# The graph structure of a deterministic automaton chosen at random: full version 

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#### Abstract

An $n$-state deterministic finite automaton over a $k$-letter alphabet can be seen as a digraph with $n$ vertices which all have $k$ labeled out-arcs. Grusho [20] proved that whp in a random $k$-out digraph there is a strongly connected component of linear size, i.e., a giant, and derived a central limit theorem. We show that whp the part outside the giant contains at most a few short cycles and mostly consists of tree-like structures, and present a new proof of Grusho's theorem. Among other things, we pinpoint the phase transition for strong connectivity.


Keywords: random digraphs; deterministic finite automaton

## 1 Introduction

### 1.1 The model and the history

The deterministic finite automaton (DFA) is widely used in computational complexity theory. Formally, a DFA is a 5 -tuple $\left(Q, \Sigma, \delta, q_{0}, F\right)$, where $Q$ is a finite set called the set of states, $\Sigma$ is a finite set called the alphabet, $\delta: Q \times \Sigma \rightarrow Q$ is the transition function, $q_{0} \in Q$ is the start state, and $F \subseteq Q$ is the set of accept states. If $q_{0}$ and $F$ are ignored, a DFA with $n$ states and a $k$-alphabet can be seen as a digraph with vertices $[n] \equiv\{1 \ldots, n\}$ in which each vertex has $k$ out-arcs labeled by $1, \ldots, k$ (a $k$-out digraph). Note that such a digraph can have self-loops and multi-arcs. For a basic introduction to DFA and its applications, see [37].

Let $\mathcal{D}_{n, k}$ denote a digraph chosen uniformly at random from all $k$-out digraphs of $n$ vertices. Equivalently $\mathcal{D}_{n, k}$ is a random $k$-out digraph of $n$ vertices with the endpoints of its $k n$ arcs chosen independently and uniformly at random.

When $k=1, \mathcal{D}_{n, k}$ is equivalent to a uniform random mapping from $[n]$ to itself, which has been well studied by Kolchin [27], Flajolet and Odlyzko [18], and Aldous and Pitman [2]. In $\mathcal{D}_{n, 1}$, the largest strongly connected component (SCC) has expected size $\Theta(\sqrt{n})$, and so does the size of the longest cycle. However, as shown later, for $k \geq 2$, the largest SCC has expected size $\Theta(n)$.

From now on we assume that $k \geq 2$. Let $\mathcal{S}_{v}$ (the spectrum of $v$ ) be the set of vertices in $\mathcal{D}_{n, k}$ that are reachable from vertex $v$, including $v$ itself. In 1973 Grusho [20] first proved that $\left(\left|\mathcal{S}_{1}\right|-\nu_{k} n\right) / \sigma_{k} \sqrt{n}$ converges in distribution to a standard normal, where $\nu_{k}$ and $\sigma_{k}$ are explicitly defined constants.

Given a set of vertices $\mathcal{S} \subseteq[n]$, call $\mathcal{S}$ closed if there are no arcs that start from vertices in $\mathcal{S}$ and end at vertices in $\mathcal{S}^{c} \equiv[n] \backslash \mathcal{S}$. Let $\mathcal{G}_{n}$ be the set of vertices in the largest closed SCC in $\mathcal{D}_{n, k}$. (If the largest closed SCC is not unique, let $\mathcal{G}_{n}$ be the vertex set of the largest closed SCC that contains the smallest vertex-label.) We call $\mathcal{G}_{n}$ the giant. Grusho also proved that $\left|\mathcal{G}_{n}\right|$ has the same limit distribution as $\left|\mathcal{S}_{1}\right|$ by showing that with high probability (whp) $\mathcal{G}_{n}$ is reachable from all vertices and that $\left|\mathcal{S}_{1}\right|-\left|\mathcal{G}_{n}\right|=o_{p}(\sqrt{n})$ (see [22] for the notation). His proof relies on a result by Sevast'yanov [35] which approximates the exploration of $\mathcal{S}_{1}$ with a Gaussian process.

In 2012 Carayol and Nicaud [10] proved a local limit theorem for $\left|\mathcal{S}_{1}\right|$ by analyzing the limit behavior of the probability that $\left|\mathcal{S}_{1}\right|=s$ for an $s$ close to $\nu_{k} n$. Their proof depends on a theorem by Korshunov [28] which says that conditioned on every vertex having in-degree at least one, the probability that $\mathcal{S}_{1}=[n]$ tends to some constant. Carayol and Nicaud derived a simple and explicit formula of this constant from their theorem. (The same formula is also proved by Lebensztayn [29] with a more analytic approach using Lagrange series.)

Lately the simple random walk (SRW) on $\mathcal{D}_{n, k}$ has gained some attention for its applications in machine learning. Addario-Berry, Balle, and Perarnau [1] studied the stationary distribution of the SRW by analyzing the distances in $\mathcal{D}_{n, k}$. They proved that the diameter and the typical distance, rescaled by $\log n$, converge in probability to explicit constants. Angluin and Chen [3] studied the rate of the convergence to the stationary distribution of the SRW. They also suggested an algorithm for learning a uniformly random DFA under Kearns' statistical query model [26].

### 1.2 Our results and a sketch of proof

A digraph can be uniquely decomposed into sCCs which form a directed acyclic graph (DAG) through a process called condensation that contracts every SCC into a single vertex while keeping all the arcs between ScCs [5]. The condensation DAG of $\mathcal{D}_{n, k}$ is denoted by $\mathcal{D}_{n, k}^{\mathrm{A}}$.

Let $\mathcal{G}_{n}^{c} \equiv[n] \backslash \mathcal{G}_{n}$, i.e., $\mathcal{G}_{n}^{c}$ is the set of vertices that are outside the giant. The structure of $\mathcal{D}_{n, k}^{\mathrm{A}}$ depends on $\mathcal{D}_{n, k}\left[\mathcal{G}_{n}^{c}\right]$, the digraph induced by $\mathcal{G}_{n}^{c}$. Our analysis shows that in $\mathcal{D}_{n, k}\left[\mathcal{G}_{n}^{c}\right]$ the total number of cycles and the number of cycles of a fixed length
both converge to Poisson distributions with constant means. So the number of cycles and the length of the longest cycle are both $O_{p}(1)$ (see [22]). Furthermore, these cycles are vertex-disjoint whp. Therefore, almost every vertex in $\mathcal{G}_{n}^{c}$ is a SCC itself and $\mathcal{D}_{n, k}^{\mathrm{A}}$ is very much like $\mathcal{D}_{n, k}$ with the giant contracted into a single vertex.

The $d$-core of an undirected graph is the maximum induced subgraph in which all vertices have degree at least $d$. Similarly the $d$-in-core of a digraph can be defined as the maximum induced sub-digraph in which all vertices have in-degree at least $d$. Let $\mathcal{O}_{n}$ denote the set of vertices in the one-in-core of $\mathcal{D}_{n, k}$. Note that $\mathcal{G}_{n} \subseteq \mathcal{O}_{n}$ since a SCC induces a sub-digraph with each vertex having in-degree at least one. Also note that cycles cannot exist outside $\mathcal{O}_{n}$, for otherwise they contradict the maximality of $\mathcal{O}_{n}$. Now assume that every vertex can reach $\mathcal{G}_{n}$, which happens whp by Grusho [20]. Then $\mathcal{D}_{n, k}$ can be divided into three layers: the center is $\mathcal{G}_{n}$; then comes $\mathcal{O}_{n} \backslash \mathcal{G}_{n}$, which consists of cycles outside $\mathcal{G}_{n}$ and paths from these cycles to $\mathcal{G}_{n}$; the outermost is $\mathcal{O}_{n}^{c} \equiv[n] \backslash \mathcal{O}_{n}$, which is acyclic.


Figure 1: Three layers of $\mathcal{D}_{n, k}$ : the giant $\mathcal{G}_{n}$; the one-in-core $\mathcal{O}_{n}$; and the whole graph.

Since there cannot be many vertices in cycles outside the giant, the middle layer $\mathcal{O}_{n} \backslash \mathcal{G}_{n}$ must be very "thin". Thus if we can prove $\left(\left|\mathcal{O}_{n}\right|-\nu_{k} n\right) / \sqrt{n}$ converges to a normal distribution, then we can also prove it for $\left|\mathcal{G}_{n}\right|$. The event $\left|\mathcal{O}_{n}\right|=s$ happens if and only if there is a set of vertices $\mathcal{S}$ with $|\mathcal{S}|=s$ such that: (a) $\mathcal{D}_{n, k}[\mathcal{S}]$, the sub-digraph induced by $\mathcal{S}$, has minimum in-degree one (surjective) and there are no arcs going from $\mathcal{S}$ to $\mathcal{S}^{c}$ (closed), which we refer to as $\mathcal{S}$ being a $k$-surjection (since $\mathcal{D}_{n, k}[\mathcal{S}]$ is equivalent to a surjective function from $[k s]$ to $[s]$ ); (b) $\mathcal{D}_{n, k}\left[\mathcal{S}^{c}\right]$ is acyclic. The probability of (a) can be computed by counting the number of surjective functions. And we are able to show that the probability of (b) converges to a constant. Note that for a fixed set $\mathcal{S}$ (a) and (b) are independent because they depend on the endpoints of two disjoint sets of arcs. Thus we can get the limit of $\mathbb{P}\left\{\mathcal{O}_{n}=\mathcal{S}\right\}$. Since the one-in-core of a digraph is unique, $\mathbb{P}\left\{\left|\mathcal{O}_{n}\right|=s\right\}=\sum_{\mathcal{S} \subseteq[n]:|\mathcal{S}|=s} \mathbb{P}\left\{\mathcal{O}_{n}=\mathcal{S}\right\}$. Thus we can finish the proof by computing the characteristic function of $\left(\left|\mathcal{O}_{n}\right|-\nu_{k} n\right) / \sqrt{n}$.

Note that although our formula for $\mathbb{P}\left\{\left|\mathcal{O}_{n}\right|=s\right\}$ is inspired by and resembles Carayol and Nicaud's formula for $\mathbb{P}\left\{\left|\mathcal{S}_{1}\right|=s\right\}$, we actually prove the result from scratch without
relying on previous work. Since we are able to derive explicit expressions of all the constants in our formula, the computation of the characteristic function becomes quite simple. Furthermore, to our knowledge this is the first self-contained proof. Thus in Section 2 we prove:

Theorem 1 (Central limit law). Let $\mathcal{Z}$ denote a standard normal random variable. Then as $n \rightarrow \infty$,

$$
\frac{\left|\mathcal{O}_{n}\right|-\nu_{k} n}{\sigma_{k} \sqrt{n}} \xrightarrow{d} \mathcal{Z}, \quad \frac{\left|\mathcal{G}_{n}\right|-\nu_{k} n}{\sigma_{k} \sqrt{n}} \xrightarrow{d} \mathcal{Z}, \quad \frac{\max _{v \in[n]}\left|\mathcal{S}_{v}\right|-\nu_{k} n}{\sigma_{k} \sqrt{n}} \xrightarrow{d} \mathcal{Z},
$$

where $\nu_{k}$ and $\sigma_{k}$ are constants defined by

$$
\nu_{k} \equiv \frac{\tau_{k}}{k}, \quad \quad \sigma_{k}^{2} \equiv \frac{\tau_{k}}{k e^{\tau_{k}}\left(1-k e^{-\tau_{k}}\right)},
$$

and $\tau_{k}$ is the unique positive solution of $1-\tau_{k} / k-e^{-\tau_{k}}=0$.
Remark. Equivalently, $\nu_{k}$ is the unique positive solution of $1-\nu_{k}=e^{-k \nu_{k}}$ and

$$
\sigma_{k}^{2}=\frac{\nu_{k}\left(1-\nu_{k}\right)}{1-k\left(1-\nu_{k}\right)} .
$$

Let $G(n, m)$ be a Erdős-Rényi random graph, i.e., a graph chosen uniformly at random from all graphs with $n$ vertices and $m$ edges [16]. It is well-known that for $k>1$, $\left|\mathcal{C}_{\text {max }}^{n}\right|$-the size of the largest component in $G(n, m=n k / 2)$-is $\left(\nu_{k}+o(1)\right) n$ whp. Moreover, $\left(\left|\mathcal{C}_{\max }^{n}\right|-\nu_{k} n\right) / \sqrt{n}$ also converges in distribution to a normal random variable with variance $\sigma_{k}^{2}$ (see, e.g., Durrett [14]). Intuitively, this is because the in-degree of a vertex in $\mathcal{D}_{n, k}$ has asymptotically a Poisson distribution of mean $k$. Thus a backward exploration process from vertex in $\mathcal{D}_{n, k}$ is approximately a Galton-Watson process with survival probability $\nu_{k}$, as is the exploration process starting from a vertex in $G(n, m=$ $n k / 2)$.

Section 3 studies the part of $\mathcal{D}_{n, k}$ outside the giant, which determines the structure of $\mathcal{D}_{n, k}^{\mathrm{A}}$ and supports the proof of Theorem 1. Our results are summarized in two theorems, where all our logarithms are natural:

Theorem 2 (Cycles outside the giant). We have:
(a) Let $L_{n}$ be the length of the longest cycle in $\mathcal{D}_{n, k}\left[\mathcal{G}_{n}^{c}\right]$. Then $L_{n}=O_{p}(1)$.
(b) Let $C_{n}$ be the number of cycles in $\mathcal{D}_{n, k}\left[\mathcal{G}_{n}^{c}\right]$. Then

$$
C_{n} \xrightarrow{d} \operatorname{Poi}\left(\log \frac{1}{1-k e^{-\tau_{k}}}\right),
$$

where $\operatorname{Poi}(x)$ denotes the Poisson distribution with mean $x$.
(c) Let $C_{n, \ell}$ be the number of cycles of length $\ell$ in $\mathcal{D}_{n, k}\left[\mathcal{G}_{n}^{c}\right]$. Then for all fixed $\ell \geq 1$,

$$
C_{n, \ell} \xrightarrow{d} \operatorname{Poi}\left(\frac{\left(k e^{-\tau_{k}}\right)^{\ell}}{\ell}\right) .
$$

Theorem 3 (Spectra outside the giant). Let $\mathcal{S}_{v}^{\prime} \equiv \mathcal{S}_{v} \cap \mathcal{G}_{n}^{c}$, i.e., $\mathcal{S}_{v}^{\prime}$ is the spectrum of $v$ in $\mathcal{D}_{n, k}\left[\mathcal{G}_{n}^{c}\right]$. Let $\operatorname{dist}(v, u)$ be the distance from $v$ to $u$, i.e., the length of the shortest directed path from $v$ to $u$. Then
(a) $\mathbb{P}\left\{\cup_{v \in \mathcal{G}_{n}^{c}}\left[\operatorname{arc}\left(\mathcal{D}_{n, k}\left[\mathcal{S}_{v}^{\prime}\right]\right)-\left|\mathcal{S}_{v}^{\prime}\right| \geq 1\right]\right\}=o(1)$, where $\operatorname{arc}(\cdot)$ denotes the number of arcs. In other words, whp every spectrum in $\mathcal{D}_{n, k}\left[\mathcal{G}_{n}^{c}\right]$ is a tree or a tree plus an extra arc.
(b) Let $S_{n} \equiv \max _{v \in \mathcal{G}_{n}^{c}}\left|\mathcal{S}_{v}^{\prime}\right|$. Let $\lambda_{k} \equiv\left(k-\tau_{k}\right)\left(\frac{\tau_{k}}{k-1}\right)^{k-1}$. Then

$$
\frac{S_{n}}{\log n} \xrightarrow{p} \frac{1}{\log \left(1 / \lambda_{k}\right)} .
$$

(c) Let $W_{n} \equiv \max _{v \in \mathcal{G}_{n}^{c}} \min _{u \in \mathcal{G}_{n}} \operatorname{dist}(v, u)$, i.e., the maximum distance to $\mathcal{G}_{n}$. Then

$$
\frac{W_{n}}{\log _{k} \log n} \xrightarrow{p} 1 .
$$

(d) Let $M_{n}$ be the length of the longest path in $\mathcal{D}_{n, k}\left[\mathcal{G}_{n}^{c}\right]$. Then

$$
\frac{M_{n}}{\log n} \xrightarrow{p} \frac{1}{\log \left(e^{\tau_{k}} / k\right)} .
$$

(e) Let $D_{n} \equiv \max _{v \in \mathcal{G}_{n}^{c}} \max _{u \in \mathcal{S}_{v}^{\prime}} \operatorname{dist}(v, u)$. Then

$$
\frac{D_{n}}{\log n} \xrightarrow{p} \frac{1}{\log \left(e^{\tau_{k} / k}\right)} .
$$

The rest of the paper gives some other results regarding this model. Section 4 shows that $\mathcal{D}_{n, k}$ exhibits a phase transition for strong connectivity. Section 5 extends some of our results to simple $k$-out digraphs. Section 6 analyzes the typical distances in $\mathcal{D}_{n, k}$ with a technique called path counting, which is very different from the method used by Addario-Berry et al. in [1]. Section 7 suggests some extensions of this model.

Remark. Lemma 9 shows that $\left|\mathcal{O}_{n}\right|-\left|\mathcal{G}_{n}\right|=O_{p}(1)$. The intuition is that a digraph with minimal in-degree and out-degree at least one is likely to have a large Scc. This phenomenon is also observed in $D(n, p)$, which is a random digraph of $n$ vertices with each possible arc existing independently with probability $p$. Pittel and Poole [33, thm. $1.3]$ showed that in $D(n, p)$ the $(1,1)$-core - the maximal induced sub-digraph in which each vertex has in-degree and out-degree at least one-differs from the largest SCC in size by at most $O\left((\log n)^{8}\right)$, whp. This intuition is also used for studying the asymptotic counts of strongly connected digraphs (see Pérez-Giménez and Wormald [34], Pittel [32]).

## 2 The size of the one-in-core

### 2.1 The law of large numbers for the one-in-core

To prove Theorem 1, we first need to narrow the range of $\left|\mathcal{O}_{n}\right|$ to close to $\nu_{k} n$.
Theorem 4 (Law of large numbers). For all fixed $\delta \in(0,1 / 2)$,

$$
\mathbb{P}\left\{\left|\mathcal{O}_{n}\right| \notin \mathcal{I}_{n}\right\} \leq \frac{1+o(1)}{n}
$$

where $\mathcal{I}_{n} \equiv\left[\nu_{k} n-n^{1 / 2+\delta}, \nu_{k} n+n^{1 / 2+\delta}\right]$.
Thus $\left|\mathcal{O}_{n}\right| / n \xrightarrow{p} \nu_{k}$, which gives the theorem its name.
Let $K_{s}$ be the number of $k$-surjections of size $s$ in $\mathcal{D}_{n, k}$. Then it suffices to show that $\mathbb{P}\left\{\sum_{s \notin \mathcal{I}_{n}} K_{s} \geq 1\right\} \leq(1+o(1)) / n$. As argued in the introduction, for a set of vertices $\mathcal{S}$ to be the one-in-core, it must also be a $k$-surjection, i.e., every vertex in $\mathcal{D}_{n, k}[\mathcal{S}]$, the sub-digraph induced by $\mathcal{S}$, must have minimum in-degree one ( $\mathcal{S}$ is surjective), and there are no arcs going from $\mathcal{S}$ to $\mathcal{S}^{c}(\mathcal{S}$ is closed). Thus

$$
\mathbb{P}\{\mathcal{S} \text { is a } k \text {-surjection }\}=\mathbb{P}\{\mathcal{S} \text { is surjective } \mid \mathcal{S} \text { is closed }\} \mathbb{P}\{\mathcal{S} \text { is closed }\} .
$$

Computing the limit of the two factors shows that:
Lemma 1. We have

$$
\mathbb{P}\left\{\sum_{s \notin \mathcal{I}_{n}} K_{s} \geq 1\right\} \leq \frac{1+o(1)}{n} .
$$

And for $s \in \mathcal{I}_{n}$

$$
\mathbb{E} K_{s} \sim \frac{1}{\sqrt{2 \pi\left(1-k e^{-\tau_{k}}\right) n}} g\left(\frac{s}{n}\right)\left[f\left(\frac{s}{n}\right)\right]^{n}
$$

where

$$
g(x) \equiv \frac{1}{\sqrt{x(1-x)}}, \quad \quad f(x) \equiv\left[\frac{x^{k-1} \gamma_{k}}{(1-x)^{(1-x) / x}}\right]^{x}
$$

and $\gamma_{k} \equiv\left(\frac{k}{e \tau_{k}}\right)^{k}\left(e^{\tau_{k}}-1\right)$.
Theorem 4 follows immediately. The proof of Lemma 1 is postponed to the appendix. (The two functions $f(x)$ and $g(x)$ are also studied by Carayol and Nicaud [10].)

### 2.2 The central limit law of the one-in-core

In this section we prove the part of Theorem 1 about $\left|\mathcal{O}_{n}\right|$. The rest of the theorem appears as corollaries in Section 3. Let $\partial \mathcal{O}_{n}=\left|\mathcal{O}_{n}\right|-\nu_{k} n$. Then $\partial \mathcal{O}_{n}$ takes values in $[n]-\nu_{k} n \equiv\left\{s: \nu_{k} n+s \in[n]\right\}$. As Theorem 4 shows, whp $\partial \mathcal{O}_{n} \leq n^{1 / 2+\delta}$ for all fixed
$\delta \in(0,1 / 2)$. Thus it suffices to consider only the probability that $\partial \mathcal{O}_{n}$ takes value in the set

$$
\mathcal{J}_{n} \equiv\left([n]-\nu_{k} n\right) \cap\left[-n^{1 / 2+\delta}, n^{1 / 2+\delta}\right]
$$

for some fixed $\delta \in(0,1 / 2)$. Thus the characteristic function of $\partial \mathcal{O}_{n} / \sqrt{n}$ is

$$
\begin{aligned}
\phi_{n}(t) & =\sum_{s \in\left([n]-\nu_{k} n\right) \backslash \mathcal{J}_{n}} e^{i t s / \sqrt{n}} \mathbb{P}\left\{\partial \mathcal{O}_{n}=s\right\}+\sum_{s \in \mathcal{J}_{n}} e^{i t s / \sqrt{n}} \mathbb{P}\left\{\partial \mathcal{O}_{n}=s\right\} \\
& =o(1)+\sum_{s \in \mathcal{J}_{n}} e^{i t s / \sqrt{n}} \mathbb{P}\left\{\partial \mathcal{O}_{n}=s\right\}
\end{aligned}
$$

Let $\mathcal{S}$ be a set of vertices with $|\mathcal{S}|=\nu_{k} n+s$ for some $s \in \mathcal{J}_{n}$. Recall that $\mathcal{O}_{n}=\mathcal{S}$ if and only if $\mathcal{S}$ is a $k$-surjection and $\mathcal{D}_{n, k}\left[\mathcal{S}^{c}\right]$ is acyclic, two events that are independent. By Theorem 5 in Section 3.2, $\mathbb{P}\left\{\mathcal{D}_{n, k}\left[\mathcal{S}^{c}\right]\right.$ is acyclic $\} \sim 1-k e^{-\tau_{k}}$. Also recall that $K_{x}$ counts the number of $k$-surjections of size $x$. It follows from Lemma 1 that

$$
\begin{aligned}
\mathbb{P}\left\{\partial \mathcal{O}_{n}=s\right\} & =\sum_{\mathcal{S} \subseteq[n]:|\mathcal{S}|=\nu_{k} n+s} \mathbb{P}\left\{\mathcal{O}_{n}=\mathcal{S}\right\} \\
& =\sum_{\mathcal{S} \subseteq[n]: \mathcal{S} \mid=\nu_{k} n+s} \mathbb{P}\{\mathcal{S} \text { is a } k \text {-surjection }\} \times \mathbb{P}\left\{\mathcal{D}_{n, k}\left[\mathcal{S}^{c}\right] \text { is acyclic }\right\} \\
& \sim\left(1-k e^{-\tau_{k}}\right) \mathbb{E} K_{\nu_{k} n+s} \\
& =\sqrt{\frac{1-k e^{-\tau_{k}}}{2 \pi}} \frac{1}{\sqrt{n}} g\left(\nu_{k}+\frac{s}{n}\right)\left[f\left(\nu_{k}+\frac{s}{n}\right)\right]^{n},
\end{aligned}
$$

where $K_{x}, f(x)$ and $g(x)$ are defined as in the previous subsection.
If $s \in \mathcal{J}_{n}$, then Lemma A6 in the appendix shows that

$$
g\left(\nu_{k}+\frac{s}{n}\right)=\left(1+O\left(\frac{|s|}{n}\right)\right) \frac{1}{\sigma_{k} \sqrt{1-k e^{-\tau_{k}}}}
$$

and

$$
f\left(\nu_{k}+\frac{s}{n}\right)=\exp \left\{-\frac{s^{2}}{2 \sigma_{k}^{2} n^{2}}\right\}+O\left(\frac{|s|^{3}}{n^{3}}\right)
$$

Therefore, choosing $\delta$ small enough, e.g., $\delta=1 / 9$, we have

$$
\begin{aligned}
\sum_{s \in \mathcal{J}_{n}} e^{i t s / \sqrt{n}} \mathbb{P}\left\{\partial \mathcal{O}_{n}=s\right\} & \sim \frac{1}{\sqrt{2 \pi \sigma_{k}^{2}}} \frac{1}{\sqrt{n}} \sum_{s \in \mathcal{J}_{n}} e^{i t s / \sqrt{n}} \exp \left\{-\frac{s^{2}}{2 \sigma_{k} n}\right\} \\
& =o(1)+\frac{1}{\sqrt{2 \pi \sigma_{k}^{2}}} \int_{-n^{\delta}}^{n^{\delta}} e^{i t x} \exp \left\{-\frac{x^{2}}{2 \sigma_{k}^{2}}\right\} \mathrm{d} x \\
& =o(1)+\frac{1}{\sqrt{2 \pi \sigma_{k}^{2}}} \int_{-\infty}^{\infty} e^{i t x} \exp \left\{-\frac{x^{2}}{2 \sigma_{k}^{2}}\right\} \mathrm{d} x \\
& =o(1)+\exp \left(\frac{\sigma_{k}^{2} t^{2}}{2}\right)
\end{aligned}
$$

Thus the characteristic function of $\partial \mathcal{O}_{n} / \sqrt{n}$ converges to $\exp \left(\sigma_{k}^{2} t^{2} / 2\right)$, the characteristic function of $\sigma_{k} \mathcal{Z}$. It follows from the central limit theorem that $\partial \mathcal{O}_{n} / \sqrt{n}$ converges to $\sigma_{k} \mathcal{Z}$ in distribution. Note that using the estimates of this section, we actually have a local limit theorem for $\left|\mathcal{O}_{n}\right|$.

