The cut-tree of large trees with small heights

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Abstract

We destroy a finite tree of size n by cutting its edges one after the other and in uniform random order. Informally, the associated cut-tree describes the genealogy of the connected components created by this destruction process. We provide a general criterion for the convergence of the rescaled cut-tree in the Gromov-Prohorov topology to an interval endowed with the Euclidean distance and a certain probability measure, when the underlying tree has branching points close to the root and height of order $o(\sqrt{n})$. In particular, we consider uniform random recursive trees, binary search trees, scale-free random trees and a mixture of regular trees. This yields extensions of a result in Bertoin [8] for the cut-tree of uniform random recursive trees and also allows us to generalize some results of Kuba and Panholzer [27] on the multiple isolation of vertices. The approach relies in the close relationship between the destruction process and Bernoulli bond percolation, which may be useful for studying the cut-tree of other classes of trees.

Key words and phrases: Random trees, destruction of trees, percolation, Gromov-Prokhorov convergence.

1 Introduction and main result

1.1 General introduction

Consider a tree T_n on a finite set of vertices, say $[n] := \{1, \dots, n\}$, rooted at 1. Imagine that we destroy it by cutting its edges one after the other, in a uniform random order. After $n-1$ steps, all edges have been destroyed and all the vertices are isolated. Meir and Moon [30, 31]

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initiated the study of such procedure by considering the number of cuts required to isolate the root, when the edges are removed from the current component containing this distinguished vertex. More precisely, they estimated the first and second moments of this quantity for two important trees families, Cayley trees and random recursive trees. Concerning Cayley trees and other families of simply generated trees, a weak limit theorem for the number of cuts to isolate the root vertex was proven by Panholzer [33] and, in greater generality by Janson [24] who also obtained the result for complete binary trees [23]. Holmgren [20, 21] extended the approach of Janson to binary search trees and to the family of split trees. For random recursive trees a limit law was obtained, first by Drmota et al. [16] and reproved using a probabilistic approach by Iksanov and Möhle [22].

We observe that during the destruction process the cut of an edge induces the partition of the subset (or block) that contains this edge into two sub-blocks of $[n]$. We then encode the destruction of T_n by a rooted binary tree, which we call the cut-tree and denote by $\mathrm{Cut}(T_n)$. The cut-tree has internal vertices given by the non-singleton connected components which arise during the destruction, and leaves which correspond to the singletons $\{1\}, \ldots, \{n\}$ (these can be identified as the vertices of T_n). More precisely, the $\mathrm{Cut}(T_n)$ is rooted at the block $[n]$, then we build it inductively: we draw an edge between a parent block B and two children blocks B' and B'' whenever an edge is removed from the subtree of T_n with set of vertices B, producing two subtrees B' and B''. See Figure 1 for an illustration.

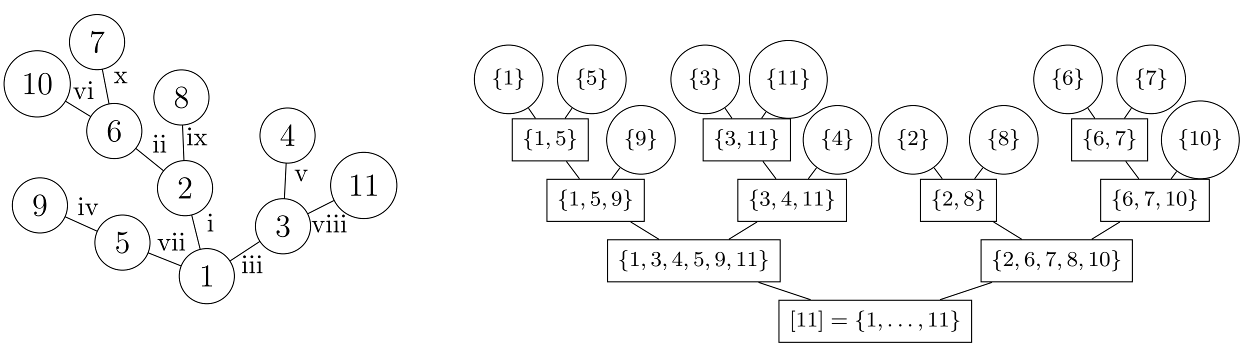

Figure 1: A tree of size eleven with the order of cuts on the left, and the corresponding cut-tree on the right

Roughly speaking, cut-trees describe the genealogy of connected components appearing in this edge-deletion process. They are especially useful in the study of the number of cuts needed to isolate any given subset of distinguished vertices, when the connected components which contain no distinguished points are discarded as soon as they appear. For instance, the number of cuts required to isolate k distinct vertices $v_1, \ldots, v_k$ coincides with the total length of the cut-tree reduced to its root and k leaves $\{v_1\}, \ldots, \{v_k\}$ minus $(k-1)$, where the length is measured as usual by the graph distance on $\mathrm{Cut}(T_n)$. This motivated the study of the cut-tree for several families of trees. Bertoin [7] considered the cut-tree of Cayley trees, more generally, Bertoin and Miermont [9] dealt with critical Galton-Watson trees with finite variance and conditioned to have size n. More recently, Bertoin [8] studied the uniform random recursive trees, Dieuleveut [14] the Galton-Watson trees with offspring distribution belonging to the domain of attraction of a stable law of index $\alpha \in (1, 2]$, and Broutin and Wang [11] the so-called p-trees. They described the asymptotic behavior (in distribution) of the cut-trees when $n \to \infty$, for these classes of trees. We stress that in [9, 14] the cut-tree slightly differs from the one defined above, and in particular [14] considered a vertex removal procedure.

On the other hand, Baur [3] has recently introduced another tree associated to the destruction process of uniform random recursive trees, called *tree of components*. Informally, one considers a dynamically version of the cutting procedure, where edges are equipped with i.i.d. exponential clocks and deleted at time given by the corresponding variable. Then, each removal of an edge gives birth to a new tree component, whose sizes and birth times are encoding by a tree-indexed process. He used this tree of components to study cluster sizes created from performing Bernoulli bond percolation on uniform random recursive trees. We do not study the tree of components in this work but, we think it would be of interest, and may be seen as a complement of the cut-tree. However, a common feature with our analysis is that, it is useful to consider a continuous time version of the destruction process.

The main purpose of this work is study the behavior of $\mathrm{Cut}(T_n)$ when the vertices of the underlying tree T_n is star-shaped. Informally, we assume that the last common ancestor of two randomly chosen vertices is close to the root, after proper rescaling, with high probability. We consider also that T_n has a small height of order $o(\sqrt{n})$, in the sense that that the distance (the number of edges) between its root 1, and a typical vertex in T_n is of this order $o(\sqrt{n})$. For instance, this is the case for uniform random recursive trees, binary search trees, scale-free random trees and regular trees; see for example Drmota [15], Barabási [2], and Mahmound and Neininger [29]. Informally, our main result provides a general criterion, depending on the nature of T_n, for the convergence in distribution of the rescaled $\mathrm{Cut}(T_n)$ when $n \to \infty$.

We next introduce the necessary notation and relevant background, which we will enable us to state our main result in Section 1.3.

1.2 Measured metric spaces and the Gromov-Prokhorov topology

We begin by introducing some basic facts about topological space of trees in which limits can be taken, and define the limit objects. A pointed metric measure space is a quadruple $(\mathcal{T}, d, \rho, \nu)$ where $(\mathcal{T}, d)$ is a separable and complete metric space, $\rho \in \mathcal{T}$ a distinguished element called the root of $\mathcal{T}$, and ν a Borel probability measure on $(\mathcal{T}, d)$. This quadruple is called a real tree if in addition, $\mathcal{T}$ is a tree, in the sense that it is a geodesic space for which any two points are connected via a unique continuous injective path up to reparametrization. This is a continuous analog of the graph-theoretic definition of a tree as a connected graph with no cycle. For sake of simplicity, we frequently write $\mathcal{T}$ to refer to a pointed metric measure space $(\mathcal{T}, d, \rho, \nu)$. We say that two measured rooted spaces $(\mathcal{T}, d, \rho, \nu)$ and $(\mathcal{T}', d', \rho', \nu')$ are isometry-equivalent if there exists a root-preserving, bijective isometry $\phi : \mathrm{supp}(\mu) \cup \{\rho\} \to \mathcal{T}'$ (here supp is the topological support) such that the image of ν by ϕ is ν'. This defines an equivalence relation between pointed metric measure spaces, and we note that representatives $(\mathcal{T}, d, \rho, \nu)$ of a given isometry-equivalence class can always be assumed to have $\mathrm{supp}(\mu) \cup \{\rho\} = \mathcal{T}$. It is also convenient to agree that for $a > 0$, $a\mathcal{T}$ denotes the same space $\mathcal{T}$ but with distance rescaled by the factor a, i.e. $(\mathcal{T}, ad, \rho, \nu)$.

