Online Ramsey Games for more than two colors

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Abstract

Consider the following one-player game played on an initially empty graph with n vertices. At each stage a randomly selected new edge is added and the player must immediately color the edge with one of r available colors. Her objective is to color as many edges as possible without creating a monochromatic copy of a fixed graph F.

We use container and sparse regularity techniques to prove a tight upper bound on the typical duration of this game with an arbitrary, but fixed, number of colors for a family of 2-balanced graphs. The bound confirms a conjecture of Marciniszyn, Spöhel and Steger and yields the first tight result for online graph avoidance games with more than two colors.

1 Introduction

Consider the following one-player game played on an initially empty graph on n vertices. In every round we insert a new edge chosen uniformly at random among all non-edges of the graph. The player, henceforth called Painter, must immediately color this edge with one of r available colors. Her objective is to avoid a monochromatic copy of some fixed graph F for as long as possible. We refer to this game as the online F-avoidance game with r colors.

We call $N_0(F, r, n)$ a threshold function for the online *F*-avoidance game with *r* colors if for every $N \ll N_0(F, r, n)$ there exists a strategy for Painter that survives for *N* rounds with high probability and if for every $N \gg N_0(r, n)$ every strategy fails to survive for *N* rounds with high probability. Note that such a threshold function always exists [10, Lemma 2.1].

This game was first studied by Friedgut, Kohayakawa, Rödl, Ruciński and Tetali, who have shown in [4] that $N_0(K_3, 2, n) = n^{4/3}$ is a threshold for the online triangle-avoidance game with two colors. In 2009 Marciniszyn, Spöhel and Steger proved the following lower bound

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Theorem 1.1 ([10]). Let *F* be a graph that is not a forest, and let $r \ge 1$. Then the online *F*-avoidance game with *r* colors has a threshold $N_0(F, r, n)$ that satisfies

$$N_0(F, r, n) \ge n^{2-1/\overline{m}_2^r(F)}$$

where $\overline{m}_r^2(F)$ is given by

$$\overline{m}_{2}^{r}(F) \coloneqq \begin{cases} \max_{H \subseteq F} \frac{e_{H}}{v_{H}} & \text{if } r = 1, \\ \max_{H \subseteq F} \frac{e_{H}}{v_{H} - 2 + 1/\overline{m}_{2}^{r-1}(F)} & \text{if } r \geq 2. \end{cases}$$

In an accompanying paper they provide matching upper bounds in the two color case for a large class of graphs, which includes cycles and cliques: **Theorem 1.2 ([11]).** *Let F be a graph that is not a forest which has a subgraph*

Theorem 1.2 ([11]). Let *F* be a graph that is not a forest which has a subgraph $F_{-} \subset F$ with $e_{F} - 1$ edges satisfying

$$m_2(F_-) \le \overline{m}_2^2(F).$$

Then the threshold for the online F-avoidance coloring game with two colors is

$$N_0(F, 2, n) = n^{2-1/\overline{m}_2^2(F)}$$

They conjecture that a similar result is true for all $r \ge 3$.

For three or more colors no tight upper bounds are known. The corresponding offline triangle avoidance game (where Painter gets to see all N edges at once) has a threshold given by $N = n^{3/2}$ [13]. Clearly this upper bound also applies to the online game. In [3] Belfrage, Mütze and Spöhel connected the probabilistic one player game to a deterministic two player game originally introduced by Kurek and Ruciński in [9]. In this version of the game the edges are no longer presented in a random order but can be chosen by a second player called Builder. They show that if there exists a winning strategy for Builder which only creates subgraphs of density at most d, then $n^{2-1/d}$ is an upper bound for the threshold of the original probabilistic game.

This technique was used by Balogh and Butterfield in [1] to improve the upper bound for the online triangle avoidance game to $n^{3/2-c_r}$ for some constant $c_r > 0$. Thus the thresholds of the online and offline games differ. Still their upper bound of $n^{2/3-c_r}$ does not match the lower bound provided by Marciniszyn, Spöhel and Steger.

Our contribution is an upper bound, which matches the lower bound Marciniszyn, Spöhel and Steger, for an arbitrary number of colors. That is we show the following:

Theorem 1.3. Let *F* be a 2-balanced graph that is not a tree which has a subgraph $F_{-} \subset F$ with $e_{F} - 1$ edges satisfying

$$m_2(F_-) \leq \overline{m}_2^2(F).$$

Then the threshold for the online F-avoidance game with r colors is

$$N_0(F,r,n) = n^{2-1/\overline{m}_2^r(F)}$$

The premise of our theorem is satisfied by a large class of graphs, which includes cycles and cliques. The condition that F is 2-balanced is used only for technical reasons. On the other hand the second condition is (in general) necessary. In [10] the authors give an example of a graph (two triangles intersecting in a single vertex) for which the above threshold is incorrect.

To go from two to more colors we prove a generalization of the KŁRconjecture. For two colors the (unmodified) KŁR-conjecture immediately tells us that Painter has to color on the order of $n^{v_{F-}}p^{e_{F-}}$ copies of F_- with the majority color (where $p \simeq n^{-1/\overline{m}_2^2}$). In expectation a *p*-fraction of those copies of F_- will form a copy of F which contains one edge colored in the secondary color. The density of those edges is roughly $n^{v_F-2}p^{e_F} = n^{-1/m(F)}$ so we may expect them to form a copy of F. A.a.s. this is indeed the case and thus Painter looses the game after $\omega(n^{2-1/\overline{m}_2^2(F)})$ edges have been presented.

To generalize this argument to three colors we want to show that there exist copies of F_{-} in the primary and secondary colors which share their missing (non)-edge. These non-edges, if they appear later, will have to be colored with the tertiary color and as before the \overline{m}_{2}^{r} density is large enough to guarantee that a.a.s. Painter will have to close a copy of *F* in the tertiary color.

To find the aforementioned copies we prove a variant of the KŁR-conjecture, which allows us find copies of F_{-} where the missing edge lies in some fixed set of (non-)edges. The proof of this statement uses the container theorem introduced by Saxton and Thomason [14] and independently by Balogh, Morris and Samotij [2].

1.1 Preliminaries and Notation

For $n \in \mathbb{N}$ let $[r] = \{1, ..., r\}$. For sets V, V' and $\varepsilon \in [0, 1]$ we write $V' \subseteq_{\varepsilon} V$ to denote that V' is a subset of V of cardinality at least $\varepsilon |V|$. We say that a statement holds asymptotically almost surely (a.a.s.) if it holds with probability 1 - o(1). The underlying uncolored graph of the game follows the random graph process $(G(n, N))_{1 \leq N \leq {n \choose 2}}$, where the edges are added in an order selected uniformly at random from the ${n \choose 2}$ possible permutations. Let $G_{n,p}$ denote the binomial random graph on n vertices where every edge is present with probability p independently of all others. If $N \simeq p{n \choose 2}$ then the two models are equivalent in terms of asymptotic properties [6]. We will thus mostly work with $G_{n,p}$.

Let *G* be a graph and let $R \subseteq V(H)$ denote an ordered subset of the vertices. We call the pair (R, G) a rooted graph. We denote the number of vertices of *G* with v_G and the number of edges with e_G . For a rooted graph we set $\overline{v}_{R,G} = v_G - |R|$ and $\overline{e}_{R,G} = e_G - e_{G[R]}$. For convenience we drop the dependence on *R* if the set of roots is obvious from the context. That is $\overline{v}_G = \overline{v}_{R,G}$ and $\overline{e}_G = \overline{e}_{R,G}$. We write $H \subseteq_R G$ do denote that *H* is a subgraph of *G* with $R \subseteq V(H)$ and H[R] = G[R]. For a rooted graph (R, F) we denote with F_- the subgraph of *F* obtained by removing all edges of F[R]. We write (e, F) to indicate that the set of roots has cardinality two (this notation does not imply that $e \in E(F)$).

For two rooted graphs (R, G), (e, F) we denote with $(R, G) \times (e, F)$ the graph obtained by attaching to every non root edge e' of (R, G) a new copy

of *F* rooted in e' (possibly removing the edge e' if it is not present in *F*). In general one can choose to orient the attached copies of *F* in two different ways. For our purposes the actual choice does not matter so we fix one based on the lexicographic ordering of the vertices.

For a collection of graphs $(R, G_1), \ldots, (R, G_k)$ which agree on R we denote with $\bigsqcup_{i=1}^{k} (R, G_i)$ the rooted graph obtained by joining pairwise disjoint copies of G_1, \ldots, G_k together at their roots.

For graphs we define the following densities (by convention 0/0 = 0):

$$d(G) = \frac{e_G}{v_G} \qquad m(G) = \max_{H \subseteq G} d(H)$$

$$d_1(G) = \frac{e_G}{v_G - 1} \qquad m_1(G) = \max_{H \subseteq G} d_1(H)$$

$$d_2(G) = \frac{e_G - 1}{v_G - 2} \qquad m_2(G) = \max_{H \subseteq G} d_2(H)$$

$$\overline{d}_2^r(G, H) = \begin{cases} \frac{e_H}{v_H} & \text{if } r = 1, \\ \frac{e_H}{v_H - 2 + 1/\overline{m_2^{r-1}(G)}} & \text{if } r \ge 2. \end{cases} \qquad \overline{m}_2^r(G) = \max_{H \subseteq G} \overline{d}_2^r(G, H)$$

We say that a graph *G* is (strictly) balanced with respect to a density function if the maximum is attained (uniquely) by *G*. We say that *G* is balanced (1-balanced, 2-balanced) if it is balanced with respect to $m(m_1, m_2)$.

One can check (see [10]) that for every graph *G* we have

$$m(G) = \overline{m}_2^1(G) < \overline{m}_2^2(G) < \cdots < \overline{m}_2^r(G) < \cdots < m_2(G).$$

Furthermore if *G* is 2-balanced then it is also balanced with respect to \overline{m}_2^r for all *r*. It is also easy to check that for every graph *G* which is not a forest

$$m_1(G) \le \overline{m}_2^2(G),$$

and that if *G* is additionally 2-balanced then it is also strictly 1-balanced. The density of a rooted graph is defined by

$$d(R,G) = \frac{\overline{e}_G}{\overline{v}_G} \qquad m(R,G) = \max_{H \subseteq G} d(R \cap V(H), H).$$

As in the unrooted case we call a rooted graph balanced if it is balanced with respect to *m*.

Assume that there exists $G' \subseteq G_{n,p}$ such that $G' \sim G - G[R]$. We say that G' is a copy of G - G[R] in $G_{n,p}$ and that the vertices of G' which correspond to the roots of (R, G) span a copy of (R, G). Observe that the edges between root vertices are immaterial. We will make heavy use of the following upper bound due to Spencer on the number of rooted graphs spanned by vertices of the random graph.

Theorem 1.4 ([15]). Let (R, G) be a rooted graph and suppose that t > m(R, G) and $p(n) = \Omega(n^{-1/t})$. Then a.a.s. in $G_{n,p}$ every |R|-tuple of vertices spans $(1 \pm o(1))\mu$ copies of (R, G) where $\mu \simeq n^{\overline{v}_G} p^{\overline{e}_G}$ is the expected number of such copies.

If *p* is below the density of (R, G) then the following easy to show upper bound will suffice:

Lemma 1.5. Suppose that (R, G) is a balanced rooted graph and that t < m(R, G). Then there exists a constant D(t) such that for $p \le n^{-1/t}$ with probability 1 - o(1) no set of |R| vertices in $G_{n,p}$ spans more than D copies of G.

Proof. Let us first prove that the balancedness of (R, G) implies that t < m(R', G) for all R' with $R \subseteq R' \subsetneq V(G)$. As (R, G) is balanced we have for $q = m^{-1/m(R,G)}$

$$n^{v_{G}-|R'|}q^{e_{G}-e_{G[R']}} = \frac{n^{v_{G}-|R|}q^{e_{G}-e_{G[R]}}}{n^{|R'|-|R|}q^{e_{G[R']}-e_{G[R]}}} = \frac{\Theta(1)}{\Omega(1)} = O(1),$$

and thus $m(R', F) \ge m(R, F) > t$.

Now the probability that a fixed set of roots spans *C* pairwise edge disjoint copies of (R', G) is at most

$$(n^{\overline{v}_{R',G}}p^{\overline{e}_{R',G}})^{C} = (n^{-\Theta(1)})^{C} = o(n^{-v_{G}})$$

provided that *C* is large enough depending on *t* and v_G . Using the union bound we conclude that a.a.s. for every $R' \supseteq R$ no set of |R'| vertices spans more than *C* pairwise edge disjoint copies of (R', G).

Fix a set of roots $R \subseteq V(G_{n,p})$ and a maximal set of edge disjoint copies of (R, G) spanned by R. Every other copy of (R, G) spanned by R must intersect these copies in some set $R' \supseteq R$. By induction R' spans at most a constant number of copies of (R', G) and since the number of choices for R' is a constant the total number of copies of (R, G) spanned by R is a constant as well. \Box

The final density of interest is a generalization of the 2-density to rooted graphs. For t > 0 we define

$$d_2(R,G,t) = \begin{cases} \frac{\overline{e}_G - 1}{v_G - 2 - t e_{G[R]}} & \text{if } v_G - 2 - t e_{G[R]} > 0, \\ \infty & \text{otherwise.} \end{cases}$$

And

$$m_2(R,G,t) = \max_{\substack{H \subseteq G\\e_H - e_{H[R]} > 1}} d_2(R \cap V(H), H, t).$$

The motivation for this definition is given in Section 1.1.2.