## 3 The structure of the directed acyclic graph

### 3.1 De-randomizing the giant

Since a SCC induces a sub-digraph in which each vertex has in-degree at least one, a closed SCC is also a $k$-surjection. Lemma 1 implies that whp all $k$-surjections are of sizes in $\mathcal{I}_{n} \equiv\left[\nu_{k} n-n^{1 / 2+\delta}, \nu_{k} n+n^{1 / 2+\delta}\right]$. When this happens, as $\nu_{k}>1 / 2$ (Lemma A1), there exists one and only one closed SCC and it is $\mathcal{G}_{n}$. And if $\mathcal{G}_{n}$ is the only closed SCC, then every vertex must be able to reach it. This can be summarized as:
Lemma 2. Whp $\left|\mathcal{G}_{n}\right| \in \mathcal{I}_{n}$ and $\mathcal{G}_{n}$ is reachable from all vertices.
Since $e^{-\tau_{k}} \equiv 1-\tau_{k} / k \equiv 1-\nu_{k}$, the above lemma implies that whp $\left|\left|\mathcal{G}_{n}^{c}\right|-e^{-\tau_{k}} n\right| \leq$ $n^{1 / 2+\delta}$. Thus the structure of $\mathcal{D}_{n, k}\left[\mathcal{G}_{n}^{c}\right]$, the sub-digraph induced by $\mathcal{G}_{n}^{c} \equiv[n] \backslash \mathcal{G}_{n}$, should be close to that of a sub-digraph induced by a fixed set of vertices whose size is close to $e^{-\tau_{k}} n$. Formally, we have:
Lemma 3. Let $f_{n}$ be a sequence of integer-valued functions on a sequence of digraphs. Let $X$ be an integer-valued random variable. If there exists a sequence $\varepsilon_{n} \rightarrow 0$ such that

$$
\sup _{\mathcal{V}_{n} \subseteq[n]:\left|\mathcal{V}_{n}\right| \in \mathcal{I}_{n}}\left\|f_{n}\left(\mathcal{D}_{n, k}\left[\mathcal{V}_{n}^{c}\right]\right), X\right\|_{\mathrm{TV}} \leq \varepsilon_{n}
$$

where $\mathcal{V}_{n}^{c} \equiv[n] \backslash \mathcal{V}_{n}$ and $\|\cdot, \cdot\|_{\text {тV }}$ denotes the total variation distance, then

$$
f_{n}\left(\mathcal{D}_{n, k}\left[\mathcal{G}_{n}^{c}\right]\right) \xrightarrow{d} X .
$$

Proof. Define the event $E_{n}=\left[\left|\mathcal{G}_{n}\right| \in \mathcal{I}_{n}\right]$. Let $m$ be an integer, let $\mathcal{V}_{n} \subseteq[n]$ be a fixed set of vertices with $\left|\mathcal{V}_{n}\right| \in \mathcal{I}_{n}$. Recall that since $\nu_{k}>1 / 2,\left|\mathcal{V}_{n}\right|>n / 2$ for large $n$. Thus the event $\left[\mathcal{G}_{n}=\mathcal{V}_{n}\right]$ depends only on the induced sub-digraph $\mathcal{D}_{n, k}\left[\mathcal{V}_{n}\right]$, which is independent of $\mathcal{D}_{n, k}\left[\mathcal{V}_{n}^{c}\right]$. Therefore the two events $\left[\mathcal{G}_{n}=\mathcal{V}_{n}\right]$ and $\left[f_{n}\left(\mathcal{D}_{n, k}\left[\mathcal{V}_{n}^{c}\right]\right)=m\right]$ are independent. Using this observation and Lemma 2, we have

$$
\begin{aligned}
& \mathbb{P}\left\{f_{n}\left(\mathcal{D}_{n, k}\left[\mathcal{G}_{n}^{c}\right]\right)=m\right\} \\
& =\mathbb{P}\left\{\left[f_{n}\left(\mathcal{D}_{n, k}\left[\mathcal{G}_{n}^{c}\right]\right)=m\right] \cap E_{n}^{c}\right\}+\mathbb{P}\left\{\left[f_{n}\left(\mathcal{D}_{n, k}\left[\mathcal{G}_{n}^{c}\right]\right)=m\right] \cap E_{n}\right\} \\
& =o(1)+\sum_{\mathcal{V}_{n} \subseteq[n]:\left|\mathcal{V}_{n}\right| \in \mathcal{I}_{n}} \mathbb{P}\left\{f_{n}\left(\mathcal{D}_{n, k}\left[\mathcal{V}_{n}^{c}\right]\right)=m \mid \mathcal{G}_{n}=\mathcal{V}_{n}\right\} \mathbb{P}\left\{\mathcal{G}_{n}=\mathcal{V}_{n}\right\} \\
& \leq o(1)+\sum_{\mathcal{V}_{n} \subseteq[n]:\left|\mathcal{V}_{n}\right| \in \mathcal{I}_{n}}\left(\mathbb{P}\{X=m\}+\varepsilon_{n}\right) \mathbb{P}\left\{\mathcal{G}_{n}=\mathcal{V}_{n}\right\} \\
& \leq o(1)+\mathbb{P}\{X=m\} .
\end{aligned}
$$

Similarly we have $\mathbb{P}\left\{f_{n}\left(\mathcal{D}_{n, k}\left[\mathcal{G}_{n}^{c}\right]\right)=m\right\} \geq \mathbb{P}\{X=m\}+o(1)$. Since this applies to all integers $m, f_{n}\left(\mathcal{D}_{n, k}\left[\mathcal{G}_{n}^{c}\right]\right) \xrightarrow{d} X$.

Corollary 1. Let $\mathcal{E}_{n}$ be a sequence of sets of digraphs. If there exists a sequence $\varepsilon_{n} \rightarrow 0$ such that

$$
\sup _{\mathcal{V}_{n} \subseteq[n]:\left|\mathcal{V}_{n}\right| \in \mathcal{I}_{n}} \mathbb{P}\left\{\mathcal{D}_{n, k}\left[\mathcal{V}_{n}^{c}\right] \notin \mathcal{E}_{n}\right\} \leq \varepsilon_{n},
$$

then whp $\mathcal{D}_{n, k}\left[\mathcal{G}_{n}^{c}\right] \in \mathcal{E}_{n}$.
Proof. This follows from the previous lemma by taking $X \equiv 1$ and $f_{n}$ to be the indicator function that a digraph is in $\mathcal{E}_{n}$.

The rest of this section proves Theorem 2 and Theorem 3. But instead of working on $\mathcal{G}_{n}^{c}$ directly, we prove similar theorems on fixed sets of vertices, and then apply the above lemma or its corollary to get the final result.

### 3.2 Cycles outside the giant

In this subsection, we show the following:
Theorem 5. Let $\omega_{n} \rightarrow \infty$ be an arbitrary sequence. There exists a sequence $\varepsilon_{n}=o(1)$ such that for all fixed sets of vertices $\mathcal{V}_{n} \subseteq[n]$ with $\left|\mathcal{V}_{n}\right| \in \mathcal{I}_{n}$, we have:
(a) Let $L_{n}^{*}$ be the length of the longest cycle in $\mathcal{D}_{n, k}\left[\mathcal{V}_{n}^{c}\right]$. Then $\mathbb{P}\left\{L_{n}^{*}>\omega_{n}\right\} \leq \varepsilon_{n}$.
(b) The probability that $\mathcal{D}_{n, k}\left[\mathcal{V}_{n}^{c}\right]$ contains vertex-intersecting cycles is at most $\varepsilon_{n}$.
(c) Let $C_{n, \ell}^{*}$ be the number of cycles of length $\ell$ in $\mathcal{D}_{n, k}\left[\mathcal{V}_{n}^{c}\right]$. Let $X_{\ell}=\operatorname{Poi}\left(\left(k e^{-\tau_{k}}\right)^{\ell} / \ell\right)$. Then for all fixed $\ell,\left\|C_{n, \ell}^{*}, X_{\ell}\right\|_{\text {TV }} \leq \varepsilon_{n}$.
(d) Let $C_{n}^{*}$ be the number of cycles in $\mathcal{D}_{n, k}\left[\mathcal{V}_{n}^{c}\right]$. Let $X=\operatorname{Poi}\left(\log \frac{1}{1-k e^{-\tau_{k}}}\right)$. Then $\left\|C_{n}^{*}, X\right\|_{\mathrm{TV}} \leq \varepsilon_{n}$. As a result, $\left|\mathbb{P}\left\{C_{n}^{*}=0\right\}-\left(1-k e^{-\tau_{k}}\right)\right| \leq 2 \varepsilon_{n}$.

Theorem 2 follows from the above theorem and Lemma 3. Our proof is inspired by Cooper and Frieze's work on the directed configuration model [12]. Note that the Cooper-Frieze model is different from that studied by us. In their model, both indegrees and out-degrees are predetermined, whereas we require all out-degrees to be $k$ but the in-degrees are random.

The intuition behind Theorem 5 is that when two cycles share vertices, they contain fewer vertices than arcs. So if we fix the "shape" of a pair of such cycles, the number of ways to label them times the probability that they both exist is $o(1)$. Thus whp cycles in $\mathcal{V}_{n}^{c}$ are vertex-disjoint and the total number of cycles has a distribution close to a sum of independent indicator random variables.

In the following proof, instead of finding the exact $\varepsilon_{n}$, we derive implicit o(1) upper bounds for probabilities and total variation distances which only requires that $\left|\mathcal{V}_{n}\right| \in \mathcal{I}_{n}$.

Lemma 4. Let $\overline{C_{n}^{*}} \equiv \sum_{1 \leq \ell \leq \omega_{n}} C_{n, \ell}^{*}$. Then $\mathbb{P}\left\{C_{n}^{*} \neq \overline{C_{n}^{*}}\right\}=o(1)$.

Proof. Define $(x)_{\ell} \equiv x(x-1) \cdots(x-\ell+1)$. Then the number of all possible cycles of length $\ell$ is $\left(\left|\mathcal{V}_{n}^{c}\right|\right)_{\ell} k^{\ell} / \ell$. (Note that we are also considering the labels on arcs, which makes the counting easier.) And the probability that such a cycle exists is $n^{-\ell}$. Recalling that $\left|\mathcal{V}_{n}^{c}\right| \in\left[e^{-\tau_{k}} n-n^{1 / 2+\delta}, e^{-\tau_{k}} n+n^{1 / 2+\delta}\right]$, we have

$$
\begin{equation*}
\mathbb{E}\left[C_{n, \ell}^{*}\right]=\frac{1}{\ell}\left(\left|\mathcal{V}_{n}^{c}\right|\right)_{\ell} k^{\ell}\left(\frac{1}{n}\right)^{\ell} \leq\left(k e^{-\tau_{k}}\left(1+O\left(n^{-1 / 2+\delta}\right)\right)\right)^{\ell} \tag{1}
\end{equation*}
$$

Since $k e^{-\tau_{k}} \equiv k-\tau_{k}<1$ (Lemma A1), there exists a constant $c_{1}<1$ such that the above is less than $c_{1}^{\ell}$ for $n$ large enough. Since $C_{n}^{*} \neq \overline{C_{n}^{*}}$ if and only if $\sum_{\ell>\omega_{n}} C_{n, \ell}^{*} \geq 1$,

$$
\mathbb{P}\left\{C_{n}^{*} \neq \overline{C_{n}^{*}}\right\}=\mathbb{P}\left\{\sum_{\ell>\omega_{n}} C_{n, \ell}^{*} \geq 1\right\} \leq \mathbb{E}\left[\sum_{\ell>\omega_{n}} C_{n, \ell}^{*}\right] \leq O\left(c_{1}^{\omega_{n}}\right)=o(1) .
$$

Since $L_{n}^{*}>\omega_{n}$ if and only if $\overline{C_{n}^{*}} \neq C_{n}^{*}$, part (a) of Theorem 5 follows. From now on let $\omega_{n}=\log \log n$. We show that:

Lemma 5. Let $X$ and $X_{\ell}$ be as in Theorem 5. Then $\left\|\operatorname{Poi}\left(\mathbb{E} \overline{C_{n}^{*}}\right), X\right\|_{\mathrm{TV}}=o(1)$. And for all $\ell \leq \omega_{n},\left\|\operatorname{Poi}\left(\mathbb{E} C_{n, \ell}^{*}\right), X_{\ell}\right\|_{\mathrm{TV}}=o(1)$.

Proof. For all $\ell \leq \omega_{n}$, by (1) we have

$$
\mathbb{E} C_{n, \ell}^{*}=\frac{1}{\ell}\left(e^{-\tau_{k}} n+O\left(n^{1 / 2+\delta}\right)\right)_{\ell} k^{\ell}\left(\frac{1}{n}\right)^{\ell}=\frac{\left(k e^{-\tau_{k}}\right)^{\ell}}{\ell}\left(1+O\left(\ell n^{-1 / 2+\delta}\right)\right)
$$

Thus

$$
\mathbb{E} \overline{C_{n}^{*}}=\sum_{1 \leq \ell \leq \omega_{n}} \mathbb{E}\left[C_{n, \ell}^{*}\right]=\log \left(\frac{1}{1-k e^{-\tau_{k}}}\right)+O\left(\omega_{n} n^{-1 / 2+\delta}\right)
$$

Therefore $\mathbb{E} \overline{C_{n}^{*}} \rightarrow \mathbb{E} X$ and $\mathbb{E} C_{n, \ell}^{*} \rightarrow \mathbb{E} X_{\ell}$, which implies the lemma.
Proof of Theorem 5. By the two previous lemmas, it suffices to show that

$$
\left\|\overline{C_{n}^{*}}, \operatorname{Poi}\left(\mathbb{E} \overline{C_{n}^{*}}\right)\right\|_{\mathrm{TV}}=o(1), \quad\left\|C_{n, \ell}^{*}, \operatorname{Poi}\left(\mathbb{E} C_{n, \ell}^{*}\right)\right\|_{\mathrm{TV}}=o(1) \quad \text { for all fixed } \ell .
$$

We prove this by using a theorem of Arratia et al. [4]. (A similar result is proved by Barbour et al. [6]). The method is known as the Chen-Stein method because it was first developed by Chen [11] who applied Stein's theory [38] on probability metrics to Poisson distributions.

Let $\mathcal{C}$ be the space of all possible cycles of length at most $\omega_{n}$ in $\mathcal{D}_{n, k}\left[\mathcal{V}_{n}^{c}\right]$. For $\alpha \in \mathcal{C}$, let $\mathcal{B}_{\alpha} \subseteq \mathcal{C}$ be the set of cycles that are vertex-intersecting with $\alpha$. Let $\mathbb{1}_{\alpha}$ be the indicator that a cycle $\alpha$ appears in $\mathcal{D}_{n, k}\left[\mathcal{V}_{n}^{c}\right]$. Define
where

$$
s_{\alpha}=\mathbb{E}\left|\mathbb{E}\left[\mathbb{1}_{\alpha} \mid \sigma\left(\mathbb{1}_{\beta}: \beta \in \mathcal{C} \backslash \mathcal{B}_{\alpha}\right)\right]-\mathbb{E} \mathbb{1}_{\alpha}\right|,
$$

and $\sigma(\cdot)$ denotes the sigma algebra generated by $(\cdot)$. Theorem 1 of Arratia et al. [4] states that

$$
\left\|\overline{C_{n}^{*}}, \operatorname{Poi}\left(\overline{\mathbb{E}} \overline{C_{n}^{*}}\right)\right\|_{\mathrm{TV}} \leq 2\left(b_{1}+b_{2}+b_{3}\right)
$$

If $\beta \in \mathcal{C} \backslash \mathcal{B}_{\alpha}$, then $\alpha$ and $\beta$ are vertex-disjoint. Thus $\mathbb{1}_{\alpha}$ and $\mathbb{1}_{\beta}$ are independent and $s_{\alpha}=0$ for all $\alpha \in \mathcal{C}$, i.e., $b_{3}=0$. It suffices to show that $b_{1}$ and $b_{2}$ are $o(1)$.

Let $|\alpha|$ denote the length of a cycle $\alpha$. Fix $\ell_{1} \leq \omega_{n}$ and $\ell_{2} \leq \omega_{n}$. There are at most $\left|\mathcal{V}_{n}^{c}\right|^{\ell_{1}} k^{\ell_{1}}$ cycles of length $\ell_{1}$. For $|\alpha|=\ell_{1}$, there are at most $\ell_{1}\left|\mathcal{V}_{n}^{c}\right|^{\ell_{2}-1} k^{\ell_{2}}$ cycles of length $\ell_{2}$ that share at least one vertex with $\alpha$. Since $\left(\left|\mathcal{V}_{n}^{c}\right|\right)^{\ell}=(1+o(1))\left(e^{-\tau_{k}} n\right)^{\ell}$ for $\ell \leq \omega_{n}$,

$$
\begin{aligned}
\sum_{\alpha \in \mathcal{C}:|\alpha|=\ell_{1}} \sum_{\beta \in \mathcal{B}_{\alpha}:|\beta|=\ell_{2}} \mathbb{E}_{\alpha} \mathbb{E}_{\beta} & \leq(1+o(1))\left[\left(e^{-\tau_{k}} n\right)^{\ell_{1}} k^{\ell_{1}}\right]\left[\ell_{1}\left(e^{-\tau_{k}} n\right)^{\ell_{2}-1} k^{\ell_{2}}\right]\left(\frac{1}{n}\right)^{\ell_{1}+\ell_{2}} \\
& =(1+o(1)) \frac{1}{e^{-\tau_{k} n}}\left[\ell_{1}\left(e^{-\tau_{k}} k\right)^{\ell_{1}}\right]\left[\left(e^{-\tau_{k}} k\right)^{\ell_{2}}\right] .
\end{aligned}
$$

Therefore

$$
\begin{aligned}
b_{1} & =\sum_{1 \leq \ell_{1} \leq \omega_{n}} \sum_{1 \leq \ell_{2} \leq \omega_{n}} \sum_{\alpha \in \mathcal{C}:|\alpha|=\ell_{1}} \sum_{\beta \in \mathcal{\mathcal { B } _ { \alpha } : | \beta | = \ell _ { 2 }}} \mathbb{E} \mathbb{1}_{\alpha} \mathbb{E}_{\beta} \\
& \leq(1+o(1)) \frac{1}{e^{-\tau_{k}} n} \sum_{\ell_{1} \geq 1} \sum_{\ell_{2} \geq 1}\left[\ell_{1}\left(k e^{-\tau_{k}}\right)^{\ell_{1}}\right]\left[\left(k e^{-\tau_{k}}\right)^{\ell_{2}}\right] \\
& \leq(1+o(1)) \frac{1}{e^{-\tau_{k} n}}\left[\sum_{\ell_{1} \geq 1} \ell_{1}\left(k e^{-\tau_{k}}\right)^{\ell_{1}}\right]\left[\sum_{\ell_{2} \geq 1}\left(k e^{-\tau_{k}}\right)^{\ell_{2}}\right]
\end{aligned}
$$

which is $O(1 / n)$ since both sums converge.
To compute $b_{2}$, we upper bound the number of pairs of vertex-intersecting cycles that could possibly appear in $\mathcal{D}_{n, k}\left[\mathcal{V}_{n}^{c}\right]$ at the same time. Let $\alpha$ and $\beta$ be such a pair. Let $V(\alpha), A(\alpha), V(\beta), A(\beta)$ be the vertex set and (labeled) arc set of $\alpha$ and $\beta$ respectively. Let $\alpha \cup \beta$ be the digraph of vertex set $V=V(\alpha) \cup V(\beta)$ and arc set $A=A(\alpha) \cup B(\beta)$. Assume that $|V|=s$ and $|A|=s+t$. Note that as $\alpha$ and $\beta$ share at least one vertex, $t \geq 1$. Since $V \subset[n]$, we can relabel the $s$ vertices in $\alpha \cup \beta$ with $[s]$ such that the order of the vertex labels is maintained. The result is a digraph with vertex set [ $s$ ] and $s+t$ arcs labeled with $[k]$. There are at most $\left(s^{2}\right)^{s+t} k^{s+t}$ such digraphs, since there are at most $s^{2}$ choices of endpoints and $k$ choices of labels for each of the $s+t$ arcs. Each digraph of this type corresponds to at most $\binom{\left|\mathcal{V}_{n}^{c}\right|}{s} \leq\left|\mathcal{V}_{n}^{c}\right|^{s}$ pairs of cycles like $\alpha$ and $\beta$. Thus there
are at most $\left|\mathcal{V}_{n}^{c}\right|^{s}\left(s^{2}\right)^{s+t} k^{s+t}$ such pairs. Summing over $s$ and $t$, we have

$$
\begin{align*}
b_{2} & \leq \sum_{1 \leq s \leq 2 \omega_{n}} \sum_{1 \leq t \leq 2 \omega_{n}}\left|\mathcal{V}_{n}^{c}\right|^{s}\left(s^{2}\right)^{s+t} k^{s+t} \mathbb{E}\left[\mathbb{1}_{\alpha} \mathbb{1}_{\beta}\right] \\
& \leq \sum_{1 \leq s \leq 2 \omega_{n}} \sum_{1 \leq t \leq 2 \omega_{n}}\left(e^{-\tau_{k}} n+n^{1 / 2+\delta}\right)^{s}\left(2 \omega_{n}\right)^{2 \times 4 \omega_{n}} k^{s+t} \frac{1}{n^{s+t}} \\
& \leq\left(2 \omega_{n}\right)^{8 \omega_{n}} \sum_{1 \leq s \leq 2 \omega_{n}} \sum_{1 \leq t \leq 2 \omega_{n}} \frac{\left(n+e^{\tau_{k}} n^{1 / 2+\delta}\right)^{s}}{n^{s}}\left(k e^{-\tau_{k}}\right)^{s} \frac{k^{t}}{n^{t}}  \tag{2}\\
& \leq O\left(\frac{1}{n}\right)\left(2 \omega_{n} k\right)^{8 \omega_{n}} \sum_{1 \leq s \leq 2 \omega_{n}} \sum_{1 \leq t \leq 2 \omega_{n}}\left(1+e^{\tau_{k}} n^{-1 / 2+\delta}\right)^{2 \omega_{n}} \quad\left(k e^{-\tau_{k}}<1 / 2\right) \\
& \leq O\left(\frac{1}{n}\right)\left(2 \omega_{n} k\right)^{8 \omega_{n}}\left(2 \omega_{n}\right)^{2}\left(1+O\left(n^{-1 / 2+\delta} \omega_{n}\right)\right) \rightarrow 0,
\end{align*}
$$

where the last step we use that $\omega_{n}=\log \log n$.
Thus part (d) of Theorem 5 for $C_{n}^{*}$ is proved. We can prove part (c) for $C_{n, \ell}^{*}$ using the same method by limiting $\mathcal{C}$ to contain only cycles of a fixed length $\ell$. Note that the above inequality shows that the probability that there exist vertex-intersecting cycles in $\mathcal{D}_{n, k}\left[\mathcal{V}_{n}^{c}\right]$ is $o(1)$, thus part (b) is also proved.

The method used above can be easily adapted to prove similar results for undirected cycles, like the following lemma which is needed in the study of spectra in $\mathcal{D}_{n, k}\left[\mathcal{G}_{n}^{c}\right]$ :

Lemma 6. Let $\psi_{n} \rightarrow \infty$ be an arbitrary sequence. There exists a sequence $\varepsilon_{n}=o(1)$ such that for all fixed sets of vertices $\mathcal{V}_{n}$ with $\left|\mathcal{V}_{n}\right| \in \mathcal{I}_{n}$, we have:
(a) The probability that $\mathcal{D}_{n, k}\left[\mathcal{V}_{n}^{c}\right]$ contains an undirected cycle of length greater than $\psi_{n}$ is at most $\varepsilon_{n}$.
(b) The probability that $\mathcal{D}_{n, k}\left[\mathcal{V}_{n}^{c}\right]$ contains vertex-intersecting undirected cycles is at most $\varepsilon_{n}$.

Proof. Let $U_{\ell}$ be the number of undirected cycles of length $\ell$ in $\mathcal{D}_{n, k}\left[\mathcal{V}_{n}^{c}\right]$. Then

$$
\mathbb{E}\left[U_{\ell}\right] \leq \frac{1}{\ell}\left(\left|\mathcal{V}_{n}^{c}\right|\right)^{\ell}(2 k)^{\ell} \frac{1}{n^{\ell}} \leq\left(2 k e^{-\tau_{k}}\left(1+n^{-1 / 2+\delta}\right)\right)^{\ell}
$$

where the 2 comes from the fact that each edge in an undirected cycle has two possible directions. Since $2 k e^{-\tau_{k}}=2\left(k-\tau_{k}\right)<1$ (Lemma A1), with exact the same argument of Lemma 4, we can show that $\mathbb{E}\left[\sum_{\ell>\psi_{n}} U_{\ell}\right]=o(1)$ for all $\psi_{n} \rightarrow \infty$. Thus (a) is proved.

Now choose $\psi_{n}=\log \log n$. Again we can show that whp there are no vertexintersecting undirected cycles of length at most $\psi_{n}$ by repeating the computation of $b_{2}$ in the proof of Theorem 5 with $k e^{-\tau_{k}}$ replaced by $2 k e^{-\tau_{k}}$ in (2).

### 3.3 Spectra outside the giant

In this section, we prove Theorem 3 (spectra outside the giant). Instead of working on $\mathcal{G}_{n}^{c}$ directly, we again prove similar results on a fixed set of vertices and then apply Lemma 3 to finish the proof.