It is well-known that the set $\mathbb{M}$ of isometry-equivalence classes of pointed metric spaces is a Polish space when endowed with the so-called Gromov-Prokhorov topology. This topology was introduced by Greven, Pfaffelhuber and Winter in [17] under the name of *Gromov-weak* topology. We also refer to Gromov's book [18], the article of Haas and Miermont [19] and references therein for background. We can then view the $\mathrm{Cut}(T_n)$ for $n \geq 1$ as a sequence random variables with values in $\mathbb{M}$ (i.e. a sequence of real random tree). For convenience, we adopt a slightly different point of view for $\mathrm{Cut}(T_n)$ than the usual for finite trees, focusing on leaves rather than internal nodes. More precisely, we set $[n]^0 = \{0, 1, \ldots, n\}$ where 0 correspond to the root $[n]$ of $\mathrm{Cut}(T_n)$ and $1, \ldots, n$ to the leaves (i.e. i is identified with the singleton $\{i\}$). We consider the random pointed metric measure space $([n]^0, \delta_n, 0, \mu_n)$ where δ_n is the random graph distance on $[n]^0$ induced by the cut-tree, 0 is the distinguished element, and μ_n is the uniform probability measure on $[n]$ extended by $\mu_n(0) = 0$. That is, μ_n is the uniform probability measure on the set of leaves of $\mathrm{Cut}(T_n)$. We point out that the combinatorial structure of the cut-tree can be recovered from $([n]^0, \delta_n, 0, \mu_n)$, so by a slight abuse of notation, sometimes we refer to $\mathrm{Cut}(T_n)$ as the latter pointed metric measure space.

Finally, we recall a convenient characterization of the Gromov-Prokhorov topology that relies on the convergence of distances between random points. A sequence $(\mathcal{T}_n, d_n, \rho_n, \nu_n)$ of pointed measure metric spaces converges in the Gromov-Prokhorov sense to an element of

$\mathbb{M}$, say $(\mathcal{T}_\infty, d_\infty, \rho_\infty, \nu_\infty)$, if and only if the following holds: for $n \in \{1, 2, \dots\} \cup \{\infty\}$, set $\xi_n(0) = \rho_n$ and let $\xi_n(1), \xi_n(2), \dots$ be a sequence of i.i.d. random variables with law ν_n, then

$$(d_n(\xi_n(i), \xi_n(j)) : i, j \geq 0) \Rightarrow (d_\infty(\xi_\infty(i), \xi_\infty(j)) : i, j \geq 0)$$

where $\Rightarrow$ means convergence in the sense of finite-dimensional distribution, $\xi_\infty(0) = \rho_\infty$ and $\xi_\infty(1), \xi_\infty(2), \dots$ is a sequence of i.i.d. random variables with law ν_∞; see for example Corollary 8 of [28]. One can interpret $(d_\infty(\xi_\infty(i), \xi_\infty(j)) : i, j \geq 0)$ as the matrix of mutual distances between the points of an i.i.d. sample of $(\mathcal{T}_\infty, d_\infty, \rho_\infty, \nu_\infty)$. Moreover, it is important to point out that by the Gromov's reconstruction theorem in [18], the distribution of the above matrix of distances characterizes $(\mathcal{T}_\infty, d_\infty, \rho_\infty, \nu_\infty)$ as an element of $\mathbb{M}$.

1.3 Main result

We first introduce notation and hypotheses which will have an important role for the rest of the work. Recall that T_n is a tree with set of vertices $[n] = \{1, \dots, n\}$, rooted at 1. We denote by u and v two independent uniformly distributed random vertices on $[n]$. Let d_n be the graph distance in T_n, and $\ell : \mathbb{N} \to \mathbb{R}_+$ be some function such that $\lim_{n\to\infty} \ell(n) = \infty$. We introduce the following hypothesis

$$\frac{1}{\ell(n)}(d_n(1, u), d_n(u, v)) \Rightarrow (\zeta_1, \zeta_1 + \zeta_2). \tag{H}$$

where ζ_1 and ζ_2 are i.i.d. variables in $\mathbb{R}_+$ with no atom at 0. This happens with ζ_i a positive constant for some important families of random trees, such as uniform recursive trees, regular trees, scale-free random trees and binary search trees (and more generally b-ary recursive trees). In Section 4, we consider a different class of examples where the variable ζ_i is not a constant, which results of the mixture of similar trees satisfying the hypothesis (H).

Remark 1. *We observe that*

$$d_n(u, v) = d_n(1, u) + d_n(1, v) - 2d_n(1, u \wedge v),$$

where $u \wedge v$ is the last common ancestor of u and v in T_n. Then, the condition (H) readily implies that $\lim_{n\to\infty} \ell(n)^{-1} d_n(1, u \wedge v) = 0$ in probability. Moreover, if for each fixed $k \in \mathbb{N}$, we denote by $L_{k,n}$ the length of the tree T_n reduced to k vertices chosen uniformly at random with replacement and its root 1, i.e. the minimal number of edges of T_n which are needed to *connect* 1 *and such vertices, we see that (H) is equivalent to*

$$\frac{1}{\ell(n)}(L_{1,n}, L_{2,n}) \Rightarrow (\zeta_1, \zeta_1 + \zeta_2).$$

We then write

$$\lambda(t) = \mathbb{E}[e^{-t\zeta_1}], \quad \text{for } t \geq 0,$$

for the Laplace transform of the random variable ζ_1. We henceforth denote

$$a = \mathbb{E}[1/\zeta_1],$$

which can be infinite. We define the bijective mapping $\Lambda : [0, \infty) \to [0, a)$ by

$$\Lambda(t) = \int_0^t \lambda(s)\mathrm{d}s, \quad \text{for } t \geq 0,$$

where $\Lambda(\infty) = \lim_{t\to\infty} \Lambda(t) = a$, and write Λ^{-1} for its inverse mapping. Observe that (H) entails that

$$\frac{1}{\ell(n)} d_n(u, v) \Rightarrow \zeta_1 + \zeta_2,$$

then we consider the next technical condition

$$\lim_{n\to\infty} \mathbb{E}\left[\frac{\ell(n)}{d_n(u, v)} \mathbf{1}_{\{u\neq v\}}\right] = \mathbb{E}\left[\frac{1}{\zeta_1 + \zeta_2}\right] < \infty. \tag{H'}$$

Theorem 2. *Suppose that (H) and (H′) hold with ℓ such that $\ell(n) = o(\sqrt{n})$. Furthermore, assume that $a < \infty$. Then as $n \to \infty$, we have the following convergence in distribution in the sense of the pointed Gromov-Prokhorov topology:*

$$\frac{\ell(n)}{n} \mathrm{Cut}(T_n) \Rightarrow I_\mu.$$

where I_μ is the pointed measure metric space given by the interval $[0, a]$, pointed at 0, equipped with the Euclidean distance, and the probability measure μ given by

$$\int_0^a f(x)\mu(\mathrm{d}x) = -\int_0^a f(x)\, \mathrm{d}\lambda \circ \Lambda^{-1}(x) \tag{1}$$

where f is a generic positive measurable function. The result still valid when $a = \infty$, and then one considers the interval $[0, \infty)$, pointed at 0, equipped with the same distance and measure.

We stress that Theorem 2 does not apply for the family of critical Galton-Watson trees conditioned to have size n considered for Bertoin and Miermont [9] and Dieuleveut [14] since they do not satisfy the condition (H), and the height of a typical vertex is not of the order $o(\sqrt{n})$. For instance, the case when T_n is a Cayley tree (conditioned Galton-Watson tree with Poisson offspring distribution), for which it is know that $\ell(n) = \sqrt{n}$ and the variable $L_{i,n}$ in Remark 1, for $i = 1, 2$, is a chi-variable with $2k$ degrees of freedom; see for example Aldous [1]. We believe that the threshold $\sqrt{n}$ appearing in this work is critical, and that for trees with larger heights (of order $\Omega(\sqrt{n})$ following Knut's definition) the limit of their rescaled cut-tree is a random tree, and not a deterministic one. For instance, in the case when $T_n^{(c)}$ is a Cayley tree of size n, it has been shown in [7] that $n^{-1/2}\mathrm{Cut}(T_n^{(c)})$ converges in distribution to a Brownian Continuum Random tree, in the sense of Gromov-Hausdorff-Prokhorov. This uses crucially a general limit theorem due to Haas and Miermont [19] for so-called Markov branching trees. This has been extended in [9] to a large family of critical Galton-Watson trees with finite variance, and by Dieuleveut [14] when the offspring distribution belongs to the domain of attraction of a stable law of index $\alpha \in (1, 2]$, both in the sense of Gromov-Prokhorov. We point out that in [14] the limit is a stable random tree of index α.