1.1.1 Szemerédi's regularity lemma for sparse graphs

Our proof relies heavily on the sparse regularity lemma and related concepts. The required definitions and theorems are briefly stated below. A more in depth introduction to the topic can be found in [5].

Definition 1.6. A bipartite graph $B = (U \cup W, E)$ is called (ε, p) -regular if for all $U' \subseteq_{\varepsilon} U$ and $W' \subseteq_{\varepsilon} W$,

$$\left|\frac{|E(U',W')|}{|U'||W'|} - \frac{|E|}{|U||W|}\right| \le \varepsilon p.$$

We write (ε) *-regular in case p equals the density* |E|/(|U||W|)*.*

The original regularity lemma of Szemerédi allows us to partition arbitrary graphs into a constant number of $(\varepsilon, 1)$ -regular pairs. Kohayakawa[7] and Rödl (unpublished) independently introduced an analogue of Szemerédi's regularity lemma which gives meaningful results for $p \rightarrow 0$. The generalization works for a class of graph which do not contain large dense spots.

Definition 1.7. Let G = (V, E) be a graph and let $0 < \eta, p \le 1$. We say that G is (η, p) -upper-uniform if for all disjoint sets $U, W \subseteq_{\eta} V$

$$|E(U, W)| \le (1 + \eta)p|U||W|.$$

We can now state Szemerédi's regularity lemma for sparse graphs. We use the second version presented in [7].

Definition 1.8. A partiton $(V_i)_0^k$ of the vertex set V is called an (ε, p) -regular partition with exceptional class V_0 if $|V_1| = |V_2| = \cdots = |V_k|, |V_0| \le \varepsilon n$, and, with the exception of at most εk^2 pairs, the pairs $(V_i, V_j), 1 \le i \le j \le k$ are (ε, p) -regular.

Theorem 1.9 (sparse regularity lemma). For any $\varepsilon > 0$ and $m_0 \ge 1$, there are constants $\eta = \eta(\varepsilon, m_0) > 0$ and $M_0 = M_0(\varepsilon, m_0) \ge m_0$ such that for any p > 0, any (η, p) -upper-uniform graph with at least m_0 vertices admits an (ε, p) -regular partition $(V_i)_{i=0}^k$ with exceptional class V_0 such that $m_0 \le k \le M_0$.

The (ε, p) -regularity of a pair does not imply any lower bounds on its density. In fact the empty graph is (ε, p) -regular for all $\varepsilon > 0$, $0 \le p \le 1$. Still it is not hard to show that if *G* has density at least αp then, for η, ε small enough, we find at least one pair V_i, V_j which is (ε, p) -regular with density at least $\alpha p/2$ (and thus $(2\varepsilon/\alpha)$ -regular).

1.1.2 A KŁR type statement for rooted graphs

Fix a graph *F* and let $(V_i)_{i \in V(F)}$ denote pairwise disjoint sets of size *n*. We call a graph *G* on the vertex set $\bigcup_{i \in V(F)} V_i$ (*F*, ε)-*regular* if for every $\{i, j\} \in E(F)$ the pair (V_i, V_j) is (ε) -regular. We denote with $\mathcal{G}(F, n, m, \varepsilon)$ the class of (F, ε) regular graphs *G* for which for every $i, j \in V(F)$

$$|E(V_i, V_j)| = \begin{cases} m & \text{if } \{i, j\} \in E(F), \\ 0 & otherwise. \end{cases}$$

A partite copy of *F* in *G* is a set of vertices $\{v_i \in V_i : i \in V(F)\}$ such that $\{v_i, v_j\} \in E(G)$ whenever $\{i, j\} \in E(F)$. In [8] Kohayakawa, Łuczak and Rödl conjectured that almost all graphs in $\mathcal{G}(F, n, m, \varepsilon)$ contain a partite copy of *F*. This conjecture, known as the KŁR-conjecture, was recently proven in full by Saxton and Thomason [14] and independently by Balogh, Morris and Samotij [2]. The following counting version is due to Saxton and Thomason.

Theorem 1.10 (KŁR conjecture, weak counting version [14]). Let *F* be a graph and let $\beta > 0$. There exists $\mu(\beta) > 0$ such that for *n* sufficiently large and $m \ge \mu^{-1}n^{2-1/m_2(F)}$ the number of graphs in $\mathcal{G}(F, n, m, \mu)$ which do not contain at least $\mu(m/n^2)^{e_F}n^{v_F}$ partite copies of *F* is at most

$$\beta^m \binom{n^2}{m}^{e_F}$$

Our main tool will be a slight generalization of this theorem: we want to count only those copies of *F* which satisfy some additional constraints. These constraints take the form of a partite hypergraph on a subset of the vertex partitions. For a rooted graph (R, F) we denote with $\mathcal{R}(R, n)$ the class of partite |R|-uniform hypergraphs on the partitions $V_1, \ldots, V_{|R|}$. Fix $G_R \in \mathcal{R}(R, n)$ and $G \in \mathcal{G}(F_-, n, m, \varepsilon)$. We denote with $T(G, G_R)$ the multi-hypergraph on $V_1, \ldots, V_{|R|}$ which contains an edge $e \in E(G_R)$ with multiplicity *k* if *G* contains exactly *k* partite copies of F_- which contain all vertices from *e*.

For our theorem to work we require that the edges of G_R are roughly distributed like partite copies of F[R] in a random |R|-partite graph. This notion is formalized in the following two definitions.

Definition 1.11. We say that $G_R \in \mathcal{R}(R, n)$ is (F, q, ε) -lower-regular if all tuples of subsets $V'_1 \subseteq_{\varepsilon} V_1, \ldots, V'_{|R|} \subseteq_{\varepsilon} V_{|R|}$ induce at least $q^{e_{F[R]}} \prod_{i \in R} |V'_i|$ edges.

Definition 1.12. We say that $G_R \in \mathcal{R}(R, n)$ is (F, q)-upper-extensible if for every induced subgraph $F' \subseteq F[R]$ the degree of all tuples from $\bigotimes_{i \in V(F')} V_i$ is at most

$$q^{e_{F[R]}-e_{F'}}n^{|R|-v_{F'}}$$

With these definitions at hand we can state our generalization of Theorem 1.10.

Theorem 1.13. Let (R, F) be a rooted graph. For every $\beta > 0$, $A \ge 1$ there exists $\alpha(A, \beta), \mu(\beta) > 0$ such that for every q(n) = o(1) the following holds: For n large enough suppose that $m \ge \alpha^{-1}n^{2-1/m_2(R,F,-\log_n q)}$ and that $G_R \in C$

For n large enough suppose that $m \ge \alpha^{-1}n^{2-1/m_2(R,F,-\log_n q)}$ and that $G_R \in \mathcal{R}(R,n)$ is (F,Aq)-upper-extensible as well as (F,q,μ) -lower-regular. Then the number of graphs G in $\mathcal{G}(F_-,n,m,\mu)$ for which $T(G,G_R)$ contains fewer than $\alpha(m/n^2)^{e_F}-q^{e_F[R]}n^{v_F}$ edges is at most

$$\beta^m \binom{n^2}{m}^{e_F - e_{F[R]}}$$

The proof follows the proof of the KŁR conjecture presented in [14] and is deferred to Section 3.

2 **Proof of Main Theorem**

We will assume that *F* is a fixed 2-balanced graph which contains an edge $e \in E(F)$ such that $m_2(F - e) \leq \overline{m}_2^2(F)$. This fixes a rooted graph (e, F). Based on the choice of *e* we now define the classes $\mathcal{F}^1, \mathcal{F}^2, \ldots$ of rooted graphs. \mathcal{F}^1 consists of a singular rooted graph: an edge rooted in its endpoints. For $k \geq 2$ we define

$$\mathcal{F}^k := \left\{ \bigsqcup_{i < k} (e, F) \times (e_i, F_i^*) \mid \forall i \colon (e_i, F_i^*) \in \mathcal{F}^{\leq i} \right\},\$$

where $\mathcal{F}^{\leq i} := \bigcup_{j \leq i} \mathcal{F}^j$. It is useful to observe that every $F^* \in \mathcal{F}^k$, $k \geq 2$ can be built by starting with a copy of *F* and then repeatedly attaching a copy of (e, F) to some edge. Since *F* is 2-balanced this implies that F^* is 2-balanced with the same 2-density (see Lemma 2.16).

If Painter employs the greedy strategy then all edges colored with her *k*-th favorite color will span a copy of the densest graph from \mathcal{F}^k . We will ultimately show that, for some $F^* \in \mathcal{F}^{\leq k}$, Painter will have to color a linear fraction of all edges which span a copy of F^* with the *k*-color (up to a permutation of the colors). To this end let us define the notion of a *dangerous* copy of F^*_- .

Definition 2.1. Let $F^* \in \mathcal{F}^k$. We say that an r-coloring of F_-^* is dangerous if the k - 1 copies of F_- whose roots were identified in the construction of F_-^* are all monochromatic and colored with pairwise different colors. We say that F_-^* is dangerous if it is colored with a dangerous coloring. We say that $F \times (e, F_-^*)$ is dangerous if all attached copies of F_-^* are colored according to the same dangerous coloring.

Assume that Painter has crated a dangerous copy of (e, F_{-}^{*}) where $F^{*} \in \mathcal{F}^{k}$. If *e* appears as an edge in a later round then Painter will be forced to color it with one of the remaining r - k + 1 colors or close a monochromatic copy of *F*. In particular if $F^{*} \in \mathcal{F}^{r}$ and Painter creates a dangerous copy of $F \times (e, F_{-}^{*})$ then Painter cannot color the inner copy of *F* (if it were to appear) without creating a monochromatic copy of *F*. The following Lemma states that Painter cannot avoid such dangerous copies of $F \times (e, F_{-}^{*})$.

Lemma 2.2. Fix a function p = p(n) satisfying $n^{-1/\overline{m}_2^r(F)} \ll p \ll n^{-1/\overline{m}_2^r(F)} \log n$. Then there exists constants c, C > 0 such that a.a.s. after Cn^2p rounds there either exists a monochromatic copy of F or we find a graph $(e, F^*) \in \mathcal{F}^r$ such that Painter has created $cn^{v_F} \left(n^{v_{F_-}^* - 2} p^{e_{F_+}^*}\right)^{e_F}$ dangerous copies of $F \times (e, F_-^*)$.

In other words at least a constant fraction of the copies of $F \times (e, F_{-}^*)$ are dangerous (for some $F^* \in \mathcal{F}^r$). Assuming the above Lemma the main result follows from a second moment argument similarly to the one presented in [11]. For completeness we restate the proof below. We shall also require the following proposition, whose proof we defer to Section 2.2.

Proposition 2.3. All rooted graphs $(e, F^*) \in \mathcal{F}^r$ satisfy

$$m(F \times (e, F^*)) \leq \overline{m}_2^r(F).$$

Proof of main theorem. We pause the game after $m = \Theta(n^2 p)$ rounds. Exploiting the asymptotic equivalence between $G_{n,m}$ and $G_{n,p}$ we consider the resulting graph to be distributed like a $G_{n,p}$.

For a graph *G* let the random variable X_G denote the number of copies of *G* in $G_{n,p}$. Let F^* denote the graph guaranteed by Lemma 2.2 and define $\tilde{F} := F \times (e, F^*)$ and $\tilde{F}_- := F \times (e, F^*_-)$. By Lemma 2.2 Painter has created $M = \Omega(X_{\tilde{F}_-})$ dangerous copies of \tilde{F}_- . We now consider these *M* copies to be fixed and for $i \in [M]$ denote with F_i the missing inner copy of *F* of the *i*-th copy of \tilde{F}_- . Observe that F_i and F_j are not required to be disjoint and may in fact be identical.

Observe that if Painter is forced to color one of the F_i in a future round then she must close a monochromatic copy of F and thus loose the game. We now show that indeed a.a.s. one of the F_i appears within the next $\Theta(n^2p)$ rounds.

Let Z_i denote the event that F_i appears and let $Z = \sum_{i=1}^M Z_i$. We have

$$\mathbf{E}[Z] = Mp^{e_F} = \Omega\left(\mathbf{E}\left[X_{\tilde{F}_{-}}\right]\right)p^{e_F} = \Omega(\mathbf{E}[X_{\tilde{F}}]) \stackrel{(*)}{=} \omega(1),$$

where (*) follows from Proposition 2.3. Furthermore

$$\begin{aligned} \operatorname{Var}[Z] &= \operatorname{E}\left[Z^{2}\right] - \operatorname{E}[Z]^{2} \\ &= \sum_{i,j} \operatorname{E}\left[Z_{i}Z_{j}\right] - \operatorname{E}[Z_{i}]\operatorname{E}\left[Z_{j}\right] \\ &= \sum_{\substack{G \subseteq F \\ e_{G} \geq 1}} \sum_{\substack{i,j \\ F_{i} \cap F_{j} \sim G}} p^{2e_{F}-e_{G}} - p^{2e_{F}} \\ &\leq \sum_{\substack{G \subseteq F \\ e_{G} \geq 1}} M_{G}p^{2e_{F}-e_{G}}, \end{aligned}$$

where M_G denotes the number of pairs of F_-^* whose (missing) inner copies of *F* intersect in a copy of *G*.