### 3.3.1 The tree-like structure of some spectra

We prove part (a) of Theorem 3. Let $\mathcal{V}_{n} \subseteq[n]$ with $\left|\mathcal{V}_{n}\right| \in \mathcal{I}_{n} \equiv\left[\nu_{k} n-n^{1 / 2+\delta}, \nu_{k} n+n^{1 / 2+\delta}\right]$ be a fixed set of vertices. For $v \in \mathcal{V}_{n}^{c} \equiv[n] \backslash \mathcal{V}_{n}$, let $\mathcal{S}_{v}^{*}$ be the spectrum of $v$ in $\mathcal{D}_{n, k}\left[\mathcal{V}_{n}^{c}\right]$, the sub-digraph induced by $\mathcal{V}_{n}^{c}$. The following lemma shows that whp every spectrum in $\mathcal{D}_{n, k}\left[\mathcal{V}_{n}^{c}\right]$ induces a sub-digraph that is a tree or a tree plus one extra arc:

Lemma 7. We have

$$
\sup _{\mathcal{V}_{n} \subseteq|n|:\left|\nu_{n}\right| \in \mathcal{I}_{n}} \mathbb{P}\left\{\cup_{v \in \mathcal{V}_{n}^{v}}\left[\operatorname{arc}\left(\mathcal{D}_{n, k}\left[\mathcal{S}_{v}^{*}\right]\right)-\left|\mathcal{S}_{v}^{*}\right| \geq 1\right]\right\}=o(1),
$$

where $\operatorname{arc}(\cdot)$ denotes the number of arcs.
Proof. For $v \in \mathcal{V}_{n}^{c}$, if $\operatorname{arc}\left(\mathcal{D}_{n, k}\left[\mathcal{S}_{v}^{*}\right]\right) \geq\left|\mathcal{S}_{v}^{*}\right|+1$, then $\mathcal{D}_{n, k}\left[\mathcal{S}_{v}^{*}\right]$ must contain at least two undirected cycles. By Lemma 6, whp all undirected cycles in $\mathcal{D}_{n, k}\left[\mathcal{S}_{v}^{*}\right]$ are vertex-disjoint. Therefore, if $\mathcal{D}_{n, k}\left[\mathcal{S}_{v}^{*}\right]$ contains two undirected cycles, then whp they are vertex-disjoint and connected by an undirected path.

Let $X_{r, s, t}$ be the number of pairs of undirected cycles of length $r$ and $s$ respectively that are connected by an undirected path of length $t$. In such a structure the number of arcs is $r+s+t$ while the number of vertices is $r+s+t-1$. Since $\left|\mathcal{V}_{n}\right| \in \mathcal{I}_{n}$, we have $\left|\mathcal{V}_{n}^{c}\right|=n-\left|\mathcal{V}_{n}\right| \in \mathcal{I}_{n}^{c} \equiv\left[e^{-\tau_{k}} n-n^{1 / 2+\delta}, e^{-\tau_{k}} n+n^{1 / 2+\delta}\right]$. Thus

$$
\mathbb{E} X_{r, s, t} \leq\left(\left|\mathcal{V}_{n}^{c}\right|\right)^{r+s+t-1}(2 k)^{r+s+t}\left(\frac{1}{n}\right)^{r+s+t} \leq O\left(\frac{1}{n}\right)\left(2 k e^{-\tau_{k}}+\frac{2 k}{n^{1 / 2-\delta}}\right)^{r+s+t}
$$

Summing over all possible $r, s$ and $t$ shows that

$$
\begin{aligned}
\sum_{1 \leq r \leq n} \sum_{1 \leq s \leq n} \sum_{1 \leq t \leq n} \mathbb{E} X_{r, s, t} & \leq O\left(\frac{1}{n}\right) \sum_{1 \leq r} \sum_{1 \leq s} \sum_{1 \leq t}\left(2 k e^{-\tau_{k}}+\frac{2 k}{n^{1 / 2-\delta}}\right)^{r+s+t} \\
& \leq O\left(\frac{1}{n}\right)\left(\sum_{1 \leq i}\left(2 k e^{-\tau_{k}}+\frac{2 k}{n^{1 / 2-\delta}}\right)^{i}\right)^{3}
\end{aligned}
$$

which is $o(1)$ since the sum in the brackets converges.

### 3.3.2 The maximum size of spectra

This section proves part (b) of Theorem 3 (the sizes of spectra outside the giant).

Lemma 8. Let $\varepsilon>0$ be a constant. Then

$$
\sup _{\mathcal{V}_{n} \subseteq[n]:\left|\mathcal{V}_{n}\right| \in \mathcal{I}_{n}} \mathbb{P}\left\{\left|\frac{\max _{v \in \mathcal{V}_{n}^{c}}\left|\mathcal{S}_{v}^{*}\right|}{\log n}-\frac{1}{\log \left(1 / \lambda_{k}\right)}\right|>\varepsilon\right\}=o(1),
$$

where $\lambda_{k} \equiv\left(k-\tau_{k}\right)\left(\frac{\tau_{k}}{k-1}\right)^{k-1}$.
The exploration of $\mathcal{D}_{n, k}\left[\mathcal{S}_{v}^{*}\right]$ can be coupled with a colouring process. Initially, colour all vertices in $\mathcal{V}_{n}$ green, all vertices in $\mathcal{V}_{n}^{c}$ yellow, and all arcs white. Then:
(i) Colour the vertex $v$ black, and colour the $k$ arcs that start from $v$ red. (Red $\operatorname{arcs}$ start from vertices in $\mathcal{S}_{v}^{*}$ but their endpoints are not determined yet.)
(ii) Pick an arbitrary red arc. Choose its endpoint uniformly at random from all the $n$ vertices. Colour this arc with the colour of its chosen endpoint vertex. (So a yellow arc goes to a vertex that is not already in $\mathcal{S}_{v}^{*}$, a black arc goes to a vertex that is already in $\mathcal{S}_{v}^{*}$.) If the chosen vertex is yellow, colour this vertex black and colour all its arcs red.
(iii) If there are no red arcs left, terminate. Otherwise go to the previous step.

In the end, $\mathcal{S}_{v}^{*}$ consists of all black vertices, and arcs that start from vertices in $\mathcal{S}_{v}^{*}$ have one of three colors: green arcs go to $\mathcal{V}_{n}$; yellow arcs form a spanning tree of $\mathcal{D}_{n, k}\left[\mathcal{S}_{v}^{*}\right]$ rooted at $v$; black arcs connect vertices in $\mathcal{S}_{v}^{*}$ but they are not part of the yellow spanning tree, so they are in cycles in $\mathcal{D}_{n, k}\left[\mathcal{S}_{v}^{*}\right]$. Figure 2 depicts the colouring process.


Figure 2: The colouring process.
We use random variables $R_{t}$ and $Y_{t}$ to track the number of red arcs and yellow vertices after the $t$-th red arc is colored. Thus $R_{0}=k$ and $Y_{0}=\left|\mathcal{V}_{n}^{c}\right|-1$. When a red arc is colored, if a yellow vertex is chosen as its endpoint, then the number of red arcs increases by $(k-1)$ and the number of yellow vertices decreases by one. Otherwise the number of red arcs decreases by one and the number of yellow vertices remains unchanged. Thus for $t \geq 1$,

$$
R_{t}=R_{t-1}+k \xi_{t}-1=k \sum_{i=1}^{t} \xi_{i}-(t-k), \quad \text { and } \quad Y_{t}=Y_{t-1}-\xi_{t}=\left|\mathcal{V}_{n}^{c}\right|-1-\sum_{i=1}^{t} \xi_{i}
$$

where $\xi_{t}$ are independent Bernoulli $Y_{t} / n$ (the probability that a yellow vertex is chosen). Let $T \equiv \min \left\{t: R_{t} \leq 0\right\}$. Then $\left|\mathcal{S}_{v}^{*}\right|=T / k$, since $T$ is the total number arcs that have been colored and $\left|\mathcal{S}_{v}^{*}\right|$ is the total number of vertices that have been colored.

Let $\left(\overline{\xi_{t}}\right)_{t \geq 1}$, be i.i.d. Bernoulli $\left(e^{-\tau_{k}}+n^{-1 / 2+\delta}\right)$. Since $Y_{t} / n \leq\left|\mathcal{V}_{n}^{c}\right| / n \leq e^{-\tau_{k}}+n^{-1 / 2+\delta}$, we have $\overline{\xi_{t}} \succeq \xi_{t}$, where $\succeq$ denotes stochastically greater than (see [36]). Therefore there exists a coupling such that $\bar{\xi}_{t} \geq \xi_{t}$ for all $t$ almost surely. Let $\bar{T}_{t} \equiv \min \{t$ : $\left.k \sum_{i=1}^{t} \bar{\xi}_{i}-(t-k) \leq 0\right\}$. Then $\bar{T} \geq T$ almost surely. (The random variable $T$ is called the total progeny of a Galton-Watson process with offspring distribution $\overline{\xi_{1}}$. For an introduction to Galton-Watson processes see [13]). It is well know that if $\mathbb{E} \overline{\xi_{1}}<1$, which is true in this case, then $\mathbb{E} \bar{T}=k /\left(1-\mathbb{E} \overline{\xi_{1}}\right)=O(1)$. Thus $\mathbb{E} T=O(1)$.

Proof of the upper bound. Let $\omega_{n}=\left\lfloor(1+\varepsilon) \log n / \log \left(1 / \lambda_{k}\right)\right\rfloor+1$. Since $\bar{T} \geq T$,

$$
\mathbb{P}\left\{T \geq k \omega_{n}\right\} \leq \mathbb{P}\left\{\bar{T} \geq k \omega_{n}\right\} \leq \mathbb{P}\left\{\frac{\sum_{i=1}^{k \omega_{n}} \overline{\xi_{i}}}{k \omega_{n}} \geq \frac{1}{k_{n}}\right\}
$$

where $k_{n}=k \omega_{n} /\left(\omega_{n}-1\right)$. Hoeffding [21] showed that

$$
\mathbb{P}\left\{\frac{\operatorname{Bin}(m, p)}{m} \geq p+x\right\} \leq\left\{\left(\frac{p}{p+x}\right)^{p+x}\left(\frac{1-p}{1-p-x}\right)^{1-p-x}\right\}^{m}
$$

where $\operatorname{Bin}(m, p)$ denotes a binomial $(m, p)$ random variable. Recalling that $\mathbb{E} \overline{\xi_{1}}=e^{-\tau_{k}}+$ $n^{-1 / 2+\delta} \equiv 1-\tau_{k} / k+n^{-1 / 2+\delta}$ and $\lambda_{k} \equiv\left(k-\tau_{k}\right)\left(\frac{\tau_{k}}{k-1}\right)^{k-1}$, it follows from Hoeffding's inequality that $\mathbb{P}\left\{T \geq k \omega_{n}\right\}$ is at most

$$
\begin{align*}
{\left[\left(\frac{\mathbb{E} \overline{\xi_{1}}}{1 / k_{n}}\right)\left(\frac{1-\mathbb{E} \overline{\xi_{1}}}{1-1 / k_{n}}\right)^{k_{n}-1}\right]^{\omega_{n}} } & =\left[\left(k-\tau_{k}\right)\left(\frac{\tau_{k}}{k-1}\right)^{k-1}+O\left(n^{-1 / 2+\delta}\right)\right]^{\omega_{n}+O(1)} \\
& =O\left(\lambda_{k}^{\omega_{n}}\right)\left(1+O\left(n^{-1 / 2+\delta}\right)\right)^{\omega_{n}} \\
& =O\left(n^{-(1+\varepsilon)}\right) \tag{3}
\end{align*}
$$

Since $k\left|\mathcal{S}_{v}^{*}\right|=T$, by the union bound

$$
\mathbb{P}\left\{\cup_{v \in \mathcal{V}_{n}^{c}}\left|\mathcal{S}_{v}^{*}\right| \geq \omega_{n}\right\} \leq n \mathbb{P}\left\{\bar{T} \geq k \omega_{n}\right\}=O\left(n^{-\varepsilon}\right)
$$

Proof of the lower bound. Let $\psi_{n} \equiv\left\lceil(1-\varepsilon) \log n / \log \left(1 / \lambda_{k}\right)\right\rceil$. To show that whp there exists a $v \in \mathcal{V}_{n}^{c}$ such that $\left|\mathcal{S}_{v}^{*}\right| \geq \psi_{n}$, pick an arbitrary yellow vertex and run the colouring process. If at least $\psi_{n}$ vertices are colored black (success) in the process then terminate. Otherwise (failure) pick another yellow vertex and repeat the colouring process until one trial succeeds. If the colouring process is repeated for at most $t_{n} \equiv\left\lfloor n /(\log n)^{3}\right\rfloor$ times, then at most $a_{n} \equiv t_{n} \psi_{n}=O\left(n /(\log n)^{2}\right)$ vertices are colored black in the end. Therefore, the probability that the number of red arcs increases after colouring one red arc is at least $\left(\left|\mathcal{V}_{n}^{c}\right|-a_{n}\right) / n$.

Let $\left(\underline{\xi}_{i}\right)_{i \geq 1}$ be i.i.d. Bernoulli $\left(\left|\mathcal{V}_{n}^{c}\right|-a_{n}-\psi_{n}\right) / n$. Let $\underline{T}=\min \left\{t: k \sum_{i=1}^{t} \underline{\xi_{i}}-(t-\right.$ $k) \leq 0\}$. Then in each of the first $t_{n}$ iterations, the probability of a success is at least
$\mathbb{P}\left\{\underline{T} \geq k \psi_{n}\right\} \geq \mathbb{P}\left\{\underline{T}=k \psi_{n}\right\}$. (For a detailed proof, see van der Hofstad's discussion of the Erdős-Rényi model [39, chap. 4.2.2].) By the hitting-time theorem of Galton-Watson processes [41],

$$
\mathbb{P}\left\{\underline{T}=k \psi_{n}\right\}=\frac{1}{\psi_{n}} \mathbb{P}\left\{k \sum_{i=1}^{k \psi_{n}} \underline{\xi_{i}}=k\left(\psi_{n}-1\right)\right\} .
$$

Since $\sum_{i=1}^{k \psi_{n}} \underline{\xi_{i}}$ is a binomial random variable, the above equals

$$
\frac{1}{\psi_{n}}\binom{k \psi_{n}}{\psi_{n}-1}\left(\frac{\left|\mathcal{V}_{n}^{c}\right|-a_{n}-\psi_{n}}{n}\right)^{\psi_{n}-1}\left(1-\frac{\left|\mathcal{V}_{n}^{c}\right|-a_{n}-\psi_{n}}{n}\right)^{k \psi_{n}-\left(\psi_{n}-1\right)} \equiv b_{n} .
$$

By Stirling's approximation [17, pp. 407]

$$
\binom{k \psi_{n}}{\psi_{n}-1}=\Theta(1)\binom{k \psi_{n}}{\psi_{n}}=\frac{1}{\Theta\left(\sqrt{\psi_{n}}\right)}\left[\frac{k}{(1-1 / k)^{k-1}}\right]^{\psi_{n}} .
$$

Recalling that $a_{n} \equiv O\left(n /(\log n)^{2}\right)$ and $\psi_{n} \equiv\left\lceil(1-\varepsilon) \log n / \log \left(1 / \lambda_{k}\right)\right\rceil$, we have, in view of $\left|\mathcal{V}_{n}^{c}\right|=e^{-\tau_{k}} n+O\left(n^{1 / 2+\delta}\right)$,

$$
\left(\frac{\left|\mathcal{V}_{n}^{c}\right|-a_{n}-\psi_{n}}{n}\right)^{\psi_{n}-1}=\left(e^{-\tau_{k}}-O\left(\frac{1}{(\log n)^{2}}\right)\right)^{\psi_{n}-1}=\Theta\left(e^{-\tau_{k} \psi_{n}}\right),
$$

and

$$
\begin{aligned}
\left(1-\frac{\left|\mathcal{V}_{n}^{c}\right|-a_{n}-\psi_{n}}{n}\right)^{k \psi_{n}-\left(\psi_{n}-1\right)} & =\left(1-e^{-\tau_{k}}+O\left(\frac{1}{(\log n)^{2}}\right)\right)^{k \psi_{n}-\left(\psi_{n}-1\right)} \\
& =\Theta\left(\left(\frac{\tau_{k}}{k}\right)^{(k-1) \psi_{n}}\right)
\end{aligned}
$$

Recall that $e^{-\tau_{k}} \equiv 1-\tau_{k} / k$. Therefore

$$
\lambda_{k} \equiv\left(k-\tau_{k}\right)\left(\frac{\tau_{k}}{k-1}\right)^{k-1}=k e^{-\tau_{k}}\left(\frac{\tau_{k}}{k-1}\right)^{k-1}=\frac{k}{(1-1 / k)^{k-1}} e^{-\tau_{k}}\left(\frac{\tau_{k}}{k}\right)^{k-1} .
$$

Putting everything together, we have

$$
b_{n}=\Theta\left(\frac{1}{\psi_{n}} \frac{1}{\sqrt{\psi_{n}}}\left[\frac{k}{(1-1 / k)^{k-1}} e^{-\tau_{k}}\left(\frac{\tau_{k}}{k}\right)^{k-1}\right]^{\psi_{n}}\right)=\Theta\left(\frac{\lambda_{k}^{\psi_{n}}}{\psi_{n}^{3 / 2}}\right)=\Theta\left(\frac{n^{-1+\varepsilon}}{\psi_{n}^{3 / 2}}\right) .
$$

So the probability that all the first $t_{n} \equiv\left\lfloor n /(\log n)^{3}\right\rfloor$ trials fail is at most

$$
\left(1-b_{n}\right)^{t_{n}} \leq \exp \left\{-b_{n} t_{n}\right\}=\exp \left\{\Theta\left(-\frac{n^{\varepsilon}}{(\log n)^{9 / 2}}\right)\right\}=o(1)
$$

By Lemma 2, whp $\mathcal{G}_{n}$ is reachable from all vertices. When this happens, $\mathcal{O}_{n} \backslash \mathcal{G}_{n}$ consists of vertices either on cycles in $\mathcal{D}_{n, k}\left[\mathcal{G}_{n}^{c}\right]$ or on paths from these cycles to $\mathcal{G}_{n}$. Since the number of such cycles and the length of the longest one of them are both $O_{p}(1)$, Lemma 8 implies that $\left|\mathcal{O}_{n}\right|-\left|\mathcal{G}_{n}\right|=O_{p}(\log n)$. Thus

$$
\frac{\left|\mathcal{G}_{n}\right|-\nu_{k} n}{\sqrt{n}}=\frac{\left|\mathcal{O}_{n}\right|-\nu_{k} n}{\sqrt{n}}-O_{p}\left(\frac{\log n}{\sqrt{n}}\right) \xrightarrow{d} \mathcal{Z}
$$

which is the second part of Theorem 1.
In fact we can show that $\left|\mathcal{O}_{n}\right|-\left|\mathcal{G}_{n}\right|=O_{p}(1)$. This seems to be obvious since in $\mathcal{D}_{n, k}\left[\mathcal{V}_{n}^{c}\right]$ the expected size of a spectrum is $O(1)$ and the number of cycles is $O_{p}(1)$. However, it is not trivial because $\mathbb{1}_{[v}$ is on a cycle] and $\left|\mathcal{S}_{v}^{*}\right|$ are not independent. For a proof using Cayley's formula, see Lemma 9 in the next section (Section 3.3.3).

We can also use Lemma 8 to show that

$$
\frac{\max _{v \in[n]}\left|\mathcal{S}_{v}\right|-\left|\mathcal{G}_{n}\right|}{\log n} \xrightarrow[\rightarrow]{p} \frac{1}{\log \left(1 / \lambda_{k}\right)},
$$

which finishes the last part of Theorem 1, i.e., $\left(\max _{v \in[n]}\left|\mathcal{S}_{v}\right|-\nu_{k} n\right) / \sigma_{k} \sqrt{n} \xrightarrow{d} \mathcal{Z}$. Let $A_{n}$ be the event that every vertex can reach $\mathcal{G}_{n}$. Assuming $A_{n}$ happens, $\mathcal{G}_{n} \subseteq \mathcal{S}_{v}$ for all $v \in[n]$. Thus for all $\varepsilon>0$,

$$
\begin{aligned}
& \mathbb{P}\left\{\left|\frac{\max _{v \in[n]}\left|\mathcal{S}_{v}\right|-\left|\mathcal{G}_{n}\right|}{\log n}-\frac{1}{\log \left(1 / \lambda_{k}\right)}\right|>\varepsilon\right\} \\
& \leq \mathbb{P}\left\{\left[\left|\frac{\max _{v \in[n]}\left|\mathcal{S}_{v}^{\prime}\right|}{\log n}-\frac{1}{\log \left(1 / \lambda_{k}\right)}\right|>\varepsilon\right] \cap A_{n}\right\}+\mathbb{P}\left\{A_{n}^{c}\right\}=o(1)
\end{aligned}
$$

Since $\left|\mathcal{S}_{1}\right| \leq \max _{v \in[n]}\left|\mathcal{S}_{v}\right|$ and whp $\left|\mathcal{S}_{1}\right| \geq\left|\mathcal{G}_{n}\right|$, we also recover Grusho's central limit law of $\left|\mathcal{S}_{1}\right|$.

### 3.3.3 The size of the middle layer

Lemma 9 and Corollary 1 imply that $\left|\mathcal{O}_{n}\right|-\left|\mathcal{G}_{n}\right|=O_{p}(1)$.
Lemma 9. Let $\omega_{n} \rightarrow \infty$ be an arbitrary sequence of nonnegative numbers. Then

$$
\sup _{\mathcal{V}_{n} \subseteq[n]\left|: \mathcal{V}_{n}\right| \in \mathcal{I}_{n}} \mathbb{P}\left\{\sum_{v \in \mathcal{C}\left(\mathcal{V}_{n}^{c}\right)}\left|\mathcal{S}_{v}^{*}\right| \geq \omega_{n}\right\}=o(1)
$$

where $\mathcal{C}\left(\mathcal{V}_{n}^{c}\right)$ denotes the set of vertices on cycles in $\mathcal{D}_{n, k}\left[\mathcal{V}_{n}^{c}\right]$, and $\mathcal{S}_{v}^{*}$ is the spectrum of $v$ in $\mathcal{D}_{n, k}\left[\mathcal{V}_{n}^{c}\right]$, the sub-digraph induced by $\mathcal{V}_{n}^{c}$.

Proof. By Theorem 5 and Lemma 7, in $\mathcal{D}_{n, k}\left[\mathcal{V}_{n}^{c}\right]$ whp: (a) there are at most $\sqrt{\omega_{n}}$ vertices on cycles, i.e., $\left|\mathcal{C}\left(\mathcal{V}_{n}^{c}\right)\right| \leq \sqrt{\omega_{n}} ;(\mathrm{b})$ every $\mathcal{S}_{v}^{*}$ induces either a tree or a tree plus one extra arc; (c) $\max _{v \in \mathcal{G}_{n}^{c}}\left|\mathcal{S}_{v}^{*}\right|=O(\log n)$. Now assume all these events happen. If $\sum_{v \in \mathcal{C}\left(\mathcal{V}_{n}^{c}\right)}\left|\mathcal{S}_{v}^{*}\right| \geq$


Figure 3: The leftmost shaded part of this figure is an $\ell$-eye.
$\omega_{n}$, then (a) implies there is at least one vertex $u \in \mathcal{C}\left(\mathcal{V}_{n}^{c}\right)$ with $\left|\mathcal{S}_{u}^{*}\right| \geq \sqrt{\omega_{n}}$. By (b), $\mathcal{S}_{u}^{*}$ induces a sub-digraph that consists of exactly one cycle and isolated trees with their roots on this cycle. If $\left|\mathcal{S}_{u}^{*}\right|=\ell$, we call the induced sub-digraph an $\ell$-eye. Note that by (c) there are no $\ell$-eyes with $\ell>(\log n)^{2}$.

Let $\mathcal{S} \subseteq \mathcal{V}_{n}^{c}$ with $|\mathcal{S}|=\ell$ be a set of vertices. If $\mathcal{S}$ induces an $\ell$-eye $\mathcal{D}_{e}$, then there are $\ell$ arcs that start and end at specific vertices in $\mathcal{S}$ decided by $\mathcal{D}_{e}$, which happens with probability $(1 / n)^{\ell}$. If $\mathcal{S}=\mathcal{S}_{u}^{*}$ for some vertex $u \in \mathcal{S}$, call $\mathcal{S}$ a partial spectrum. For $\mathcal{S}$ to be a partial spectrum, the other $(k-1) \ell$ arcs that start from $\mathcal{S}$ must end at $\mathcal{V}_{n}$, which happens with probability $\left(\left|\mathcal{V}_{n}\right| / n\right)^{(k-1) \ell}$. So the probability that $\mathcal{S}$ induces a fixed $\mathcal{D}_{e}$ and $\mathcal{S}$ is a partial spectrum is $(1 / n)^{\ell}\left(\left|\mathcal{V}_{n}\right| / n\right)^{(k-1) \ell}$.

By Cayley's formula [7], there are $\ell^{\ell-1}$ ways that $\mathcal{S}$ can form a rooted tree. In such a tree, there are at most $\ell^{2}$ ways to add an extra arc to make it an $\ell$-eye. In a vertexlabeled $\ell$-eye, there are at most $k^{\ell}$ ways to label the arcs. So the number of $\ell$-eyes can be induced by $\mathcal{S}$ is less than $\ell^{\ell-1} \ell^{2} k^{\ell}$. And there are $\binom{\left|\mathcal{V}_{n}^{c}\right|}{\ell}$ ways to choose $\mathcal{S}$.

