perfect matching.

Proof. Let $\varepsilon, \varepsilon^{\prime}>0$ be fixed, small quantities (with $\varepsilon \ll \varepsilon^{\prime}$ ) that we shall choose later. We generate $\mathcal{B}$ in two steps: we first choose a random set sequence $\mathcal{B}^{\prime}=\left(B_{i}^{\prime}\right)_{i=1}^{m} \in S(n, m,(1+\delta) p)$, and then obtain $\mathcal{B}$ from $\mathcal{B}^{\prime}$ by deleting each element from each set $B_{i}^{\prime}$ independently with probability $\delta^{\prime}=$ $\delta /(1+\delta)$.

Note first that for any $i, j$, the distribution of the intersection $\left|A_{i} \cap B_{j}^{\prime}\right|$ is stochastically dominated by a binomial $\operatorname{Bi}(n q, p(1+\delta))$. So for fixed $\varepsilon^{\prime}>0$, it follows from Proposition 3 that we have $\left|A_{i} \cap B_{j}^{\prime}\right|<\varepsilon^{\prime} \log m$ for all $i$ and $j$, with failure probability $O\left(n^{-K}\right)$, provided $\varepsilon$ is small enough in terms of $\varepsilon^{\prime}$. We may therefore assume from now on that this event occurs, and condition on the choice of $\mathcal{B}^{\prime}$ (so $\mathcal{B}^{\prime}$ is fixed and $\mathcal{B}$ is still random).

Now consider the bipartite graph $H$. We need to prove that $H$ has a perfect matching. We shall apply Hall's condition to $\mathcal{B}$, so it is enough to show that for every subset $S \subset \mathcal{B}$ we have $\left|\Gamma_{H}(S)\right| \geq|S|$.

Consider $B_{i}^{\prime} \in \mathcal{B}^{\prime}$, and let $\mathcal{A}_{i}^{\prime}=\left(A_{j} \cap B_{i}^{\prime}\right)_{j=1}^{m}$ be the restriction of $\mathcal{A}$ to $B_{i}^{\prime}$. Then every vertex belongs to at most $d$ sets from $\mathcal{A}_{i}^{\prime}$ and $\max _{j}\left|A_{j} \cap B_{i}^{\prime}\right|<$ $\varepsilon^{\prime} \log m$, so provided $\varepsilon^{\prime}$ is sufficiently small we can apply Lemma 4 to deduce that with failure probability $O\left(e^{-m^{\gamma / 3}}\right)$ the set $B_{i}$ is disjoint from at least $m^{1-\gamma / 4}$ sets from $\mathcal{A}_{i}^{\prime}$. This occurs independently for each $i$ (recall that we are conditioning on $\left.\mathcal{B}^{\prime}\right)$, so with failure probability $O\left(m e^{-m^{\gamma / 3}}\right)=O\left(n^{-K}\right)$ every vertex in $\mathcal{B}$ has degree at least $m^{1-\gamma / 4}$ in $H$, and so Hall's condition holds for every $S \subset \mathcal{B}$ with $|S|<m^{1-\gamma / 4}$.

Now consider an element $A_{i} \in \mathcal{A}$. Each $B_{j}^{\prime}$ meets $A_{i}$ in at most $\varepsilon^{\prime} \log m$ vertices, and so each $B_{j}$ independently is disjoint from $A_{i}$ with probability at least $\left(\delta^{\prime}\right)^{\varepsilon^{\prime} \log m}>m^{-\gamma / 6}$, provided $\varepsilon^{\prime}$ is sufficiently small. The number of $B_{j}$ disjoint from $A_{i}$ is thus a binomial with expectation at least $m^{1-\gamma / 6}$ and so, by (11), is at least $m^{1-\gamma / 6} / 2>m^{1-\gamma / 4}$, with failure probability $O\left(e^{-m^{1-\gamma / 6 / 8}}\right)$. So with failure probability $O\left(m e^{-m^{1-\gamma / 6} / 8}\right)=O\left(n^{-K}\right)$ every vertex in $\mathcal{A}$ has degree at least $m^{1-\gamma / 4}$ in $H$, and so Hall's condition holds for every $S \subset \mathcal{B}$ with $|S|>m-m^{1-\gamma / 4}$.