Fix $G \subseteq F$ and let H denote a graph obtained as the union of two copies of \tilde{F}_- whose (missing) inner copies intersect in a copy of G. Let T denote their intersection and T_+ the graph obtained by adding the missing edges of the inner copy of G to T. Observe that $T_+ \subseteq \tilde{F}$ and thus by Proposition 2.3 $E[X_{T_+}] = \omega(1)$.

$$\mathbf{E}[X_H] = \Theta\left(\frac{\mathbf{E}\left[X_{\tilde{F}_{-}}\right]^2}{\mathbf{E}[X_T]}\right) = \Theta\left(\frac{\mathbf{E}\left[X_{\tilde{F}_{-}}\right]^2 p^{e_G}}{\mathbf{E}\left[X_{T_{+}}\right]}\right) = o\left(\mathbf{E}\left[X_{\tilde{F}_{-}}\right]^2 p^{e_G}\right).$$

Since for every $G \subseteq F$ the number of choices for H is constant we have (over the first $\Theta(n^2p)$ rounds) $\mathbb{E}[M_G] = o\left(\mathbb{E}\left[X_{\tilde{F}_-}\right]^2 p^{e_G}\right)$ and thus by first moment method $M_G = o\left(\mathbb{E}\left[X_{\tilde{F}_-}\right]^2 p^{e_G}\right)$ a.a.s.

This implies (over the second set of $\Theta(n^2p)$ rounds) $\operatorname{Var}[Z] = o(\operatorname{E}[Z]^2)$ and thus $Z \ge 1$ a.a.s.

2.1 Proof of Lemma 2.2

Fix a function p = p(n) which satisfies $n^{-1/\overline{m}_2^r(F)} \ll p \ll n^{-1/\overline{m}_2^r(F)} \log n$. We will divide the game into a constant number of phases. In each phase we sample a copy of the binomial random graph $G_{n,p}$ and present its edges to Painter in random order (edges already presented in a previous phase are ignored). A.a.s. in each phase at most $\Theta(n^2p)$ edges are presented. Denote with $G_{n,p}^k$ the colored graph after k phases. We implicitly assume that $G_{n,p}^k$ does not contain a monochromatic copy of F.

As a main step in the proof we will show that for every set *S* of at most r-2 colors Painter must create a graph $G \in \mathcal{G}(K_t, \tilde{n}, m, \varepsilon)$ monochromatic in some color from $[r] \setminus S$ after a constant number of phases. In general we cannot expect that $m = \Omega(n^2 p)$ (for example a greedy Painter will only produce a single color class with this density). Instead we will require that $m = \Theta(n^{v_{F*}}p^{e_{F*}})$ for some $F^* \in \mathcal{F}^{|S|+1}$. To retain some control on these graphs we introduce the concept of an F^* -spanning subgraph.

Definition 2.4. For a rooted graph $(e, F^*) \in \mathcal{F}^k$ we say that a subgraph $G \subseteq G_{n,p}^k$ is F^* -spanning if for every edge $e \in E(G)$ there exists $F^*(e) \sim F^*$ in $G_{n,p}^k$ such that

- 1. the endpoints of e are the roots of $(e, F^*(e))$,
- 2. all non root vertices of $F^*(e)$ lie outside of V(G) and,
- 3. $F^*(e)$ and $F^*(e')$ are edge disjoint for all $e' \in E(G) \setminus \{e\}$.

We shall see later that an F^* -spanning subgraph behaves like a $G_{n,q}$ with $q = n^{v_{F^*}-2}p^{e_{F^*}}$ in the sense that we obtain bounds on its maximum degree as well as exponential upper bounds on the number of edges between linear sized vertex sets.

We are now in a position to state the main Lemma of this subsection.

Lemma 2.5. Fix a set *S* of at most r - 2 colors and an integer $t \ge 2$. Then there exists a positive integer *k* and a constant $\delta > 0$ such that for every $\varepsilon > 0$ there exists $\eta > 0$ such that for $p = \omega \left(n^{-1/\overline{m}_2^r(F)} \right)$ a.a.s. in $G_{n,p}^k$ we find a subgraph $G \in \mathcal{G}(K_t, \tilde{n}, m, \varepsilon)$ which is monochromatic in some color from $[r] \setminus S$ and F^* -spanning in $G_{n,p}^k$ where $F^* \in \mathcal{F}^{\leq |S|+1}$, $m \ge \eta n^{v_{F^*}} p^{e_{F^*}}$ and $\tilde{n} \ge \eta n$.

Furthermore for every choice of \tilde{n} , m and graphs $F^* \in \mathcal{F}^{\leq |S|+1}$ and G' with $|E(G')| = \omega(n)$ the probability that the statement nominates \tilde{n} , m, F^* and $G \supseteq G'$ is at most

$$\left(\frac{m}{\tilde{n}^2\delta}\right)^{|E(G')|}$$

It is crucial that in the probability bound we loose only a constant factor (the δ) independently of the requested regularity (as opposed to the density of *G* which depends on $\eta(\varepsilon)$).

The next lemma states that this is the density guaranteed by Lemma 2.5 has the right order of magnitude in the sense that if we forbid |S| colors then the resulting graph should have high enough density for Painter to loose the game with r - |S| colors.

Lemma 2.6. Every $F^* \in \mathcal{F}^k$ satisfies

$$n^{v_{F^*}-2}n^{-e_{F^*}/\overline{m}_2^r(F)} > n^{-1/\overline{m}_2^{r-k+1}(F)}$$

provided that $k \leq r$.

Proof. The proof proceeds by induction on k. The singular graph in \mathcal{F}^1 consists of a single edge and thus the statement holds for k = 1.

For $k \ge 2$ let F_1^*, \ldots, F_{k-1}^* denote the graphs used during the construction of $F^* \in \mathcal{F}^k$. Writing $p_i = n^{-1/\overline{m}_2^i(F)}$ we have

$$n^{v_{F^*}-2}p_r^{e_{F^*}} = p_r \prod_{i < k} n^{v_F-2} \left(n^{v_{F^*_i}-2} p_r^{e_{F^*_i}} \right)^{e_F-1} \stackrel{(*)}{\geq} p_r \prod_{i < k} n^{v_F-2} p_{r-i+1}^{e_F-1}, \quad (1)$$

where (*) follows from the induction hypothesis.

By definition of \overline{m}_2^i we have

$$n^{v_F-2}p_{r-i+1}^{e_F} \ge p_{r-i}$$

and thus (1) is at least

$$p_r \prod_{i < k} p_{r-i} / p_{r-i+1} = p_{r-k+1}.$$

We will give a detailed proof of Lemma 2.5 below. Before that we will walk through the main argument and state a number of auxiliary lemmas.

Assume that (by induction) we have found graphs G_1, G_2, \ldots, G_k with

$$G_i \in \mathcal{G}\left(F_{-}, \tilde{n}, \Theta\left(n^{-1/\overline{m}_2^{r-i+1}(F)}\right), \varepsilon\right),$$

which are monochromatic in pairwise different colors. Assume furthermore that G_i is F^i -spanning for some $F^i \in \mathcal{F}^i$ and that the partitions V_a, V_b corresponding to the missing edge of $F_- = F - \{\{a, b\}\}$ are the same for all G_i . Through repeated application of Theorem 1.13 we will be able to count the number of copies of $\bigsqcup_{i \leq k} (e, F_-)$ in $\bigsqcup_{i \leq k} G_i$ (where the *i*-th copy of F_- is to be from G_i).

We expect to find roughly

$$n^{2} \prod_{i \le k} n^{v_{F}-2} n^{-(e_{F}-1)/\overline{m}_{2}^{r-i+1}(F)} = n^{2} \prod_{i \le k} \frac{n^{-1/\overline{m}_{2}^{r-i}(F)}}{n^{-1/\overline{m}_{2}^{r-i+1}(F)}} = n^{2-1/\overline{m}_{2}^{r-k}(F)+1/\overline{m}_{2}^{r}(F)}$$
(2)

such graphs. The 2-density of *F* is strictly above $\overline{m}_2^r(F)$. Since *F* is 2-balanced we have $m_2(F) = m(e, F) = m(e, F_-)$ and Lemma 1.5 implies that every pair from $V_a \times V_b$ spans at most a constant number of copies of (e, F_-) . Thus the number of pairs in $V_a \times V_b$ which span a copy of F_- in each of the graphs G_i is of the same order of magnitude as (2).

Out of these pairs roughly $n^{2-1/\overline{m}^{r-k}(F)}$ will appear as actual edges if we present Painter with another set of n^2p edges. If Painter wants to avoid a monochromatic copy of *F* then she is forced to color these edges with colors distinct from those used in G_1, \ldots, G_k . Furthermore since all the G_i were F^i -spanning these edges all span a copy of

$$\bigsqcup_{i\leq k} (e,F)\times (e,F^i)=F^*\in \mathcal{F}^{k+1}.$$

We are below the 2-density of F^* (which equals that F) and therefore the following Lemma tells us that this edge set can be turned into an F^* -spanning subgraph by discarding a negligible number of edges.

Lemma 2.7. Suppose that *F* is a 2-balanced graph and that $F_1, F_2 \sim F$ intersect in at least one, but not all edges. Then $n^{v_F}p^{e_F} \gg n^{v_{F_1 \cup F_2}}p^{e_{F_1 \cup F_2}}$ provided that $p = o\left(n^{-1/m_2(F)}\right)$.

Proof. For a graph *H* write $X_H = n^{v_H} p^{e_H}$. Let $G = F_1 \cap F_2$. Since *F* is 2-balanced and $v_F, v_G \ge 2$ we have

$$\frac{X_F}{n^2 p} = \left(\frac{p}{n^{-1/m_2(F)}}\right)^{e_F - 1}$$
 and $\frac{X_G}{n^2 p} \ge \left(\frac{p}{n^{-1/m_2(F)}}\right)^{e_G - 1}$

For $p = o\left(n^{-1/m_2(F)}\right)$ we obtain

$$\frac{X_F}{X_{F_1 \cup F_2}} = \frac{X_F}{\frac{X_F^2}{X_G}} = \frac{X_G}{X_F} \ge \left(\frac{p}{n^{-1/m_2(F)}}\right)^{e_G - e_F} = \omega(1),$$

as desired.

Finally we will want to apply the sparse regularity lemma to this F^* -spanning subgraph. For this we need it to be upper-uniform, which is confirmed in the following lemma.

Lemma 2.8. Suppose that $p = \omega(n^{-1/\overline{m}_2^r(F)})$. Let $(e, F^*) \in \mathcal{F}^{\leq r-1}$. Then for every $\eta > 0$ a.a.s. every F^* -spanning subgraph G of G(n, p) with at least ηn vertices is $(\eta, n^{v_{F^*}-2}p^{e_{F^*}})$ -upper-uniform.

Proof. The lemma follows from the following extension of the standard Chernoff bound:

Theorem 2.9 ([12]). Let X_1, \ldots, X_n be a sequence of not necessarily independent Bernoulli-distributed random variables which satisfy $\Pr[\bigwedge_{i \in S} X_i] \leq q^{|S|}$ for all subsets $S \subseteq [n]$. Then for $0 < \varepsilon \leq 1$

$$\Pr\left[\sum_{i=1}^{n} X_i \ge (1+\varepsilon)qn\right] \le e^{-nq\varepsilon^2/3}.$$

For fixed vertex sets $V_1, V_2 \subseteq_{\eta^2} V$ let $G \subseteq G(G_{n,p})$ denote a (canonical) (e, F^*) -spanning graph in $G_{n,p}$ which maximizes the number of edges between V_1 and V_2 . For $e \in E(K_n[V_1, V_2])$ let X_e denote the indicator random variable for the event $e \in G$. We have for every set S

$$\Pr\left[\bigwedge_{e\in S} X_e\right] \leq \Pr\left[S \text{ is } (e, F^*) \text{-spanning in } G_{n,p}\right] \leq \left(n^{v_{F^*}-2} p^{e_{F^*}}\right)^{|S|}.$$

And thus by Theorem 2.9

$$\Pr\Big[|E_G(V_1, V_2)| \ge (1+\eta)|V_1||V_2|n^{v_{F^*}-2}p^{e_{F^*}}\Big] \le e^{-\Theta(n^{v_{F^*}}p^{e_{F^*}})}.$$

Since

$$n^{v_{F^*}} p^{e_{F^*}} \overset{Lemma \ 2.6}{\gg} n^{2-1/\overline{m}_2^1(F)} \ge n^{2-1/\overline{m}_2^1(F)}$$

a union bound over at most 4^n choices for V_1 and V_2 proves the Lemma.

We thus obtain an F^* -spanning graph $G \in \mathcal{G}(K_2, \tilde{n}, \Theta(n^{v_{F^*}}p^{e_{F^*}}), \varepsilon)$. Repeating the argument a constant number of times (by exposing more edges inside one of the two partitions of *G*) one can obtain a monochromatic graph $G_{k+1} \in \mathcal{G}(K_t, \tilde{n}, \Theta(n^{v_{F^*}}p^{e_{F^*}}), \varepsilon)$ as required to finish the induction.