Let $X_{\ell}$ be the number of $\ell$-eyes induced by partial spectra. Recall that $\nu_{k} \equiv \tau_{k} / k=$ $1-e^{-\tau_{k}}$. Thus $\left|\mathcal{V}_{n}\right| \in \mathcal{I}_{n} \equiv\left[\nu_{k} n-n^{1 / 2+\delta}, \nu_{k} n+n^{1 / 2+\delta}\right]$ implies that $\left|\mathcal{V}_{n}^{c}\right| \leq e^{-\tau_{k}} n+n^{1 / 2+\delta}$. So for $\ell \leq(\log n)^{2}$, by the above arguments,

$$
\begin{aligned}
\mathbb{E} X_{\ell} & \leq\binom{\left|\mathcal{V}_{n}^{c}\right|}{\ell} \ell^{\ell-1} \ell^{2} k^{\ell}\left(\frac{1}{n}\right)^{\ell}\left(\frac{\left|\mathcal{V}_{n}\right|}{n}\right)^{(k-1) \ell} \\
& \leq \frac{\left(e^{-\tau_{k}} n+n^{1 / 2+\delta}\right)^{\ell}}{(\ell / e)^{\ell}} \ell^{\ell+1} k^{\ell}\left(\frac{1}{n}\right)^{\ell}\left(\frac{\tau_{k}}{k}+n^{-1 / 2+\delta}\right)^{(k-1) \ell} \\
& =\left[e\left(e^{-\tau_{k}}+n^{-1 / 2+\delta}\right) k\left(\frac{\tau_{k}}{k}+n^{-1 / 2+\delta}\right)^{k-1}\right]^{\ell} \ell \\
& =\left(1+O\left(\ell n^{-1 / 2+\delta}\right)\right)\left(k e^{1-\tau_{k}}\left(\frac{\tau_{k}}{k}\right)^{k-1}\right)^{\ell} \ell \\
& \equiv\left(1+O\left(\ell n^{-1 / 2+\delta}\right)\right) \rho_{k}^{\ell} \ell .
\end{aligned}
$$

By Lemma A1, $\rho_{k}<1$. Since $\sqrt{\omega_{n}} \rightarrow \infty$,

$$
\sum_{\sqrt{\omega_{n}} \leq \ell \leq(\log n)^{2}} \mathbb{E} X_{\ell} \leq\left[1+O\left(\frac{(\log n)^{2}}{n^{1 / 2-\delta}}\right)\right] \sum_{\sqrt{\omega_{n}} \leq \ell}^{\infty} \ell\left(\rho_{k}\right)^{\ell}=o(1)
$$

Thus whp there are no $\ell$-eyes induced by partial spectra with $\ell \in\left[\sqrt{\omega_{n}},(\log n)^{2}\right]$.

### 3.3.4 The distance to the giant

This subsection proves part (c) of Theorem 3.
Lemma 10. For all $\varepsilon>0$,

$$
\sup _{\mathcal{V}_{n} \subseteq\left[n!:\left|\mathcal{V}_{n}\right| \in \mathcal{I}_{n}\right.} \mathbb{P}\left\{\left|\frac{\max _{v \in \mathcal{V}_{n}^{c}} W_{v}^{*}}{\log _{k} \log n}-1\right|>\varepsilon\right\}=o(1),
$$

where $W_{v}^{*} \equiv \min _{u \in \mathcal{V}_{n}} \operatorname{dist}(v, u)$, i.e., $W_{v}^{*}$ is the length of the shortest path from $v$ to $\mathcal{V}_{n}$.
Let $v \in \mathcal{V}_{n}^{c}$ be a vertex. If $W_{v}^{*}>1$, then all neighbors of $v$ are in $\mathcal{V}_{n}^{c}$, and most likely there are $k$ of them. So $\mathbb{P}\left\{W_{v}^{*}>1\right\} \approx\left(\left|\mathcal{V}_{n}^{c}\right| / n\right)^{k} \approx e^{-\tau_{k} k}$. If $W_{v}^{*}>2$, then the neighbors of $v$ 's neighbors are all in $\mathcal{V}_{n}^{c}$, and most likely there are $k^{2}$ of them. So $\mathbb{P}\left\{W_{v}^{*}>2\right\} \approx\left(\left|\mathcal{V}_{n}^{c}\right| / n\right)^{k+k^{2}} \approx e^{-\tau_{k}\left(k+k^{2}\right)}$. Repeating this argument shows that $\mathbb{P}\left\{W_{v}^{*}>x\right\} \approx \exp \left\{-\tau_{k}\left(k+k^{2} \ldots k^{x}\right)\right\}=e^{-\tau_{k} \Theta\left(k^{x}\right)}$, which is $o(1 / n)$ when $x \geq(1+$ ع) $\log _{k} \log n$.

To make the above intuition rigorous, the colouring process defined in the previous subsection needs to be slightly modified. Let $v$ be the vertex where the process has started. When choosing a red arc to colour, instead of choosing one arbitrarily from all red arcs, choose one arbitrarily from those that are closest to $v$. Thus at the end, the yellow arcs consist of not just a spanning tree but a breadth-first-search (bfs) spanning tree of $\mathcal{D}_{n, k}\left[\mathcal{S}_{v}^{*}\right]$. If $\mathcal{V}_{n}$ (the set of green vertices) is contracted into a single green vertex, then the green arcs together with yellow arcs form a DAG. Let $\mathcal{T}_{v}$ denote this DAG. Then $W_{v}^{*}$ is the length of the shortest path from $v$ to the green vertex contracted from $\mathcal{V}_{n}$. Figure 4 shows an example of $\mathcal{T}_{v}$.


Figure 4: An example of $\mathcal{T}_{v}$.

Proof. Let $\omega_{n}=\left\lfloor(1+\varepsilon) \log _{k} \log n\right\rfloor$. Call the arcs whose endpoints are at distance $i$ to $v$ the $i$-th layer of $\mathcal{T}_{v}$. The event $W_{v}^{*}>\omega_{n}$ implies that the first $\omega_{n}$ layers of $\operatorname{arcs}$ in $\mathcal{T}_{v}$ are all yellow arcs and thus they form a tree of height $\omega_{n}$. By Lemma 7, whp there are no $v \in \mathcal{V}_{n}^{c}$ such that $\mathcal{D}_{n, k}\left[\mathcal{S}_{v}^{*}\right]$ contains more than one black arc. Thus
whp in every $\mathcal{T}_{v}$ all internal (non-leaf) vertices except at most one have out degree $k$. Let $A_{n}$ denote this event. Assuming $A_{n}$ happens, $W_{v}^{*}>\omega_{n}$ implies that there are at least $\Theta\left(k^{\omega_{n}}\right)=\Theta(\log n)^{1+\varepsilon}$ yellow arcs in the first $\omega_{n}$ layers of $\mathcal{T}_{v}$. Thus in the colouring process, the first $\Theta(\log n)^{1+\varepsilon}$ arcs choose their endpoints in $\mathcal{V}_{n}^{c}$. The probability that this happens is at most $\left(\left|\mathcal{V}_{n}^{c}\right| / n\right)^{\Theta(\log n)^{1+\varepsilon}}$. Since $\left|\mathcal{V}_{n}\right| \in \mathcal{I}_{n},\left|\mathcal{V}_{n}^{c}\right|=n-\left|\mathcal{V}_{n}\right| \leq e^{-\tau_{k}} n+n^{1 / 2+\delta}$. Then by the union bound,

$$
\begin{aligned}
\mathbb{P}\left\{\cup_{v \in \mathcal{V}_{n}^{c}}\left[W_{v}^{*}>\omega_{n}\right]\right\} & \leq \sum_{v \in \mathcal{V}_{n}^{c}} \mathbb{P}\left\{\left[W_{v}^{*}>\omega_{n}\right] \cap A_{n}\right\}+\mathbb{P}\left\{A_{n}^{c}\right\} \\
& \leq n\left(\left|\mathcal{V}_{n}^{c}\right| / n\right)^{\Theta(\log n)^{1+\varepsilon}}+o(1) \\
& \leq n\left(e^{-\tau_{k}}+n^{-1 / 2+\delta}\right)^{\Theta(\log n)^{1+\varepsilon}}+o(1)=o(1) .
\end{aligned}
$$

Thus whp $\max _{v \in \mathcal{V}_{n}^{c}} W_{v}^{*} \leq \omega_{n}$.
Let $\psi_{n}=\left\lceil(1-\varepsilon) \log _{k} \log n\right\rceil$. To show that whp there is a vertex $v$ with $W_{v}^{*} \geq \psi_{n}$, run the colouring process starting from an arbitrary yellow vertex $v$ until either an arc is colored black or green (failure), or the first $\psi_{n}-1$ layers of $\mathcal{T}_{v}$ are colored yellow (success). So to succeed, the first $\psi_{n}-1$ layers of $\mathcal{T}_{v}$ form a full $k$-ary tree, i.e., the first $k+k^{2}+\cdots+k^{\psi_{n}-1}=\Theta\left(k^{\psi_{n}}\right)=\Theta(\log n)^{1-\varepsilon}$ arcs must be colored yellow. If the process fails, we pick another yellow vertex and try again until one trial succeeds. Since the colouring process stops before colouring the $\psi_{n}$ layer of $\mathcal{T}_{v}$, each trial colors at most $\Theta\left(k^{\psi_{n}}\right)=\Theta(\log n)^{1-\varepsilon}$ vertices black. If the process is tried at most $\left\lceil n /(\log n)^{2}\right\rceil$ times, then at most $b_{n} \equiv\left\lceil n /(\log n)^{2}\right\rceil O(\log n)^{1-\varepsilon}=O\left(n /(\log n)^{1+\varepsilon}\right)$ vertices are colored black. Therefore, each arc has probability at least $\left(\left|\mathcal{V}_{n}^{c}\right|-b_{n}\right) / n$ to be colored yellow during the first $\left\lceil n /(\log n)^{2}\right\rceil$ trials. Since $\left|\mathcal{V}_{n}\right| \in \mathcal{I}_{n},\left|\mathcal{V}_{n}^{c}\right|=n-\left|\mathcal{V}_{n}\right| \geq e^{-\tau_{k}} n-n^{1 / 2+\delta}$. Thus the probability to succeed in one trial is at least

$$
\left(\frac{\left|\mathcal{V}_{n}^{c}\right|-b_{n}}{n}\right)^{O(\log n)^{1-\varepsilon}} \geq\left[e^{-\tau_{k}}-O\left(\frac{1}{(\log n)^{1+\varepsilon}}\right)\right]^{O(\log n)^{1-\varepsilon}}=e^{-O(\log n)^{1-\varepsilon}} .
$$

Therefore, the probability that the first $\left\lceil n /(\log n)^{2}\right\rceil$ trials fail is at most

$$
\left(1-e^{-O(\log n)^{1-\varepsilon}}\right)^{\left\lceil n /(\log n)^{2}\right\rceil} \leq \exp \left\{-e^{-O(\log n)^{1-\varepsilon}} \frac{n}{(\log n)^{2}}\right\}=o(1)
$$

Thus whp $\max _{v \in \mathcal{V}_{n}^{c}} W_{v}^{*} \geq \psi_{n}$.

### 3.3.5 The longest path outside the giant

This subsection proves (d) and (e) of Theorem 3.
Lemma 11. For all $\varepsilon>0$, we have:

$$
\sup _{\mathcal{V}_{n} \subseteq[n]:\left|\mathcal{V}_{n}\right| \in \mathcal{I}_{n}} \mathbb{P}\left\{\left|\frac{m\left(\mathcal{V}_{n}^{c}\right)}{\log n}-\frac{1}{\log \left(e^{\tau_{k}} / k\right)}\right|>\varepsilon\right\}=o(1),
$$

where $m\left(\mathcal{V}_{n}^{c}\right)$ denotes the length of the longest path in $\mathcal{D}_{n, k}\left[\mathcal{V}_{n}^{c}\right]$; and

$$
\sup _{\mathcal{V}_{n} \subseteq[n]:\left|\mathcal{V}_{n}\right| \in \mathcal{I}_{n}} \mathbb{P}\left\{\left|\frac{d\left(\mathcal{V}_{n}^{c}\right)}{\log n}-\frac{1}{\log \left(e^{\tau_{k}} / k\right)}\right|>\varepsilon\right\}=o(1) .
$$

where $d\left(\mathcal{V}_{n}^{c}\right)$ denotes the maximal distance between two connected vertices in $\mathcal{D}_{n, k}\left[\mathcal{V}_{n}^{c}\right]$.
Since $m\left(\mathcal{V}_{n}^{c}\right) \geq d\left(\mathcal{V}_{n}^{c}\right)$, it suffices to prove the upper bound for $m\left(\mathcal{V}_{n}^{c}\right)$ and the lower bound for $d\left(\mathcal{V}_{n}^{c}\right)$.

Proof of the upper bound. Let $\omega_{n}=(1+\varepsilon) \log n / \log \left(e^{\tau_{k}} / k\right)$. Let $X_{\ell}$ be the number of labeled paths of length $\ell$ in $\mathcal{D}_{n, k}\left[\mathcal{V}_{n}^{c}\right]$. There are less than $\left|\mathcal{V}_{n}^{c}\right|^{\ell+1} k^{\ell}$ possible such paths. Each of them exists with probability $(1 / n)^{\ell}$. Recall that $\left|\mathcal{V}_{n}\right| \in \mathcal{I}_{n}$ implies $\left|\mathcal{V}_{n}^{c}\right| \leq e^{-\tau_{k}} n+n^{1 / 2+\delta}$. Thus

$$
\mathbb{E} X_{\ell} \leq\left|\mathcal{V}_{n}^{c}\right|^{\ell+1} k^{\ell}\left(\frac{1}{n}\right)^{\ell} \leq\left(e^{-\tau_{k}} n+n^{1 / 2+\delta}\right)\left(k e^{-\tau_{k}}+k n^{-1 / 2+\delta}\right)^{\ell}
$$

Since $k e^{-\tau_{k}}<1$ (Lemma A1), for $n$ large enough,

$$
\sum_{\omega_{n}<\ell<\left|\mathcal{V}_{n}^{c}\right|} \mathbb{E} X_{\ell} \leq n \sum_{\omega_{n}<\ell}\left(k e^{-\tau_{k}}+k n^{-1 / 2+\delta}\right)^{\ell}=O\left(n\left(k e^{-\tau_{k}}\right)^{\omega_{n}}\right)=O\left(n^{-\varepsilon}\right)
$$

Thus $\mathbb{P}\left\{m\left(\mathcal{V}_{n}^{c}\right)>\omega_{n}\right\}=O\left(n^{-\varepsilon}\right)$.
Proof of the lower bound. Let $\psi_{n} \equiv\left\lceil(1-\varepsilon) \log n / \log \left(1 / k e^{-\tau_{k}}\right)\right\rceil$. To show there are two vertices at distance within $\left[\psi_{n}, \infty\right)$, pick an arbitrary yellow vertex $v$ and run the colouring process until either a vertex at distance $\psi_{n}$ from $v$ has been colored (success), or $\left\lceil(\log n)^{2}\right\rceil$ vertices have been colored (failure), or the process terminates because all vertices that are reachable from $v$ in $\mathcal{D}_{n, k}\left[\mathcal{V}_{n}^{c}\right]$ has been discovered (failure). If the process fails, we pick another yellow vertex and try again until one trial succeeds.

If at most $t_{n} \equiv\left\lfloor n /(\log n)^{4}\right\rfloor$ trials are made, then at most $\left\lceil(\log n)^{2}\right\rceil t_{n}=O\left(n /(\log n)^{2}\right)$ vertices are colored. So in the first $t_{n}$ trials, when an arc is colored, the probability that it is colored yellow is at least $\mu_{n} \equiv\left(\left|\mathcal{V}_{n}^{c}\right|-O\left(n /(\log n)^{2}\right)\right) / n=e^{-\tau_{k}}-O\left(1 /(\log n)^{2}\right)$. Let $\left(Z_{m}\right)_{m \geq 0}$ be a Galton-Watson process with offspring distribution $\operatorname{Bin}\left(k, \mu_{n}\right)$ and $Z_{0}=1$. In other words, $Z_{m+1}=\sum_{j=1}^{Z_{m}} X_{m, j}$, where $\left(X_{m, j}\right)_{m \geq 0, j \geq 1}$ are i.i.d. $\operatorname{Bin}\left(k, \mu_{n}\right)$. Then the probability that one trial succeeds is at least $\mathbb{P}\left\{Z_{\psi_{n}}>0\right\}$ minus the probability that in a trial $\left\lceil(\log n)^{2}\right\rceil$ vertices have been colored, which is $O\left(n^{-1-\varepsilon}\right)$ by (3) in Lemma 8.

Let $\varphi_{m}(y)=\mathbb{E} y^{Z_{m}}$, i.e., $\varphi_{m}(y)$ is the probability generating function of $Z_{m}$. Thus $\mathbb{P}\left\{Z_{m}=0\right\}=\varphi_{m}(0)$. Since $k e^{-\tau_{k}}<1 / 2$ (Lemma A1), for $n$ large enough $k \mu_{n}<1 / 2$. So we can apply Lemma A7 in the appendix to show that

$$
\varphi_{m}(0) \leq 1-\left(k \mu_{n}\right)^{m}+\left(1-\frac{1}{2^{m}}\right)\left(k \mu_{n}\right)^{m+1}<1-\frac{1}{2}\left(k \mu_{n}\right)^{m}, \quad \text { for all } m \geq 0
$$

Recalling that $\psi_{n} \equiv\left\lceil(1-\varepsilon) \log n / \log \left(1 / k e^{-\tau_{k}}\right)\right\rceil$,

$$
\mathbb{P}\left\{Z_{\psi_{n}}>0\right\}=1-\varphi_{\psi_{n}}(0)>\frac{1}{2}\left(k e^{-\tau_{k}}-O\left(\frac{1}{(\log n)^{2}}\right)\right)^{\psi_{n}}=\Omega\left(n^{-1+\varepsilon}\right)
$$

So the probability that one trial succeeds is $\Omega\left(n^{-1+\varepsilon}\right)-O\left(n^{-1-\varepsilon}\right)=\Omega\left(n^{-1+\varepsilon}\right)$. (The $O\left(n^{-1-\varepsilon}\right)$ term is the probability that one trial colors too many vertices.) Thus the probability that the first $t_{n} \equiv\left\lfloor n /(\log n)^{4}\right\rfloor$ trials fail is at most

$$
\left(1-\Omega\left(n^{-1+\varepsilon}\right)\right)^{t_{n}} \leq \exp \left\{-\Omega\left(\frac{1}{n^{1-\varepsilon}}\left\lfloor\frac{n}{(\log n)^{4}}\right\rfloor\right)\right\}=\exp \left\{-\Omega\left(\frac{n^{\varepsilon}}{(\log n)^{4}}\right)\right\}=o(1)
$$

Therefore whp $d\left(\mathcal{V}_{n}^{c}\right) \geq \psi_{n}$.

## 4 Phase transition in strong connectivity

Now instead of assuming that $k$ is fixed, let $k \rightarrow \infty$ as $n \rightarrow \infty$. Let $K$ be a fixed integer. We can construct $\mathcal{D}_{n, k}$ by first generating $\mathcal{D}_{n, K}$ and then adding arcs with labels in $\{K+1, \ldots, k\}$ into it. By Lemma 2, for all $\varepsilon>0$, there exists a $K$ depending only on $\varepsilon$ such that whp in $\mathcal{D}_{n, K}$ the largest closed SCC has size at least $(1-\varepsilon) n$ and is reachable from all vertices. Since adding arcs can only increase the size of this SCC, whp $\mathcal{D}_{n, k}$ has a SCC of size at least $(1-\varepsilon) n$ that is reachable from all vertices.
In fact, if $k$ increases fast enough, then whp $\mathcal{D}_{n, k}$ is strongly connected. More precisely, $\mathcal{D}_{n, k}$ exhibits a phase transition for strong connectivity similar to the analogous event in the Erdős-Rényi model [15].

Theorem 6. If $k-\log n \rightarrow-\infty$, then whp $\mathcal{D}_{n, k}$ is not strongly connected. If $k-\log n \rightarrow$ $\infty$, then whp $\mathcal{D}_{n, k}$ is strongly connected.

If there is a vertex with in-degree zero, then obviously the digraph is not strongly connected. Thus the following lemma proves the lower bound in Theorem 6.

Lemma 12. If $k-\log n \rightarrow-\infty$, whp $\mathcal{D}_{n, k}$ contains a vertex of in-degree zero.
Proof. Let $\omega_{n}=\log n-k$. For vertex $i \in[n]$, let $X_{i}$ be the indicator that $i$ has in-degree zero. Let $N=\sum_{i=1}^{n} X_{i}$. We use second moment method to show that $N \geq 1$ whp.

To have $X_{1}=1, n k$ arcs need to avoid vertex 1 as their endpoints. Thus

$$
\mathbb{E} X_{1}=\left(1-\frac{1}{n}\right)^{n k} \geq e^{-n k\left(1 / n+1 / n^{2}\right)}=e^{-k(1+1 / n)}=\left(\frac{e^{\omega_{n}}}{n}\right)^{1+1 / n}
$$

Since by assumption $\omega_{n} \rightarrow \infty, \mathbb{E} N=n \mathbb{E} X_{1}=e^{\omega_{n}(1+1 / n)} / n^{1 / n} \rightarrow \infty$.
To have $X_{1} X_{2}=1, n k$ arcs need to avoid vertices 1 and 2 as their endpoints. Thus $\mathbb{E} X_{1} X_{2}=(1-2 / n)^{n k}$. Therefore

$$
\frac{\mathbb{E}\left[X_{1} X_{2}\right]}{\left(\mathbb{E}\left[X_{1}\right]\right)^{2}}=\frac{(1-2 / n)^{n k}}{(1-1 / n)^{2 n k}}=\left(\frac{n^{2}-2 n}{n^{2}-2 n+1}\right)^{n k}=\left(1-\frac{1}{(n-1)^{2}}\right)^{n k} \rightarrow 1
$$

since $n k /(n-1)^{2}=o(1)$. Thus

$$
1 \leq \frac{\mathbb{E}\left[N^{2}\right]}{(\mathbb{E} N)^{2}}=\frac{\mathbb{E} N+n(n-1) \mathbb{E}\left[X_{1} X_{2}\right]}{(\mathbb{E} N)^{2}} \leq \frac{1}{\mathbb{E} N}+\frac{\mathbb{E}\left[X_{1} X_{2}\right]}{\left(\mathbb{E} X_{1}\right)^{2}} \rightarrow 1 .
$$

Therefore $\mathbb{P}\{N=0\} \leq \operatorname{Var}(N) /(\mathbb{E} N)^{2}=\mathbb{E}\left[N^{2}\right] /(\mathbb{E} N)^{2}-1 \rightarrow 0$.

Given a set of vertices $\mathcal{S}$, if there are no arcs that start from $\mathcal{S}^{c} \equiv[n] \backslash \mathcal{S}$ and end at $\mathcal{S}$, then call $\mathcal{S}$ a non-leaf. If $\mathcal{D}_{n, k}$ is not strongly connected, then there must exist a non-leaf set of vertices $\mathcal{S}$ with $|\mathcal{S}|<n$. Thus the following lemma implies the upper bound in Theorem 6.
Lemma 13. If $k-\log n \rightarrow+\infty$, whp there does not exist a non-leaf set of vertices $\mathcal{S}$ with $|\mathcal{S}|<n$.

Proof. By the argument at the beginning of this subsection, whp $\mathcal{D}_{n, k}$ contains a SCC of size at least $n / 2$ that is reachable form all vertices. So if $|\mathcal{S}| \geq n / 2$, then $\mathcal{S}$ contains part of this SCC and cannot be a non-leaf. Thus it suffices to prove the lemma for $\mathcal{S}$ with $|\mathcal{S}|<n / 2$.

Let $\omega_{n}=k-\log n$. For $s \in[\lfloor n / 2\rfloor]$, let $X_{s}$ be the number of non-leaf sets of vertices of size $s$. Thus

$$
\begin{equation*}
\mathbb{E} X_{s}=\binom{n}{s}\left(1-\frac{s}{n}\right)^{k(n-s)} \leq\binom{ n}{s} e^{-k s(1-s / n)} \tag{4}
\end{equation*}
$$

Therefore for $s<n / \log n$,

$$
\mathbb{E} X_{s} \leq \frac{n^{s}}{s!} e^{-k s(1-s / n)} \leq \frac{1}{s!}\left(\frac{n}{e^{k(1-s / n)}}\right)^{s} \leq \frac{1}{s!}\left(\frac{n}{\left(n e^{\omega_{n}}\right)^{1-1 / \log n}}\right)^{s} \equiv \frac{\alpha_{n}^{s}}{s!}
$$

By assumption $\omega_{n} \rightarrow \infty$. Thus $\alpha_{n} \equiv n^{1 / \log n} / e^{\omega_{n}(1-1 / \log n)}=e^{1-\omega_{n}(1-1 / \log n)}=o(1)$. Therefore,

$$
\sum_{1 \leq s<n / \log n} \mathbb{E} X_{s} \leq \sum_{1 \leq s} \frac{\alpha_{n}^{s}}{s!}=e^{\alpha_{n}}-1=o(1)
$$

On the other hand, it follows from (4) that for $n / \log n \leq s<n / 2$,

$$
\mathbb{E} X_{s} \leq\left(\frac{e n}{s}\right)^{s} e^{-k s(1-s / n)}=\left(\frac{e n}{s e^{k(1-s / n)}}\right)^{s} \leq\left(\frac{e n}{\frac{n}{\log n} e^{k / 2}}\right)^{s}=\left(\frac{e \log n}{\left(n e^{\omega_{n}}\right)^{1 / 2}}\right)^{s} \equiv \beta_{n}^{s}
$$

Since $\beta_{n}=e \log n /\left(n e^{\omega_{n}}\right)^{1 / 2}=o(1)$,

$$
\sum_{n / \log n \leq s<n / 2} \mathbb{E} X_{s} \leq \sum_{1 \leq s} \beta_{n}^{s}=O\left(\beta_{n}\right)=o(1)
$$

Thus $\mathbb{P}\left\{\sum_{1 \leq s<n / 2} X_{s} \geq 1\right\} \leq \sum_{1 \leq s<n / 2} \mathbb{E} X_{s}=o(1)$.