On the other hand, it has been shown in [8] for a uniform random recursive tree $T_n^{(r)}$ of size n that upon rescaling the graph distance of $\mathrm{Cut}(T_n^{(r)})$ by a factor $n^{-1}\ln n$, the latter converges in probability in the sense of pointed Gromov-Hausdorff-Prokhorov distance to the unit interval $[0, 1]$ equipped with the Euclidean distance and the Lebesgue measure, and pointed at 0. The basic idea in [8] for establishing the result for uniform random recursive trees relies crucially on a coupling due to Iksanov and Möhle [22] that connects the destruction process in this family of trees with a remarkable random walk. However, this coupling is not fulfilled in general for the trees we are interested in, and we thus have to use a fairly different route.

Loosely speaking, our approach relies on the introduction of a continuous version of the cutting down procedure, where edges are equipped with i.i.d. exponential random variables and removed at a time given by the corresponding variable. Following Bertoin [6] we represent the destruction process up to a certain finite time as a Bernoulli bond-percolation, allowing us to relate the tree components with percolation clusters. We then develop the ideas in [6] used to analyze cluster sizes in supercritical percolation, and study the asymptotic behavior of the process that counts the number of edges which are remove from the root as time passed, which is closely related with the distance induced by the cut-tree.

The plan of the rest of this paper is as follows. Section 2 is devoted to the continuous-time version of the destruction procedure on a general random tree, which will play a crucial role in our analysis of the cut-tree. We then establish our main result Theorem 2 in Section 3. In Section 4, we provide some examples of trees that fulfill the hypotheses (H) and (H'). Then in Section 5 we present some applications on the isolation of multiple vertices, which extend the results of Kuba and Panholzer [27], and Baur and Bertoin [4] for uniform random recursive trees. Section 6 is devoted to the proof of a technical result about the shape of scale-free random trees, which may be of independent interest.

2 Cutting down in continuous time

The purpose of this section is to study the destruction dynamics on a general sequence of random trees T_n. We consider a continuous time version of the destruction process in which edges are removed independently one of the others at a given rate. We establish the link with Bernoulli bond-percolation and deduce some properties related to the destruction process, which will be relevant for the proof of Theorem 2.

Recall that for each fixed $k \in \mathbb{N}$, we denote by $L_{k,n}$ the length of the tree T_n reduced to k vertices chosen uniformly at random with replacement and its root 1. Recall also the Remark 1 and then consider the following weaker version of the hypothesis (H),

$$\frac{1}{\ell(n)} L_{k,n} \Rightarrow \zeta_1 + \cdots + \zeta_k, \tag{H_k}$$

where $\zeta_1, \ldots$ is a sequence of i.i.d. variables in $\mathbb{R}_+$ with no atom at 0, and the convergence in (H_k) is in the sense of one-dimensional distribution, i.e. for each fixed k. We stress that the hypothesis (H) implies (H_k) for $k = 1, 2$.

We then present the continuous time version of the destruction process. We attach to each edge e of T_n an independent exponential random variable $\mathbf{e}(e)$ of parameter $1/\ell(n)$, and we delete it at time $\mathbf{e}(e)$. After the $(n-1)$th edge has been deleted, the tree has been destructed, and the process ends. Rigorously, let $e_1, \ldots, e_{n-1}$ denote the edges of T_n listed in the increasing order of their attached exponential random variables, i.e. such that $\mathbf{e}(e_1) < \cdots < \mathbf{e}(e_{n-1})$. Then at time $\mathbf{e}(e_1)$, the first edge e_1 is removed from T_n, and T_n splits into two subtrees, say τ_n^1 and τ_n^*, where τ_n^1 contains the root 1. Next, if e_2 connects two vertices in τ_n^* then at time $\mathbf{e}(e_2)$, τ_n^* splits in two tree components. Otherwise, τ_n^1 splits in two subtrees after removing the edge e_2. We iterate in an obvious way until all the vertices of T_n have been isolated.

Define $p_n(t) = \exp(-t/\ell(n))$ for $t \geq 0$, and observe that the probability that a given edge has not yet been removed at time t in the continuous time destruction process is $p_n(t)$. Thus, the configuration observed at time t is precisely that resulting from a Bernoulli bond percolation on T_n with parameter $p_n(t)$. Further, Bertoin [5] proved that when the hypothesis (H_k) is fulfilled for $k = 1, 2$, the percolation parameter $p_n(t)$ corresponds to the supercritical regime, in the sense that with high probability, there exists a giant cluster, that is of size (number of vertices) comparable to that of the entire tree. Thus focusing on the evolution of the tree component which contains the root 1, we write $X_n(t)$ for its size at time $t \geq 0$; plainly $X_n(t) \leq n$. We shall establish the following limit theorem which is an improvement of Corollary 1 (i) in [5].

Proposition 3. *Suppose that (H_k) holds for $k = 1, 2$. Then, we have that*

$$\lim_{n\to\infty} \sup_{s\geq 0} |n^{-1}X_n(s) - \lambda(s)| = 0 \qquad \textit{in probability.} \tag{2}$$

Proof. It follows from Corollary 1(i) in [5] that for $t \geq 0$

$$\lim_{n\to\infty} n^{-1}X_n(t) = \lambda(t) \qquad \text{in probability,}$$

where $\lambda(t) = \mathbb{E}(e^{-t\zeta_1})$ for $t \geq 0$, when ever (H_k) holds for $k = 1, 2$. Then by the diagonal procedure, we may extract from an arbitrary increasing sequence of integers a subsequence, say $(n_l)_{l\in\mathbb{N}}$, such that with probability one,

$$\lim_{l\to\infty} n_l^{-1}X_{n_l}(s) = \lambda(s) \qquad \text{for all rational } s \geq 0.$$

As $s \to X_n(s)$ decreases, and $s \to \lambda(s)$ is continuous, the above convergence holds uniformly on $[0, t]$ for an arbitrary fixed $t > 0$, i.e.

$$\lim_{l\to\infty} \sup_{0\leq s\leq t} |n_l^{-1}X_{n_l}(s) - \lambda(s)| = 0 \qquad \text{a.s..} \tag{3}$$

On the other hand, we observe that $\lim_{s\to\infty}\lambda(s) = 0$. Then for $\varepsilon > 0$, we can find $t_\varepsilon > 0$ and $N(\varepsilon) > 0$ such that

$$\sup_{s>t_\varepsilon} |n_l^{-1}X_{n_l}(s) - \lambda(s)| < \varepsilon \qquad \text{for } n_l > N(\varepsilon), \quad \text{a.s.,}$$

and therefore, our claim follows by combining (3) and the above observation. $\square$

It is interesting to recall that the reciprocal of Proposition 3 holds. More precisely, Corollary 1 (ii) in [5] shows that (H_k), for $k = 1, 2$, form a necessarily and sufficient condition for (2).

In order to make the connexion with the discrete destruction process introduced at the beginning of this work, which is the one we are interested in, we now turn our attention to the number $R_n(t)$ of edges of the current root component which have been removed up to time t in the procedure described above. We observe that every jump of the process $R_n = (R_n(t) : t \geq 0)$ corresponds to removing an edge from the root component according to the discrete destruction process. We interpret the latter as a continuous time version of a random algorithm introduced by Meir and Moon [30, 31] for the isolation of the root. Recall also that

$$\Lambda(t) = \int_0^t \lambda(s)\mathrm{d}s, \qquad \text{for } t \geq 0.$$

Lemma 4. *Suppose that (H_k) holds for $k = 1, 2$, with ℓ such that $\ell(n) = o(\sqrt{n})$. Then, we have for every fixed $t > 0$*

$$\lim_{n\to\infty} \sup_{0\leq s\leq t} \left| \frac{\ell(n)}{n} R_n(s) - \Lambda(s) \right| = 0 \qquad \textit{in probability.}$$