This argument can be repeated as long as $|S| \leq r - 2$. One could hope to iterate one more time and find a graph $G_r \in \mathcal{G}(F_-, \tilde{n}, \Theta(n^{2-1/\overline{m}_2^1(F)}), \varepsilon)$. This approach is bound to fail. The density of G_r is (in general) not above the 2-density of F_- so we cannot hope to find copies of F_- in G_r . Instead we

find graphs G_1, \ldots, G_{r-1} , where $G_i \in \mathcal{G}(F \times (e, F_-), \tilde{n}, \Theta(n^{2-1/\overline{m}^{r-i+1}(F)}), \varepsilon)$, whose *inner* partitions agree and use Theorem 1.13 to show directly that many v_F -tuples span copies of $F \times (e, F_-)$ in all the G_i .

Before formalizing the above we need two more auxiliary lemmas. The first one asserts that the density of G_i is indeed large enough to apply Theorem 1.13.

Lemma 2.10. Suppose that *F* is a 2-balanced graph, which contains an edge *e* such that $m_2(F - \{e\}) \le \overline{m}_2^2(F)$. Then for all $r \ge k \ge 2$

$$\overline{m}_{2}^{k}(F) \geq m_{2}(e, F, +1/\overline{m}_{2}^{k}(F) - 1/\overline{m}_{2}^{r}(F)),
\overline{m}_{2}^{k}(F) \geq m_{2}(V(F), F \times (e, F), +1/\overline{m}_{2}^{k}(F) - 1/\overline{m}_{2}^{r}(F)).$$

Secondly Theorem 1.13 requires the (hyper)-graph to be upper-extensible. For us this hypergraph will consist of all pairs (all v_F -tuples) which already span a copy of F_- (of $F \times (e, F_-)$) in all graphs G_1, \ldots, G_i . By Theorem 1.4 it suffices to show that we are above the rooted density of the corresponding graphs:

Lemma 2.11. Let $F^* \in \mathcal{F}^k$ where $k \geq 2$. Then

$$m_1(F_-^*) < \overline{m}_2^k(F)$$

and for every $V_0 \subsetneq V(F)$

$$m(V_0, (V_0, F) \times (e, F_-^*)) < \overline{m}_2^k(F).$$

We can now state the proof of Lemma 2.5.

Lemma 2.5. Fix a set *S* of at most r - 2 colors and an integer $t \ge 2$. Then there exist a positive integer *k* and a constant $\delta > 0$ such that for every $\varepsilon > 0$ there exists $\eta > 0$ such that for $p = \omega \left(n^{-1/\overline{m}_2^r(F)} \right)$ a.a.s. in $G_{n,p}^k$ we find a subgraph $G \in \mathcal{G}(K_t, \tilde{n}, m, \varepsilon)$ which is monochromatic in some color from $[r] \setminus S$ and F^* -spanning in $G_{n,p}^k$ where $F^* \in \mathcal{F}^{\leq |S|+1}, m \geq \eta n^{v_{F^*}} p^{e_{F^*}}$ and $\tilde{n} \geq \eta n$.

Furthermore for every choice of \tilde{n} , m and graphs $F^* \in \mathcal{F}^{\leq |S|+1}$ and G' with $|E(G')| = \omega(n)$ the probability that the statement nominates \tilde{n} , m, F^* and $G \supseteq G'$ is at most

$$\left(\frac{m}{\tilde{n}^2\delta}\right)^{|E(G')|}$$

Proof. The proof follows by induction on |S| and t. For t = 2 and $S = \emptyset$ we apply the sparse regularity lemma (Theorem 1.9) to the majority color class (in that case $F^* = (e, e)$ is just an edge - the unique graph in \mathcal{F}^1).

t **step:** Fix t > 2 and a set *S*. We will apply the induction hypothesis (for $t \leftarrow 2$ and $S \leftarrow S$) $K = rt |\mathcal{F}^{\leq |S|+1}|$ times. Let k', δ' denote the absolute constants guaranteed for $t \leftarrow 2$. Denote with ε_i the value which we will use for ε in the *i*-th application of the induction hypothesis and let $\eta_i(\varepsilon_i)$ denote the guaranteed constant. Our choice for ε_i will depend only on the constants ε_i, η_i where j > i and on the requested ε .

Apply the induction hypothesis once to $G_{n,p}^{k'}$ for $t \leftarrow 2$ and obtain an (ε_1) -regular graph $G_1 \in \mathcal{G}(K_2, \tilde{n}_1, m_1, \varepsilon_1)$. Let $V_1 \subset V(G)$ denote one

of its vertex partitions. We then ask painter to color another $G_{n,p}^{k'}$ but we look only at the subgraph induced by V_1 (which is distributed like a $G_{|V_1|,p}^{k'}$). Since $|V_1| \ge \eta_1 n$ we have $p = \omega \left(|V_1|^{-1/\overline{m}_2^r(F)} \right)$ and thus we can apply the induction hypothesis a second time to obtain an (ε_2) regular graph G_2 whose edges are fully contained in V_1 . We repeat this procedure K times and obtain a sequence of sets $V_1 \supset V_2 \supset \cdots \supset V_K$ and nested graphs G_1, \ldots, G_K , where $G_i \in \mathcal{G}(K_2, \tilde{n}_i, m_i, \varepsilon_i)$. Every such G_i nominates a color and a graph $F_i^* \in \mathcal{F}^{\leq |S|+1}$ out of at most $r|F_{\leq |S|+1}|$ choices. By the pigeonhole principle we may thus fix a subset $T \subseteq [K]$ of size t such that all graphs G_i with $i \in T$ nominate the same color and the same graph $F^* \in \mathcal{F}^{\leq |S|+1}$.

Let $\tilde{n} \coloneqq \tilde{n}_K$ denote the size of the vertex partitions of G_K . We arbitrarily pick sets $\overline{V_i}$ of size \tilde{n} such that

$$\overline{V_1} \subseteq V(G_1) \setminus V_1,$$
$$\vdots$$
$$\overline{V_K} \subseteq V(G_K) \setminus V_K.$$

Finally set $\overline{V_{K+1}} = V_K$. These sets are pairwise disjoint and for every pair i < j we have $\overline{V_j} \subseteq V_{j-1} \subseteq V_i$. Therefore the sets $\overline{V_i}, \overline{V_j}$ are subsets of the two partitions of G_i and for ε_i small enough, depending on ε , $\eta_{i+1}, \ldots, \eta_K$, the induced bipartite graph $G_i[\overline{V_i}, \overline{V_j}]$ is $(\varepsilon/2)$ -regular with at least half the density. Let

$$m = \min_{\substack{i,j \in T \\ i < j}} \left| E\left(G_i\left[\overline{V}_i, \overline{V}_j\right]\right) \right|$$

and pick for every $i, j \in T$, i < j a subgraph $G_{i,j} \subset G_i[\overline{V}_i, \overline{V}_j]$ with exactly *m* edges u.a.r. among all subgraphs with *m* edges. By Lemma 2.6 we have

$$m = \Omega(n^{v_{F^*}} p^{e_{F^*}}) \gg n^{2-1/\overline{m}_2^{r-|S|+1}(F)} \ge n^{2-1/\overline{m}_2^{r-|S|+1}(F)}$$

and thus these graphs $G_{i,i}$ will be (ε) -regular with high probability. Since

$$ilde{n} \ge n \prod_{i \in [K]} \eta_i$$
 and $m \ge \eta_k ilde{n}^{v_F^*} p^{e_F^*}$

we may set

$$G := \bigcup_{\substack{i,j\in T\\i< i}} G_{i,j} \in \mathcal{G}(K_t, \tilde{n}, m, \varepsilon).$$

Furthermore we claim that the graphs $F^*(e)$ guaranteed by the invocations of the induction hypothesis are pairwise edge disjoint. This is because for $e \in E(G_i)$ the graph $F^*(e)$ has no edges inside V_i , but for j > i the graphs $F^*(e')$, $e' \in E(G_j)$ lie completely inside V_i . Thus *G* is F^* -spanning.

Finally we have to calculate the probability that $G' \subseteq G$ for some graph G' with $\omega(n)$ edges. To do so fix \tilde{n} , m and the sets \overline{V}_i among $2^{\Theta(n)}$

possibilities. We may assume that all edges of G' go between the sets \overline{V}_i . Since G is the (disjoint) union of random subgraphs of $G_i[\overline{V}_i, \overline{V}_j]$ and $|E(G_i[\overline{V}_i, \overline{V}_j])| \geq \tilde{n}^2(m_i/2\tilde{n}_i^2)$ whose density is at least half as large as the density of G_i the probability that $G' \subseteq G$ is then at most

$$\begin{split} &\prod_{i< j\in T} \Pr\left[G'[\overline{\nabla}_i, \overline{\nabla}_j] \subseteq G_i[\overline{\nabla}_i, \overline{\nabla}_j]\right] \left(\frac{m}{|E(G_i[\overline{\nabla}_i, \overline{\nabla}_j])|}\right)^{|E(G'[\overline{\nabla}_i, \overline{\nabla}_j])|} \\ &\leq \prod_{i< j\in T} \left(\frac{m_i}{\tilde{n}_i^2 \delta'} \frac{m}{|E(G_i[\overline{\nabla}_i, \overline{\nabla}_j])|}\right)^{|E(G'[\overline{\nabla}_i, \overline{\nabla}_j])|} \\ &\leq \prod_{i< j\in T} \left(\frac{2m}{\delta' \tilde{n}^2}\right)^{|E(G'[\overline{\nabla}_i, \overline{\nabla}_j])|} = \left(\frac{2m}{\delta' \tilde{n}^2}\right)^{|E(G')|}. \end{split}$$

Allowing some room for a union bound over the choices for \tilde{n} , m and the sets \overline{V}_i we may fix $\delta = \delta'/3$, $\eta = \eta_K \prod_{i \in [K]} \eta_i^{v_{F^*}}$ and k = Kk'.

S **step:** Fix a nonempty set *S* of at most r - 2 colors and assume that the statement holds for all sets containing fewer than |S| colors. Our goal is to show that then the statement holds for *S* and t = 2.

Similarly to what we did in the induction step for *t* we apply the induction hypothesis |S| times in a nested fashion. As before let k', δ' denote absolute constants for which the induction hypothesis holds for $t \leftarrow v_F - 1$ and all subsets of *S*. Denote with ε_i the value which we will use for ε in the *i*-th application of the induction hypothesis and let $\eta_i(\varepsilon_i)$ denote the guaranteed constant. Again ε_i will depend only on ε_j , η_j , where j > i. Crucially ε_i will not depend on the requested ε and $\varepsilon_{|S|}$ will be an absolute constant. Finally for the *i*-th invocation we will pick *S* as the set of colors of G_1, \ldots, G_{i-1} (thus $S \leftarrow \emptyset$ for i = 1).

As before we obtain monochromatic graphs $G_1, \ldots, G_{|S|}$, such that $G_i \in \mathcal{G}(K_{v_F-1}, \tilde{n}_i, m_i, \varepsilon_i)$ and $V(G_i) \subseteq V_{i-1}$ for $V_0 = V$ and where V_i is an arbitrary partition of G_i .

Assume that one of the G_i is monochromatic in a color from $[r] \setminus S$. The density of G_i is in $\Theta(n^{v_{F_i^*}-2}p^{e_{F_i^*}})$, where the constant does not depend on ε (since $\varepsilon_1, \ldots, \varepsilon_{|S|}$ do not depend on ε). Furthermore by Lemma 2.8 G_i is $(o(1), n^{v_{F_i^*}-2}(|S|k'p)^{e_{F_i^*}})$ -upper-uniform. Thus we can apply the sparse regularity lemma (Theorem 1.9) to G_i and obtain a graph from $\mathcal{G}(K_2, \tilde{n}, m, \varepsilon)$ whose density is of the same order as the density of G_i and we are done.

Otherwise all of the G_i are monochromatic in distinct colors of S. We want to show that in $V_{|S|}$ there are many pairs of vertices which span a copy of (e, F_-) in each of the G_i . To this end we define the auxiliary directed graphs A_i . Let A_0 denote the complete directed graph on V. For i = 1, ..., |S| the vertex set of A_i is V_i and we connect two vertices $x \neq y \in V_i$ if $(x, y) \in E(A_{i-1})$ and if (x, y) span a partite copy of F_- in

 G_i (partite with respect to the non root vertices, *x* and *y* lie in the same partition).

Define $(e, F^*) := \bigsqcup_{j \in [|S|]} (e, F) \times (e, F_i^*) \in \mathcal{F}^{|S|+1}$. By definition of $A_{|S|}$ every edge $e \in E(A_{|S|})$ spans a copy of (e, F_-) in each of the G_i . Furthermore since every edge of G_i spans a copy of (e, F_i^*) the edge e spans a copy of (e, F_i^*) . Finally observe that since the G_i are monochromatic in pairwise different colors of S the edge e (if it would be presented to Painter in some later round) has to be colored with some color from $[r] \setminus S$.

We will need the following auxiliary claim about the density of $A_{|S|}$ whose proof we defer.