## 5 The simple digraph model, the number of self-loops and multiple arcs

A simple digraph is one in which there are no self-loops and there is no more than one arc from one vertex to another. Let $\mathcal{D}_{n, k}^{*}$ denote a simple $k$-out digraph with $n$ vertices chosen uniformly at random from all such digraphs. $\mathcal{D}_{n, k}^{*}$ can be viewed as $\mathcal{D}_{n, k}$ restricted to the event that $\mathcal{D}_{n, k}$ is simple. This section proves the following theorem:

Theorem 7. The probability that $\mathcal{D}_{n, k}$ is simple converges to $e^{-k-\binom{k}{2}}$ as $n \rightarrow \infty$.
Theorem 7 can be proved directly as follows. Let $\mathbb{1}_{v}$ be the indicator that the $k$ arcs starting from vertex $v$ do not end at $v$ and do not end at the same vertex. Then

$$
\mathbb{P}\left\{\mathbb{1}_{v}=1\right\}=\frac{(n-1)(n-2) \cdots(n-k)}{n^{k}}=1-\frac{k(k+1)}{2 n}+O\left(\frac{1}{n^{2}}\right) .
$$

Since $\mathcal{D}_{n, k}$ is simple if and only if $\cap_{v=1}^{n}\left[\mathbb{1}_{v}=1\right]$ happens, we have

$$
\begin{aligned}
\mathbb{P}\left\{\mathcal{D}_{n, k} \text { is simple }\right\} & =\mathbb{P}\left\{\cap_{v=1}^{n}\left[\mathbb{1}_{v}=1\right]\right\}=\prod_{v=1}^{n} \mathbb{P}\left\{\mathbb{1}_{v}=1\right\} \\
& =\left(1-\frac{k(k+1)}{2 n}+O\left(\frac{1}{n^{2}}\right)\right)^{n} \rightarrow e^{-k(k+1) / 2}=e^{-k-\binom{k}{2}} .
\end{aligned}
$$

However, we can say more about self-loops and multiple arcs between vertices. Let $\mathcal{I} \equiv[n] \times[k]$. For $(v, i) \in \mathcal{I}$, define the random variable $\mathbb{1}_{v, i}$ to be the indicator that the arc with label $i$ starting from vertex $v$ forms a self-loop. Let $\mathcal{J} \equiv\{(v, i, j) \in$ $[n] \times[k] \times[k]: i<j\}$. For $(v, i, j) \in \mathcal{J}$, define the random variable $\mathbb{1}_{v, i, j}$ to be the indicator that the two arcs starting from vertex $v$ with labels $i$ and $j$ both end at the same vertex. Let $S_{n}=\sum_{\alpha \in \mathcal{I}} \mathbb{1}_{\alpha}$ and $M_{n}=\sum_{\alpha \in \mathcal{J}} \mathbb{1}_{\alpha}$. Then $\left[S_{n}=0\right] \cap\left[M_{n}=0\right]$ if and only if $\mathcal{D}_{n, k}$ is simple.

Lemma 14. Let $S$ and $M$ be two independent Poisson random variables of means $k$ and $\binom{k}{2}$ respectively. Then $\left(S_{n}, M_{n}\right) \xrightarrow{d}(S, M)$ as $n \rightarrow \infty$. In fact,

$$
\left\|\left(S_{n}, M_{n}\right),(S, M)\right\|_{\mathrm{TV}}=O\left(\frac{1}{n}\right)
$$

Indeed the lemma implies that as $n \rightarrow \infty$,

$$
\mathbb{P}\left\{\mathcal{D}_{n, k} \text { is simple }\right\}=\mathbb{P}\left\{S_{n}=M_{n}=0\right\} \rightarrow \mathbb{P}\{S=0\} \mathbb{P}\{M=0\}=e^{-k} e^{-\binom{k}{2}} .
$$

Remark. Bollobás [9] proved a theorem similar to Lemma 14 for the configuration model (see also Bollobás [8, sec. 2.4]). Many authors have extended this result under various conditions, see, e.g., McKay [30], McKay and Wormald [31], Janson [23, 24]. Our proof uses Stein's method, which may also be applied to self-loops and multiple edges in the configuration model to get proofs shorter than previous ones.

Proof of Lemma 14. We use the Chen-Stein method [11]. Since the probability that an arc forms a self-loop is $1 / n$,

$$
\mathbb{E} S_{n}=\sum_{(v, i) \in \mathcal{I}} \mathbb{E}_{v, i}=k n \frac{1}{n}=k
$$

Thus $\mathbb{E} S=k=\mathbb{E} S_{n}$. Since the probability that two arcs with the same start point have the same endpoint is also $1 / n$,

$$
\mathbb{E} M_{n}=\sum_{v \in[n]} \sum_{1 \leq i<j \leq k} \mathbb{E}_{v, i, j}=n\binom{k}{2} \frac{1}{n}=\frac{k(k-1)}{2} .
$$

Thus $\mathbb{E} M=k(k-1) / 2=\mathbb{E} M_{n}$.
For $\alpha \in \mathcal{I} \cup \mathcal{J}$, let

$$
\mathcal{B}_{\alpha}=\left\{\beta \in \mathcal{I} \cup \mathcal{J}: \mathbb{1}_{\beta} \text { and } \mathbb{1}_{\alpha} \text { are dependent }\right\} .
$$

(Note that $\mathbb{1}_{\alpha} \in \mathcal{B}_{\alpha}$.) Define

$$
b_{1} \equiv \sum_{\alpha \in \mathcal{I} \cup \mathcal{J}} \sum_{\beta \in \mathcal{B}_{\alpha}} \mathbb{E}\left[\mathbb{1}_{\alpha}\right] \mathbb{E}\left[\mathbb{1}_{\beta}\right], \quad b_{2} \equiv \sum_{\alpha \in \mathcal{I} \cup \mathcal{J}} \sum_{\beta \in \mathcal{B}_{\alpha}: \alpha \neq \beta} \mathbb{E}\left[\mathbb{1}_{\alpha} \mathbb{1}_{\beta}\right], \quad b_{3} \equiv \sum_{\alpha \in \mathcal{I} \cup \mathcal{J}} s_{\alpha},
$$

where

$$
s_{\alpha}=\mathbb{E}\left|\mathbb{E}\left[\mathbb{1}_{\alpha} \mid \sigma\left(\mathbb{1}_{\beta}: \beta \in[\mathcal{I} \cup \mathcal{J}] \backslash \mathcal{B}_{\alpha}\right)\right]-\mathbb{E}_{\alpha}\right| .
$$

By [11, thm. 2], if $b_{1}+b_{2}+b_{3} \rightarrow 0$, then $\left(S_{n}, M_{n}\right) \xrightarrow{d}(S, M)$. Since $\mathbb{1}_{\alpha}$ is independent of the random variables $\mathbb{1}_{\beta}$ with $\beta \in[\mathcal{I} \cup \mathcal{J}] \backslash \mathcal{B}_{\alpha}$, we have $s_{\alpha}=0$ and thus $b_{3}=0$.

For $(v, i) \in \mathcal{I}, \mathbb{1}_{v, i}$ depends on the random variables $\mathbb{1}_{v, r, s}$ with $1 \leq r<s \leq k$ and $i \in\{r, s\}$, of which there are $k-1$. Thus $\left|\mathcal{B}_{v, i}\right|=1+(k-1)=k<2 k$. For $(v, i, j) \in \mathcal{J}, \mathbb{1}_{v, i, j}$ depends on $\mathbb{1}_{v, i}$ and $\mathbb{1}_{v, j}$. It also depends on the random variables $\mathbb{1}_{v, r, s}$ with $1 \leq r<s \leq k$ and $\{r, s\} \cap\{i, j\} \neq \emptyset$, of which there are $2(k-1)-1=2 k-3$. Thus $\left|\mathcal{B}_{v, i, j}\right|=2+2 k-3<2 k$. So for all $\alpha \in \mathcal{I} \cup \mathcal{J},\left|\mathcal{B}_{\alpha}\right|<2 k$. Therefore

$$
\begin{aligned}
b_{1} & =\sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{B}_{\alpha}} \mathbb{E}\left[\mathbb{1}_{\alpha}\right] \mathbb{E}\left[\mathbb{1}_{\beta}\right]+\sum_{\alpha \in \mathcal{J}} \sum_{\beta \in \mathcal{B}_{\alpha}} \mathbb{E}\left[\mathbb{1}_{\alpha}\right] \mathbb{E}\left[\mathbb{1}_{\beta}\right] \\
& <n k \times 2 k \times \frac{1}{n} \times \frac{1}{n}+n\binom{k}{2} \times 2 k \times \frac{1}{n} \times \frac{1}{n}=O\left(\frac{1}{n}\right) .
\end{aligned}
$$

Consider $(v, i) \in \mathcal{I}$. If $\beta \in \mathcal{B}_{v, i} \cap \mathcal{I}$, then $\beta=(v, i)$. If $\beta \in \mathcal{B}_{v, i} \cap \mathcal{J}$, then $\beta=(v, r, s)$ for some $(r, s)$ with $i \in\{r, s\}$. Then $\mathbb{1}_{v, i} \mathbb{1}_{v, r, s}=1$ if and only if the two arcs starting from vertex $v$ labeled $r$ and $s$ respectively both end at $v$. Thus $\mathbb{E}\left[\mathbb{1}_{v, i} \mathbb{1}_{v, r, s}\right]=1 / n^{2}$. Therefore

$$
b_{2, \mathcal{I}} \equiv \sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{B}_{\alpha} ; \beta \neq \alpha} \mathbb{E}\left[\mathbb{1}_{\alpha} \mathbb{1}_{\beta}\right]=\sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{B}_{\alpha} \cap \mathcal{J}} \mathbb{E}\left[\mathbb{1}_{\alpha} \mathbb{1}_{\beta}\right]<n k \times 2 k \times \frac{1}{n^{2}}=O\left(\frac{1}{n}\right) .
$$

Consider $(v, r, s) \in \mathcal{J}$. If $(v, i) \in \mathcal{B}_{v, r, s}$, then $(v, r, s) \in \mathcal{B}_{v, i}$. Thus by the above argument $\mathbb{E}\left[\mathbb{1}_{v, r, s} \mathbb{1}_{v, i}\right]=1 / n^{2}$. If $(v, i, j) \in \mathcal{B}_{v, r, s}$ and $(i, j) \neq(r, s)$, then $|\{r, s\} \cup\{i, j\}|=$ 3. So $\mathbb{1}_{v, r, s \mathbb{1}_{v, i, j}}=1$ iff the three arcs starting from vertex $v$ with labels in $\{r, s\} \cup\{i, j\}$ all end at the same vertex. Thus $\mathbb{E}\left[\mathbb{1}_{v, r, s} \mathbb{1}_{v, i, j}\right]=1 / n^{2}$. Therefore

$$
b_{2, \mathcal{J}} \equiv \sum_{\alpha \in \mathcal{J}} \sum_{\beta \in \mathcal{B}_{\alpha}: \beta \neq \alpha} \mathbb{E}\left[\mathbb{1}_{\alpha} \mathbb{1}_{\beta}\right]<n\binom{k}{2} \times 2 k \times \frac{1}{n^{2}}=O\left(\frac{1}{n}\right) .
$$

Thus $b_{2} \equiv b_{2, \mathcal{I}}+b_{2, \mathcal{J}}=O(1 / n)$.

Corollary 2. Let $\mathcal{E}$ be a set of digraphs. If $\mathcal{D}_{n, k} \in \mathcal{E}$ whp, then $\mathcal{D}_{n, k}^{*} \in \mathcal{E}$ whp.
Proof. We have

$$
\mathbb{P}\left\{\mathcal{D}_{n, k}^{*} \notin \mathcal{E}\right\}=\mathbb{P}\left\{\mathcal{D}_{n, k} \notin \mathcal{E} \mid \mathcal{D}_{n, k} \text { is simple }\right\} \leq \frac{\mathbb{P}\left\{\mathcal{D}_{n, k} \notin \mathcal{E}_{n}\right\}}{\mathbb{P}\left\{\mathcal{D}_{n, k} \text { is simple }\right\}} \rightarrow 0
$$

This corollary implies that all previous results in the form of "whp $\mathcal{D}_{n, k} \ldots$ ". can be automatic translated into "whp $\mathcal{D}_{n, k}^{*} \ldots$. . For example, the statement of Theorem 3 with $\mathcal{D}_{n, k}$ replaced by $\mathcal{D}_{n, k}^{*}$ is still true.

Corollary 3. Let $\mathcal{D}_{n, k}^{* *}$ be a digraph chosen uniformly at random from all simple and arc-unlabeled $k$-out digraphs with $n$ vertices. If whp $\mathcal{D}_{n, k}$ has property $\boldsymbol{P}$ where $\boldsymbol{P}$ does not depend on arc-labels, then whp $\mathcal{D}_{n, k}^{* *}$ has property $\boldsymbol{P}$.

Proof. Note that: (a) for each digraph in the space of $\mathcal{D}_{n, k}^{* *}$, there $(k!)^{n}$ ways to arc-label it to get $(k!)^{n}$ different digraphs in the space of $\mathcal{D}_{n, k}^{*} ;(\mathrm{b})$ no two different arc-unlabeled digraphs can be turned into the same digraph by arc-labeling. So there exists a $(k!)^{n}$ -to-one surjective mapping from the space of $\mathcal{D}_{n, k}^{*}$ to the space of $\mathcal{D}_{n, k}^{* *}$. Thus $\mathcal{D}_{n, k}^{* *}$ can be viewed as $\mathcal{D}_{n, k}^{*}$ with arc labels removed. Since $\mathbf{P}$ does not depend on arc-labels, it follows from Corollary 2 that whp $\mathcal{D}_{n, k}^{* *}$ has property $\mathbf{P}$.

## 6 The typical distance

The typical distance $H_{n}$ of $\mathcal{D}_{n, k}$ is the distance between two vertices $v_{1}$ and $v_{2}$ chosen uniformly at random. If $v_{1}$ cannot reach $v_{2}$, then $H_{n}=\infty$. Addario-Berry et al. [1] proved that conditioned on $H_{n}<\infty, H_{n} / \log _{k} n \xrightarrow{p} 1$. This section ${ }^{1}$ gives an alternative proof using the path counting technique invented by van der Hofstad [40, chap. 3.5].

Theorem 8 (The typical distance). For all $\varepsilon>0$,

$$
\mathbb{P}\left\{\left.\left|\frac{H_{n}}{\log _{k} n}-1\right|>\varepsilon \right\rvert\, H_{n}<\infty\right\}=o(1) .
$$

By Theorem $1,\left|\mathcal{S}_{v_{1}}\right| / n \xrightarrow{p} \nu_{k}$, where $\mathcal{S}_{v_{1}}$ is the spectrum of $v_{1}$. Thus $\mathbb{P}\left\{H_{n}<\infty\right\}=$ $\mathbb{P}\left\{v_{2} \in \mathcal{S}_{v_{1}}\right\} \rightarrow \nu_{k}>0$. Therefore

$$
\begin{aligned}
\mathbb{P}\left\{H_{n}<(1-\varepsilon) \log _{k} n \mid H_{n}<\infty\right\} & =\frac{\mathbb{P}\left\{H_{n}<(1-\varepsilon) \log _{k} n\right\}}{\mathbb{P}\left\{H_{n}<\infty\right\}} \\
& \sim \frac{1}{\nu_{k}} \mathbb{P}\left\{H_{n}<(1-\varepsilon) \log _{k} n\right\},
\end{aligned}
$$

[^0]and
\[

$$
\begin{aligned}
\mathbb{P}\left\{H_{n}>(1+\varepsilon) \log _{k} n \mid H_{n}<\infty\right\} & =\frac{\mathbb{P}\left\{(1+\varepsilon) \log _{k} n<H_{n}<\infty\right\}}{\mathbb{P}\left\{H_{n}<\infty\right\}} \\
& \sim \frac{1}{\nu_{k}} \mathbb{P}\left\{(1+\varepsilon) \log _{k} n<H_{n}<\infty\right\} \equiv \frac{\mathbb{P}\left\{B_{n}\right\}}{\nu_{k}} .
\end{aligned}
$$
\]

Thus it suffices to show that $\mathbb{P}\left\{H_{n}<(1-\varepsilon) \log _{k} n\right\}$ and $\mathbb{P}\left\{B_{n}\right\}$ are both $o(1)$.
Lemma 15 (Lower bound of the typical distance). For all $\varepsilon>0$,

$$
\mathbb{P}\left\{H_{n}<(1-\varepsilon) \log _{k} n\right\}=o(1) .
$$

Proof. Let $N_{\ell}$ denote the number of paths from $v_{1}$ to $v_{2}$ of length $\ell$. Consider such a path without labels on internal vertices and arcs. There are at most $n^{\ell-1}$ ways to label its internal vertices and there are at most $k^{\ell}$ ways to label its arcs. And the probability that such a labeled path appears is $(1 / n)^{\ell}$. Thus

$$
\mathbb{E} N_{\ell} \leq n^{\ell-1} k^{\ell}\left(\frac{1}{n}\right)^{\ell}=\frac{k^{\ell}}{n} .
$$

Let $\omega_{n}=(1-\varepsilon) \log _{k} n$. Then

$$
\sum_{\ell<\omega_{n}} \mathbb{E} N_{\ell} \leq \sum_{\ell<\omega_{n}} \frac{k^{\ell}}{n}=\frac{O\left(k^{\omega_{n}}\right)}{n}=\frac{O\left(n^{1-\varepsilon}\right)}{n}=o(1) .
$$

Thus $\mathbb{P}\left\{H_{n}<\omega_{n}\right\}=\mathbb{P}\left\{\sum_{\ell<\omega_{n}} N_{\ell} \geq 1\right\}=o(1)$.
The rest of this section is organized as follows: Subsection 6.1 shows that if $v_{1}$ can reach $v_{2}$ but only through a very long path, then it is very likely that $v_{1}$ can reach a lot of vertices and a lot of vertices can reach $v_{2}$. Subsection 6.2 computes a lower bound of the probability that there is a path of specific length from one large set of vertices to another large set of vertices. Finally, subsection 6.3 shows that these results together imply the upper bound in Theorem 8, i.e., $\mathbb{P}\left\{B_{n}\right\}=o(1)$.

### 6.1 Comparison to Galton-Watson processes

Let $\mathcal{S}_{m}^{+}(v)$ and $\mathcal{S}_{m}^{-}(v)$ be the sets of vertices at distance exactly $m$ from or to vertex $v$ respectively. Let $\mathcal{S}_{\leq m}^{+}(v)$ and $\mathcal{S}_{\leq m}^{-}(v)$ be the sets of vertices at distance at most $m$ from or to vertex $v$ respectively. The following proposition shows that for fixed $m$, we can perfectly couple $\left(\left|\mathcal{S}_{t}^{+}\left(v_{1}\right)\right|,\left|\mathcal{S}_{t}^{-}\left(v_{2}\right)\right|\right)_{t=0}^{m}$ with two independent Galton-Watson processes. It is inspired by a similar result of the configuration model by van der Hofstad [40, sec. 5.2], but the coupling method used here is new.

Proposition 1. Let $\left(S_{t}\right)_{t \geq 0}$ be a Galton-Watson process with a binomial offspring distribution $\operatorname{Bin}(k n, 1 / n)$. For all fixed $m \geq 1$, there exists a coupling

$$
\left[\left(k^{t}, Y_{t}\right)_{t=0}^{m},\left(Y_{t}^{+}, Y_{t}^{-}\right)_{t=0}^{m}\right],
$$

of $\left(k^{t}, S_{t}\right)_{t=0}^{m}$ and $\left(\left|\mathcal{S}_{t}^{+}\left(v_{1}\right)\right|,\left|\mathcal{S}_{t}^{-}\left(v_{2}\right)\right|\right)_{t=0}^{m}$, such that

$$
\mathbb{P}\left\{\left(k^{t}, Y_{t}\right)_{t=0}^{m} \neq\left(Y_{t}^{+}, Y_{t}^{-}\right)_{t=0}^{m}\right\}=o(1)
$$

Proof. We construct an incremental sequence of random digraphs, denoted by $\left(\mathcal{D}_{n, k}^{[t]}\right)_{t \geq 0}$, through a signal spreading process. Let $\mathcal{D}_{n, k}^{[0]}$ be a digraph of vertex set $[n]$ that has no arcs. Without loss of generality, let $v_{1}=1$ and $v_{2}=2$. At time 0 , put a $\oplus$ signal at $v_{1}$ and put a $\ominus$ signal at $v_{2}$.

If a $\oplus$ signal reaches a vertex $v$ at time $t$, then at time $t+1 / 3$ the vertex $v$ grows $k$ out-arcs labeled $1, \ldots, k$ from itself and to $k$ endpoints chosen independently and uar from all the $n$ vertices. Then the $\oplus$ signal splits into $k \oplus$ signals and each of them picks a different newly-grown out-arc and travels along the arc's direction to reach its endpoint at time $t+1$.

If a $\ominus$ signal reaches a vertex $v$ at time $t$, then at time $t+2 / 3$ the vertex $v$ grows a random number $X$ in-arcs from itself to $X$ random vertices as follows: Let $\left(X_{i, j}\right)_{i \in[n], j \in[k]}$ be i.i.d. Bernoulli $1 / n$ random variables. If $X_{i, j}=1$, then $v$ grows an in-arc from itself to vertex $i$ with label $j$. Thus in total $X \equiv \sum_{i \in[n], j \in[k]} X_{i, j}$ in-arcs are grown from $v$. Then the $\ominus$ signal splits into $X \ominus$ signals and each of them picks a different newly-grown in-arc and travels against the arc's direction to reach its starting vertex at time $t+1$. If $X=0$, then the $\ominus$ signal vanishes.

Let $\mathcal{D}_{n, k}^{[t]}$ be the digraph generated in the above process at time $t$. Let $\mathcal{Y}_{t}^{+}$and $\mathcal{Y}_{t}^{-}$be the sets of vertices that are visited by $\oplus$ and $\ominus$ signals at time $t$ respectively. Let $\mathcal{Y}_{\leq t}^{+}$ and $\mathcal{Y}_{\leq t}^{-}$be the sets of vertices that have been visited by $\oplus$ and $\ominus$ signals before time $t+1$ respectively. At time $t$, if a signal visits a vertex in $\left[\mathcal{Y}_{\leq t-1}^{+} \cup \mathcal{Y}_{\leq t-1}^{-}\right]$or if two signals visit the same vertex, then we say a collision happens. Let $T$ be the first time when a collision happens.

Table 1 lists the types of events that make a collision happen. Three of them need special attention for reasons to be clear soon. First, if multiple $\ominus$ signals visit the same vertex $v$, then multiple arcs with the same label and $v$ as the starting point may grow. If this happens we pick an arbitrary arc among them and call the others duplicate. Second, a $\oplus$ signal may visit a vertex in $\mathcal{Y}_{\leq T-1}^{-}$through a newly-grown out-arc. Finally, a $\ominus$ signal may visit a vertex in $\mathcal{Y}_{\leq T-1}^{+}$through a newly-grown in-arc. We also call the newly-grown arcs being passed by in these two cases biased.

| Signals visit the same vertex |  |  | Signals visit $\mathcal{Y}_{\leq t-1}^{+}$ |  | Signals visit $\mathcal{Y}_{\leq t-1}^{-}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\oplus \rightarrow 0+-\oplus$ | $\ominus \rightarrow 0+-\oplus$ | $\ominus \rightarrow 0+-\ominus$ | $\oplus \rightarrow \rightarrow \underset{\substack{\infty \\ \rightarrow 0}}{\substack{0}}$ | $\bigcirc \rightarrow \rightarrow$ ¢ | $\oplus \rightarrow \infty$ | $\theta \rightarrow+\sum_{0}^{0}$ |

Table 1: Events that lead to a collision. Three special types of events are marked.
We construct a random $k$-out graph $\widehat{\mathcal{D}}_{n, k}$ as follows: First remove all duplicate and all biased arcs in $\mathcal{D}_{n, k}^{[T]}$. Then for each pair $(v, i) \in[n] \times[k]$, if vertex $v$ does not have an out-arc labeled $i$, then add such an out-arc with its endpoint chosen uar from $[n] \backslash \mathcal{Y}_{\leq T-1}^{-}$. Denote the result digraph by $\widehat{\mathcal{D}}_{n, k}$.

The seemingly complicated $\widehat{\mathcal{D}}_{n, k}$ is nothing but $\mathcal{D}_{n, k}$ in disguise. In $\mathcal{D}_{n, k}$, the endpoints of the arcs are chosen uar and simultaneously. In $\widehat{\mathcal{D}}_{n, k}$, the endpoints of the arcs are still chosen uar but in several steps. First we mark the arcs whose end (start) vertices are at distance $t$ to $v_{1}\left(\right.$ from $v_{2}$ ) for $t=1, \ldots, T$. To have $\widehat{\mathcal{D}}_{n, k} \stackrel{\mathcal{L}}{=} \mathcal{D}_{n, k}$, obviously duplicate arcs must be removed. The biased arcs also cause trouble as their endpoints are chosen non-uniformly. For example, if at time $T$ a $\oplus$ signal visits a vertex in $\mathcal{Y}_{\leq T-1}^{-}$, then an in-arc is added to a vertex whose in-arcs have already been decided by time $T-1$. Thus biased arcs must also be removed. Finally, we add arcs that are still missing in $\widehat{\mathcal{D}}_{n, k}$ and choose their endpoints uar from $[n] \backslash \mathcal{Y}_{\leq T-1}^{-}$, i.e., from these vertices whose in-arcs have not yet been marked. Thus we have $\widehat{\mathcal{D}}_{n, k} \stackrel{\mathcal{L}}{=} \mathcal{D}_{n, k}$. Let $Y_{t}^{+}$and $Y_{t}^{-}$be the number of vertices in $\widehat{\mathcal{D}}_{n, k}$ at distance $t$ from $v_{1}$ and to $v_{2}$ respectively. Then

$$
\left(Y_{t}^{+}, Y_{t}^{-}\right)_{t=0}^{m} \stackrel{\mathcal{L}}{=}\left(\left|\mathcal{S}_{t}^{+}\left(v_{1}\right)\right|,\left|\mathcal{S}_{t}^{-}\left(v_{2}\right)\right|\right)_{t=0}^{m} .
$$

A $\oplus$ signal always splits into $k \oplus$ signals after it arrives at a vertex. Thus at a non-negative integer time $t$ there are in total $k^{t} \oplus$ signals. On the other hand, the number of $\ominus$ signals at time $t$, denoted by $Y_{t}$, is random. Each time a $\ominus$ signal splits, it splits into $\operatorname{Bin}(k n, 1 / n)$ signals. Because the splits are mutually independent, $\left(Y_{t}\right)_{t \geq 0}$ has the same distribution as $\left(S_{t}\right)_{t \geq 2} 0$, the Galton-Watson process with offspring distribution $\operatorname{Bin}(k n, 1 / n)$.