Proof. We denote by $X_n = (X_n(t) : t \geq 0)$ the process of the size of the root cluster. The dynamics of the continuous time destruction process show that the counting process R_n grows at rate $\ell(n)^{-1}(X_n - 1)$, which means rigorously that the predictable compensator of $R_n(t)$ is absolutely continuous with respect to the Lebesgue measure with density $\ell(n)^{-1}(X_n(t) - 1)$. In other words,

$$M_n(t) = R_n(t) - \int_0^t \ell(n)^{-1}(X_n(s) - 1)\mathrm{d}s$$

is a martingale; note also that its jumps $|M_n(t) - M_n(t-)|$ have size at most 1. Since there are at most $n - 1$ jumps up to time t, the bracket of M_n can be bounded by $[M_n]_t \leq n - 1$. By Burkholder–Davis–Gundy inequality, we have that

$$\mathbb{E}[|M_n(t)|^2] \leq n - 1,$$

and in particular, since we assumed that $\ell(n) = o(\sqrt{n})$,

$$\lim_{n\to\infty} \mathbb{E}\left[\left|\frac{\ell(n)}{n} M_n(t)\right|^2\right] = 0. \tag{4}$$

On the other hand, since (H_k) holds for $k = 1, 2$, Proposition 3 and dominated convergence entail

$$\lim_{n\to\infty} \frac{\ell(n)}{n} \int_0^t \ell(n)^{-1}(X_n(s) - 1)\mathrm{d}s = \int_0^t \lambda(s)\mathrm{d}s \quad \text{in probability.}$$

Hence from (4) we have that

$$\lim_{n\to\infty} \frac{\ell(n)}{n} R_n(t) = \Lambda(t) \quad \text{in probability,}$$

and since $t \to R_n(t)$ increases, by the diagonal procedure as in the proof of Proposition 3, our claim follows. □

We continue our analysis of the destruction process, and prepare the ground for the main result of this section, which is the estimation of the number of steps in the algorithm for the isolating the root which are needed to disconnect (and not necessarily isolate) a vertex chosen uniformly at random from the root component. We start by studying the analogous quantity in continuous time. For each fixed $n \in \mathbb{N}$, we denote by $u_1, u_2, \dots$ a sequence of i.i.d. vertices in $[n] = \{1, \dots, n\}$ with the uniform distribution. Next, for every $i \in \mathbb{N}$, we write $\Gamma_i^{(n)}$ the first instant when the vertex u_i is disconnected from the root component. We shall establish the following limit theorem in law.

Proposition 5. *Suppose that (H_k) holds for $k = 1, 2$. Then as $n \to \infty$, the random vector*

$$(\Gamma_i^{(n)} : i \geq 1) \Rightarrow (\gamma_i : i \geq 1)$$

in the sense of finite-dimensional distribution, where $\gamma_1, \gamma_2, \dots$ are i.i.d. random variables in $\mathbb{R}_+$ with distribution given by $\mathbb{P}(\gamma_1 > t) = \lambda(t)$ for $t \geq 0$.

Proof. We observe that for every $j \in \mathbb{N}$ and $t_1, \dots, t_j \geq 0$, there is the identity

$$\mathbb{P}(\Gamma_1^{(n)} > t_1, \dots, \Gamma_j^{(n)} > t_j) = \mathbb{P}(u_1 \in T_n^{(1)}(t_1), \dots, u_j \in T_n^{(1)}(t_j)),$$

where $T_n^{(1)}(t)$ denotes the subtree at time t which contains the root 1. Recall that $u_1, \dots, u_j$ are i.i.d. uniformly distributed vertices, which are independent of the destruction process.

On the other hand, for $t \geq 0$ the variable $n^{-1}X_n(t)$ is the proportion of vertices in the root component at time t, and represents the conditional probability that a vertex of T_n chosen uniformly at random belongs to the root component at time t. We thus have

$$\mathbb{P}(\Gamma_1^{(n)} > t_1, \dots, \Gamma_j^{(n)} > t_j) = \mathbb{E}\left[n^{-j} \prod_{i=1}^{j} X_n(t_i)\right].$$

Since (H_k) holds for $k = 1, 2$, we conclude from Proposition 3 that

$$\lim_{n\to\infty} \mathbb{P}(\Gamma_1^{(n)} > t_1, \dots, \Gamma_j^{(n)} > t_j) = \prod_{i=1}^{j} \lambda(t_i),$$

which establishes our claim. $\square$

We are now in position to state the main result of this section. We provide a non-trivial limit in distribution for the number $Y_i^{(n)}$ of cuts (in the algorithm for isolating the root) which are needed to disconnect a vertex chosen uniformly at random, say u_i, from the root component.

Corollary 6. *Suppose that (H_k) holds for $k = 1, 2$, with ℓ such that $\ell(n) = o(\sqrt{n})$. Then as $n \to \infty$, we have that*

$$\left(\frac{\ell(n)}{n} Y_i^{(n)} : i \geq 1\right) \Rightarrow (Y_i : i \geq 1)$$

in the sense of finite-dimensional distribution, where $Y_1, Y_2, \dots$ are i.i.d. random variables on $[0, a)$ where $a = \Lambda(\infty)$, and with distribution given by

$$\mathbb{E}[f(Y_1)] = -\int_0^a f(x) \mathrm{d}\lambda \circ \Lambda^{-1}(x), \tag{5}$$

where f is a generic positive measurable function.

Proof. Recall that $R_n(t)$ denotes the number of edges of the root component which have been removed up to time t in the continuous procedure described above. We recall also that $\Gamma_i^{(n)}$ denotes the first instant when the vertex u_i, chosen uniformly at random, has been disconnected from the root component. Hence we have the following identity,

$$Y_i^{(n)} = R_n(\Gamma_i^{(n)}) \quad \text{for } i \in \mathbb{N}.$$

It follows from Lemma 4 and Proposition 5 that

$$\lim_{n\to\infty}\left(\frac{\ell(n)}{n}R_n(\Gamma_i^{(n)})-\Lambda(\Gamma_i^{(n)})\right)=0 \quad \text{in probability,}$$

and therefore, as $n\to\infty$, we have that

$$\left(\frac{\ell(n)}{n}Y_i^{(n)}:i\geq 1\right)\Rightarrow(\Lambda(\gamma_i):i\geq 1)$$

in the sense of finite-dimensional distribution, where $\gamma_1,\gamma_2,\dots$ are i.i.d. random variables in $\mathbb{R}_+$ with distribution given by $\mathbb{P}(\gamma_1>t)=\lambda(t)$. Finally, we only need to verify that the law of $\Lambda(\gamma_1)$ is given by (5). We observe that by dominated convergence λ is differentiable, and we denote by λ' its derivative. Then for f a generic positive measurable function that

$$\mathbb{E}\left[f\left(\Lambda(\gamma_1)\right)\right] \;=\; -\int_0^\infty f\left(\Lambda(x)\right)\lambda'(x)\mathrm{d}x.$$

On the other hand, we observe that Λ is an increasing continuous and differentiable function whose derivative is never 0. Hence

$$\begin{aligned}\mathbb{E}\left[f\left(\Lambda(\gamma_1)\right)\right] &= -\int_0^{\Lambda(\infty)} f\left(x\right)\frac{\lambda'\circ\Lambda^{-1}(x)}{\lambda\circ\Lambda^{-1}(x)}\mathrm{d}x\\ &= -\int_0^{\Lambda(\infty)} f\left(x\right)\mathrm{d}\lambda\circ\Lambda^{-1}(x),\end{aligned}$$

which completes the proof. □

Corollary 6 will have a crucial role in the proof of Theorem 2. This result will enable us to get a precise estimate of distances in the cut-tree.

Finally, let $N^{(u)}(n)$ be the number of remaining cuts that is needed to isolate a vertex chosen uniformly at random, say u, once it has been disconnected from the root component. The next proposition establishes a criterion which ensures that $N^{(u)}(n)$ is small compared to $n/\ell(n)$ with high probability. This technical ingredient will be useful later on in the proof of Theorem 2.