Claim 2.12. For every integer $i \leq |S|$ and $\kappa > 0$ and small enough $\varepsilon_1, \ldots, \varepsilon_i$ there exists $\gamma(\kappa, \varepsilon_1, \ldots, \varepsilon_i) > 0$ such that a.a.s. for all disjoint and equi-sized subsets $X, Y \subseteq_{\kappa} V_i$ the induced bipartite subgraph $A_i[X, Y]$ contains at least $\gamma|X||Y|\prod_{i=1}^i p_i$ edges, where

$$p_j = n^{v_F - 2} \left(n^{v_{F_j^*} - 2} p^{e_{F_j^*}} \right)^{e_F - 1}.$$

We invoke the claim for $i \leftarrow |S|$ and $\kappa \leftarrow 1/4$ to lower bound the number of pairs $x, y \in V_{|S|}$ which span a dangerous copy of F^* by

$$\frac{\gamma}{2} \left(\frac{\left| V_{|S|} \right|}{2} \right)^2 \prod_{j=1}^{|S|} p_i \ge \gamma' n^{v_{F^*}} p^{e_{F^*}-1} \overset{Lemma \ 2.6}{\gg} \frac{n^{2-1/\overline{m}_2^{r-|S|}(F)}}{p} \gg \frac{n}{p},$$

where γ is the constant guaranteed by the claim and $\gamma' = \gamma (\prod_i \eta_i)^2 / 8$ is an absolute constant, which in particular does not depend on ε .

We then present another $G_{n,p}$ to Painter. Painter will be forced to color at least a p/r-fraction of the edges in $A_{|S|}$ with some color from $[r] \setminus S$ (or create a monochromatic copy of F). Thus we obtain a monochromatic set of $\gamma' n^{v_{F*}} p^{e_{F*}}/r$ edges which all span a copy of F^* . Next we remove all edges whose copies of F^* intersect. Lemma 2.7 together with Markov's inequality implies that with probability 1 - o(1) we remove only $o(n^{v_{F*}} p^{e_{F*}})$ edges.

So we are left with a (e, F^*) -spanning set of at least $\gamma' n^{v_{F^*}} p^{e_{F^*}}/2 \gg n$ edges E'. By Lemma 2.8 E' is $(o(1), n^{v_{F^*}-2}((|S|k'+1)p)^{e_{F^*}})$ -upperuniform. Therefore we may apply the sparse regularity lemma to E' and obtain a graph from $\mathcal{G}(K_2, \tilde{n}, m, \varepsilon)$ whose density is a constant fraction of the density of E'.

Finally the probability that a fixed set of *s* edges is (e, F^*) spanning is at most $(n^{v_{F^*}-2}((|S|k'+1)p)^{e_{F^*}})^s$. Since $m/\tilde{n}^2 = \Omega(n^{v_{F^*}-2}p^{e_{F^*}})$ (not depending on ε) the probability bound holds for some δ .

It remains to prove Claim 2.12. We proceed by induction on *i*. A_0 is complete and thus the base case i = 0 holds vacuously. So let $i \ge 1$

and fix some $\kappa > 0$. Denote the lower bound on the density of A_{i-1} guaranteed by the induction by

$$q = \Omega\left(\prod_{j < i} p_j\right) \gg \frac{n^{-1/\overline{m}_2^{r-i+1}(F)}}{p}.$$

We define $\beta = (\delta'/(3e))^{e_{F_{-}}}$ and

$$A = \frac{\prod_{j=1}^{i-1} p_j}{q\kappa \prod_{j=1}^{i} \eta_j} = \Theta(1).$$

Let $\mu(\beta)$ and $\alpha(A,\beta)$ denote the constant guaranteed by Theorem 1.13 when invoked with $(R,F) \leftarrow (e,F)$, $\beta \leftarrow \beta$ and $A \leftarrow A$. Fix disjoint equi-sized sets $X, Y \subseteq_{\kappa} V_i$. Write $m = \lceil |X| |Y| m_i / (2\tilde{n}_i^2) \rceil$ and pick any subgraph $G'_i \subseteq G_i$ from $\mathcal{G}(F_-, |X|, m, \mu)$ such that its partitions which correspond to the roots of F_- are X and Y (taking suitable vertex sets and a random subset of m edges from each partition succeeds with probability $1 - 2^{-\Theta(m)}$).

If $T(G', A_{i-1}[X, Y])$ contains at least

$$\alpha\left(\frac{m}{|X|^2}\right)^{e_{F_-}}|X|^{v_F}q = \Theta\left(n^2\prod_{j\leq i}p_j\right)$$

edges, then since by Lemma 1.5 every edge *e* spans at most a constant number of copies of F_- the density of $A_i[X, Y]$ is of the correct order of magnitude.

Otherwise we want to apply Theorem 1.13 with $G \leftarrow G'$ and $G_R \leftarrow A_{i-1}[X, Y]$ (viewed as an undirected graph) and $q \leftarrow q$. To apply Theorem 1.13 it suffices to check the following:

1. The number of edges in G' is in

$$\Omega\left(n^{v_{F_{i}^{*}}}p^{e_{F_{i}^{*}}}\right) \overset{Lemma \ 2.6}{\gg} n^{2-1/\overline{m}_{2}^{r-i+1}(F)} \overset{Lemma \ 2.10}{\geq} n^{2-1/m_{2}(e,F,-\log_{n}q)},$$

since for *n* large enough $-\log_n q \le 1/\overline{m}_2^{r-i+1}(F) - 1/\overline{m}_2^r(F)$.

- 2. If we invoke the induction hypothesis with say $\kappa \leftarrow \kappa \eta_i \mu$ then $A_{i-1}[X, Y]$ is (F, μ, q) -lower-regular.
- 3. To see that it is also (F, Aq)-upper-extensible observe that every edge of A_{i-1} spans a copy of of $(e, F'_{-}) := \bigsqcup_{j < i} (e, F_{-}) \times (e, F_{j}^{*})$. So for upper uniformity it suffices to bound the number of copies of F'_{-} spanned by a single vertex. But our p is such that we are above the 1-density of F'_{-} (Lemma 2.11). Thus this number is concentrated around its expectation (Theorem 1.4), which is upper bounded by

$$n^{v_{F'^*}-1}p^{e_{F'^*}} = n\prod_{j< i} p_j \le |X| \frac{\prod_{j< i} p_j}{\kappa \prod_{j\le i} \eta_j} = Aq|X|.$$

Therefore we can apply Theorem 1.13 and *G*' must come from a set of at most $2 \int_{a}^{b} e^{-m} dx = 2 \int_{a}^{b} e^{-m} dx$

$$\beta^m \binom{|X|^2}{m}^{e_{F_-}} \leq \beta^m \left(\frac{e|X|^2}{m}\right)^{e_{F_-}m} \leq \beta^m \left(\frac{2e\tilde{n}_i^2}{m_i}\right)^{e_{F_-}}$$

graphs. But then for our choice of β the probability that $G' \subseteq G_i$ is in o(1).

The proof of Lemma 2.2 proceeds similarly to the proof of Claim 2.12 from the previous lemma. The only difference is that we replace A_i with the hypergraph of v_F -tuples which span a copy of $F \times (e, F_-)$ in each of the graphs G_i .

Lemma 2.2. Fix a function p = p(n) satisfying $n^{-1/\overline{m}_2^r(F)} \ll p \ll n^{-1/\overline{m}_2^r(F)} \log n$. Then there exists constants c, C > 0 such that a.a.s. after Cn^2p rounds there either exists a monochromatic copy of F or we find a graph $(e, F^*) \in \mathcal{F}^r$ such that Painter has created $cn^{v_F} \left(n^{v_{F^*_{-2}}}p^{e_{F^*_{-2}}}\right)^{e_F}$ dangerous copies of $F \times (e, F^*_{-})$.

Proof. Let $t = v_F - 1$ and let k, δ be such that Lemma 2.5 holds for all $S \subseteq [r]$, $|S| \leq r - 2$. Let $\varepsilon_1, \ldots, \varepsilon_{r-1}$ denote constants whose value we will determine later in reverse order (that is ε_i will depend on $\varepsilon_{i+1}, \ldots, \varepsilon_{r-1}$.)

We ask Painter to color an instance of $G_{n,p}^k$. Applying Lemma 2.5 with $t \leftarrow t, S \leftarrow \emptyset, \varepsilon \leftarrow \varepsilon_1$ we obtain a constant $\eta_1(\varepsilon_1)$, a graph $F_1^* \in \mathcal{F}^1$ and an F_1^* -spanning graph $G_1 \in \mathcal{G}(K_t, \tilde{n}_1, m_1, \varepsilon_1)$ monochromatic in some color s_1 . Pick one of the vertex partitions of G_1 arbitrarily and call it V_1 . We now present Painter with a second instance of $G_{n,p}^k$ but only consider the subgraph induced by V_1 which is distributed like a $G_{\tilde{n}_1,p}^k$. Invoking Lemma 2.5 a second time with $\varepsilon \leftarrow \varepsilon_2$ and $S \leftarrow \{s_1\}$ we obtain a second graph $G_2 \in \mathcal{G}(K_t, \tilde{n}_2, m_2, \varepsilon_2)$. We repeat this procedure r - 1 times and obtain

- 1. sets $V = V_0 \supset V_1 \supset \cdots \supset V_{r-1}$ such that $|V_i| \ge \eta_i |V_{i-1}|$ for $i \in [r-1]$,
- 2. graphs $F_i^* \in \mathcal{F}^{\leq i}$ where $i \in [r-1]$,
- 3. monochromatic graphs $G_i \subseteq \mathcal{G}(K_t, \tilde{n}_i, m_i, \varepsilon_i) \subseteq G_{n,p}^{i,k}[V_{i-1}]$ in pairwise different colors, where $\tilde{n}_i = |V_i|, m_i \ge \eta_i \tilde{n}^{v_{F_i^*}} p^{e_{F_i^*}}$ such that G_i is F_i^* -spanning in $G_{n,p}^{i,k}[V_{i-1}]$.

Furthermore for every graph G' with $\omega(n)$ edges we have

$$\Pr[G' \subseteq G_i] \le \left(\frac{m_i}{\delta \tilde{n}_i^2}\right)^{|E(G')|}$$

Observe that this probability is over the phases $(i - 1) \cdot k + 1, \dots, i \cdot k$ and that G_i is fixed after the first $i \cdot k$ phases.

Let A_0 denote the complete directed v_F -uniform hypergraph on $V = V_0$. We identify the edges of A_0 with a (hypothetical) copy of F in V (depending on the automorphisms of F different (directed) edges might represent the same copy of F). Now for $i \in [r-1]$ let A_i denote the directed v_F -uniform hypergraph on $V_i \subseteq V_{i-1}$ where $e \in E(A_i)$ if $e \in E(A_{i-1})$ and additionally the edges of e (when viewed as a graph $F(e) \sim F$) are the roots of pairwise edge disjoint copies of F_- in G_i .

Define

$$\begin{split} \tilde{F}_{i}^{*} &= \bigsqcup_{j < i} (e, F) \times (e, F_{j}^{*}) \in \mathcal{F}^{i}, \\ p_{i} &= n^{v_{F}-2} \left(n^{v_{F_{i}^{*}}-2} p^{e_{F_{i}^{*}}} \right)^{e_{F}-1}, \\ q_{i} &= \prod_{j \in [i]} p_{i} = n^{v_{\bar{F}_{i+1}}-2} p^{e_{\bar{F}_{i+1}}-1}. \end{split}$$

Since G_i is F_i^* -spanning the edges of F(e) (for $e \in A_i$) are not only roots of pairwise edge disjoint copies of F_- but even of (still pairwise edge disjoint) copies of $F_- \times (e, F_i^*)$. Furthermore these copies are disjoint from those certifying membership in A_1, \ldots, A_{i-1} . Thus for every $e \in E(A_i)$ the edges of F(e) are the roots of pairwise edge disjoint copies of $(\tilde{F}_{i+1})_-$.

Claim 2.13. Suppose that $X_1, \ldots, X_{v_F} \subseteq V_i$ are mutually disjoint and of size \tilde{n} . Then $A_i[X_1, \ldots, X_{v_F}]$, when viewed as a undirected hypergraph from $\mathcal{R}(V(F), \tilde{n})$, is $(F, (n/\tilde{n})^{v_F}q_i)$ -upper-extensible provided that $i \leq r - 1$.

Proof of claim. $q_0 = 1$ and thus the claim holds vacuously for i = 0. For $i \ge 1$ fix some $\sigma \subseteq V(A_i)$. If there exists e such that $\sigma \subsetneq e \in E(A_i)$ then σ fixes some $V' \subsetneq V(F)$ and the degree of σ is at most the number of copies of $(V', F) \times (e, (\tilde{F}_{i+1}^*)_{-})$ rooted in σ . By Lemma 2.11

$$m(V', (V', F) \times (e, (\tilde{F}_{i+1}^*)_{-})) < \overline{m}_2^{i+1}(F) \le \overline{m}_2^r(F)$$

and thus by Theorem 1.4 this number is concentrated around its expectation which is at most $n^{v_F - |V'|} q_i^{e_F - e_{F[V']}}$ as required for the upper-extensibility of A_i .