Assume that $T>m$. Then the part of $\widehat{\mathcal{D}}_{n, k}$ within distance $m$ from $v_{1}$ or to $v_{2}$ is determined by $\mathcal{D}_{n, k}^{[m]}$. Thus for $t \leq m$, in $\widehat{\mathcal{D}}_{n, k}$ a vertex is at distance $t$ from $v_{1}$ if and only if it has a $\oplus$ signal at time $t$ and a vertex is at distance $t$ to $v_{2}$ if and only if it has a $\ominus$ signal at time time $t$. This implies that $\left(k^{t}, Y_{t}\right)_{t=0}^{m}=\left(Y_{t}^{+}, Y_{t}^{-}\right)_{t=0}^{m}$. Thus to finish the proof, it suffices to show the following lemma:
Lemma 16. For all fixed integers $m \geq 1$, whp $T>m$.
The intuition is that since $m$ is fixed, for $t<m$, most likely $\left|\mathcal{Y}_{\leq t}^{+} \cup \mathcal{Y}_{\leq t}^{-}\right|$is small. Thus it is unlikely that a collision happens at time $t+1$. See the end of this subsection for a detailed proof.

Corollary 4. Let $\omega_{n} \rightarrow \infty$ be an arbitrary sequence. Let $M, \delta, \varepsilon$ be three arbitrary positive numbers. Let $\psi_{n} \equiv\left\lfloor(1+\varepsilon) \log _{k} n\right\rfloor$. Let

$$
A_{n}(M, m) \equiv\left[M \leq\left|\mathcal{S}_{m}^{+}\left(v_{1}\right)\right|\right] \cap\left[M \leq\left|\mathcal{S}_{m}^{-}\left(v_{2}\right)\right|\right] \cap\left[\left|\mathcal{S}_{\leq m}^{-}\left(v_{2}\right)\right| \leq \omega_{n}\right] .
$$

Then there exists $m \geq 1$ such that

$$
\limsup _{n \rightarrow \infty} \mathbb{P}\left\{A_{n}^{c}(M, m) \cap\left[\psi_{n}<H_{n}<\infty\right]\right\}<\delta
$$

Proof. Let $\left(k^{t}, Y_{t}\right)_{t=0}^{m}$ be the coupling of $\left(\left|\mathcal{S}_{t}^{+}\left(v_{1}\right)\right|,\left|\mathcal{S}_{t}^{-}\left(v_{2}\right)\right|\right)_{t=0}^{m}$ constructed in Proposition 1. Thus $\left(Y_{t}\right)_{t \geq 0}$ is a Galton-Watson process with $\operatorname{Bin}(k n, 1 / n)$ offspring distribution, i.e., $Y_{0}=1$ and $Y_{t}=\sum_{i=1}^{Y_{t-1}} X_{t, i}$ for $t \geq 1$, where $X_{t, i}$ 's are i.i.d. $\operatorname{Bin}(k n, 1 / n)$. Since $\mathbb{E} X_{1,1}=k>1$, the survival probability of this process is a constant $\eta>0$ (see [39,
thm. 3.1]). For the same reason, $Y_{t} / k^{t} \rightarrow Y_{\infty}$ almost surely for some random variable $Y_{\infty}$ (see [39, thm. 3.9]). Since $\mathbb{E}\left[X_{1,1}^{2}\right]<\infty$, by the Kesten-Stigum Theorem [39, thm. 3.10], $\mathbb{P}\left\{Y_{\infty}>0\right\}=\eta$. Thus by the Bounded Convergence Theorem [13, thm. 1.5.3],

$$
\lim _{m \rightarrow \infty} \mathbb{P}\left\{Y_{m}>M\right\}=\lim _{m \rightarrow \infty} \mathbb{P}\left\{\frac{Y_{m}}{k^{m}}>\frac{M}{k^{m}}\right\}=\mathbb{P}\left\{Y_{\infty}>0\right\}=\eta
$$

For the same reason $\mathbb{P}\left\{Y_{m} \geq 1\right\} \rightarrow \eta$ as $m \rightarrow \infty$. Thus

$$
\lim _{m \rightarrow \infty} \mathbb{P}\left\{1 \leq Y_{m}<M\right\}=\lim _{m \rightarrow \infty}\left(\mathbb{P}\left\{Y_{m} \geq 1\right\}-\mathbb{P}\left\{Y_{m} \geq M\right\}\right)=0
$$

Thus we can choose $m$ large enough such that $\mathbb{P}\left\{1 \leq Y_{m}<M\right\}<\delta / 2$ and that $k^{m} \geq M$.
Recall that $B_{n} \equiv\left[\psi_{n}<H_{n}<\infty\right]$. When $n$ is large enough, $\psi_{n}>m$. Thus $B_{n}$ implies that $\left|\mathcal{S}_{m}^{+}\left(v_{1}\right)\right| \geq 1$. Define the event

$$
C_{n} \equiv\left[\left(k^{t}, Y_{t}\right)_{t=0}^{m}=\left(\left|\mathcal{S}_{t}^{+}\left(v_{1}\right)\right|,\left|\mathcal{S}_{t}^{-}\left(v_{2}\right)\right|\right)_{t=0}^{m}\right] .
$$

By Proposition 1, $\mathbb{P}\left\{C_{n}^{c}\right\}=o(1)$ as $n \rightarrow \infty$. Therefore

$$
\begin{aligned}
\mathbb{P}\left\{A_{n}(M, m)^{c} \cap B_{n}\right\} & \leq \mathbb{P}\left\{C_{n}^{c}\right\}+\mathbb{P}\left\{A_{n}(M, m)^{c} \cap C_{n} \cap B_{n}\right\} \\
& \leq o(1)+\mathbb{P}\left\{\left[k^{m}<M\right] \cup\left[1 \leq Y_{m}<M\right] \cup\left[\omega_{n}<\sum_{t=0}^{m} Y_{t}\right]\right\} \\
& \leq o(1)+\mathbb{P}\left\{k^{m}<M\right\}+\mathbb{P}\left\{1 \leq Y_{m}<M\right\}+\mathbb{P}\left\{\omega_{n}<\sum_{t=0}^{m} Y_{t}\right\} \\
& =o(1)+0+\delta / 2+o(1),
\end{aligned}
$$

where the last equality is due to our choice of $m$ and that $\mathbb{E}\left[\sum_{t=0}^{m} Y_{t}\right]=\sum_{t=0}^{m} k^{t}=$ $O(1)$.

Proof of Lemma 16. Recall that $\mathcal{Y}_{t}^{+}$and $\mathcal{Y}_{t}^{-}$are the sets of vertices that are reached at time $t$ by a $\oplus$ signal or $\ominus$ signal respectively. Let $\mathcal{M}_{m-1}=\cup_{t=0}^{m-1}\left[\mathcal{Y}_{t}^{+} \cup \mathcal{Y}_{t}^{-}\right]$. Define event $A_{m} \equiv \cap_{i \in[4]} E_{m, i}$ where $E_{m, i}$ 's are defined as follows:

- $E_{m, 1}$ - The out-arcs that grow from vertices in $\mathcal{Y}_{m-1}^{+}$all end at different vertices in $[n] \backslash \mathcal{M}_{m-1}$. Thus at time $m$ all $\oplus$ signals visit different vertices and these vertices have never been visited by signals before.
- $E_{m, 2}$ - There are no in-arcs that grow from vertices in $\mathcal{Y}_{m-1}^{-}$that have starting vertices in $\mathcal{M}_{m-1} \cup \mathcal{Y}_{m}^{+}$. Thus at time $m$ all $\ominus$ signals visit vertices that have never been visited by signals before and that are not reached by $\oplus$ signals at time $m$.
- $E_{m, 3}$ - There are no two in-arcs that grow from vertices in $\mathcal{Y}_{m-1}^{-}$that have the same starting vertex. Thus at time $m$ all $\ominus$ signals reach different vertices.
- $E_{m, 4}-\left|\mathcal{Y}_{m}^{-}\right| \leq(\log n)^{m}$.

The event $A_{t}$ implies that no collision happens at time $t$. Thus $\cap_{t=0}^{m} A_{t}$ implies that no collision has happened by time $m$, and thus $T>m$. We show by induction that $\mathbb{P}\left\{\cap_{t=0}^{m} A_{t}\right\}=1-o(1)$.

Since $\left|\mathcal{Y}_{0}^{-}\right|=1$ and there are no arc-growing before time $0, \mathbb{P}\left\{A_{0}\right\}=1$, which is the induction basis. Now assume that $\mathbb{P}\left\{\cap_{t=0}^{m-1} A_{t}\right\}=1-o(1)$. Then

$$
\mathbb{P}\left\{\cap_{t=0}^{m} A_{t}\right\}=\mathbb{P}\left\{A_{m} \mid \cap_{t=0}^{m-1} A_{t}\right\} \mathbb{P}\left\{\cap_{t=0}^{m-1} A_{t}\right\}=\mathbb{P}\left\{A_{m} \mid \cap_{t=0}^{m-1} A_{t}\right\}(1-o(1)) .
$$

Thus it suffices to show that

$$
\mathbb{P}\left\{A_{m}^{c} \mid \cap_{t=0}^{m-1} A_{t}\right\}=\mathbb{P}\left\{\left[\cup_{i \in[4]} E_{m, i}^{c}\right] \mid \cap_{t=0}^{m-1} A_{t}\right\} \leq \sum_{i \in[4]} \mathbb{P}\left\{E_{m, i}^{c} \mid \cap_{t=0}^{m-1} A_{t}\right\}=o(1)
$$

The event $\cap_{t=0}^{m-1} A_{t}$ implies that

$$
\left|\mathcal{M}_{m-1}\right| \leq \sum_{t=1}^{m-1}\left|\mathcal{Y}_{t}^{+}\right|+\sum_{t=1}^{m-1}\left|\mathcal{Y}_{t}^{-}\right| \leq \sum_{t=1}^{m-1} k^{t}+\sum_{t=1}^{m-1}(\log n)^{t}=O(\log n)^{m}
$$

For $E_{m, 1}$ to happen, the $k^{m}$ arcs that grow out of $\mathcal{Y}_{m-1}^{+}$must end at different vertices in $[n] \backslash \mathcal{M}_{m-1}$. Thus

$$
\mathbb{P}\left\{E_{m, 1} \mid \cap_{t=0}^{m-1} A_{t}\right\}=\prod_{0 \leq i<k^{m}}\left[\frac{n-\left|\mathcal{M}_{m-1}\right|-i}{n}\right] \geq\left[1-\frac{O(\log n)^{m}}{n}\right]^{k^{m}}=1-o(1) .
$$

For $E_{m, 2}$ to happen, the vertices in $\mathcal{Y}_{m-1}^{-}$cannot grow in-arcs that have starting vertex in in $\mathcal{M}_{m-1} \cup \mathcal{Y}_{m}^{+} . \cap_{t=0}^{m-1} A_{t}$ implies that $\left|\mathcal{Y}_{m-1}^{-}\right| \leq(\log n)^{m-1}$. Since deterministically $\left|\mathcal{Y}_{m}^{+}\right|=k^{m},\left|\mathcal{M}_{m-1} \cup \mathcal{Y}_{m}^{+}\right|=O(\log n)^{m}$. Thus the number of in-arcs that need to not grow at time $m-1 / 3$ to make sure that $E_{m, 2}$ happens is at most

$$
k\left|\mathcal{Y}_{m-1}^{-}\right|\left|\mathcal{M}_{m-1} \cup \mathcal{Y}_{m}^{+}\right|=O(\log n)^{2 m}
$$

Since an in-arc does not grow with probability $1-1 / n$,

$$
\mathbb{P}\left\{E_{m, 2} \mid \cap_{t=0}^{m-1} A_{t}\right\} \geq\left(1-\frac{1}{n}\right)^{O(\log n)^{2 m}}=1-o(1)
$$

Let $X_{v}$ be the number of in-arcs that grow from $\mathcal{Y}_{m-1}^{-}$and that have starting vertex $v$. Conditioned on $\mathcal{Y}_{m-1}^{-}, \quad X_{v} \stackrel{\mathcal{L}}{=} \operatorname{Bin}\left(k\left|\mathcal{Y}_{m-1}^{-}\right|, 1 / n\right)$. Since $\cap_{t=0}^{m-1} A_{t}$ implies $\left|\mathcal{Y}_{m-1}^{-}\right| \leq$ $(\log n)^{m-1}$,

$$
\begin{aligned}
\mathbb{P}\left\{X_{v} \leq 1 \mid \cap_{t=0}^{m-1} A_{t}\right\} & \geq \mathbb{P}\left\{\operatorname{Bin}\left(k(\log n)^{m-1}, \frac{1}{n}\right) \leq 1\right\} \\
& =\left(1-\frac{1}{n}\right)^{k(\log n)^{m-1}}+k(\log n)^{m-1} \frac{1}{n}\left(1-\frac{1}{n}\right)^{k(\log n)^{m-1}-1} \\
& =1-O\left(\frac{(\log n)^{2(m-1)}}{n^{2}}\right)
\end{aligned}
$$

Since for two different vertices $u$ and $v, X_{u}$ and $X_{v}$ depend on disjoint set of arcs, $\left(X_{u}\right)_{u \in[n]}$ are mutually independent. Thus

$$
\begin{aligned}
\mathbb{P}\left\{E_{m, 3} \mid \cap_{t=0}^{m-1} A_{t}\right\} & =\mathbb{P}\left\{\cap_{v \in[n]}\left[X_{v} \leq 1\right] \mid \cap_{t=0}^{m-1} A_{t}\right\} \\
& \geq\left(1-O\left(\frac{(\log n)^{2(m-1)}}{n^{2}}\right)\right)^{n}=1-o(1) .
\end{aligned}
$$

Since $\left(\left|\mathcal{Y}_{t}^{-}\right|\right)_{t \geq 1}$ is a Galton-Watson process with a $\operatorname{Bin}(k n, 1 / n)$ offspring distribution, $\mathbb{E}\left|\mathcal{Y}_{m}^{-}\right|=k^{m}$. Thus $\mathbb{P}\left\{\left|\mathcal{Y}_{m}^{-}\right|>(\log n)^{m}\right\}=o(1)$. Therefore

$$
\mathbb{P}\left\{E_{m, 4}^{c} \mid \cap_{t=0}^{m-1} A_{t}\right\} \equiv \mathbb{P}\left\{\left|\mathcal{Y}_{m}^{-}\right|>(\log n)^{m} \mid \cap_{t=0}^{m-1} A_{t}\right\} \leq \frac{\mathbb{P}\left\{\left|\mathcal{Y}_{m}^{-}\right|>(\log n)^{m}\right\}}{\mathbb{P}\left\{\cap_{t=0}^{m-1} A_{t}\right\}}=o(1)
$$

where the last equality is due to the induction assumption that $\mathbb{P}\left\{\cap_{t=0}^{m-1} A_{t}\right\}=1-$ $o(1)$.

### 6.2 Path counting

For three disjoint sets of vertices $\mathcal{A}, \mathcal{B}, \mathcal{C} \subseteq[n]$, let $N_{\ell}$ denote the number of paths of length $\ell$ that start from $\mathcal{A}$ and end at $\mathcal{B}$, and that have all internal vertices in $\mathcal{C}$. In the next subsection, we use the second moment method to lower bound $\mathbb{P}\left\{N_{\ell} \geq 1\right\}$, which requires estimates of $\mathbb{E}\left[N_{\ell}\right]$ and $\operatorname{Var}\left(N_{\ell}\right)$. The following lemma does so by using the path counting technique [40, chap. 3.5].
Proposition 2. Let $\omega$, $\ell$ and $M$ be three positive integers, possibly depending on $n$. Let $\mathcal{A}, \mathcal{B}, \mathcal{C} \subseteq[n]$ be disjoint sets of vertices with $|\mathcal{A}|=|\mathcal{B}|=M \geq 1$ and $|\mathcal{C}| \geq n-\omega$. There exist constants $C_{1}$ and $C_{2}$ such that

$$
\begin{equation*}
\mathbb{E} N_{\ell} \geq \frac{k^{\ell} M^{2}}{n}\left(1-\frac{(\omega+\ell) \ell}{n}\right) \tag{5}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{Var}\left(N_{\ell}\right) \leq \mathbb{E} N_{\ell}+C_{1} \frac{k^{2 \ell} M^{3}}{n^{2}}+C_{2} \frac{k^{2 \ell} M^{4} \ell^{4}}{n^{3}} \tag{6}
\end{equation*}
$$

Proof of (5). Note that if $n \leq(\omega+\ell) \ell$, then (5) is trivially true. So we assume that $n>(\omega+\ell) \ell$. We simplify by contracting $\mathcal{A}$ and $\mathcal{B}$ into to two special vertices $v_{a}$ and $v_{b}$. The vertex $v_{a}$ has out-degree $k M$ and the vertex $v_{b}$ has probability $M / n$ to be chosen as the endpoint of each arc. Consider an unlabeled path of length $\ell \geq 1$ from $v_{a}$ to $v_{b}$. There are $k M$ ways to label the first arc. There are $k^{\ell-1}$ ways to label the other arcs. Recall that $(x)_{y} \equiv(x-1)(x-2) \cdots(x-y+1)$. There are $(|\mathcal{C}|)_{\ell-1}$ ways to label the internal vertices of the path. The probability that a vertex-and-arc labeled path of length $\ell$ from $v_{a}$ to $v_{b}$ exists is $(1 / n)^{\ell-1}(M / n)$. Thus

$$
\begin{aligned}
\mathbb{E} N_{\ell} & =(k M) k^{\ell-1}(|\mathcal{C}|)_{\ell-1}\left(\frac{1}{n}\right)^{\ell-1}\left(\frac{M}{n}\right) \\
& \geq \frac{k^{\ell} M^{2}}{n}\left(1-\frac{\omega+\ell}{n}\right)^{\ell} \geq \frac{k^{\ell} M^{2}}{n}\left(1-\frac{(\omega+\ell) \ell}{n}\right),
\end{aligned}
$$

where the last step is because $(1-x)^{y} \geq 1-x y$ when $x \geq 0, y \geq 1$.

Proof of (6). Let $\mathcal{L}$ be the space of all possible arc-and-vertex labeled paths of length $\ell$ from $v_{a}$ to $v_{b}$ through $\mathcal{C}$. In other words, if $\alpha \in \mathcal{C}$, then

$$
\alpha=\left(v_{0}^{[\alpha]} \equiv v_{a}, a_{0}^{[\alpha]}, v_{1}^{[\alpha]}, a_{1}^{[\alpha]}, \ldots, v_{\ell-1}^{[\alpha]}, a_{\ell-1}^{[\alpha]}, v_{\ell}^{[\alpha]} \equiv v_{b}\right),
$$

where $a_{0}^{[\alpha]}, \ldots, a_{\ell-1}^{[\alpha]}$ are arc labels and $v_{1}^{[\alpha]}, \ldots, v_{\ell-1}^{[\alpha]}$ are different vertex labels in $\mathcal{C}$. For $\alpha \in \mathcal{L}$, let $\mathbb{1}_{\alpha}$ be the indicator that $\alpha$ appears. Given two paths $\alpha, \beta \in \mathcal{L}$, call them arc-disjoint if there does not exist an $i$ such that $v_{i}^{[\alpha]}=v_{i}^{[\beta]}$ and $a_{i}^{[\alpha]}=a_{i}^{[\beta]}$. If two paths $\alpha$ and $\beta$ are arc-disjoint, then $\mathbb{1}_{\alpha}$ and $\mathbb{1}_{\beta}$ are independent, since they depend on the endpoints of two disjoint sets of arcs. Let $\alpha \sim \beta$ denote that $\alpha$ and $\beta$ are not arc-disjoint and that $\alpha$ and $\beta$ can both appear simultaneously. Then

$$
\begin{aligned}
\operatorname{Var}\left(N_{\ell}\right) & =\sum_{\alpha, \beta \in \mathcal{L}}\left(\mathbb{E}\left[\mathbb{1}_{\alpha} \mathbb{1}_{\beta}\right]-\mathbb{E}\left[\mathbb{1}_{\alpha}\right] \mathbb{E}\left[\mathbb{1}_{\beta}\right]\right) \\
& \leq \sum_{\alpha, \beta \in \mathcal{L}} \mathbb{1}_{[\alpha \sim \beta]}\left[\mathbb{E}\left[\mathbb{1}_{\alpha} \mathbb{1}_{\beta}\right]-\mathbb{E}\left[\mathbb{1}_{\alpha}\right] \mathbb{E}\left[\mathbb{1}_{\beta}\right]\right] \\
& \leq \mathbb{E} N_{\ell}+\sum_{\alpha, \beta \in \mathcal{L}} \mathbb{1}_{[\alpha \sim \beta]} \mathbb{1}_{[\alpha \neq \beta]} \mathbb{E}\left[\mathbb{1}_{\alpha} \mathbb{1}_{\beta}\right] \\
& \equiv \mathbb{E} N_{\ell}+I .
\end{aligned}
$$

To bound $I$, we use a technique called path counting. Consider two paths $\alpha, \beta \in \mathcal{L}$ with $\alpha \sim \beta$ and $\alpha \neq \beta$. First colour all vertices and arcs in $\alpha$ and $\beta$ white. Then colour all vertices and arcs shared by $\alpha$ and $\beta$ black. After this, $\alpha$ and $\beta$ both contain the same number, say $m$, of white paths separated by black paths (possibly a single black vertex). Since both $\alpha$ and $\beta$ start and end with black paths, each of them contains $m+1$ black paths. Define:

1. $\vec{x}_{m+1}=\left(x_{1}, \ldots, x_{m+1}\right)$, where $x_{i} \geq 0$ denotes the length of the $i$-th black path in $\alpha$.
2. $\vec{s}_{m}=\left(s_{1}, \ldots, s_{m}\right)$, where $s_{i}>0$ denotes the length of the $i$-th white path in $\alpha$.
3. $\vec{t}_{m}=\left(t_{1}, \ldots, t_{m}\right)$, where $t_{i}>0$ denotes the length of the $i$-th white path in $\beta$.
4. $\vec{o}_{m+1}=\left(o_{1}, \ldots, o_{m+1}\right)$ records the order in which black paths appear in $\beta$. Note that $o_{1} \equiv 1, o_{m+1} \equiv m+1$, and $\left(o_{2}, \ldots, o_{m}\right)$ is a permutation of $\{2, \ldots, m\}$.

Define the shape of $\alpha$ and $\beta$ by $\operatorname{Sh}(\alpha, \beta) \equiv\left(\vec{x}_{m+1}, \vec{s}_{m}, \vec{t}_{m}, \vec{o}_{m+1}\right)$.
Let $r$ be the number of arcs shared by $\alpha$ and $\beta$, i.e., $r \equiv \sum_{i=1}^{m+1} x_{i}$. Since $\alpha \sim \beta$ and $\alpha \neq \beta, 1 \leq r<\ell$. Thus there are $\ell-r$ white arcs in $\alpha$. Since each white path contains at least one white arc, there are at most $\ell-r$ white paths in $\alpha$, i.e., $m \leq \ell-r$. As $\alpha$ and $\beta$ must differ by at least one arc, $m \geq 1$. Let $\mathcal{S}_{m, r}$ denote the set of shapes of two paths in $\mathcal{L}$ that share $r$ arcs and each contains $m$ white paths. Then $I$ can be expressed as a sum over $r, m$ and $\mathcal{S}_{m, r}$ by

$$
I=\sum_{1 \leq r<\ell} \sum_{1 \leq m \leq \ell-r} \sum_{\sigma \in \mathcal{S}_{m, r}} \sum_{\alpha, \beta \in \mathcal{L}} \mathbb{1}_{[\operatorname{Sh}(\alpha, \beta)=\sigma]} \mathbb{E}\left[\mathbb{1}_{\alpha} \mathbb{1}_{\beta}\right] \equiv \sum_{1 \leq m<\ell} \sum_{1 \leq r<\ell-m} \sum_{\sigma \in \mathcal{S}_{m, r}} J_{m, r, \sigma} .
$$



Figure 5: A pair of paths and their shape.

Now fix $m, r$ and a shape $\sigma=\left(\vec{x}_{m+1}, \vec{s}_{m}, \vec{t}_{m}, \vec{o}_{m+1}\right) \in \mathcal{S}_{m, r}$. Consider arcs in two paths $\alpha, \beta \in \mathcal{L}$ with $\mathcal{S}(\alpha, \beta)=\sigma$. Call those starting from $v_{a} a$-arcs, those ending at $v_{b} b$-arcs, and other arcs middle-arcs. Let $z_{a} \equiv \mathbb{1}_{\left[x_{1}=0\right]}$ and $z_{b} \equiv \mathbb{1}_{\left[x_{m+1}=0\right]}$. In other words, $z_{a}$ is the indicator that $\alpha$ and $\beta$ do not share an $a$-arc, and $z_{b}$ is the indicator that they do not share a $b$-arc. Then $\alpha$ and $\beta$ contain $1+z_{a} a$-arcs and $1+z_{b} b$-arcs. Since $\alpha$ and $\beta$ are both of length $\ell$ and they share $r$ arcs, they contain $2 \ell-r \operatorname{arcs}$ in total. Thus they contain $2 \ell-r-\left(1+z_{a}\right)-\left(1+z_{b}\right)=2 \ell-r-z_{a}-z_{b}-2$ middle-arcs.