We have now shown that $H$ has minimal degree at least $m^{1-\gamma / 4}$. All that remains is to verify Hall's condition for sets $S \subset \mathcal{B}$ of size between $m^{1-\gamma / 4}$ and $m-m^{1-\gamma / 4}$. Let $t \in\left[m^{1-\gamma / 4}, m-m^{1-\gamma / 4}\right]$ : we shall bound the probability that there is any subset of $\mathcal{B}$ of size $t$ with $t$ or fewer neighbours in $\mathcal{A}$. Suppose that $S \subset \mathcal{B}$ has size $t$ and $T \subset \mathcal{A}$ has size $m-t$. For any fixed $B_{i}^{\prime} \in S$, the set sequence $\mathcal{A}^{\prime}=\left(A \cap B_{i}^{\prime}\right)_{A \in T}$ has $\max _{A^{\prime} \in \mathcal{A}^{\prime}}\left|A^{\prime}\right| \leq \varepsilon^{\prime} \log m$ and every vertex belongs to at most $d$ sets from $\mathcal{A}^{\prime}$, where $d=m^{1-\gamma} \leq\left|\mathcal{A}^{\prime}\right|^{1-\gamma / 4}$. So by Lemma 4, the probability that $B_{i}$ intersects every set in $T$ is at most $\exp \left(-(m-t)^{\gamma / 12}\right)$. Thus the probability that (in the graph $\left.H\right) S$ has no neighbours in $T$ is at most $\exp \left(-t \cdot(m-t)^{\gamma / 12}\right)$. Since there are at most $n^{2 t}=\exp (2 t \log n)$ choices for the pair $(S, T)$, we deduce that the probability that there is any set $S$ of size $t$ with at most $t$ neighbours is bounded by $\exp (2 t \log n) \exp \left(-t \cdot(m-t)^{\gamma / 12}\right)=O\left(n^{-(K+1)}\right)$, uniformly in $t$. Summing over $t$, we see that Hall's condition holds with failure probability $O\left(n^{-K}\right)$.

We conclude by noting that we can choose first $\varepsilon^{\prime}$ and then $\varepsilon$ sufficiently small for the estimates above to hold.

We are now ready to prove Theorem 2.
Proof of Theorem 圆. Let $\eta=\gamma / 2, t=\lceil(K+2) / \eta\rceil$, and let $G_{1}$ have vertex set $V$ and $G_{2}$ have vertex set $W$. We begin by finding a partition of $V$ into sets $V_{1}, V_{2}, \ldots$ of size $\Theta\left(n^{\eta}\right)$ such that:

- $V_{i}$ is an independent set in $G_{1}$ for every $i$,
- Every vertex in $V$ has fewer than $t$ neighbours in each set $V_{j}$.

Indeed, we first colour $V$ randomly with $n^{1-\eta}$ colours, giving each vertex a colour selected uniformly at random and independently. It follows from (1) and (2) that, with failure probability $O\left(n^{-K}\right)$, every colour class has size $(1+o(1)) n^{\eta}$. Consider a vertex $v \in V$, say with degree $d$. Then by assumption $d \leq q n \leq n^{1-\gamma}$. So the probability that $v$ has a set of $t$ neighbours, all with the same colour, is at most

$$
\binom{d}{t}\left(1 / n^{1-\eta}\right)^{t-1} \leq d^{t} n^{\eta-t+1} \leq n^{1+t \eta-t \gamma}=n^{1-t \eta}=O\left(n^{-(K+1)}\right)
$$

It follows that, with failure probability $O\left(n^{-K}\right)$, no vertex has $t$ neighbours of the same colour. Each colour class now induces a subgraph with maximum degree less than $t$, so we can apply the Hajnal-Szemerédi Theorem [7] to each class, splitting it into $O(t)$ independent sets of (almost) the same size. The vertex classes are now independent, have size $\Theta\left(n^{\eta}\right)$, and no vertex has $t$ neighbours in any other class.