Proposition 7. *Assume that (H) and (H′) hold with ℓ such that $\ell(n)=o(\sqrt{n})$. Then we have*

$$\lim_{n\to\infty}\frac{\ell(n)}{n}N^{(u)}(n)=0 \quad \textit{in probability.}$$

Proof. We write $R_n^{(u)}(t)$ for the number of edges that have been removed up to time t from the tree component containing the vertex u, and Γ_n the first instant when the vertex u has been disconnected from the root cluster; in particular,

$$\lim_{t\to\infty} R_n^{(u)}(\Gamma_n+t)-R_n^{(u)}(\Gamma_n)=N^{(u)}(n).$$

Let $X_n^{(u)}(t)$ be the size of the subtree containing the vertex u at time t. Since each edge is removed with rate $\ell(n)^{-1}$, independently of the other edges, the process

$$M_n^{(u)}(t)=R_n^{(u)}(\Gamma_n+t)-R_n^{(u)}(\Gamma_n)-\int_0^t \ell(n)^{-1}(X_n^{(u)}(\Gamma_n+s)-1)\mathrm{d}s, \quad t\geq 0,$$

is a purely discontinuous martingale with terminal value

$$\lim_{t\to\infty} M_n^{(u)}(t)=N^{(u)}(n)-\int_0^\infty \ell(n)^{-1}(X_n^{(u)}(\Gamma_n+s)-1)\mathrm{d}s.$$

Further, its bracket can be bounded by $[M_n^{(u)}]_t\leq n-1$. Then since we assume that $\ell(n)=o(\sqrt{n})$,

$$\lim_{n\to\infty}\mathbb{E}\left[\left|\frac{\ell(n)}{n}N^{(u)}(n)-\int_0^\infty n^{-1}(X_n^{(u)}(\Gamma_n+s)-1)\mathrm{d}s\right|^2\right]=0.$$

Therefore, it only remains to prove that

$$\lim_{n\to\infty}\mathbb{E}\left[\int_0^\infty n^{-1}(X_n^{(u)}(\Gamma_n+s)-1)\mathrm{d}s\right]=0. \tag{6}$$

Let $T_n^{(u)}(s)$ denote the subtree at time s which contains the vertex u. We observe that

$$\begin{aligned}\mathbb{E}\left[\int_0^\infty n^{-1}(X_n^{(u)}(\Gamma_n+s)-1)\mathrm{d}s\right] &= \mathbb{E}\left[\int_0^\infty n^{-1}(X_n^{(u)}(s)-1)\mathbf{1}_{\{\Gamma_n\leq s\}}\mathrm{d}s\right]\\ &= \mathbb{E}\left[\int_0^\infty n^{-1}(X_n^{(u)}(s)-1)\mathbf{1}_{\left\{1\notin T_n^{(u)}(s)\right\}}\mathrm{d}s\right].\end{aligned}$$

We note that a vertex v chosen uniformly at random in $[n]$ and independent of u belong to the same cluster at time t if and only if no edge on the path form u and v has been removed at time t. Recall that the probability that a given edge has not yet been removed at time t is $\exp(-t/\ell(n))$ in the continuous time destruction process. Recall that d_n denotes the graph distance in T_n, and $u \wedge v$ the last common ancestor of u and v. Then, we have that

$$\begin{aligned}\mathbb{E}\left[n^{-1}(X_n^{(u)}(t)-1)\mathbf{1}_{\{1\notin T^{(u)}(t)\}}\right] &= n^{-1}\mathbb{E}\left[\sum_{i\in[n]\setminus u}\mathbf{1}_{\left\{i\in T_n^{(u)}(t),\, 1\notin T_n^{(u)}(t)\right\}}\right] \\ &= \mathbb{E}\left[\left(e^{-\frac{d_n(u,v)}{\ell(n)}t}-e^{-\frac{L_{2,n}}{\ell(n)}t}\right)\mathbf{1}_{\{v\neq u\}}\right],\end{aligned}$$

where $L_{2,n}$ is the length of the tree T_n reduced to the vertex u, v and its root. Then,

$$\mathbb{E}\left[\int_0^\infty n^{-1}(X_n^{(u)}(\Gamma_n+s)-1)\mathrm{d}s\right] = \mathbb{E}\left[\left(\frac{\ell(n)}{d_n(u,v)}-\frac{\ell(n)}{L_{2,n}}\right)\mathbf{1}_{\{v\neq u\}}\right]. \tag{7}$$

On the other hand, since

$$\frac{\ell(n)}{L_{2,n}}\mathbf{1}_{\{v\neq u\}} \leq \frac{\ell(n)}{d_n(u,v)}\mathbf{1}_{\{v\neq u\}},$$

it is not difficult to see from Remark 1 that the assumption (H') implies that

$$\lim_{n\to\infty}\mathbb{E}\left[\frac{\ell(n)}{L_{2,n}}\mathbf{1}_{\{v\neq u\}}\right] = \mathbb{E}\left[\frac{1}{\zeta_1+\zeta_2}\right] < \infty.$$

Therefore, we get (6) by letting $n\to\infty$ in (7). □

3 Proof of Theorem 2

In this section, we prove our main result, Theorem 2. We stress that during the proof we consider that the tree T_n is a deterministic tree. This will clearly imply the result for random trees. In this direction, we recall that we view the $\mathrm{Cut}(T_n)$ as the pointed metric measure space $([n]^0, \delta_n, 0, \mu_n)$, where 0 corresponds to the root and $1, \ldots, n$ to the leaves, δ_n the graph distance induced by the cut-tree, and μ_n the uniform probability measure on $[n]$ with $\mu_n(0) = 0$. We assume that $a = \Lambda(\infty) < \infty$. We then recall that I_μ denotes the pointed measure metric space given by the interval $[0, a]$, pointed at 0, equipped with the Euclidean distance, and the probability measure μ given in (1), i.e.

$$\int_0^a f(x)\mu(\mathrm{d}x) = -\int_0^a f(x)\,\mathrm{d}\lambda\circ\Lambda^{-1}(x),$$

where f is a generic positive measurable function. We stress that in the case $a=\infty$ the proof follows along the same lines as that of $a<\infty$. Then, I_μ denotes the pointed measure metric space given by the interval $[0,\infty)$, pointed at 0, equipped with the Euclidean distance and the measure μ.

We recall that to establish weak convergence in the sense induced by the Gromov-Prokhorov topology, we shall prove the convergence in distribution of the rescaled distances of $\mathrm{Cut}(T_n)$. Specifically, for every $n \in \mathbb{N}$, set $\xi_n(0) = 0$ and consider a sequence $(\xi_n(i))_{i\geq 1}$ of i.i.d. random variables with law μ_n. We will prove that

$$\left(\frac{\ell(n)}{n}\delta_n(\xi_n(i),\xi_n(j)) : i,j \geq 0\right) \Rightarrow (\delta(\xi(i),\xi(j)) : i,j \geq 0)$$

in the sense of finite-dimensional distribution, where $\xi(0) = 0$ and $(\xi(i))_{i\geq 1}$ is a sequence of i.i.d. random variables on $\mathbb{R}_+$ with law μ. Furthermore, $\delta(\xi(i),\xi(j)) = |\xi(i) - \xi(j)|$ since δ is the Euclidean distance, and in particular, $\delta(0,\xi(i)) = \xi(i)$.

The key idea of the proof relies in the relationship between the distance in $\mathrm{Cut}(T_n)$, and the number of cuts needed to disconnect certain number of vertices in T_n. Indeed, the height of the leaf $\{i\}$ in $\mathrm{Cut}(T_n)$ is precisely the number of cuts needed to isolate the vertex i in T_n. Therefore, it will be convenient to think in $(\xi_n(i))_{i\geq 1}$ as a sequence of i.i.d. vertices in $[n]$, with the uniform distribution.

Proof of Theorem 2. We observe that for $i \geq 1$,

$$\delta_n(\xi_n(0),\xi_n(i)) = \delta_n(0,\xi_n(i))$$

is precisely the number of cuts which are needed to isolate the vertex $\xi_n(i)$. For each $n \in \mathbb{N}$, we denote by $\delta_n^{(1)}(0,\xi_n(i))$ the number of cuts which are needed to disconnect the vertex $\xi_n(i)$ from the root component, and by $\eta(\xi_n(i))$ the remaining number of cuts which are needed to isolate the vertex $\xi_n(i)$ after it has been disconnected. Clearly, we have

$$\delta_n(0,\xi_n(i)) - \delta_n^{(1)}(0,\xi_n(i)) = \eta(\xi_n(i)).$$

Since the condition (H') holds, Proposition 7 implies that $\lim_{n\to\infty} n^{-1}\ell(n)\eta(\xi_n(i)) = 0$ in probability for $i \geq 1$. Therefore, the assumption (H) entails according to Corollary 6 that

$$\left(\frac{\ell(n)}{n}\delta_n(0,\xi_n(i)) : i \geq 0\right) \Rightarrow (\xi(i) : i \geq 0)$$

in the sense of finite-dimensional distribution. Essentially, we follow the same argument to show that the preceding also holds jointly with

$$\left(\frac{\ell(n)}{n}\delta_n(\xi_n(i),\xi_n(j)) : i,j\geq 1\right) \Rightarrow (\delta(\xi(i),\xi(j)) : i,j\geq 1) \tag{8}$$

which is precisely our statement.