Recall that black paths are shared by $\alpha$ and $\beta$. Since the $i$-th black path is of length $x_{i}$, it contains $x_{i}+1$ black vertices. So the number of vertices shared by the two paths is $\sum_{i=1}^{m+1}\left(x_{i}+1\right)=r+m+1$. Therefore in total there are $2(\ell+1)-r-m-1$ vertices in the two paths, and among them $2 \ell-r-m-1$ are internal vertices.

The above argument shows that, given two unlabeled path of the shape $\sigma$, there are at most $n^{2 \ell-r-m-1}$ ways to choose the internal vertices. There are at $\operatorname{most}(k M)^{1+z_{a}}$ ways to label $a$-arcs. There are $k^{2 \ell-r-z_{a}-z_{b}-2}$ ways to label middle-arcs. There are at most $k^{1+z_{b}}$ ways to label $b$-arcs. Thus

$$
\begin{aligned}
|\{(\alpha, \beta) \in \mathcal{L} \times \mathcal{L}: \operatorname{Sh}(\alpha, \beta)=\sigma\}| & \leq n^{2 \ell-r-m-1}(k M)^{1+z_{a}} k^{2 \ell-r-z_{a}-z_{b}-2} k^{z_{b}+1} \\
& =n^{2 \ell-r-m-1} M^{1+z_{a}} k^{2 \ell-r} .
\end{aligned}
$$

And the probability that a pair of paths with shape $\sigma$ does appear is

$$
\left(\frac{1}{n}\right)^{1+z_{a}}\left(\frac{1}{n}\right)^{2 \ell-r-z_{a}-z_{b}-2}\left(\frac{M}{n}\right)^{1+z_{b}}=\frac{M^{1+z_{b}}}{n^{2 \ell-r}}
$$

Together,

$$
\begin{align*}
J_{m, r, \sigma} & \equiv \sum_{\alpha, \beta \in \mathcal{L}} \mathbb{1}_{[\operatorname{Sh}(\alpha, \beta)=\sigma]} \mathbb{E}\left[\mathbb{1}_{\alpha} \mathbb{1}_{\beta}\right] \leq n^{2 \ell-r-m-1} M^{1+z_{a}} k^{2 \ell-r} \frac{M^{1+z_{b}}}{n^{2 \ell-r}} \\
& =\frac{k^{2 \ell-r} M^{2+z_{a}+z_{b}}}{n^{m+1}} \equiv K_{m, r, z_{a}, z_{b}} . \tag{7}
\end{align*}
$$

Let $\mathcal{S}_{m, r, z_{a}, z_{b}}$ be the set of shapes with parameters $m, r, z_{a}, z_{b}$. Then we have $\mathcal{S}_{m, r}=$
$\cup_{z_{a}, z_{b} \in\{0,1\}} \mathcal{S}_{m, r, z_{a}, z_{b}}$, where the sets in the union are disjoint. Thus

$$
\begin{aligned}
I & =\sum_{1 \leq m<\ell} \sum_{z_{a}, z_{b} \in\{0,1\}} \sum_{1 \leq r<\ell-m} \sum_{\sigma \in \mathcal{S}_{m, r}, z_{a}, z_{b}} J_{m, r, \sigma} \\
& \leq \sum_{1 \leq m<\ell} \sum_{z_{a}, z_{b} \in\{0,1\}} \sum_{1 \leq r<\ell-m}\left|\mathcal{S}_{m, r, z_{a}, z_{b}}\right| K_{m, r, z_{a}, z_{b}} \\
& =\sum_{z_{a}, z_{b} \in\{0,1\}} \sum_{1 \leq r<\ell-m}\left|\mathcal{S}_{1, r, z_{a}, z_{b}}\right| K_{1, r, z_{a}, z_{b}}+\sum_{2 \leq m<\ell} \sum_{z_{a}, z_{b} \in\{0,1\}} \sum_{1 \leq r<\ell-m}\left|\mathcal{S}_{m, r, z_{a}, z_{b}}\right| K_{m, r, z_{a}, z_{b}} \\
& \equiv I^{[1]}+I^{\text {² }} .
\end{aligned}
$$

By counting the choices of $\vec{x}_{m+1}, \vec{s}_{m}, \vec{t}_{m}, \vec{o}_{m+1}$, we can upper bound $\left|\mathcal{S}_{m, r, z_{a}, z_{b}}\right|$ :
Lemma 17. If $m \geq z_{a}+z_{b}$, then

$$
\begin{equation*}
\left|\mathcal{S}_{m, r, z_{a}, z_{b}}\right|=(r+1)^{m-z_{a}-z_{b}}\binom{\ell-r-1}{m-1}\binom{\ell-r-1}{m-1}(m-1)!. \tag{8}
\end{equation*}
$$

If $m<z_{a}+z_{b}$, then $\left|\mathcal{S}_{m, r, z_{a}, z_{b}}\right|=0$.
Proof of Lemma 17. First consider $m \geq 2$, which implies that $m \geq z_{a}+z_{b}$. When $z_{a}=1, x_{1}=0$. When $z_{b}=1, x_{m+1}=0$. Thus the number of ways to choose $\vec{x}_{m+1}$ equals the number of ways to choose $m+1-z_{a}-z_{b} \geq 1$ ordered non-negative integers such that they sum to $r$, which is well known to be $(r+1)^{m-z_{a}-z_{b}}$, which explains the first factor in (8). Similarly the second term and the third term are the numbers of ways to choose $\vec{s}_{m}$ and $\vec{t}_{m}$ respectively. The last term is the number of ways to choose $\vec{o}_{m+1}$ since $o_{2}, \ldots, o_{m}$ is a permutation of $\{2, \ldots, m\}$.

Now assume $m=1$. If $z_{a}+z_{b} \leq m=1$, the above argument still works. If $z_{a}+z_{b}>1$, then $z_{a}=z_{b}=1$. In other words, the two paths do not share arcs at the beginning and at the end, and they must meet at least one internal vertex. So in this shape, there must be at least two white sub-paths in each of the two paths, i.e., $m \geq 2$, which is a contradiction. Therefore, $S_{1, r, 1,1}=\emptyset$.

Lemma 18. $I^{[1]} \leq 6 k^{2 \ell} M^{3} / n^{2}$.
Proof of Lemma 18. By (7) and the above lemma,

$$
\begin{aligned}
\sum_{1 \leq r<\ell-1}\left|\mathcal{S}_{1, r, 0,0}\right| \times K_{1, r, 0,0} & =\sum_{1 \leq r<\ell-1}(r+1)\left[\binom{\ell-r-1}{0}\right]^{2} 0!\frac{k^{2 \ell-r} M^{2}}{n^{2}} \\
& \leq \frac{k^{2 \ell} M^{2}}{n^{2}} \sum_{1 \leq r} \frac{r+1}{k^{r}} \leq \frac{k^{2 \ell} M^{2}}{n^{2}}\left[\sum_{1 \leq r} \frac{1}{2^{r}}+\sum_{1 \leq r} \frac{r}{2^{r}}\right] \\
& =\frac{k^{2 \ell} M^{2}}{n^{2}}\left(1+\frac{1}{2}+\sum_{2 \leq r} \frac{r}{2^{r}}\right) \leq 4 \frac{k^{2 \ell} M^{2}}{n^{2}},
\end{aligned}
$$

where the last step is because $\sum_{2 \leq r} r / 2^{r} \leq \int_{1}^{\infty} x / 2^{x} \mathrm{~d} x \leq 2$. Similarly,

$$
\begin{aligned}
\sum_{1 \leq r<\ell-1}\left|\mathcal{S}_{1, r, 0,1}\right| \times K_{1, r, 0,1} & =\sum_{1 \leq r<\ell-1}\left|\mathcal{S}_{1, r, 1,0}\right| \times K_{1, r, 1,0} \\
& =\sum_{1 \leq r<\ell-1}(r+1)^{0}\left[\binom{\ell-r-1}{0}\right]^{2} 0!\frac{k^{2 \ell-r} M^{3}}{n^{2}} \\
& \leq \frac{k^{2 \ell} M^{3}}{n^{2}} \sum_{1 \leq r} \frac{1}{k^{r}} \\
& \leq \frac{k^{2 \ell} M^{3}}{n^{2}} \sum_{1 \leq r} \frac{1}{2^{r}}=\frac{k^{2 \ell} M^{3}}{n^{2}} .
\end{aligned}
$$

Also by Lemma $17, \mathcal{S}_{1, r, 1,1}=\emptyset$. Thus

$$
\begin{aligned}
I^{[1]} & \equiv \sum_{z_{a}, z_{b} \in\{0,1\}} \sum_{1 \leq r<\ell-1}\left|\mathcal{S}_{1, r, z_{a}, z_{b}}\right| \times K_{1, r, z_{a}, z_{b}} \\
& \leq 4 \frac{k^{2 \ell} M^{2}}{n^{2}}+2 \frac{k^{2 \ell} M^{3}}{n^{2}}+0 \leq 6 \frac{k^{2 \ell} M^{3}}{n^{2}} .
\end{aligned}
$$

Lemma 19. $I^{[\geq 2]}=4 \ell^{4} k^{2 \ell} M^{4} / n^{3}$.
Proof of Lemma 19. By Lemma 17, for $r \in[1, \ell)$,

$$
\begin{aligned}
& \sum_{z_{a}, z_{b} \in\{0,1\}}\left|\mathcal{S}_{m, r, z_{a}, z_{b}}\right| \times K_{m, r, z_{a}, z_{b}} \\
= & \sum_{z_{a}, z_{b} \in\{0,1\}}(r+1)^{m-z_{a}-z_{b}}\left[\binom{\ell-r-1}{m-1}\right]^{2}(m-1)!\frac{k^{2 \ell-r} M^{2+z_{a}+z_{b}}}{n^{m+1}} \\
\leq & \ell^{m} \frac{\ell^{2(m-1)}}{(m-1)!} \frac{k^{2 \ell-r}}{n^{m+1}} \sum_{z_{a}, z_{b} \in\{0,1\}} M^{2+z_{a}+z_{b}} \\
\leq & \frac{\ell^{3 m-2} k^{2 \ell-r}}{(m-1)!n^{m+1}} 4 M^{4} .
\end{aligned}
$$

Thus

$$
\begin{aligned}
\sum_{1 \leq r<\ell-m} \sum_{z_{a}, z_{b} \in\{0,1\}}\left|\mathcal{S}_{m, r, z_{a}, z_{b}}\right| \times K_{m, r, z_{a}, z_{b}} & \leq \sum_{1 \leq r<\ell-m} \frac{\ell^{3 m-2} k^{2 \ell-r}}{(m-1)!n^{m+1}} 4 M^{4} \\
& \leq \frac{\ell^{3 m-2} k^{2 \ell}}{(m-1)!n^{m+1}} 4 M^{4} \sum_{1 \leq r} \frac{1}{k^{r}} \\
& \leq \frac{\ell^{3 m-2} k^{2 \ell}}{(m-1)!n^{m+1}} 4 M^{4}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
I^{[\geq 2]} & \equiv \sum_{2 \leq m<\ell} \sum_{1 \leq r<\ell-m} \sum_{z_{a}, z_{b} \in\{0,1\}}\left|\mathcal{S}_{m, r, z_{a}, z_{b}}\right| \times K_{m, r, z_{a}, z_{b}} \\
& \leq \sum_{2 \leq m} \frac{\ell^{3 m-2} k^{2 \ell}}{(m-1)!n^{m+1}} 4 M^{4} \\
& \leq \frac{\ell k^{2 \ell} 4 M^{4}}{n^{2}} \sum_{2 \leq m} \frac{\ell^{3(m-1)}}{n^{m-1}(m-1)!} \\
& \leq \frac{\ell k^{2 \ell} 4 M^{4}}{n^{2}}\left(\exp \left\{\frac{\ell^{3}}{n}\right\}-1\right) \leq 4 \frac{\ell^{4} k^{2 \ell} M^{4}}{n^{3}} .
\end{aligned}
$$

By Lemma 18 and Lemma 19,

$$
I=I^{[1]}+I^{[\geq 2]} \leq 6 \frac{k^{2 \ell} M^{3}}{n^{2}}+4 \frac{\ell^{4} k^{2 \ell} M^{4}}{n^{3}} .
$$

Thus $\operatorname{Var}\left(N_{\ell}\right) \leq \mathbb{E}\left[N_{\ell}\right]+I=\mathbb{E}\left[N_{\ell}\right]+6 k^{2 \ell} M^{3} / n^{2}+4 \ell^{4} k^{2 \ell} M^{4} / n^{3}$.

### 6.3 Finishing the proof of Theorem 8

Proof of the upper bound of the typical distance. We can assume $\varepsilon<1 / 2$. Recall that $\psi_{n} \equiv\left\lfloor(1+\varepsilon) \log _{k} n\right\rfloor$ and that $B_{n}=\left[\psi_{n}<H_{n}<\infty\right]$. As argued at the beginning of this section, to finish the proof of Theorem 8 , it suffices to show that $\mathbb{P}\left\{B_{n}\right\}=o(1)$.

Let $\omega_{n} \equiv \psi_{n}$. Let $M, m$ be two positive integers which are picked later. Recall that $\mathcal{S}_{i}^{+}(v)$ and $\mathcal{S}_{i}^{-}(v)$ are the sets of vertices at distance exactly $i$ from or to vertex $v$ respectively, and that $\mathcal{S}_{\leq i}^{+}(v)$ and $\mathcal{S}_{\leq i}^{-}(v)$ are the sets of vertices at distance at most $i$ from or to $v$ respectively. The following argument shows that by properly choosing $M$ and $m$, the probability that there exists a path of length exactly $\psi_{n}-2 m$ from $\mathcal{S}_{m}^{+}\left(v_{1}\right)$ to $\mathcal{S}_{\leq m}^{-}\left(v_{2}\right)$ is at least $1-\delta$ for $n$ large enough, where $\delta>0$ is arbitrary and fixed.


Figure 6: $\mathcal{S}_{\leq m-1}^{+}\left(v_{1}\right), \mathcal{S}_{m}^{+}\left(v_{1}\right)$, and $\mathcal{S}_{\leq m}^{-}\left(v_{2}\right)$.
Let the event $A_{n}(M, m)$ be defined as in Corollary 4, i.e.,

$$
A_{n}(M, m) \equiv\left[M \leq\left|\mathcal{S}_{m}^{+}\left(v_{1}\right)\right|\right] \cap\left[M \leq\left|\mathcal{S}_{m}^{-}\left(v_{2}\right)\right|\right] \cap\left[\left|\mathcal{S}_{\leq m}^{-}\left(v_{2}\right)\right| \leq \omega_{n}\right] .
$$

Since each vertex has out-degree exactly $k \geq 2$, deterministically,

$$
\left|\mathcal{S}_{\leq m-1}^{+}\left(v_{1}\right)\right| \leq 1+k+\cdots+k^{m-1}<k^{m}, \quad\left|\mathcal{S}_{m}^{+}\left(v_{1}\right)\right| \leq k^{m} .
$$

Since $\psi_{n}>2 m$ for $n$ large enough, $B_{n}$ implies $\mathcal{S}_{\leq_{m}}^{+}\left(v_{1}\right)$ and $\mathcal{S}_{<_{m}}^{-}\left(v_{2}\right)$ are disjoint. Thus the event $A_{n}(M, m) \cap B_{n}$ implies that $\left(\mathcal{S}_{\leq m-1}^{+}\left(v_{1}\right), \overline{\mathcal{S}}_{m}^{+}\left(v_{1}\right), \mathcal{S}_{m}^{-}\left(v_{2}\right), \mathcal{S}_{\leq m-1}^{-}\left(v_{2}\right)\right) \in \mathcal{A}$, where $\mathcal{A}$ is a set of quadruples of disjoint sets of vertices defined by

$$
\begin{aligned}
\mathcal{A} \equiv\left\{\left(\mathcal{S}_{1}, \mathcal{S}_{2}, \mathcal{S}_{3}, \mathcal{S}_{4}\right):\right. & v_{1} \in \mathcal{S}_{1} ; v_{2} \in \mathcal{S}_{4} ; \\
& \left.\left|\mathcal{S}_{1}\right|<k^{m} ; M \leq\left|\mathcal{S}_{2}\right| \leq k^{m} ; M \leq\left|\mathcal{S}_{3}\right| ;\left|\mathcal{S}_{3} \cup \mathcal{S}_{4}\right| \leq \omega_{n}\right\} .
\end{aligned}
$$

For $\overrightarrow{\mathcal{S}}=\left(\mathcal{S}_{1}, \mathcal{S}_{2}, \mathcal{S}_{3}, \mathcal{S}_{4}\right) \in \mathcal{A}$, define the event

$$
A_{n}^{\prime}(\overrightarrow{\mathcal{S}}) \equiv\left[\mathcal{S}_{\leq m-1}^{+}\left(v_{1}\right)=\mathcal{S}_{1}\right] \cap\left[\mathcal{S}_{m}^{+}\left(v_{1}\right)=\mathcal{S}_{2}\right] \cap\left[\mathcal{S}_{m}^{-}\left(v_{2}\right)=\mathcal{S}_{3}\right] \cap\left[\mathcal{S}_{\leq m-1}^{-}\left(v_{2}\right)=\mathcal{S}_{4}\right] .
$$

Thus $\left[B_{n} \cap A_{m}(M, m)\right] \subseteq \cup_{\overrightarrow{\mathcal{S}} \in \mathcal{A}}\left[B_{n} \cap A_{n}^{\prime}(\overrightarrow{\mathcal{S}})\right]$ and the events in the union are disjoint.
Now fix a $\overrightarrow{\mathcal{S}} \in \mathcal{A}$. Let $\mathcal{A}_{\overrightarrow{\mathcal{S}}}$ and $\mathcal{B}_{\overrightarrow{\mathcal{S}}}$ be arbitrary subsets of $\mathcal{S}_{2}$ and $\mathcal{S}_{3}$ respectively with $\left|\mathcal{A}_{\overrightarrow{\mathcal{S}}}\right|=M$ and $\left|\mathcal{B}_{\overrightarrow{\mathcal{S}}}\right|=M$. Let $N_{\overrightarrow{\mathcal{S}}}$ be the number of paths of length $\psi_{n}-2 m$ that start from $\mathcal{A}_{\overrightarrow{\mathcal{S}}}$ and end at $\mathcal{B}_{\overrightarrow{\mathcal{S}}}$, and that contain internal vertices only in $\mathcal{C}_{\overrightarrow{\mathcal{S}}} \equiv[n] \backslash \cup_{i \in[4]} \mathcal{S}_{i}$. Thus there are $\left|\mathcal{C}_{\overrightarrow{\mathcal{S}}}\right|=n-\left|\cup_{i \in[4]} \mathcal{S}_{i}\right| \geq n-\left(\omega_{n}+2 k^{m}\right)$ vertices that can be internal vertices of these paths. By (5) of Proposition 2,

$$
\begin{aligned}
\mathbb{E} N_{\overrightarrow{\mathcal{S}}} & \geq \frac{k^{\psi_{n}-2 m} M^{2}}{n}\left(1-\frac{\left(\omega_{n}+2 k^{m}+\psi_{n}-2 m\right)\left(\psi_{n}-2 m\right)}{n}\right) \\
& \geq \frac{k^{(1+\varepsilon) \log _{k}(n)-1-2 m} M^{2}}{n}\left(1-\frac{2 \psi_{n}^{2}}{n}\right) \geq \frac{n^{\varepsilon} M^{2}}{k^{2 m+1}} \frac{1}{2},
\end{aligned}
$$

for $n$ large enough. By (6) of Proposition 2,

$$
\begin{aligned}
\operatorname{Var}\left(N_{\overrightarrow{\mathcal{S}}}\right) & \leq \mathbb{E} N_{\overrightarrow{\mathcal{S}}}+C_{1} \frac{k^{2\left(\psi_{n}-2 m\right)} M^{3}}{n^{2}}+C_{2} \frac{k^{2\left(\psi_{n}-2 m\right)} M^{4}\left(\psi_{n}-2 m\right)^{4}}{n^{3}} \\
& \leq \mathbb{E} N_{\overrightarrow{\mathcal{S}}}+C_{1} \frac{n^{2(1+\varepsilon)} M^{3}}{n^{2} k^{4 m}}+C_{2} \frac{n^{2(1+\varepsilon)} M^{4} \psi_{n}^{4}}{n^{3} k^{4 m}} \\
& \leq \mathbb{E} N_{\overrightarrow{\mathcal{S}}}+C_{1} \frac{n^{2 \varepsilon} M^{3}}{k^{4 m}}+C_{3} \frac{M^{4}}{k^{4 m}} \frac{(\log n)^{4}}{n^{1-2 \varepsilon}},
\end{aligned}
$$

where $C_{3}$ is a constant that does not depend on $M$ or $m$. Thus

$$
\begin{aligned}
\mathbb{P}\left\{N_{\overrightarrow{\mathcal{S}}}=0\right\} & \leq \frac{\operatorname{Var}\left(N_{\overrightarrow{\mathcal{S}}}\right)}{\left(\mathbb{E} N_{\overrightarrow{\mathcal{S}}}\right)^{2}} \leq \frac{2 k^{2 m+1}}{n^{\varepsilon} M^{2}}+\frac{C_{1} n^{2 \varepsilon} M^{3} k^{-4 m}}{\left(n^{\varepsilon} M^{2} 2^{-1} k^{-2 m-1}\right)^{2}}+\frac{C_{3} M^{4}(\log n)^{4} n^{2 \varepsilon-1} k^{-4 m}}{\left(n^{\varepsilon} M^{2} 2^{-1} k^{-2 m-1}\right)^{2}} \\
& \leq \frac{2 k^{2 m+1}}{n^{\varepsilon} M^{2}}+\frac{4 k^{2} C_{1}}{M}+\frac{4 k^{2} C_{3}(\log n)^{4}}{n} .
\end{aligned}
$$

Later $m$ is chosen solely depending on $M$. Thus we can pick $M$ large enough such that for $n$ large enough, $\mathbb{P}\left\{N_{\overrightarrow{\mathcal{S}}}=0\right\} \leq \delta / 2$ for all $\overrightarrow{\mathcal{S}} \in \mathcal{A}$.

If $H_{n}>\psi_{n}$, then there cannot exist paths of length $\psi_{n}-2 m$ from $\mathcal{S}_{m}^{+}\left(v_{1}\right)$ to $\mathcal{S}_{m}^{-}\left(v_{2}\right)$. Thus $B_{n} \cap A_{n}^{\prime}(\overrightarrow{\mathcal{S}})$ implies that $\left[N_{\overrightarrow{\mathcal{S}}}=0\right] \cap A_{n}^{\prime}(\overrightarrow{\mathcal{S}})$. A crucial observation is that

$$
\mathbb{P}\left\{N_{\overrightarrow{\mathcal{S}}}=0 \mid A_{n}^{\prime}(\overrightarrow{\mathcal{S}})\right\} \leq \mathbb{P}\left\{N_{\overrightarrow{\mathcal{S}}}=0\right\} .
$$

This is because $A_{n}^{\prime}(\overrightarrow{\mathcal{S}})$ implies that arcs starting from vertices in $\mathcal{C}_{\overrightarrow{\mathcal{S}}}$ cannot choose vertices in $\mathcal{S}_{\leq m-1}^{-}\left(v_{2}\right)=\mathcal{S}_{4}$ as their endpoints. Whereas when we compute $\mathbb{P}\left\{N_{\overrightarrow{\mathcal{S}}}=0\right\}$ without any condition, arcs starting from vertices in $\mathcal{C}_{\overrightarrow{\mathcal{S}}}$ are allowed to choose all vertices as their endpoints. Thus some of these arcs are possibly "wasted" by choosing their endpoints in $\mathcal{S}_{4}$. This increases the probability that $N_{\overrightarrow{\mathcal{S}}}=0$. Thus

$$
\begin{aligned}
\mathbb{P}\left\{B_{n} \cap A_{n}^{\prime}(\overrightarrow{\mathcal{S}})\right\} & \leq \mathbb{P}\left\{\left[N_{\overrightarrow{\mathcal{S}}}=0\right] \cap A_{n}^{\prime}(\overrightarrow{\mathcal{S}})\right\}=\mathbb{P}\left\{N_{\overrightarrow{\mathcal{S}}}=0 \mid A_{n}^{\prime}(\overrightarrow{\mathcal{S}})\right\} \mathbb{P}\left\{A_{n}^{\prime}(\overrightarrow{\mathcal{S}})\right\} \\
& \leq \mathbb{P}\left\{N_{\overrightarrow{\mathcal{S}}}=0\right\} \mathbb{P}\left\{A_{n}^{\prime}(\overrightarrow{\mathcal{S}})\right\} \leq \frac{\delta}{2} \mathbb{P}\left\{A_{n}^{\prime}(\overrightarrow{\mathcal{S}})\right\}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\mathbb{P}\left\{B_{n} \cap A_{n}(M, m)\right\} & \leq \sum_{\overrightarrow{\mathcal{S}} \in \mathcal{A}} \mathbb{P}\left\{B_{n} \cap A_{n}^{\prime}(\overrightarrow{\mathcal{S}})\right\} \leq \frac{\delta}{2} \sum_{\overrightarrow{\mathcal{S}} \in \mathcal{A}} \mathbb{P}\left\{A_{n}^{\prime}(\overrightarrow{\mathcal{S}})\right\} \\
& \leq \frac{\delta}{2} \mathbb{P}\left\{\left(\mathcal{S}_{\leq m-1}^{+}\left(v_{1}\right), \mathcal{S}_{m}^{+}\left(v_{1}\right), \mathcal{S}_{m}^{-}\left(v_{2}\right), \mathcal{S}_{\leq m-1}^{-}\left(v_{2}\right)\right) \in \mathcal{A}\right\} \leq \frac{\delta}{2} .
\end{aligned}
$$

By Corollary 4, we can choose $m$ depending on $M$ such that for $n$ large enough, $\mathbb{P}\left\{B_{n} \cap A_{n}^{c}(M, m)\right\}<\delta / 2$. Thus

$$
\limsup _{n \rightarrow \infty} \mathbb{P}\left\{B_{n}\right\}=\limsup _{n \rightarrow \infty}\left(\mathbb{P}\left\{B_{n} \cap A_{n}(M, m)\right\}+\mathbb{P}\left\{B_{n} \cap A_{n}^{c}(M, m)\right\}\right) \leq \delta
$$

## 7 Extensions

Addario-Berry et al. [1] also proved that the diameter of the giant component divided by $\log n$ converges in probability to $1 / \log (k)+1 / \log \left(1 / \lambda_{k}\right)$. Recall that the longest path outside the giant divided by $\log n$ converges in probability to $1 / \log \left(1 / \lambda_{k}\right)$. This seems to be a strong indication that it might be possible to derive a new proof for the diameter of the giant.