Reordering if necessary, we may assume that $\left|V_{1}\right| \geq\left|V_{2}\right| \geq \cdots$. Now let $W=W_{1} \cup W_{2} \cup \cdots$ be an arbitrary partition of $W$ (chosen before revealing $G_{2}$ ) such that $\left|W_{i}\right|=\left|V_{i}\right|$ for every $i$. We construct a bijection between $V$ and $W$ that defines a packing (i.e., does not map any edge of $G_{1}$ to an edge of $G_{2}$ ) by constructing suitable bijections between $V_{i}$ and $W_{i}$ for $i=1,2, \ldots$.

For $i=1$, we choose an arbitrary bijection between $V_{1}$ and $W_{1}$. (Recall that $V_{1}$ is independent.) For $i>1$, we set $S_{i}=\bigcup_{j<i} V_{j}$ and $T_{i}=\bigcup_{j<i} W_{j}$, and suppose that we have found a bijection $\varphi_{i}: S_{i} \rightarrow T_{i}$. The neighbourhoods of vertices in $V_{i}$ and $W_{i}$ define set sequences $\mathcal{A}=\left(\Gamma(v) \cap S_{i}\right)_{v \in V_{i}}$ in $S_{i}$ and $\mathcal{B}=\left(\Gamma(v) \cap T_{i}\right)_{v \in W_{i}}$ in $T_{i}$, and the bijection $\varphi_{i}$ allows us to identify $S_{i}$ and $T_{i}$. We now check that these two set sequences satisfy the conditions of Lemma 5. which we will then apply to obtain a bijection between $V_{i}$ and $W_{i}$. Let

$$
\begin{aligned}
\widetilde{n} & =\left|S_{i}\right|
\end{aligned}=\left|T_{i}\right|=\Theta\left(i n^{\eta}\right), ~, ~=~\left(V _ { i } \left|=\left|W_{i}\right|=\Theta\left(n^{\eta}\right), ~ \$\right.\right.
$$

and note that $|\mathcal{A}|=|\mathcal{B}|=\widetilde{m}$ and $\widetilde{m} \in\left[\widetilde{n}^{\eta / 2}, \widetilde{n}\right]$. By construction of the partition $\left(V_{j}\right)_{j \geq 1}$, no vertex belongs to $t$ sets from $\mathcal{A}$, as each vertex in $S_{i}$ has fewer than $t$ neighbours in $V_{i}$. Let $\widetilde{q}=\max _{A \in \mathcal{A}}|A| / \widetilde{n}$. Each set in $\mathcal{A}$
has size at most $q n$ and so $\widetilde{q} \leq q n / \widetilde{n}=O\left(q n^{1-\eta} / i\right)$. The set sequence $\mathcal{B}$ is random with $\mathcal{B} \in S(\widetilde{n}, \widetilde{m}, p)$, and depends only on the edges between $W_{i}$ and $T_{i}$. Furthermore,

$$
p \widetilde{q} \widetilde{n} \leq p \cdot(q n / \widetilde{n}) \cdot \widetilde{n}=p q n \leq \varepsilon \log n=O(\varepsilon \log \widetilde{n}) .
$$

We can therefore apply Lemma 5, to deduce that if $\varepsilon$ is sufficiently small then with failure probability $O\left(n^{-(K+1)}\right)$ there is a bijection between the two set sequences for which the corresponding pairs are disjoint; this corresponds to a bijection between $V_{i}$ and $W_{i}$ so that there are no common edges in the bipartite graphs between $\left(V_{i}, S_{i}\right)$ and $\left(W_{i}, T_{i}\right)$ where $S_{i}$ and $T_{i}$ are identified by $\varphi_{i}$. Extending $\varphi_{i}$ with this bijection, we obtain a bijection $\varphi_{i+1}: S_{i+1} \rightarrow T_{i+1}$.