Claim 2.14. For every integer $i \leq r-1$ and $\kappa > 0$ and small enough $\varepsilon_1, \ldots, \varepsilon_i$ there exists $\gamma(\kappa, \varepsilon_1, \ldots, \varepsilon_i) > 0$ such a.a.s. for all pairwise disjoint equi-sized sets $X_1, \ldots, X_{v_F} \subseteq_{\kappa} V_i$ the number of directed edges in $E(A_i[X_1, \ldots, X_{v_F}])$ is at least $\gamma|X_1|^{v_F}q_i^{e_F}$.

Invoking the claim for i = r - 1 and say $\kappa = 1/(2v_F)$ proves the Lemma.

To prove the claim we will proceed by induction on *i*. For i = 0 the statement holds vacuously with $\gamma = 1$ since A_0 is complete. For $i \ge 1$ we fix equi-sized and pairwise disjoint sets $X_1, \ldots, X_{v_F} \subseteq_{\kappa} V_i$ of size \tilde{n} . Define

$$m = \left[\frac{\tilde{n}^2 m_i}{2e_F \tilde{n}_i^2}\right]$$
 and $\beta = \left(\frac{\delta}{4e_F e}\right)^{e_F \cdot (e_F - 1)}$

Let $\mu = \mu(\beta)$ be given by Theorem 1.13 (invoked with $(R, F) \leftarrow (V(F), F \times (e, F_{-}))$) and let $\gamma' = \gamma(\kappa \eta_i \mu, \varepsilon_1, \dots \varepsilon_{i-1})$ denote the constant guaranteed by the induction hypothesis. Finally define

$$A = \frac{1}{\gamma' \left(\kappa \prod_{j \in [i]} \eta_j \right)^{v_F}}.$$

 $F \times (e, F_-)$ is *t*-partite and thus for ε_i small enough depending on μ we can, using standard techniques, find a graph $G'_i \subseteq G_i$ from $\mathcal{G}(F \times (e, F_-), \tilde{n}, m, \mu)$ such that the vertex partitions corresponding to to the vertices of the (missing) inner copy of *F* are the sets X_1, \ldots, X_{v_F} .

If the number of edges in $T(G'_i, A_{i-1}[X_1, ..., X_{v_F}])$ is at least

$$\Theta\left(\left(\frac{m}{\tilde{n}^2}\right)^{e_F\cdot(e_F-1)}\tilde{n}^{v_F+e_F\cdot(v_F-2)}q_{i-1}^{e_F}\right)=\Theta\left(p_i^{e_F}n^{v_F}q_{i-1}^{e_F}\right)=\Theta(n^{v_F}q_i^{e_F}),$$

then we are done, since by Lemma 2.16 $(V(F), F \times (e, F_{-}))$ is balanced with density $m_2(F)$ and thus Lemma 1.5 the multiplicity of all edges is at most a constant.

Otherwise we invoke the induction hypothesis with $\kappa \leftarrow \kappa \eta_i \mu$ to deduce that $A_{i-1}[X_1, \ldots, X_{v_F}]$ is (F, q, μ) -lower-regular where $q = \gamma' q_{i-1}$. We have

$$\left(\frac{n}{\tilde{n}}\right)^{v_F} q_{i-1} \le \frac{q_{i-1}}{(\kappa \prod_{j \in [i]} \eta_j)^{v_F}} = Aq$$

and therefore by Claim 2.13 the hypergraph $A_{i-1}[X_1, ..., X_{v_F}]$ is (F, Aq)-upperextensible. Finally

$$m = \Omega\left(n^{v_{F_{i}^{*}}}p^{e_{F_{i}^{*}}}\right) \overset{Lemma \ 2.6}{\gg} n^{2-1/\overline{m}_{2}^{r-i+1}(F)} \overset{Lemma \ 2.10}{\geq} n^{2-1/m_{2}(V(F),F \times (e,F),q_{i})}$$

$$\approx n^{2-1/m_{2}(V(F),F \times (e,F),q)}$$

and thus we may apply apply Theorem 1.13 with $q \leftarrow q$, $G \leftarrow G'_i$ and $G_R \leftarrow A_{i-1}[X_1, \ldots, X_{v_F}]$ to deduce that G'_i is from a set graphs of size at most

$$\begin{split} \beta^{m} \binom{\tilde{n}^{2}}{m}^{e_{F} \cdot (e_{F}-1)} &\leq \beta^{m} \left(\frac{e\tilde{n}^{2}}{m}\right)^{m \cdot e_{F} \cdot (e_{F}-1)} \\ &\leq \beta^{m} \left(\frac{e2e_{F}\tilde{n}_{i}^{2}}{m_{i}}\right)^{m \cdot e_{F} \cdot (e_{F}-1)} \\ &= \left(\frac{\tilde{n}_{i}^{2}}{2\delta m_{i}}\right)^{m \cdot e_{F} \cdot (e_{F}-1)}, \end{split}$$

Since $m \gg n$ a union bound over the $2^{\Theta(n)}$ choices for the sets X_1, \ldots, X_{v_F} together with the bound

$$\Pr[G' \subseteq G_i] \le \left(\frac{m_i}{\tilde{n}_i^2 \delta}\right)^{|E(G')|}$$

from Lemma 2.5 guarantees that a.a.s. no such subgraph G'_i exists.

2.2 Auxiliary Lemmas

In Section 2 we stated a number of auxiliary statements without proof (namely Proposition 2.3, Lemma 2.10 and Lemma 2.11). The proofs of these statements are somewhat technical and are given in this section.

We start with the proof of Lemma 2.10 for which we need the following simple bound.

Lemma 2.15. Every graph *F* on at least 3 vertices and with at least one edge satisfies $m_2(F) \le 2m(F)$.

Proof. One checks that the statement holds for all graphs on 3 vertices. If *F* is 2-balanced and contains at least 4 vertices then

$$m_2(F) = \frac{e-1}{v-2} \le \frac{e}{v-2} \le 2\frac{e}{v} = 2d(F) \le 2m(F).$$

Otherwise let $F' \subseteq F$ denote a graph that attains the m_2 density of F. By the above

$$m_2(F) = m_2(F') \le 2m(F') \le 2m(F).$$

Lemma 2.10. Suppose that *F* is a 2-balanced graph, which contains an edge *e* such that $m_2(F - \{e\}) \le \overline{m}_2^2(F)$. Then for all $r \ge k \ge 2$

$$\overline{m}_2^k(F) \ge m_2(e, F, +1/\overline{m}_2^k(F) - 1/\overline{m}_2^r(F)),$$

$$\overline{m}_2^k(F) \ge m_2(V(F), F \times (e, F), +1/\overline{m}_2^k(F) - 1/\overline{m}_2^r(F)).$$

Proof. Write $p = n^{-1/\overline{m}_2^k(F)}$ and $q = n^{-1/\overline{m}_2^k(F)+1/\overline{m}_2^r(F)}$. The first inequality is equivalent to

$$\min_{\substack{(R,F')\subseteq (e,F)\\\overline{e}_{t'}>1}} n^{v_{F'}-2} p^{\overline{e}_{F'}-1} q^{e_{F'[R]}} \ge 1.$$

Fix such a rooted graph (R, F'). If $|R| \le 1$ then $F' \subseteq F - \{e\}$ and $e_{F'[R]} = 0$. Since $p \ge n^{-1/\overline{m}_2^2(F)} \ge n^{-1/m_2(F-\{e\})}$ we have

$$n^{v_{F'}-2}p^{e_{F'}-1} \ge 1.$$

Otherwise |R| = 2 and without loss of generality $e_{F'[R]} = 1$ and $\overline{e}_{F'} = e_{F'} - 1$. We rewrite the above as

$$n^{v_{F'}-2}p^{e_{F'}-2}q = \frac{n^{v_{F'}-2}p^{e_{F'}}}{pn^{-1/\overline{m}_2^r(F)}} \ge \frac{n^{v_{F'}-2}p^{e_{F'}}}{n^{-2/m_2(F)}}.$$

By definition of \overline{m}_2^k we have $n^{v_{F'}-2}p^{e_{F'}} \ge n^{-1/\overline{m}_2^{k-1}(F)}$. Together with

$$n^{2/m_2(F)} \stackrel{Lemma \ 2.15}{\geq} n^{1/m(F)} \geq n^{1/\overline{m}_2^{k-1}(F)}$$

this implies the desired bound.

The second inequality is equivalent to

$$\min_{\substack{(R,H)\subseteq (V(F),F\times(e,F))\\\overline{e}_H\geq 1}} n^{v_H-2} p^{\overline{e}_H} q^{e_{H[R]}} \geq 1.$$

For $e \in E(F)$ let $F_e \subseteq H$ denote the graph isomorphic to a subgraph of F which is attached to the root e in H. There must exist at least on edge e' such

that $F_{e'}$ contains at least one non root edge. For such an edge we apply the first inequality to obtain

$$n^{v_{F_{e'}}-2}p^{e_{F_{e'}}-e_{F_{e'}[e']}-1}q^{e_{F_{e'}[e']}} \ge 1.$$

Thus the minimization is at least

$$n^{v_{H[R]-2}} \prod_{e \in E(F) \setminus \{e'\}} n^{v_{F_e} - v_{F_e[e]}} p^{e_{F_e} - e_{F_e[e]}} q^{e_{F_e[e]}}.$$

If for some $e \in E(F)$ we have $v_{F_e[e]} \leq 1$ then F_e is isomorphic to a subgraph of F_- and

$$u^{v_{F_e}-v_{F_e[e]}}p^{e_{F_e}-e_{F_e[e]}}q^{e_{F_e[e]}} \ge n^{v_{F_e}-1}p^{e_{F_e}} \ge 1,$$

since we are above the 1-density of F_- . In particular if H does not contain at least two root vertices then we are done. If $v_{F_r[e]} = 2$ then

$$n^{v_{F_e}-v_{F_e[e]}}p^{e_{F_e}-e_{F_e[e]}}q^{e_{F_e[e]}} \ge n^{v_{F_e}-2}p^{e_{F_e}-1}q = \frac{n^{v_{F_e}-2}p^{e_{F_e}}}{n^{-1/\overline{m}_2^r(F)}} \\ \ge \frac{n^{-1/\overline{m}_2^{k-1}(F)}}{n^{-1/\overline{m}_2^r(F)}} \ge \frac{n^{-1/m(F)}}{n^{-1/m_2(F)}} \ge n^{-1/m_2(F)}.$$

Thus the original minimization reduces to

$$\min_{\substack{F' \subseteq F \\ v_{F'} \ge 2}} n^{v_{F'}-2} n^{-(e_{F'}-1)/m_2(F)}$$

which is at least 1 by definition of $m_2(F)$.

Proposition 2.3 concern the density of graphs in the class \mathcal{F}^k . Every graph $F^* \in \mathcal{F}^k$ (for $k \ge 2$) can be constructed by starting with a copy of F and repeatingly attaching copies of (e, F) to some edge. Since F is 2-balanced one may expect that graphs constructed by this procedue will be also be 2-balanced. The following lemma establishes that this is indeed the case.

Lemma 2.16. Suppose that G and H are two 2-balanced graphs such that $G \cap H$ is a single edge. If G and H both have 2-density d then $G \cup H$ is also 2-balanced with density d.

Similarly if (R,G) and (R,H) are balanced rooted graphs of density d with $V(G) \cap V(H) = R$ then $(R, G \cup H)$ is balanced with density d.

Proof. Let $p = n^{-1/d}$ and pick an induced subgraph $F \subseteq G \cup H$ with $e_F \ge 1$. Write G' = F[V(G)] and H' = F[V(H)]. Without loss of generality we have $e_{G'} \ge 1$ and

$$n^{v_F-2}p^{e_F-1} = n^{v_{G'}-2}p^{e_{G'}-1}n^{v_{H'}-v_{H'\cap G'}}p^{e_{H'}-e_{H'\cap G'}} \geq 1,$$

since $d = m_2(H) \ge m_1(H) \ge m(H)$ and since $H' \cap G'$ is either an edge, a vertex or empty. Thus $m_2(G \cup H) \le d$. Furthermore

$$n^{v_{G\cup H}-2}p^{e_{G\cup H}-1} = n^{v_{G}-2}p^{e_{G}-1}n^{v_{H}-2}p^{e_{H}-1} = 1 \cdot 1,$$

which implies $m_2(G \cup H) = d$ and that $G \cup H$ is balanced with respect to the 2-density.

The second claim can be proved in a similar fashion.

Thus every $F^* \in \mathcal{F}^k$, where $k \ge 2$, is 2-balanced with 2-density $m_2(F)$. Similarly $F \times (e, F^*)$ is 2-balanced (and thus balanced)

Proposition 2.3. All rooted graphs $(e, F^*) \in \mathcal{F}^r$ satisfy

$$m(F \times (e, F^*)) \leq \overline{m}_2^r(F).$$

Proof. Fix $F^* \in \mathcal{F}^r$, where $r \ge 2$. As noted above $F \times (e, F^*)$ is balanced. Therefore it suffices to check that

$$n^{v_{F\times(e,F^{*})}}n^{-e_{F\times(e,F^{*})}/\overline{m}_{2}^{r}(F)} = n^{v_{F}}\left(n^{v_{F^{*}}-2}n^{-e_{F^{*}}/\overline{m}_{2}^{r}(F)}\right)^{e_{F}}$$

$$\stackrel{(2.6)}{\geq}n^{v_{F}}n^{-e_{F}/\overline{m}_{2}^{1}(F)} \geq 1.$$

It remains to prove Lemma 2.11. To do so we require two more auxiliary lemmas.