In this direction, for $i,j\geq 1$, we denote by $\delta_n^{(2)}(\xi_n(i),\xi_n(j))$ the number of cuts which are needed to isolate the vertices $\xi_n(i)$ and $\xi_n(j)$. We also write $\delta_n^{(3)}(\xi_n(i),\xi_n(j))$ for the number of cuts (in the algorithm for isolating the root) until for the first time, the vertices $\xi_n(i)$ and $\xi_n(j)$ are disconnected. Hence from the description of the cut-tree, it should be plain that

$$\delta_n(\xi_n(i),\xi_n(j)) = (\delta_n^{(2)}(\xi_n(i),\xi_n(j))+1) - (\delta_n^{(3)}(\xi_n(i),\xi_n(j))-1). \tag{9}$$

Next we observe that

$$\delta_n^{(3)}(\xi_n(i),\xi_n(j)) - \min(\delta_n^{(1)}(0,\xi_n(i)),\delta_n^{(1)}(0,\xi_n(j))) \leq \eta(\xi_n(i))+\eta(\xi_n(j)),$$

and

$$\delta_n^{(2)}(\xi_n(i),\xi_n(j)) - \max(\delta_n^{(1)}(0,\xi_n(i)),\delta_n^{(1)}(0,\xi_n(j))) \leq \eta(\xi_n(i))+\eta(\xi_n(j)).$$

Since the assumption (H) and (H') hold, it follows from Proposition 7 that

$$\lim_{n\to\infty}\frac{\ell(n)}{n}\left(\eta(\xi_n(i))+\eta(\xi_n(j))\right) = 0 \qquad \text{in probability.}$$

Moreover, Corollary 6 implies that

$$\left(\frac{\ell(n)}{n}\delta_n^{(3)}(\xi_n(i),\xi_n(j)) : i,j\geq 1\right) \Rightarrow \left(\min(\xi(i),\xi(j)) : i,j\geq 1\right),$$

and

$$\left(\frac{\ell(n)}{n}\delta_n^{(2)}(\xi_n(i),\xi_n(j)) : i,j\geq 1\right) \Rightarrow \left(\max(\xi(i),\xi(j)) : i,j\geq 1\right)$$

hold jointly. Therefore, since δ is the Euclidean distance, the convergence in (8) follows from the identity (9). $\square$

4 Examples

In this section, we present some examples of trees that fulfilled the conditions of Theorem 2. But first, we observe that when the hypotheses of the latter are satisfied with $\zeta_1 \equiv 1$, the probability measure μ given in (1) corresponds to the Lebesgue measure on the unit interval $[0, 1]$. The above follows from the fact that $\lambda(t) = e^{-t}$ for all $t \geq 0$. Then we have the following interesting consequence of Theorem 2.

Corollary 8. *Suppose that (H) and (H′) hold, with $\zeta_1 \equiv 1$ and ℓ such that $\ell(n) = o(\sqrt{n})$. Then as $n \to \infty$, we have the following convergence in the sense of the pointed Gromov-Prokhorov topology:*

$$\frac{\ell(n)}{n}\mathrm{Cut}(T_n) \Rightarrow I_1.$$

where I_1 is the pointed measure metric space given by the unit interval $[0, 1]$, pointed at 0, equipped with the Euclidean distance and the Lebesgue measure.

A natural example is the class of random trees with logarithmic heights, i.e. which fulfill hypothesis (H) with $\ell(n) = c \ln n$ for some $c > 0$, such as binary search trees, regular trees, uniform random recursive trees, and more generally scale-free random trees. We are now going to prove that (H') is also satisfied for the previous families of trees and therefore their rescaled cut-tree converges in the sense of Gromov-Prokhorov topology to I_1.

1. Binary search trees. A popular family of random trees used in computer science for sorting and searching data is the binary search tree. More precisely, a binary search tree is a binary tree in which each vertex is associated to a key, where the keys are drawn randomly from an ordered set, we say $\{1, \ldots, n\}$, until the set is exhausted. The first key is associated to the root. The next key is placed at the left child of the root if it is smaller than the root's key and placed to the right if it is larger. Then one proceeds progressively, inserting key by key. When all the keys are placed one gets a binary tree with n vertices. For further details, see e.g. [15]. Theorem S1 in Devroye [12] shows that the hypothesis (H) holds with $\ell(n) = 2 \ln n$. Hence in order to be in the framework of Corollary 8 all that we need is to check that this family of trees fulfills the hypothesis (H'), namely

$$\lim_{n\to\infty} \mathbb{E}\left[\frac{2\ln n}{d_n(u, v)}\mathbf{1}_{\{u\neq v\}}\right] = \frac{1}{2},$$

where u and v are two vertices chosen uniformly at random with replacement from the binary search tree of size n. In this direction, we pick $0 < \varepsilon < (2\ln 2)^{-1}$ and consider the function

ϕ_ε given by $\phi_\varepsilon = 0$ on $[0,\varepsilon]$, $\phi_\varepsilon = 1$ on $[2\varepsilon,\infty)$, and ϕ_ε linear on $[\varepsilon, 2\varepsilon]$. We observe that

$$\lim_{n\to\infty} \mathbb{E}\left[\frac{2\ln n}{d_n(u,v)}\phi_\varepsilon\left(\frac{d_n(u,v)}{2\ln n}\right)\mathbf{1}_{\{u\neq v\}}\right] = \frac{1}{2}\phi_\varepsilon\left(\frac{1}{2}\right).$$

Further, we note that $\phi_\varepsilon(1/2) \to 1$ as $\varepsilon \to 0$. Then, it is enough to prove that

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\mathbb{E}\left[\left(\frac{2\ln n}{d_n(u,v)} - \frac{2\ln n}{d_n(u,v)}\phi_\varepsilon\left(\frac{d_n(u,v)}{2\ln n}\right)\right)\mathbf{1}_{\{u\neq v\}}\right] = 0, \tag{10}$$

in order to show (H'). We write $X^i(n,k)$ for the number of vertices at distance $k \geq 1$ from the vertex i in a binary search tree of size n. Then

$$\begin{aligned}\mathbb{E}\left[\left(\frac{2\ln n}{d_n(u,v)} - \frac{2\ln n}{d_n(u,v)}\phi_\varepsilon\left(\frac{d_n(u,v)}{2\ln n}\right)\right)\mathbf{1}_{\{u\neq v\}}\right] &\leq \mathbb{E}\left[\frac{2\ln n}{d_n(u,v)}\mathbf{1}_{\{d_n(u,v)\leq 2\varepsilon\ln n,\, u\neq v\}}\right] \\ &\leq \frac{2\ln n}{n^2}\sum_{i=1}^{n}\sum_{k=1}^{\lfloor 2\varepsilon\ln n\rfloor}\frac{1}{k}\mathbb{E}[X^i(n,k)].\end{aligned}$$

Since each vertex in a binary search tree has at most two descendants, we observe that $\mathbb{E}[X^i(n,k)] \leq 3\cdot 2^{k-1}$. Then

$$\begin{aligned}\mathbb{E}\left[\left(\frac{2\ln n}{d_n(u,v)} - \frac{2\ln n}{d_n(u,v)}\phi_\varepsilon\left(\frac{d_n(u,v)}{2\ln n}\right)\right)\mathbf{1}_{\{u\neq v\}}\right] &\leq \frac{3\ln n}{n}\sum_{k=1}^{\lfloor 2\varepsilon\ln n\rfloor}\frac{2^k}{k} \\ &\leq \frac{6\ln n}{n}2^{2\varepsilon\ln n},\end{aligned}$$

and therefore we get (10) by letting $n\to\infty$ and $\varepsilon\to 0$.

More generally, one can consider a generalization of the binary search trees, namely the b-ary recursive trees and check that these fulfill the conditions of Corollary 8 with $\ell(n) = \frac{b}{b-1}\ln n$; we refer to Devroye [13].