It follows that, with failure probability $O\left(n^{-K}\right)$, we succeed at every step and construct the desired bijection.

Finally in this section, we note that Theorem 2 can be used to pack several random graphs.

Corollary 6. Let $\gamma, K>0$, let $\delta \in(0,1)$, and let $t$ be a positive integer. Then there exists $\varepsilon>0$ such that the following holds. Let $p_{0}(n), \ldots, p_{t}(n)$ satisfy

- $\max _{i} p_{i} \leq 1-\delta$
- $p_{0} \leq n^{-\gamma}$
- $\max _{i<j} p_{i} p_{j} n \leq \varepsilon \log n$.

Let $G_{0}$ be a graph of order $n$ with maximal degree at most $p_{0} n$ and, for $i=2, \ldots, t$, let $G_{i} \in \mathcal{G}\left(n, p_{i}\right)$. Then with failure probability $O\left(n^{-K}\right)$ there is a packing of $G_{0}, \ldots, G_{t}$.

Proof. We may assume that $p_{1} \leq \cdots \leq p_{t}$. Thus, by the second and third conditions above, we have $\sum_{i=0}^{t-1} p_{i}=\bar{O}\left(n^{-\min \{\gamma, 1 / 3\}}\right)$. We first pack $G_{0}$ and $G_{1}$, then add in the remaining graphs one at a time, applying Theorem 2 at each stage. Thus at the $i$ th stage we have packed $G_{0}, \ldots, G_{i}$ to obtain a graph $H_{i}$ : it follows easily from Proposition 3 that with high probability the maximum degree condition of Theorem 2 is satisfied by $H_{i}$ (with a slightly smaller $\gamma$ ). Provided $\varepsilon$ is sufficiently small, we get that with failure probability $O\left(n^{-K}\right)$ we can pack $H_{i}$ with $G_{i+1}$.

## 3 Packing hypergraphs

In this section, we will prove Theorem 1 .
Proof of Theorem 1. Note that the case $k=2$ follows immediately from Theorem 2, so we can assume $k \geq 3$. Let $\eta=1 / 5, t=15 k$, and let $\varepsilon, \varepsilon^{\prime}>0$ be small constants and $K, K^{\prime}$ large constants; we will choose $\varepsilon, \varepsilon^{\prime}$ and $K, K^{\prime}$ later. (In fact, we will first choose $\varepsilon^{\prime} ; K^{\prime}$ will be determined by $\varepsilon^{\prime}$; we then choose $K$ and finally $\varepsilon$.) We may assume that $q \leq p$, and so in particular $q=O\left(\sqrt{\log n / n^{k-1}}\right)<n^{-1 / 2}$ (for large $\left.n\right)$. We may also assume that $q \geq$ $\varepsilon \log n / n^{k-1}$, or increase to this value.

Our argument will follow a similar strategy to Theorem 2, but there are some additional complications. It will be helpful to reveal the edges of $G_{1}$ and $G_{2}$ in several steps. This time we let $V$ be the vertex set of $G_{2}$ and $W$ the vertex set of $G_{1}$.

We first generate a partition of $V$ into sets $V_{1}, V_{2}, \ldots$ by colouring $V$ randomly with $n^{1-\eta}$ colours, giving each vertex a colour selected uniformly at random and independently. It follows from (1) and (2) that, with failure probability $o(1)$, every colour class $V_{i}$ has size $(1+o(1)) n^{\eta}$, so we may assume that this holds. Reordering if necessary, we may assume that $\left|V_{1}\right| \geq\left|V_{2}\right| \geq$ $\cdots$. Let $W=W_{1} \cup W_{2} \cup \cdots$ be a random partition of $W$ such that $\left|W_{i}\right|=\left|V_{i}\right|$ for every $i$. For $i \geq 1$, we set $S_{i}=\bigcup_{j<i} V_{j}$ and $T_{i}=\bigcup_{j<i} W_{j}$ (note that $S_{1}=T_{1}=\emptyset$; also $S_{L}=V$ and $T_{L}=W$, where $L=n^{1-\eta}+1$ ).