Lemma 2.17. Let (R,G), (e,H) be rooted graphs. Suppose that (e,H) is balanced and that for some t > 0

$$m_1(H-e) \le t,$$

 $\overline{v}_H - \frac{\overline{e}_H}{t} \ge -\frac{1}{m(R,G)}.$

Then

$$m(R,(R,G)\times(e,H-e))\leq t.$$

Proof. Let $p = n^{-1/t}$ and let $(R, F) = (R, G) \times (e, H - e)$. It suffices to show that

$$\min_{(R,F')\subseteq (R,F)} n^{\overline{v}_{F'}-|R|} p^{\overline{e}_{F'}} \ge 1.$$
(3)

Fix a graph $(R, F') \subseteq (R, F)$ which attains the minimum. Let $H_1, \ldots, H_k \sim H - e$ denote the (canonical) copies of H - e in F and write $H'_i = F' \cap H_i$. The above term can be rewritten as

$$n^{v_{F'\cap G}-|R|}\prod_{i\in [k]}n^{v_{H'_i}-v_{H'_i}\cap G}p^{e_{H'_i}}$$

If for some i we have $v_{H'_i \cap G} \in \{0,1\}$ then $t \ge m_1(H-e) \ge m(H-e)$ implies

$$n^{v_{H'_i}-v_{H'_i\cap G}}p^{e_{H'_i}} \ge 1.$$

We may thus assume that for such *i* we have $H'_i = H_i \cap G$.

Otherwise $v_{H'_i \cap G} = 2$. If $t \ge m(e, H)$ then the above bound holds as well and in particular (3) is satisfied. If t < m(e, H) then, since (e, H) is balanced, the minimum of $n^{v_{H'_i}-2}p^{e_{H'_i}}$ is attained for $H'_i = H_i$. Thus the minimization reduces to

$$\min_{\substack{(R,G')\subseteq (R,G)}} n^{\overline{v}_{G'}} \left(n^{v_H-2} p^{e_H-1} \right)^{\overline{e}_{G'}} = \min_{\substack{(R,G')\subseteq (R,G)}} n^{\overline{v}_{G'}} \left(n^{v_H-2-(e_H-1)/t} \right)^{\overline{e}_G}$$
$$\geq \min_{\substack{(R,G')\subseteq (R,G)}} n^{\overline{v}_{G'}} n^{-\overline{e}_{G'}/m(R,G)} = 1.$$

Lemma 2.18. Let $r \ge 2$ and suppose that G is a 2-balanced graph with density at least 1. Then

$$\max_{R \subsetneq V} m(R,G) \le v - 1 < \left(\frac{1}{m(G)} - \frac{1}{\overline{m}_2^r(G)}\right)^{-1}.$$

Proof. m(R, G) is monotone increasing under edge addition. Thus for the first inequality it suffices to consider the case $G = K_v$. We have

$$m(R, K_v) = \frac{\binom{v}{2} - \binom{|R|}{2}}{v - |R|} \le \frac{\binom{v}{2} - \binom{v-1}{2}}{v - (v-1)} = v - 1.$$

For the second inequality we use $\overline{m}_2^r(G) < m_2(G)$ and the fact that since *G* is 2-balanced it is also balanced to obtain

$$\frac{1}{m(G)} - \frac{1}{\overline{m}_2^r(G)} < \frac{1}{m(G)} - \frac{1}{m_2(G)} = \frac{v}{e} - \frac{v-2}{e-1}.$$
(4)

Maximizing (4) subject to $v \le e$ we see that the maximum is attained whenever v = e. Thus the above is at most

$$1 - \frac{v - 2}{v - 1} = \frac{1}{v - 1}.$$

Lemma 2.11. Let $F^* \in \mathcal{F}^k$ where $k \ge 2$. Then

$$m_1(F_-^*) < \overline{m}_2^k(F)$$

and for every $V_0 \subsetneq V(F)$

$$m(V_0, (V_0, F) \times (e, F_-^*)) < \overline{m}_2^k(F).$$

Proof. Let $F^* \in \mathcal{F}^k$. We have

$$n^{v_{F^*}-1}n^{-e_{F^*}/\overline{m}_2^k(F)} \stackrel{Lemma \ 2.6}{\geq} n^{1-1/\overline{m}_2^1(F)} \geq 1$$

and thus $d_1(F^*) \leq \overline{m}_2^k(F)$. F^* is 2-balanced and thus strictly 1-balanced. Therefore we obtain the inequality

$$m_1(F_-^*) < m_1(F^*) = d_1(F^*) \le \overline{m}_2^k(F),$$

which proves the first part of Lemma 2.11.

For the second part we want to apply Lemma 2.17 with $(R, G) \leftarrow (V_0, F)$, $(e, H) \leftarrow (e, F^*)$ and $t \leftarrow \overline{m}_2^k(F) - \varepsilon$, where $\varepsilon > 0$ is a small constant such that $m_1(F_-^*) \leq \overline{m}_2^k - \varepsilon$. Since F^* is 2-balanced (e, F^*) is also balanced. We have choosen ε such that $m_1(F_-^*) \leq t$. The final premise of Lemma 2.17 is established by

$$v_{F^*} - 2 - \frac{e_{F^*} - 1}{\overline{m}_2^k(F)} \stackrel{Lemma \ 2.6}{\geq} - \frac{1}{\overline{m}_2^1(F)} + \frac{1}{\overline{m}_2^k(F)} \stackrel{Lemma \ 2.18}{>} - \frac{1}{m(V_0, G)}$$

Thus we can apply Lemma 2.17 which proves the last property.

3 Proof of Theorem 1.13

The proof of the theorem follows the proof of the KŁR-conjecture by Saxton, Thomason in [14] and relies on their container theorem:

Definition 3.1. Let G be an r-graph of order n and average degree d. Let $\tau > 0$. Given $v \in V(G)$ and $2 \le j \le r$, let

$$d^{(j)}(v) = \max\{d(\sigma) : v \in \sigma \subset V(G), |\sigma| = j\}.$$

If d > 0 we define δ_i by the equation

$$\delta_j \tau^{j-1} n d = \sum_v d^{(j)}(v).$$

Then the co-degree function $\delta(G, \tau)$ is defined by

$$\delta(G,\tau) = 2^{\binom{r}{2}-1} \sum_{j=2}^{r} \delta_j 2^{-\binom{j-1}{2}}.$$

If d = 0 we define $\delta(G, \tau) = 0$.

Theorem 3.2 ([14], Corollary 3.6). Let \mathcal{E} be an r-graph on the vertex set [n]. Let $0 < \varepsilon, \tau < 1/2$. Suppose that τ satisfies $\delta(\mathcal{E}, \tau) \leq \varepsilon/12r!$. Then there exists a constant c = c(r), and a function $C: \mathcal{P}([n])^s \to \mathcal{P}[n]$ where $s \leq c \log(1/\varepsilon)$, with the following properties. Let $\mathcal{T} = \{(T_1, \ldots, T_S) \in \mathcal{P}([n])^s: |T_i| \leq c\tau n\}$, and let $\mathcal{C} = \{C(T): T \in \mathcal{T}\}$. Then

- 1. for every $I \subset [n]$ for which $e(\mathcal{E}[I]) \leq \varepsilon \tau^r e(\mathcal{E})$ there exists $T = (T_1, \ldots, T_s) \in \mathcal{T} \cap P(I)^s$ with $I \subset C(T)$,
- 2. $e(\mathcal{E}[C]) \leq \varepsilon e(\mathcal{E})$ for all $C \in \mathcal{C}$.

For a graph *F* we denote with $K_{F,n}$ the v_F -partite graph with vertex partitions V_1, \ldots, V_{v_F} of size *n*, such that $K_{F,n}[V_i, V_j]$ is complete if $\{i, j\} \in E(F)$ and empty otherwise.

For a rooted graph (R, F) and $G_R \in \mathcal{R}(R, n)$ we denote with $\mathcal{E}(G_R, F)$ the hypergraph whose vertices are the edges of $K_{F_{-},n}$ and whose edges form (when seen as subgraphs of $K_{F_{-},n}$) a partite copy of F_{-} whose roots induce an edge in G_R .

To proof Theorem 1.13 we will apply Theorem 3.2 to $\mathcal{E}(G_R, F)$. The first step is to obtain a bound on the co-degree function.

Lemma 3.3. Let (R, F) be a rooted graph with $e_{F_-} > 1$. Let $0 < \gamma, q(n) \le 1 \le A$.

Then for *n* sufficiently large every hypergraph $G_R \in \mathcal{R}(R, n)$ which is (F, Aq)-upper-extensible satisfies

$$\delta\Big(\mathcal{E}(G_R,F),\gamma^{-1}n^{-1/m_2(R,F,-\log_n(q))}\Big) \leq \gamma e_{F_-} 2^{e_{F_-}^2} \frac{n^{|R|}(Aq)^{e_{F[R]}}}{|E(G_R)|}.$$

Proof. Let σ denote a set of vertices of $\mathcal{E} = \mathcal{E}(G_R, F)$. We identify σ with the set of edges from $K_{F_{-,n}}$ which it represents. If the degree of σ is non zero this set of edges is a graph $F' \subset K_{F_{-,n}}$ which is isomorphic to some subgraph of F_{-} . The degree of F' is the number of ways we can extend F' to a partite copy of F_{-} in $K_{F_{-,n}}$ whose roots form an edge in G_R .

Since *G* is (F, Aq)-upper-extensible we have

$$d(F') \le n^{v_F - v_{F'}} (Aq)^{e_{F[R]} - e_{F[R \cap V(F')]}}$$

For $j \ge 2$ and an edge $e \in E(K_{F_{-,n}})$ the quantity $d^{(j)}(e)$ is the maximum of d(F') over all F' with $e \in F'$ and |F'| = j. Thus

$$d^{(j)}(e) \le n^{v_F - v_{F_j}} (Aq)^{e_{F[R]} - e_{F[R \cap V(F_j)]}}$$

where

$$F_j = \underset{e(F')-e(F'[R])=j}{\operatorname{arg\,max}} n^{-v_{F'}} (Aq)^{-e_{F'[R]}}.$$

Observe that $F_j[R] = F[R \cap V(F_j)]$. Let $t = -\log_n(q)$ and $\tau = \gamma^{-1}n^{-1/m_2(R,F,t)}$. Using $m_2(R, F, t) \ge d_2(R \cap V(F_j), F_j, t)$ we obtain

$$\begin{aligned} \frac{1}{\tau^{j-1}} &= \gamma^{j-1} \Big(n^{1/m_2(R,F,t)} \Big)^{(j-1)} \\ &\leq \gamma^{j-1} \Big(n^{1/d_2 \big(R \cap V(F_j),F_j,t \big)} \Big)^{(j-1)} \\ &= \gamma^{j-1} n^{v_{F_j}-2} q^{e_{F_j[R]}}. \end{aligned}$$

The number of edges in \mathcal{E} is $|E(G_R)|n^{v_F-|R|}$. Thus for $j \ge 2$ we have

$$\delta_{j} = \frac{\sum_{e} d^{(j)}(e)}{\tau^{j-1}e_{F_{-}}|E(\mathcal{E})|} \le \frac{e_{F_{-}}n^{2}n^{v_{F}-v_{F_{j}}}(Aq)^{e_{F[R]}-e_{F_{j}[R]}}}{\tau^{j-1}e_{F_{-}}|E(G_{R})|n^{v_{F}-|R|}} \le \gamma^{j-1}\frac{n^{|R|}A^{e_{F[R]}-e_{F_{j}[R]}}q^{e_{F[R]}}}{|E(G_{R})|}$$

Finally we obtain

$$\delta(\mathcal{E},\tau) = 2^{\binom{e_{F_{-}}}{2}-1} \sum_{j=2}^{e_{F_{-}}} \delta_{j} 2^{-\binom{j-1}{2}} \le e_{F_{-}} 2^{e_{F_{-}}^{2}} \gamma \frac{n^{|R|} (Aq)^{e_{F[R]}}}{|E(G_{R})|}.$$

as claimed.

Having bounded the co-degree function we can obtain a collection of containers for $\mathcal{E}(G_R, F)$ which do not induce many edges in $\mathcal{E}(G_R, F)$. Viewing our contains as subgraphs of $K_{F_{-,n}}$ this means that they contain few copies of F_{-} whose roots induce an edge in G_R . To prove a KŁR-type statement we want our containers to be sparse subgraphs of $K_{F_{-,n}}$. The following two lemmas establish that if G_R is lower-regular then the containers obtained by Theorem 3.2 are indeed sparse.

Lemma 3.4. Let (R, F) denote a rooted graph. For every $\delta > 0$ there exists $\varepsilon > 0$ such that for all $p \ge \delta$ the following holds. Suppose that $G_R \in \mathcal{R}(R, n)$ is (F, q, ε) -lower-regular and that the bipartite graphs of $G \subseteq K_{F_{-},n}$ are (ε) -regular with density at least p then

$$|E(T(G, G_R))| \ge (1 - \delta) p^{e_{F_-}} q^{e_{F[R]}} n^{v_F}.$$

Proof. Observe that the density p is at least δ , which is a constant. Therefore *G* is a (dense) regular graph and standard counting arguments apply.