2. Scale free random trees. The scale-free random trees form a family of random trees that grow following a preferential attachment algorithm, and are used commonly to model complex real-word networks; see Barabási and Albert [2]. Specifically, fix a parameter $\alpha \in (-1,\infty)$, and start for $n = 1$ from the tree $T_1^{(\alpha)}$ on $\{1,2\}$ which has a single edge connecting 1 and 2. Suppose that $T_n^{(\alpha)}$ has been constructed for some $n\geq 2$, and for every $i\in\{1,\ldots,n+1\}$, denote by $g_n(i)$ the degree of the vertex i in $T_n^{(\alpha)}$. Then conditionally given $T_n^{(\alpha)}$, the tree $T_{n+1}^{(\alpha)}$ is built by adding an edge between the new vertex $n+2$ and a vertex v_n

in $T_n^{(\alpha)}$ chosen at random according to the law

$$\mathbb{P}\left(v_n = i | T_n^{(\alpha)}\right) = \frac{g_n(i) + \alpha}{2n + \alpha(n+1)}, \qquad i \in \{1, \ldots, n+1\}.$$

We observe that when one lets $\alpha \to \infty$ the algorithm yields an uniform recursive tree. It is not difficult to check that the condition (H) in Corollary 8 is fulfilled with $\ell(n) = \frac{1+\alpha}{2+\alpha} \ln n$; see for instance [10]. Then, it only remains to check the hypothesis (H'). We only prove the latter when $\alpha = 0$, the general case follows similarly but with longer computations. We then follow the same route as the case of the binary search trees. Pick $\varepsilon > 0$ and consider the same function ϕ_ε that we defined previously. Therefore, it is enough to show that

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \mathbb{E}\left[\left(\frac{\ln n}{2d_n(u,v)} - \frac{\ln n}{2d_n(u,v)} \phi_\varepsilon\left(\frac{2d_n(u,v)}{\ln n}\right)\right) \mathbf{1}_{\{u \neq v\}}\right] = 0,$$

where u and v are two independent uniformly distributed random vertices on $T_n^{(0)}$. We observe that

$$\mathbb{E}\left[\left(\frac{\ln n}{2d_n(u,v)} - \frac{\ln n}{2d_n(u,v)} \phi_\varepsilon\left(\frac{2d_n(u,v)}{\ln n}\right)\right) \mathbf{1}_{\{u \neq v\}}\right] \leq \mathbb{E}\left[\frac{\ln n}{2d_n(u,v)} \mathbf{1}_{\{d_n(u,v) \leq \frac{1}{2}\varepsilon \ln n, \, u \neq v\}}\right]. \quad (11)$$

We write $Z^i(n,k)$ for the number of vertices at distance $k \geq 1$ from the vertex i. Then,

$$\begin{aligned} \mathbb{E}\left[\frac{\ln n}{2d_n(u,v)} \mathbf{1}_{\{u \neq v\}} \mathbf{1}_{\{d_n(u,v) \leq \frac{1}{2}\varepsilon \ln n\}}\right] &\leq \frac{\ln n}{n^2} \sum_{i=1}^{n+1} \sum_{k=1}^{\lfloor \frac{1}{2}\varepsilon \ln n \rfloor} \frac{1}{k} \mathbb{E}[Z^i(n,k)] \\ &\leq \frac{\ln n}{n^2} z^{-\frac{1}{2}\varepsilon \ln n} \sum_{i=1}^{n+1} \mathbb{E}[G_n^i(z)], \end{aligned}$$

for $z \in (0,1)$, where $G_n^i(z) = \sum_{k=0}^{\infty} z^k Z^i(n, k+1)$. We claim the following.

Lemma 9. *There exists $z_0 \in (0,1)$ such that we have that*

$$\mathbb{E}[G_n^i(z_0)] \leq e^{\frac{1+z_0}{2}} n^{\frac{1+z_0}{2}}, \qquad \textit{for } i \geq 1 \textit{ and } n \geq 1.$$

The proof of the above lemma relies in the recursive structure of the scale-free random tree and for now it is convenient to postpone its proof to Section 6. We then consider z_0 such that the result of Lemma 9 holds and $0 < \varepsilon < (z_0 - 1)(\ln z_0)^{-1}$. Then

$$\mathbb{E}\left[\frac{\ln n}{2d_n(u,v)} \mathbf{1}_{\{u \neq v\}} \mathbf{1}_{\{d_n(u,v) \leq \frac{1}{2}\varepsilon \ln n\}}\right] \leq e^{\frac{1+z_0}{2}} z_0^{-\frac{1}{2}\varepsilon \ln n} n^{-\frac{1-z_0}{2}} \ln n$$

and therefore, the right-hand side in (11) tends to 0 as $n \to \infty$.

Similarly, one can easily check that the uniform random recursive trees fulfill the hypotheses of Corollary 8 with $\ell(n) = \ln n$; see Chapter 6 in [15].

3. Merging of regular trees. Our next example provides a method to build trees that fulfill the conditions of Theorem 2 and where the random variable ζ_1 in hypothesis (H) is not a constant. Basically, the procedure consists on gluing trees which satisfy the assumptions of Corollary 8. In this example, we consider a mixture of regular trees but one may consider other families of trees as well. For a fixed integer $r \geq 1$, let $(d_i)_{i=1}^r$ denote a positive sequence of integers. Next, for $i = 1, \dots, r$, let $h_i(m) : \mathbb{R}_+ \to \mathbb{R}_+$ be a function with $\lim_{m\to\infty} h_i(m) = \infty$. Moreover, we assume that

$$d_1^{h_1(m)} \sim d_2^{h_2(m)} \sim \cdots \sim d_r^{h_r(m)},$$

when $m \to \infty$. Then, let $T_{n_i}^{(d_i)}$ be a complete d_i-regular tree with height $\lfloor h_i(m) \rfloor$. Since there are d_i^j vertices at distance $j = 0, 1, \dots, \lfloor h_i(m) \rfloor$ from the root, its size is given by

$$n_i = n_i(m) = d_i(d_i^{\lfloor h_i(m) \rfloor} - 1)/(d_i - 1).$$

In particular, one can check that the assumptions in Theorem 2 are fulfilled with $\ell(n_i) = \ln n_i$. We now imagine that we merge all the r regular trees into one common root which leads us to a new tree $T_n^{(d)}$ of size $n = \sum_{i=1}^r n_i + 1 - r$. Then, we observe that the probability that a vertex of $T_n^{(d)}$ chosen uniformly at random belongs to the tree $T_{n_i}^{(d_i)}$ converges when $m \to \infty$ to $1/r$. Then, one readily checks that this new tree satisfies the hypothesis (H) with $\ell(n) = \ln n$ and ζ_i a random variable uniformly distributed in the set $\{1/\ln d_1, \dots, 1/\ln d_r\}$. Furthermore, since the number of descendants of each vertex is bounded, it is not difficult to see that also fulfills the condition (H'). Therefore, Theorem 2 implies that $n^{-1} \ln n \, \mathrm{Cut}(T_n^{(d)})$ converges in distribution in the sense of pointed Gromov-Prokhorov to the element $I_{\mu^{(d)}}$ of $\mathbb{M}$, which corresponds to the interval $[0, a)$, pointed at 0, equipped with the Euclidean distance, and the probability measure $\mu^{(d)}$ given by (1) with $\lambda(t) = \frac{1}{r} \sum_{i=1}^r e^{-\frac{t}{\ln d_i}}$ for $t \geq 0$.

5 Applications

We now present a consequence of Theorem 2 which generalizes a result of Kuba and Panholzer [27], and its recent multi-dimensional extension shown by Baur and Bertoin [4] on the isolation of multiple vertices in uniform random recursive trees. Let $u_1, u_2, \dots$ denote a sequence of i.i.d. uniform random variables in $[n] = \{1, \ldots, n\}$. We write $Z_{n,j}$ for the number of cuts which are needed to isolate $u_1, \ldots, u_j$ in T_n. We have the following convergence which extends Corollary 4 in [4].

Corollary 10. *Suppose that (H) and (H′) hold with ℓ such that $\ell(n) = o(\sqrt{n})$. Then as $n \to \infty$, we have that*

$$\left(\frac{\ell(n)}{n} Z_{n,j} : j \geq 1 \right) \Rightarrow (\max(U_1, U_2, \ldots, U_j) : j \geq 1)$$

in the sense of finite-dimensional distributions, where $U_1, U_2, \ldots$ is a sequence of i.i.d. random variables with law μ given in (1).

Proof. For a fixed integer $j \geq 1$, $u_1, \ldots, u_j$ are j independent uniform vertices of T_n, or equivalently, the singletons $\{u_1\}, \ldots, \{u_j\}$ form a sequence of j i.i.d. leaves of $\mathrm{Cut}(T_n)$ distributed according the uniform law. Denote by $\mathcal{R}_{n,j}$ the subtree of $\mathrm{Cut}(T_n)$ spanned by its root and j i.i.d. leaves chosen according to the uniform distribution on $[n]$. Similarly, write $\mathcal{R}_j$ for the subtree of I_μ spanned by 0 and j i.i.d. random variables with law μ, say $U_1, \ldots, U_j$. We adopt the framework of Aldous [1], and see both reduced trees as combinatorial trees structure with edge lengths. Therefore, Theorem 2 entails that $n^{-1}\ell(n)\mathcal{R}_{n,j}$ converges weakly in the sense of Gromov-Prokhorov to $\mathcal{R}_j$ as $n \to \infty$. In particular, we have the convergence of the lengths of those reduced trees,

$$\left(\frac{\ell(n)}{n} |\mathcal{R}_{n,1}|, \ldots, \frac{\ell(n)}{n} |\mathcal{R}_{n,j}| \right) \Rightarrow (|\mathcal{R}_1|, \ldots, |\mathcal{R}_j|) \,.$$

It is sufficient to observe that $|\mathcal{R}_j| = \max(U_1, \ldots, U_j)$. □

In particular, when the hypotheses (H) and (H') hold with $\zeta_1 \equiv 1$, we observe from Corollary 8 that the variables $U_1, U_2, \ldots$ have the uniform distribution on $[0, 1]$, and moreover, $\frac{\ell(n)}{n} Z_{n,j}$ converges in distribution to a beta$(j, 1)$ random variable.