As before, we will construct a bijection between $V$ and $W$ by constructing bijections between $V_{i}$ and $W_{i}$ for $i=1,2, \ldots$ However, we need to be a little more careful than in the graph case, as there are more ways for edges to intersect the classes $V_{i}$ and $W_{i}$. For $j=1, \ldots, k$, and any $i$, we say that an edge is of type $j$ for $V_{i}$ or $W_{i}$ if it has $j$ vertices in $V_{i}$ or $W_{i}$, and the remaining $k-j$ vertices in $S_{i}$ or $T_{i}$.

We now reveal all type 1 edges in $G_{2}$. For a $(k-1)$-set $A \subset S_{i}$, the probability that $V_{i}$ contains $t$ vertices $v$ such that $A \cup\{v\}$ is an edge of $G_{2}$ is at most

$$
\binom{2 n^{\eta}}{t} q^{t}=O\left(n^{\eta t-t / 2}\right)=o\left(n^{-k}\right)
$$

It follows that, with high probability, for every integer $i$ and every $(k-1)$-set $A \subset S_{i}, V_{i}$ contains fewer than $t$ vertices that can be added to $A$ to obtain an edge of $G_{2}$. In other words, each $(k-1)$-set in $S_{i}$ is contained in fewer than $t$ type 1 edges for $V_{i}$.

For each vertex $v \in V_{i}$, we define the type 1 neighbourhood of $v$ to be the ( $k-1$ )-uniform hypergraph on $S_{i}$ with edge set

$$
\left\{A \subset S_{i}: A \cup\{v\} \text { is a type } 1 \text { edge for } V_{i}\right\} ;
$$

similarly, for vertices in $W_{i}$, the type 1 neighbourhood is a $(k-1)$-uniform hypergraph on $T_{i}$.

At the first step of the partitioning process, we take a random bijection between $V_{1}$ and $W_{1}$. The expected number of common edges is at most $p q n^{k \eta}=o(1)$, and so with high probability there are no common edges.

Now consider a later stage of the partitioning process: suppose we have constructed a bijection $\varphi_{i}: S_{i} \rightarrow T_{i}$ and wish to extend this to a bijection $\varphi_{i}: S_{i+1} \rightarrow T_{i+1}$. In constructing our bijection, we will only consider edges of type 1 and 2 ; we will consider edges of type 3 at the end of the argument.

We first consider type 1 edges in $V_{i}$ and $W_{i}$. For each $v \in V_{i}$, we consider the type 1 neighbourhood of $v$ as a subset of $S_{i}^{(k-1)}$ (rather than as a $k$ uniform hypergraph on $S_{i+1}$ ). The collection of type 1 neighbourhoods of vertices in $V_{i}$ then defines a set sequence $\mathcal{A}$ of subsets of $S_{i}^{(k-1)}$; similarly, the collection of type 1 neighbourhoods of vertices in $W_{i}$ defines a set sequence $\mathcal{B}$ of subsets of $T_{i}^{(k-1)}$; and the bijection $\varphi_{i}$ allows us to identify $S_{i}^{(k-1)}$ and $T_{i}^{(k-1)}$. As in the proof of Theorem 2, we wish to apply Lemma 5, so we need to check that its conditions are satisfied.

Let

$$
\begin{aligned}
\widetilde{n} & =\left|S_{i}^{(k-1)}\right|=\left|T_{i}^{(k-1)}\right|=\Theta\left(i^{k-1} n^{\eta(k-1)}\right), \\
\tilde{m} & =\left|V_{i}\right|=\left|W_{i}\right|=(1+o(1)) n^{\eta}
\end{aligned}
$$