Recall that $\mathcal{D}_{n, k}^{*}$ is a simple $k$-out digraph with $n$ vertices chosen uniformly at random from all such digraphs. Section 5 proved that if whp $\mathcal{D}_{n, k}$ has property $\mathbf{P}$, then whp $\mathcal{D}_{n, k}^{*}$ has property $\mathbf{P}$. But results like Theorem 1, the central limit law of the one-in-core, cannot be transferred to $\mathcal{D}_{n, k}^{*}$ automatically. We believe that it might be possible to achieve get the same result for $\mathcal{D}_{n, k}^{*}$ following the line of Janson and Luczak's treatment of the configuration model [25].

A natural generalization of $\mathcal{D}_{n, k}$ is to have a deterministic out-degree sequence, as in the directed configuration model, instead of requiring each vertex to have out-degree exactly $k$. With some constraints on the out-degree sequence, most of our results should
hold for this generalized model. Furthermore, we could let each vertex choose its outdegree independently at random from an out-degree distribution. Again by adding some restrictions on the out-degree distribution, most of our results should still hold.

The problem of generating a uniform random surjective function with fixed domain size is an open problem. Theorem 1 implies a simple algorithm for choosing a $[\mathrm{km}] \rightarrow[\mathrm{m}]$ surjective function uniformly at random. Let $n=\left\lceil m / \nu_{k}\right\rceil$. Then we generate a $\mathcal{D}_{n, k}$. If $\left|\mathcal{O}_{n}\right|=m$, i.e., if the one-in-core in $\mathcal{D}_{n, k}$ contains $m$ vertices, then $\mathcal{D}_{n, k}\left[\mathcal{O}_{n}\right]$ is equivalent to a uniform random sample of a $[k m] \rightarrow[\mathrm{m}]$ surjective function. Otherwise we try again until $\left|\mathcal{O}_{n}\right|=m$. Theorem 1 shows that $\mathbb{P}\left\{\left|\mathcal{O}_{n}\right|=m\right\}=\Theta(1 / \sqrt{m})$. Thus the expected number of $\mathcal{D}_{n, k}$ needed to be generated is $\Theta(\sqrt{m})$. Since generating a $\mathcal{D}_{n, k}$ takes $\Theta(m)$ time, the expected running time of the whole algorithm is $\Theta\left(m^{3 / 2}\right)$. But we believe that $\Theta(m)$ should be achievable.

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## Appendix

## 1. Inequalities for constants

Lemma A1. Assume that $k \geq 2$.
(a) There exists exactly one $\tau_{k}>0$ such that $1-\tau_{k} / k-e^{-\tau_{k}}=0$;
(b) $0<k-\tau_{k}<1 / 2$;
(c) $1 / 2<1-\frac{1}{2 k}<\nu_{k} \equiv \tau_{k} / k<1$;
(d) $\lambda_{k} \equiv\left(k-\tau_{k}\right)\left(\frac{\tau_{k}}{k-1}\right)^{k-1}<\lambda_{k}^{\prime} \equiv\left(k-\tau_{k}\right) e^{1-k+\tau_{k}}<1$;
(e) $\gamma_{k} \equiv\left(\frac{k}{e \tau_{k}}\right)^{k}\left(e^{\tau_{k}}-1\right)<1$;
(f) $\rho_{k} \equiv k e^{1-\tau_{k}}\left(\frac{\tau_{k}}{k}\right)^{k-1}<1$;
(g) $\lambda_{k}=\Theta\left(k e^{-k}\right)$ as $k \rightarrow \infty$.

Proof. Let $\eta(x)=1-x / k-e^{-x}$. Since $\eta^{\prime \prime}(x)=-e^{-x}<0, \eta(x)$ is strictly concave. Since $\eta(k-1 / 2)>0$, and $\eta(k)<0, \eta(x)=0$ must have exactly one positive solution and this solution must be in $(k-1 / 2, k)$. Thus (a) and (b) are proved. (c) follows since $\tau_{k} / k>1-1 / k \geq 1 / 2$. For (d) note that $\lambda_{k}<\lambda_{k}{ }^{\prime}$ as $1-x<e^{-x}$ for all $x \neq 0$. For $\lambda_{k}^{\prime}<1$ note that

$$
\log \lambda_{k}{ }^{\prime}=\log \left(k-\tau_{k}\right)+1-\left(k-\tau_{k}\right)=\log \left[1-\left(1-\left(k-\tau_{k}\right)\right)\right]+1-\left(k-\tau_{k}\right)<0,
$$

since $\log (1-x)<-x$ for all $x \in(0,1)$.
For (e), first use $\tau_{k} / k \equiv 1-e^{-\tau_{k}}$ to get

$$
\gamma_{k}=\frac{1}{e^{k}\left(1-e^{-\tau_{k}}\right)^{k}} e^{\tau_{k}}\left(1-e^{-\tau_{k}}\right)=e^{\tau_{k}-k}\left(1-e^{-\tau_{k}}\right)^{1-k}
$$

Then use $k e^{-\tau_{k}} \equiv k-\tau_{k}$ to get

$$
\begin{aligned}
\log \gamma_{k} & =\tau_{k}-k+(1-k) \log \left(1-e^{-\tau_{k}}\right) \\
& =\left(\tau_{k}-k\right)+\log \left(1-e^{-\tau_{k}}\right)-k \log \left(1-e^{-\tau_{k}}\right) \\
& <\left(\tau_{k}-k\right)-e^{-\tau_{k}}+k\left(e^{-\tau_{k}}+e^{-2 \tau_{k}}\right) \\
& =\left(\tau_{k}-k\right)+\left(k-\tau_{k}\right)+e^{-\tau_{k}}\left(k-\tau_{k}-1\right)<0,
\end{aligned}
$$

since $-x>\log (1-x)>-x-x^{2}$ for all $x \in(0,1 / 2)$ and $e^{-\tau_{k}}=1-\nu_{k} \in(0,1 / 2)$.
For (f), use $\tau_{k}<k$ from (a) to get

$$
\begin{equation*}
\tau_{k} \equiv k\left(1-e^{-\tau_{k}}\right)<k\left(1-e^{-k}\right) . \tag{9}
\end{equation*}
$$

Therefore,

$$
\frac{\tau_{k}}{k} \equiv 1-e^{-\tau_{k}}<1-\exp \left\{-k\left(1-e^{-k}\right)\right\} .
$$

Again by (a), $\tau_{k}>k-1 / 2$. Thus

$$
\begin{equation*}
\tau_{k} \equiv k\left(1-e^{-\tau_{k}}\right)>k\left(1-e^{-k+\frac{1}{2}}\right) . \tag{10}
\end{equation*}
$$

Therefore,

$$
k e^{-\tau_{k}}<k \exp \left\{-k\left(1-e^{-k+\frac{1}{2}}\right)\right\} .
$$

The above bounds imply that

$$
\rho_{k} \equiv k e^{1-\tau_{k}}\left(\frac{\tau_{k}}{k}\right)^{k-1}<k \exp \left\{1-k\left(1-e^{-k+\frac{1}{2}}\right)\right\}\left(1-\exp \left\{-k\left(1-e^{-k}\right)\right\}\right)^{k-1}
$$

Using this bound, numeric computations show that $\rho_{2}<0.945651$. When $k \geq 3$, the above upper bound is less than

$$
k \exp \left\{1-k\left(1-e^{-\frac{5}{2}}\right)\right\},
$$

which takes its maximal value at $k=3$ for $k \in[3, \infty)$. This maximal value is about 0.52 . Thus $\rho_{k}<1$ for all $k \geq 2$.

By (9) and (10), $k-\tau_{k}=k e^{-k+O(1)}$ and $\tau_{k} / k=1-e^{-k+O(1)}$ as $k \rightarrow \infty$. Therefore

$$
\begin{aligned}
\lambda_{k} & \equiv\left(k-\tau_{k}\right)\left(\frac{\tau_{k}}{k-1}\right)^{k-1} \\
& =\left(k-\tau_{k}\right)\left(\frac{\tau_{k}}{k}\right)^{k-1}\left(\frac{k}{k-1}\right)^{k-1} \\
& =k e^{-k+O(1)}\left(1-e^{-k+O(1)}\right)^{k-1} e(1+o(1))=k e^{-k+O(1)} .
\end{aligned}
$$

Thus (g) is proved.

## 2. The sizes of $k$-surjections

In this section we prove Lemma 1. Recall that $K_{s}$ is the number of $k$-surjections of size $s$ in $\mathcal{D}_{n, k}$. We first deal the case that $s$ is small:

Lemma A2. $\mathbb{P}\left\{K_{1} \geq 1\right\} \leq 1 / n^{k-1} \leq 1 / n$.
Proof. A single vertex is a $k$-surjection if and only if all its $k$ arcs are self-loops. Thus

$$
\mathbb{P}\left\{K_{1} \geq 1\right\} \leq \sum_{v \in[n]} \mathbb{P}\{v \text { has only self-loops }\}=n\left(\frac{1}{n}\right)^{k} \leq \frac{1}{n^{k-1}} \leq \frac{1}{n}
$$

Lemma A3. $\mathbb{P}\left\{\sum_{2 \leq s \leq a n} K_{s} \geq 1\right\}=o(1 / n)$, for all fixed $a \in\left(0, e^{-1 /(k-1)}\right)$.
Proof. We can choose $\varepsilon \in(0,1)$ such that $2(k-1)(1-\varepsilon)>1$ since $k \geq 2$. Let $J=\{2, \ldots,\lfloor a n\rfloor\}$. Then

$$
\begin{aligned}
\mathbb{P}\left\{\sum_{s \in J} K_{s} \geq 1\right\} & \leq \sum_{s \in J} \sum_{\mathcal{S} \subseteq \mid n]:|\mathcal{S}|=s} \mathbb{P}\{\mathcal{S} \text { is closed }\} \\
& =\sum_{s \in J}\binom{n}{s}\left(\frac{s}{n}\right)^{k s} \\
& \leq \sum_{s \in J}\left(\frac{e n}{s}\right)^{s}\left(\frac{s}{n}\right)^{k s} \quad \text { (Stirling's approximation) } \\
& =\sum_{2 \leq s \leq n^{\varepsilon}}\left[e\left(\frac{s}{n}\right)^{k-1}\right]^{s}+\sum_{n^{\varepsilon}<s<a n}\left[e\left(\frac{s}{n}\right)^{k-1}\right]^{s} \\
& \leq\left[e\left(\frac{n^{\varepsilon}}{n}\right)^{k-1}\right]^{2} \sum_{2 \leq s+2}\left[e\left(\frac{n^{\varepsilon}}{n}\right)^{k-1}\right]^{s}+\sum_{n^{\varepsilon}<s}\left(e \times a^{k-1}\right)^{s} \\
& =O\left(n^{-2(k-1)(1-\varepsilon)}\right)+O\left(\left(e a^{k-1}\right)^{n^{\varepsilon}}\right),
\end{aligned}
$$

where both terms are $o(1 / n)$ due to our choice of $\varepsilon$ and $a$.
When $s$ is large, we need to take into account the probability that $\mathcal{S}$ is surjective. Let $\left\{\begin{array}{l}x \\ y\end{array}\right\}$ denote Stirling's number of the second kind, i.e., the number of ways to put $x$ balls into $y$ unordered bins such that there are no empty bins [17, pp. 64]. Then

$$
\mathbb{P}\{\mathcal{S} \text { is surjective } \mid \mathcal{S} \text { is closed }\}=\frac{\left\{\begin{array}{c}
k s \\
s
\end{array}\right\} s!}{s^{k s}},
$$

where the numerator is the number of ways to choose endpoints for the $k s \operatorname{arcs}$ in $\mathcal{S}$ so that minimum in-degree is one, and the denominator is the total number of ways to choose endpoints for $k s \operatorname{arcs}$ in $\mathcal{S}$. Thus
$\mathbb{P}\{\mathcal{S}$ is a $k$-surjection $\}=\mathbb{P}\{\mathcal{S}$ is surjective $\mid \mathcal{S}$ is closed $\} \mathbb{P}\{\mathcal{S}$ is closed $\}$

$$
=\frac{\left\{\begin{array}{c}
k s \\
s
\end{array}\right\} s!}{s^{k s}}\left(\frac{s}{n}\right)^{k s}=\frac{\left\{\begin{array}{c}
k s \\
s
\end{array}\right\} s!}{n^{k s}} .
$$

Good [19] established an asymptotic estimation of Stirling's numbers of the second kind

$$
\left\{\begin{array}{c}
k s \\
s
\end{array}\right\} \sim \frac{(k s)!}{s!} \frac{\left(e^{\tau_{k}}-1\right)^{s}}{\tau_{k}^{k s} \sqrt{2 \pi k s\left(1-k e^{-k}\right)}} .
$$

Applying this and Stirling's approximation for factorials, we have

$$
\begin{align*}
\mathbb{P}\{\mathcal{S} \text { is a } k \text {-surjection }\} & \sim \frac{(k s)!}{s!} \frac{\left(e^{\tau_{k}}-1\right)^{s}}{\tau_{k}^{k s} \sqrt{2 \pi k s\left(1-k e^{-k}\right)}} \frac{s!}{n^{k s}} \\
& \sim \frac{1}{\sqrt{1-k e^{-\tau_{k}}}}\left[\left(\frac{s}{n}\right)^{k} \gamma_{k}\right]^{s}, \tag{11}
\end{align*}
$$

where $\gamma_{k} \equiv\left(k / e \tau_{k}\right)^{k}\left(e^{\tau_{k}}-1\right)<1$ (see Lemma A1).

Lemma A4. There exists a constant $b \in\left(\nu_{k}, 1\right)$ such that $\mathbb{P}\left\{\sum_{b n \leq s \leq n} K_{s} \geq 1\right\}=$ $o(1 / n)$.

Proof. Let $b>\nu_{k}$ be a constant decided later. If $|\mathcal{S}|=s \in[b n, n]$, then by (11)

$$
\mathbb{P}\{\mathcal{S} \text { is a } k \text {-surjection }\}=O\left(\left[\left(\frac{s}{n}\right)^{k} \gamma_{k}\right]^{s}\right) \leq O\left(\gamma_{k}^{s}\right) \leq O\left(\gamma_{k}^{b n}\right) .
$$

Since $b>\nu_{k}>1 / 2($ Lemma A1),

$$
\binom{n}{s} \leq\binom{ n}{b n}=O\left(\frac{1}{\sqrt{n}}\left[\frac{1}{b^{b}(1-b)^{1-b}}\right]^{n}\right)
$$

Therefore

$$
\mathbb{P}\left\{K_{s} \geq 1\right\} \leq\binom{ n}{s} \mathbb{P}\{\mathcal{S} \text { is a } k \text {-surjection }\} \leq O\left(\left[\frac{\gamma_{k}^{b}}{b^{b}(1-b)^{1-b}}\right]^{n}\right)
$$

Since the quantity in the square brackets goes to $\gamma_{k}<1$ as $b \rightarrow 1$, we can pick a $b$ close enough to one such that $\mathbb{P}\left\{\sum_{b n \leq s \leq n} K_{s} \geq 1\right\}=o(1 / n)$.

Let $a \in\left(0, \nu_{k}\right)$ and $b \in\left(\nu_{k}, 1\right)$ be two constants such that the upper bounds in Lemma A3 and A4 hold. If $|\mathcal{S}|=x n$ with $x \in(a, b)$ and $x n$ integer-valued, then by (11) and Stirling's approximation

$$
\begin{align*}
\mathbb{E} K_{x n} & =\binom{n}{x n} \mathbb{P}\{\mathcal{S} \text { is a } k \text {-surjection }\} \\
& \sim \frac{1}{\sqrt{2 \pi x(1-x) n}}\left[\frac{1}{(x)^{x}(1-x)^{1-x}}\right]^{n} \frac{1}{\sqrt{1-k e^{-\tau_{k}}}}\left(x^{k} \gamma_{k}\right)^{x n} \\
& =\frac{1}{\sqrt{2 \pi\left(1-k e^{\left.-\tau_{k}\right) n}\right.}} g(x)[f(x)]^{n} \tag{12}
\end{align*}
$$

where

$$
g(x) \equiv \frac{1}{\sqrt{x(1-x)}}, \quad \quad f(x) \equiv\left[\frac{x^{k-1} \gamma_{k}}{(1-x)^{(1-x) / x}}\right]^{x}
$$

Lemma A5. For all fixed $a \in\left(0, \nu_{k}\right), b \in\left(\nu_{k}, 1\right)$ and $\delta \in(0,1 / 2), \mathbb{P}\left\{\sum_{s \in J} K_{s} \geq 1\right\}=$ $o(1 / n)$, where $J=\left[a n, \nu_{k} n-n^{\frac{1}{2}+\delta}\right] \cup\left[\nu_{k} n+n^{\frac{1}{2}+\delta}, b n\right]$.

Proof. Let $h(x) \equiv \log f(x)$. Lemma A6 shows that as $x \rightarrow \nu_{k}$,

$$
h(x)=-\frac{\left(x-\nu_{k}\right)^{2}}{2 \sigma_{k}^{2}}+O\left(\left|x-\nu_{k}\right|^{3}\right)
$$

and that $h(x)$ is strictly increasing on $\left(a, \nu_{k}\right)$ and strictly decreasing on $\left(\nu_{k}, b\right)$. It follows from $\left|s / n-\nu_{k}\right|>n^{-1 / 2+\delta}$ that $h(s / n) \leq-n^{2 \delta-1} / 2 \sigma_{k}^{2}+O\left(n^{3 \delta-3 / 2}\right)$. As for $g(x)$, it is
bounded on ( $a, b$ ). Thus by (12) and Markov's inequality

$$
\begin{aligned}
\log \left(n^{2} \mathbb{P}\left\{K_{s} \geq 1\right\}\right) & \leq \log \left(n^{2} \mathbb{E} K_{s}\right) \\
& =\log \left(n^{2} O\left(n^{-1 / 2}\right) f\left(\frac{s}{n}\right)^{n}\right) \\
& =O(\log n)+n h\left(\frac{s}{n}\right) \\
& \leq O(\log n)-\frac{n^{2 \delta}}{2 \sigma_{k}^{2}}+O\left(n^{3 \delta-1 / 2}\right),
\end{aligned}
$$

which goes to $-\infty$. In other words, $\mathbb{P}\left\{K_{s} \geq 1\right\}=o\left(1 / n^{2}\right)$. So $\mathbb{P}\left\{\sum_{s \in J} K_{s} \geq 1\right\}=$ $o(1 / n)$.

Lemma 1 follows immediately from Lemma A2, A3, A4, and A5.

## 3. Special functions

Lemma A6. Let $f(x), g(x)$ and $h(x)$ be defined as in the previous subsection. Let $\nu_{k}$, $\tau_{k}$ and $\sigma_{k}$ be as in Lemma A1. Then
(a) As $x \rightarrow \nu_{k}, g(x)=g\left(\nu_{k}\right)+O\left(\left|x-\nu_{k}\right|\right)=\left(1+O\left(\left|x-\nu_{k}\right|\right)\right) /\left(\sigma_{k} \sqrt{1-k e^{-\tau_{k}}}\right)$.
(b) $h(x)$ and $f(x)$ are strictly increasing on $\left(1-\frac{1}{k}, \nu_{k}\right)$ and strictly decreasing on $\left(\nu_{k}, 1\right)$.
(c) As $x \rightarrow \nu_{k}$,

$$
h(x)=h\left(\nu_{k}\right)+O\left(\left|x-\nu_{k}\right|^{3}\right)=-\frac{\left(x-\nu_{k}\right)^{2}}{2 \sigma_{k}^{2}}+O\left(\left|x-\nu_{k}\right|^{3}\right),
$$

which implies that

$$
f(x)=e^{h(x)}=\exp \left\{-\frac{\left(x-\nu_{k}\right)^{2}}{2 \sigma_{k}^{2}}\right\}+O\left(\left|x-\nu_{k}\right|^{3}\right)
$$

Proof. For (a), recall that $\sigma_{k}^{2} \equiv \tau_{k} /\left(k e^{\tau_{k}}\left(1-k e^{-\tau_{k}}\right)\right)$. Thus $\sigma_{k}^{2}\left(1-k e^{-\tau_{k}}\right)=\nu_{k}\left(1-\nu_{k}\right)$. Then $g\left(\nu_{k}\right)=1 / \sqrt{\nu_{k}\left(1-\nu_{k}\right)}=1 / \sigma_{k} \sqrt{1-k e^{-\tau_{k}}}$. Since $g^{\prime}(x)$ is bounded around $\nu_{k}$, by Taylor's theorem,

$$
g(x)=g\left(\nu_{k}\right)+O\left(\left|x-\nu_{k}\right|\right)=\left(1+O\left(\left|x-\nu_{k}\right|\right)\right) \frac{1}{\sigma_{k} \sqrt{1-k e^{-\tau_{k}}}}, \quad \text { as } x \rightarrow \nu_{k} .
$$

Let $r(x)=\log \left(f(x)^{1 / x}\right)=h(x) / x$. Using $\tau_{k} / k \equiv 1-e^{-\tau_{k}} \equiv \nu_{k}$ shows that

$$
\gamma_{k}=\left(\frac{1}{e \nu_{k}}\right)^{k} e^{\tau_{k}} \nu_{k}=\nu_{k}^{-k+1} e^{-k+\tau_{k}}=\nu_{k}^{-k+1}\left(e^{-\tau_{k}}\right)^{\left(k-\tau_{k}\right) / \tau_{k}}=\nu_{k}^{-k+1}\left(1-\nu_{k}\right)^{\left(1-\nu_{k}\right) / \nu_{k}}
$$

Then $r\left(\nu_{k}\right)=\log \left(\nu_{k}^{k-1}\left(1-\nu_{k}\right)^{\left(\nu_{k}-1\right) / \nu_{k}} \gamma_{k}\right)=\log (1)=0$,

$$
r^{\prime}(x)=\frac{k}{x}+\frac{1}{x^{2}} \log (1-x), \quad \text { and } \quad r^{\prime \prime}(x)=-\frac{k}{x^{2}}-\frac{2 \log (1-x)}{x^{3}}-\frac{1}{x^{2}(1-x)}
$$

Therefore $r^{\prime}\left(\nu_{k}\right)=0$ and $r^{\prime \prime}\left(\nu_{k}\right)=-1 /\left(\nu_{k} \sigma_{k}^{2}\right)$.
Since $h(x)=x r(x)$,

$$
h^{\prime}(x)=r(x)+x r^{\prime}(x), \quad h^{\prime \prime}(x)=2 r^{\prime}(x)+x r^{\prime \prime}(x)=\frac{k}{x}-\frac{1}{x(1-x)}
$$

Thus $h\left(\nu_{k}\right)=0, h^{\prime}\left(\nu_{k}\right)=0$ and $h^{\prime \prime}\left(\nu_{k}\right)=-1 / \sigma_{k}^{2}$. Also recalling that $1-\frac{1}{k}<1-\frac{1}{2 k}<$ $\nu_{k}<1$ (Lemma A1), $h(x)$ is strictly concave on ( $1-\frac{1}{k}, 1$ ), reaching maximum at $\nu_{k}$. Thus (b) is proved. The two asymptotic equations in (c) follow from Taylor's theorem.

## 4. Probability generating functions of Galton-Watson processes

Lemma A7. Let $\mu \in\left(0, \frac{1}{2 k}\right)$ be a constant where $k \geq 2$. Let $\left(Z_{m}\right)_{m \geq 0}$ be a GaltonWatson process with $Z_{0} \equiv 1$ and offspring distribution $\operatorname{Bin}(k, \mu)$. Let $\varphi_{m}(y) \equiv \mathbb{E} y^{Z_{m}}$. Then

$$
\varphi_{m}(0) \leq 1-(k \mu)^{m}+\left(1-\frac{1}{2^{m}}\right)(k \mu)^{m+1} .
$$

Proof. We use induction. Let $c_{m}=1-1 / 2^{m}$. For $m=1$,

$$
\varphi_{1}(y)=\mathbb{E} y^{Z_{1}}=(1-\mu(1-y))^{k} .
$$

Since $\mu>0$ and $k \geq 2$, by Taylor's theorem,

$$
\varphi_{1}(0)=(1-\mu)^{k} \leq 1-k \mu+\frac{(k \mu)^{2}}{2}=1-k \mu+c_{1}(k \mu)^{2} .
$$

It is well known that for $m>1, \varphi_{m}(y)=\varphi_{1}\left(\varphi_{m-1}(y)\right)$ (see [13]). Assuming the lemma holds for $m$, then

$$
\begin{aligned}
\varphi_{m+1}(0) & =\varphi_{1}\left(\varphi_{m}(0)\right)=\left(1-\mu\left(1-\varphi_{m}(0)\right)\right)^{k} \\
& \leq\left(1-\mu\left((k \mu)^{m}-c_{m}(k \mu)^{m+1}\right)\right)^{k} \\
& \leq 1-k \mu\left((k \mu)^{m}-c_{m}(k \mu)^{m+1}\right)+\frac{k^{2}}{2} \mu^{2}\left((k \mu)^{m}-c_{m}(k \mu)^{m+1}\right)^{2} \\
& =1-(k \mu)^{m+1}+c_{m}(k \mu)^{m+2}+\frac{(k \mu)^{m}}{2}\left(1-c_{m} k \mu\right)^{2}(k \mu)^{m+2} \\
& \leq 1-(k \mu)^{m+1}+c_{m+1}(k \mu)^{m+2},
\end{aligned}
$$

since $k \mu<1 / 2$ and $c_{m+1}=c_{m}+1 / 2^{m+1}$.


[^0]:    ${ }^{1}$ In a shorter version of this paper, this section is omitted.