We only sketch of the proof: using standard arguments we find roughly $n^{v_F - |R|} p^{e_{F[V(F)\setminus R]}}$ tuples in $X_{i \in V(F)\setminus R} V_i$ whose common neighborhoods into the partitions G_R are roughly as large as expected (in particular they are of linear size). Thus for ε small enough the (F, q, ε) -lower-regularity of G_R guarantees that every one of these tuples extends to roughly $n^{|R|} p^{e_F - e_{F[V(F)\setminus R]}} q^{e_{F[R]}}$ copies of F_- whose roots form an edge in G_R . Every such copy of F_- contributes on edge to the multi-hypergraph $T(G, G_R)$ and we obtain the desired bound.

Lemma 3.5. Let (R, F) be a rooted graph with $e_{F_-} > 1$. Let $\delta > 0$ be small enough and let $A \ge 1$. Then there exists $c, \varepsilon(\delta), R(\delta), \gamma(\delta, A)$ such that the following is true. Suppose that $\tau(n), q(n) \in o(1)$ satisfy $\tau \ge \gamma^{-1}n^{-1/m_2(R,F,-\log_n q)}$. If $G_R \in \mathcal{R}(R,n)$ is (F, Aq)-upper-extensible and (F, q, ε) -lower-regular then for nlarge enough there exists a collection C of subgraphs of $K_{F_-,n}$ such that

- 1. for every $G \subseteq K_{F_{-,n}}$ for which $e(T(G, G_H)) \leq \varepsilon \tau^{e_{F_{-}}} q^{e_{F[R]}} n^{v_F}$ there exists $T_1, \ldots, T_s \subseteq G$ with $G \subset C(T_1, \ldots, T_s) \in C$, $e(T_i) \leq c \tau n^2$ and $s \leq c \log(A^{e_{F[R]}}/\varepsilon)$,
- 2. for every $C \in C$ there exists $\{i, j\} \in E(F_{-})$ and equitable partitions $V_i = V_{i,1} \cup \cdots \cup V_{i,r}$ and $V_j = V_{j,1} \cup \cdots \cup V_{j,r}$ where $r \leq R(\delta)$ such that for at least $r^2/2e_{F_{-}}$ pairs $x, y \in [r]$ we have $e(C[V_{i,x}, V_{j,y}]) \leq \delta|V_{i,x}||V_{j,y}|$.

Proof. The constants $c, \mu(\delta), R(\mu), \varepsilon(\delta, \mu, R)$ and $\gamma(\varepsilon, A)$ will be determined later. Let $\varepsilon' = \varepsilon q^{e_{F[R]}} n^{|R|} / e(G_R)$ and $\mathcal{E} = \mathcal{E}(G_R, F)$. Since G_R is (F, Aq)-upper-extensible we can invoke Lemma 3.3 to obtain the bound

$$\delta(\mathcal{E},\tau) \le \gamma e_{F_{-}} 2^{e_{F_{-}}^{2}} \frac{n^{|R|} (Aq)^{e_{F[R]}}}{e(G_{H})} = \gamma e_{F_{-}} 2^{e_{F_{-}}^{2}} \frac{\varepsilon' A^{e_{F[R]}}}{\varepsilon}.$$

For $\gamma(\varepsilon, A)$ small enough we obtain

$$\delta(\mathcal{E},\tau) \leq \frac{\varepsilon'}{12r!},$$

which is what we need to apply Theorem 3.2 with $\mathcal{E} \leftarrow \mathcal{E}$, $\varepsilon \leftarrow \varepsilon'$, $\tau \leftarrow \tau$, $r \leftarrow e_{F_-}$ to obtain a collection of containers \mathcal{C} . We will now show that these containers (when viewed as subgraphs of $K_{F_-,n}$) satisfy the conditions of our Lemma.

So let $G \subseteq K_{F_{-},n}$ with

$$e(T(G,G_R)) \leq \varepsilon \tau^{e_{F-}} q^{e_{F[R]}} n^{v_F} = \varepsilon' \tau^{e_{F-}} n^{v_F-|R|} e(G_R) = \varepsilon' \tau^{e_{F-}} e(\mathcal{E}).$$

Define I = E(G) and observe that $e(\mathcal{E}[I]) = e(T(G, G_R))$ and thus $e(\mathcal{E}[I]) \leq \varepsilon' \tau^{e_F} - e(\mathcal{E})$. Therefore we obtain $T_1, \ldots, T_s \subseteq G \subseteq C(T_1, \ldots, T_s) \in C$ with $e(T_i) \leq c' \tau v(\mathcal{E}) = c \tau n^2$ and $s \leq c \log(1/\varepsilon')$. Since G_R is (F, Aq)-upper-extensible we have

$$\log\left(\frac{1}{\varepsilon'}\right) = \log\left(\frac{e(G_R)}{\varepsilon q^{e_{F[R]}} n^{|R|}}\right) \le \log\left(\frac{A^{e_{F[R]}}}{\varepsilon}\right),$$

which proves the bound on *s*.

It remains to show that every $C \in C$ contains a sparse partition (in the sense of 2.). To this end, for μ small enough depending on δ , consider a μ -regular partition of C which refines the initial partition $(V_i)_{i \in V(F)}$. For every i we obtain a partition $V_i = V_{i,1} \cup \cdots \cup V_{i,r}$ for some $r \leq R(\mu)$. Now consider $C_x = C\left[V_{1,x_1} \cup \cdots \cup V_{v_F,x_{v_F}}\right]$ for some $x \in [r]^{v_F}$. For ε small enough depending on R, μ the |R|-graph $G_{R,x} = G_R[V(C_x)]$ is (F, μ, q) -lower-regular. Thus if all pairs in C_x are μ -regular with density at least δ then by Lemma 3.4 for μ small enough depending on δ

$$e(T(C_x, G_{R,x})) \geq (1-\delta)\delta^{e_{F_-}}q^{e_{F[R]}} \left(\frac{n}{2r}\right)^{v_F}.$$

But $e(T(C_x, G_{R,x}))$ is at most

$$e(\mathcal{E}[C]) \leq \varepsilon' e(\mathcal{E}) = \varepsilon' n^{v_F - |R|} e(G_R) = \varepsilon n^{v_F} q^{e_{F[R]}},$$

which is a contradiction for ε small enough depending on R, δ .

Thus for every *x* there exists $\{i, j\} \in E(F)$ such that $C \left[V_{i,x_i}, V_{j,x_j} \right]$ is either sparse or not μ -regular. By the pigeonhole principle at least an $1/e_F$ -fraction of the *x* nominate the same edge $\{i, j\}$ and every pair $V_{i,a}, V_{j,b}$ is nominated by at most r^{v_F-2} different *x*. Finally at most an μ -fraction of these pairs is not μ -regular. Therefore we have found *i*, *j* such that at least $r^2/(2e_{F_-})$ of the pairs $V_{i,.}, V_{j,.}$ have density at most δ .

The proof of Theorem 1.13 now follows from a standard counting argument:

Theorem 1.13. Let (R, F) be a rooted graph. For every $\beta > 0$, $A \ge 1$ there exists $\alpha(A, \beta), \mu(\beta) > 0$ such that for every q(n) = o(1) the following holds:

For *n* large enough suppose that $m \ge \alpha^{-1}n^{2-1/m_2(R,F,-\log_n q)}$ and that $G_R \in \mathcal{R}(R,n)$ is (F,Aq)-upper-extensible as well as (F,q,μ) -lower-regular. Then the number of graphs G in $\mathcal{G}(F_-,n,m,\mu)$ for which $T(G,G_R)$ contains fewer than $\alpha(m/n^2)^{e_F}-q^{e_F[R]}n^{v_F}$ edges is at most

$$\beta^m \binom{n^2}{m}^{e_F - e_{F[R]}}$$

Proof. The proof will require a number of constants which will be fixed during the proof. Their dependencies are as follows: $\delta(\beta)$, $\varepsilon(\delta)$, $R(\delta)$, $\gamma(\delta, A)$, $\hat{s}(A, \varepsilon)$, $\eta(\hat{s}, \gamma)$, $\alpha(\varepsilon, \gamma, \eta)$, $\mu(\varepsilon, R)$.

We invoke Lemma 3.5 with $\delta \leftarrow \delta$, $A \leftarrow A$ and obtain constants c, $\varepsilon(\delta)$, $R(\delta)$ and $\gamma(\delta, A)$.

Fix $\tau = \eta m/n^2$. For α small enough depending on γ and η we have $\tau \geq \gamma^{-1} n^{-1/m_2(R,F,-\log_n q)}$ and for α small enough depending on ε and η we have

$$\varepsilon\tau^{e_{F_-}}q^{e_{F[R]}}n^{v_F} \geq \alpha(m/n^2)^{e_{F_-}}q^{e_{F[R]}}n^{v_F}.$$

Therefore for $\mu \leq \varepsilon$ small enough Lemma 3.5 guarantees the existence of a container $T_1, \ldots, T_s \subseteq G \subseteq C(T_1, \ldots, T_s)$ with $s \leq c \log(A^{e_{F[R]}}/\varepsilon) =: \hat{s}$ whenever $G \in \mathcal{G}(F_-, n, m, \mu)$ does not satisfy $e(T(G, G_R)) > \alpha(m/n^2)^{e_{F-}} q^{e_{F[R]}} n^{v_F}$.

To count all such graphs *G* we fix $T = (T_1, ..., T_s)$ and then pick $G \subseteq K_{F_{-,n}}$ u.a.r. among all graphs with exactly *m* edges in each bipartite graph. Following [14] we define the following events

$$E_T: T_1 \cup \cdots \cup T_s \subseteq G \subseteq C(T) \text{ and } G \in \mathcal{G}(F_-, n, m, \mu),$$

$$F_T: T_1 \cup \cdots \cup T_s \subseteq G,$$

$$G_T: G \subseteq C(T) \text{ and } G \in \mathcal{G}(F_-, n, m, \mu).$$

We firstly show that $\sum_T \Pr[F_T] \leq 2^m$. Note that this is the expected number of tuples $T \subseteq G$. The maximum number of tuples $T \subseteq G$ is at most

$$\sum_{|T_1|,\dots,|T_{\hat{s}}|} \prod_{i \leq \hat{s}} {e_{F_-}m \choose |T_i|} \leq (c\tau n^2)^{\hat{s}} {e_{F_-}m \choose c\tau n^2}^{\hat{s}}$$
$$\leq (c\tau n^2)^{\hat{s}} (\frac{ee_{F_-}m}{c\tau n^2})^{\hat{s}c\tau n^2}$$
$$= (c\eta m)^{\hat{s}} (\frac{ee_{F_-}}{c\eta})^{\hat{s}c\eta m}$$
$$\leq 2^m.$$

for η small enough depending on \hat{s} .

Secondly we show that $\Pr[G_T | F_T] \leq (\beta/2)^m$ and thus

$$\sum_{T} \Pr[E_T] = \sum_{T} \Pr[G_T \mid F_T] \Pr[F_T] \le \beta^m,$$

which implies the Theorem.

For fixed *T* and *C*(*T*) let $\{i, j\} \in E(F_{-})$ and $V_i = V_{i,1} \cup \cdots \cup V_{i,r}$ and $V_j = V_{j,1} \cup \cdots \cup V_{j,r}$ be given by property (2) of Lemma 3.5. For $\mu(R)$ small enough we use the (μ) -regularity of $G[V_i, V_j]$ to require

$$\left|G\left[V_{i,x}, V_{j,x}\right]\right| \ge (1-\mu)\left(\frac{n}{r}\right)^2 \frac{m}{n^2} \ge m/2r^2$$

for every $x, y \in [r]$ while $|C[V_{i,x}, V_{j,x}]| \le \delta(\frac{n}{r})^2$ for at least $r^2/2e_{F_-}$ choices of x, y. Let $C' = \bigcup C[V_{i,x}, V_{j,x}]$ where the union runs over $r^2/2e_{F_-}$ sparse pairs. We have $|C'| \le \delta n^2/2e_{F_-}$ and for G to be (μ) -regular we require $|G \cap C'| \ge m/4e_{F_-}$. We conclude for $\eta(\gamma, c)$ small enough

$$\begin{aligned} \Pr[G_T \mid F_T] &\leq \Pr[\left|G \cap C'\right| \geq m/4e_{F_-} \mid F_T] \\ &\leq \Pr\left[\left|(G - T) \cap C'\right| \geq m/4e_{F_-} - \hat{s}c\tau n^2\right] \\ &\leq \Pr\left[\left|(G - T) \cap C'\right| \geq m/6e_{F_-}\right] \\ &\leq \left(\frac{\delta \frac{n^2}{2e_{F_-}}}{\frac{m}{6e_{F_-}}}\right) \left(\frac{m}{n^2}\right)^{m/6e_{F_-}} \leq (3e\delta)^{m/6e_{F_-}} \leq \left(\frac{\beta}{2}\right)^m, \end{aligned}$$

for $\eta(\hat{s})$ and $\delta(\beta)$ small enough.

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