and note that $|\mathcal{A}|=|\mathcal{B}|=\tilde{m}$ and $\tilde{m} \in\left[\tilde{n}^{\eta / k}, \tilde{n}\right]$.
By construction of the partition $\left(V_{j}\right)_{j \geq 1}$, no element of $S_{i}^{(k-1)}$ is contained in $t$ sets from $\mathcal{A}$, as each $(k-1)$-set $A \subset S_{i}$ is contained in fewer than $t$ type 1 edges for $V_{i}$. The size of each set in $\mathcal{A}$ has distribution $\operatorname{Bi}(\widetilde{n}, q)$. Choose a small $\varepsilon^{\prime}>0$, let $K^{\prime}=2 /\left(\eta \varepsilon^{\prime}\right)$, and then choose a large $K$. Let $\widetilde{q}=\max \left\{K q, \varepsilon^{\prime}(\log \widetilde{n}) / \widetilde{n}\right\}$. It follows from Proposition 3 that, provided $K$ is large enough (depending on $K^{\prime}$ ), every set in $\mathcal{A}$ has size at most $\tilde{n} \widetilde{q}$, with failure probability at most

$$
\widetilde{m} e^{-K^{\prime} \tilde{n} \tilde{q}} \leq n e^{-K^{\prime} \varepsilon^{\prime} \log \widetilde{n}} \leq n^{1-K^{\prime} \varepsilon^{\prime} \eta}=o(1 / n) .
$$

Furthermore, since $\widetilde{n} \leq n^{k-1}$, by choosing $\varepsilon$ small enough we get

$$
p K q \leq K \varepsilon \frac{\log n^{k-1}}{n^{k-1}} \leq \varepsilon^{\prime} \frac{\log \widetilde{n}}{\widetilde{n}}
$$

and hence $p \widetilde{q} \leq \varepsilon^{\prime}(\log \widetilde{n}) / \widetilde{n}$. We can therefore apply Lemma 5, to deduce that if $\varepsilon^{\prime}$ is sufficiently small then with failure probability $O\left(n^{-2}\right)$ we get the following:

- a bijection $\varphi^{*}: V_{i} \rightarrow W_{i}$ such that the corresponding pairs in the two set sequences are disjoint. This corresponds to a bijection between $V_{i}$ and $W_{i}$ so that there are no collisions between type 1 edges for $V_{i}$ and $W_{i}$. Also:
- for all distinct $u, v \in V_{i}$ and $x, y \in W_{i}$, a bijection

$$
\varphi^{* *}: V_{i} \backslash\{u, v\} \rightarrow W_{i} \backslash\{x, y\}
$$

such that there are no collisions of type 1 edges for $V_{i}$ and $W_{i}$, except possibly for edges containing $u, v, x$ or $y$.

The mapping $\varphi^{*}$ deals with collisions between type 1 edges. However, we must also consider type 2 edges for $V_{i}$ and $W_{i}$. We do not reveal type 2 edges at this stage, but only the number of collisions between type 2 edges created by the mapping $\varphi^{*}$. There are at most $n^{k-2+2 \eta}$ type 2 edges for $V_{i}$ and $W_{i}$, and so the probability that $\varphi^{*}$ maps any type 2 edge for $V_{i}$ in $G_{2}$ to a type 2 edge in $G_{1}$ is at most $p q n^{k-2+2 \eta} \leq \log n / n^{1-2 \eta}$; the probability that there are at least two collisions is $O\left(\log ^{2} n / n^{2-2 \eta}\right)=o(1 / n)$ (which is small enough to ignore). If there are no collisions, then we use $\varphi^{*}$ to extend $\varphi_{i}$.

This leaves the case when there is one collision between type 2 edges. We reveal the edge where this occurs: say $A \cup\{u, v\}$ maps to $A \cup\{x, y\}$ under $\varphi^{*}$. We thus condition on the existence of these two edges in $G_{2}$ and $G_{1}$, and on this being the only collision. We shall show the existence of another mapping $\varphi^{* *}$ from $V_{i}$ to $W_{i}$ that avoids collisions for both type 1 and type 2 edges with probability at least $1-O(\log n / \sqrt{n})$. Then the probability that we get collisions for both $\varphi^{*}$ and $\varphi^{* *}$ is $O\left(\left(\log n / n^{1-2 \eta}\right) \cdot \log n / \sqrt{n}\right)$, which is $o(1 / n)$.

Let $D=\lceil 6(\log \widetilde{n}) / \delta\rceil$. We choose distinct vertices $x_{1}, \ldots, x_{D}, y_{1} \ldots, y_{D}$ in $W_{i}$ such that the type 1 neighbourhood of $u$ is edge-disjoint from the type

1 neighbourhoods of $x_{1}, \ldots, x_{D}$, and the type 1 neighbourhood of $v$ is edgedisjoint from the type 1 neighbourhoods of $y_{1}, \ldots, y_{D}$ (the existence of these vertices follows from the minimal degree condition on $H$ in Lemma (5).

We reveal the edges $A \cup\left\{x_{\ell}, y_{\ell}\right\}$ for each $\ell \leq D$ : since $p \leq 1-\delta$, it follows that with probability $1-o(1 / n)$ there is some $\ell$ such that $A \cup\left\{x_{\ell}, y_{\ell}\right\}$ is not present in $G_{1}$. We then use the appropriate mapping $\varphi^{* *}$ from $V_{i} \backslash\{u, v\}$ to $W_{i} \backslash\left\{x_{\ell}, y_{\ell}\right\}$ that we found above, and extend it by $\operatorname{setting} \varphi^{* *}(u)=x_{\ell}$ and $\varphi^{* *}(v)=y_{\ell}$ so that we have a mapping from $V_{i}$ to $W_{i}$. The mapping $\varphi^{* *}$ does not cause any collision of type 1 edges. Finally, we reexamine the type 2 edges for collisions. We have ensured that $A \cup\{u, v\}$ does not collide with anything; the probability of a collision involving any edge of form $A \cup\left\{x_{j}, y_{j}\right\}$ is at most $q D=O(\log n / \sqrt{n})$; and the probability of any other collision is at most $\log n / n^{1-2 \eta}=O(1 / \sqrt{n})$, as before. (More formally: we have conditioned on the edges $A \cup\left\{x_{j}, y_{j}\right\}$, on the event that a particular pair of type 2 edges collide, and the event that no other collisions occur. If we resample all type 2 edges that are not in the colliding pair or of form $A \cup\left\{x_{j}, y_{j}\right\}$, the number of collisions under $\varphi^{* *}$ stochastically dominates the number before resampling, giving the same bound.) So the probability that $\varphi^{* *}$ yields a collision is $O(\log n / \sqrt{n})$, as required.

It follows that, with probability $1-o(1 / n)$, we are able to find a good bijection between $V_{i}$ and $W_{i}$, and extend $\varphi_{i}$ to $\varphi_{i+1}$. Continuing in this way, we find a bijection from $V$ to $W$ in which there are no collisions between type 1 or 2 edges for any $V_{i}, W_{i}$.

Finally, we reveal all edges of type 3 or more. There are at most $n^{k-2+2 \eta}$ possible edges of type 3 or more, and so the probability that any of these is an edge in both hypergraphs is at most $p q n^{k-2+2 \eta}=o(1)$. The algorithm therefore succeeds with probability $1-o(1)$.

## 4 Conclusion

We conclude by mentioning a few open questions.

- The bound in Theorem 1 is sharp to within a constant factor. It is natural to expect that there is some $c=c(k)>0$ such that almost surely a pair of random $k$-uniform hypergraphs $G_{1}, G_{2} \in \mathcal{G}(n, k, p)$ are packable if $p<(c-\varepsilon) \sqrt{\log n / n^{k-1}}$ and are unpackable if $p>$ $(c+\varepsilon) \sqrt{\log n / n^{k-1}}$. Is this correct? If so, what is the value of $c$ ?
- What happens with the results above if we take $G_{1}=G_{2}$ ? We would expect this to make no difference.
- All our examples of unpackable $k$-uniform hypergraphs $G_{1}, G_{2}$ have $\Delta\left(G_{1}\right) \Delta\left(G_{2}\right)=\Omega\left(n^{k-1}\right)$. What is the correct bound here?


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