As another application, for $j \geq 2$ we consider the algorithm for isolating the vertices $u_1, \ldots, u_j$ with a slight modification, we discard the emerging tree components which contain at most one of these j vertices. We stop the algorithm when the j vertices are totally disconnected from each other, i.e. lie in j different tree components. We write $W_{n,2}$ for the number of steps of this algorithm until for the first time $u_1, \ldots, u_j$ do not longer belong to the same tree component, moreover $W_{n,3}$ for the number of steps until the first time, the j vertices are spread out over three distinct tree components, and so on, up to $W_{n,j}$, the number of steps until the j vertices are totally disconnected. We have the following consequence of Corollary 3, which extends Corollary 4 in [4].

Corollary 11. *Suppose that (H) and (H′) hold with ℓ such that $\ell(n) = o(\sqrt{n})$. Then as $n \to \infty$, we have that*

$$\left(\frac{\ell(n)}{n} W_{n,2}, \ldots, \frac{\ell(n)}{n} W_{n,j}\right) \Rightarrow \left(U_{(1,j)}, \ldots, U_{(j-1,j)}\right),$$

where $U_{(1,j)} \leq U_{(2,j)} \leq \cdots \leq U_{(j-1,j)}$ denote the first $j-1$ order statistics of an i.i.d. sequence $U_1, \ldots, U_j$ of random variables with law μ given in (1).

Proof. Recall the notation of Corollary 3, and write $Y_i^{(n)}$ for the number of cuts which are needed to disconnect the vertex u_i from the root component. We then observe that if we write $Y_{1,j}^{(n)} \leq Y_{2,j}^{(n)} \leq \cdots \leq Y_{j-1,j}^{(n)}$ for the first order statistics of the sequence of random variables $Y_1^{(n)}, \ldots, Y_j^{(n)}$, it follows from Proposition 7 that

$$\lim_{n\to\infty} \frac{\ell(n)}{n} (W_{n,i} - Y_{i-1,j}^{(n)}) = 0 \qquad \text{in probability.}$$

Therefore, our claim follows immediately from Corollary 3. □

As before, when (H) and (H') hold with $\zeta_1 \equiv 1$, the variables $U_1, U_2, \ldots$ have the uniform distribution on $[0,1]$, and then, $\frac{\ell(n)}{n} W_{n,j}$ converges in distribution to a beta$(1,j)$ random variable, and $\frac{\ell(n)}{n} W_{n,j}$ converges in distribution to a beta$(j-1,2)$ law.

6 Proof of Lemma 9

The purpose of this final section is to establish Lemma 9. The proof relies on the recursive structure of the scale-free random trees, and our guiding line is similar to that in [25] and [26]. We recall that we only consider the case when the parameter α of the scale-free random tree is zero, but that the general case can be treated similarly.

Recall that the construction of the scale-free tree starts at $n = 1$ from the tree $T_1^{(0)}$ on $\{1,2\}$ which has a single edge connecting 1 and 2. Suppose that $T_n^{(0)}$ has been constructed for some $n \geq 2$, then conditionally given $T_n^{(0)}$, the tree $T_{n+1}^{(0)}$ is built by adding an edge between the new vertex $n+2$ and a vertex v_n in $T_n^{(0)}$ chosen at random according to the law

$$\mathbb{P}\left(v_n = i | T_n^{(0)}\right) = \frac{g_n(i)}{2n}, \quad i \in \{1, \ldots, n+1\}.$$

where $g_n(i)$ denotes the degree of the vertex i in $T_n^{(0)}$. Let $Z^i(n,k)$ denote the number of vertices at distance $k \geq 0$ from the vertex i after the n-th step. We are interested in the expectation of the generating function

$$G_n^i(z) = \sum_{k=0}^{\infty} Z^i(n,k+1)z^k, \quad n \geq 1,$$

for $z \in (0,1)$. In particular, $G_n^1(\cdot)$ is the so-called height profile function; see Katona [25, 26] for several results related to this function. To compute $\mathbb{E}[G_n^i(z)]$ we use the evolution process of the construction of $T_n^{(0)}$ and conditional expectation. Let $\mathcal{F}_n$ denote the σ-field generated by the first n steps in the procedure. The number of vertices at distance k from i increases by one or does not change. Then for $n \geq i-1$,

$$\mathbb{E}[Z^i(n+1,1)|\mathcal{F}_n] = (Z^i(n,1)+1)\frac{Z^i(n,1)}{2n} + Z^i(n,1)\left(1 - \frac{Z^i(n,1)}{2n}\right) = \frac{2n+1}{2n}Z^i(n,1),$$

and for $k > 1$ we have

$$\begin{aligned}
&\mathbb{E}[Z^i(n+1,k)|\mathcal{F}_n] \\
&\quad = (Z^i(n,k)+1)\frac{Z^i(n,k)+Z^i(n,k-1)}{2n} + Z^i(n,k)\left(1 - \frac{Z^i(n,k)+Z^i(n,k-1)}{2n}\right) \\
&\quad = \frac{2n+1}{2n}Z^i(n,k) + \frac{1}{2n}Z^i(n,k-1),
\end{aligned}$$

where $Z^1(0,k) = 0$ and $Z^i(i-2,k) = 0$ for $2 \leq i \leq n+1$. Taking the expectation this leads to the recurrence relation

$$\mathbb{E}[G_{n+1}^i(z)] = \frac{2n+1+z}{2n}\mathbb{E}[G_n^i(z)].$$

Since $G_1^1(z) = G_1^2(z) = 1$, the above recursive formula leads to

$$\mathbb{E}[G_n^1(z)] = \mathbb{E}[G_n^2(z)] = \prod_{j=1}^{n-1} \frac{2j+1+z}{2j}, \tag{12}$$

and for $3 \leq i \leq n+1$

$$\mathbb{E}[G_n^i(z)] = \left(\prod_{j=i-1}^{n-1} \frac{2j+1+z}{2j}\right)\mathbb{E}[G_{i-1}^i(z)]. \tag{13}$$

with the convention that $\prod_{j=n}^{n-1} \frac{2j+1+z}{2j} = 1$. We point out that $G_n^i(z) = 0$ for $n \leq i-2$. We have the following technical result which will be crucial in the proof of Lemma 9.

Lemma 12. *For $2 \le i \le n$, we have that*

$$\begin{aligned}
&\mathbb{E}[G_n^i(z)Z^i(n,1)] \\
&\quad = \left(\prod_{j=i-1}^{n-1} \frac{2j+2+z}{2j}\right)\mathbb{E}[G_{i-1}^i(z)] + \sum_{k=i-1}^{n-1}\left(\prod_{j=k+1}^{n-1}\frac{2j+2+z}{2j}\right)\frac{1}{2k}\mathbb{E}[Z^i(k,1)],
\end{aligned}$$

and

$$\mathbb{E}[G_n^1(z)Z^1(n,1)] = \prod_{j=1}^{n-1}\frac{2j+2+z}{2j} + \sum_{k=1}^{n-1}\left(\prod_{j=k}^{n-1}\frac{2j+2+z}{2j}\right)\frac{1}{2k}\mathbb{E}[Z^1(k,1)].$$

Proof. We only prove the case when $2 \le i \le n$, the case $i = 1$ follows exactly by the same argument. For $n \ge i-1 \ge 1$, we observe that $G_{n+1}^i(z) = G_n^i(z) + K_n^i(z)$ where

$$\mathbb{P}(K_n^i(z) = z^{k-1}|\mathcal{F}_n) = \begin{cases} \frac{Z^i(n,k)+Z^i(n,k-1)}{2n} & k > 1 \\ \frac{Z^i(n,1)}{2n} & k = 1, \end{cases}$$

and $Z^i(n+1,1) = Z^i(n,1) + B_n^i$ where

$$\mathbb{P}(B_n^i = 1|\mathcal{F}_n) = 1 - \mathbb{P}(B_n^i = 0|\mathcal{F}_n) = \frac{Z^i(n,1)}{2n}.$$

This yields

$$\mathbb{E}(K_n^i(z)|\mathcal{F}_n) = \frac{1+z}{2n}G_n^i(z), \quad \text{and} \quad \mathbb{E}(B_n^i|\mathcal{F}_n) = \mathbb{E}(K_n^i(z)B_n^i|\mathcal{F}_n) = \frac{Z^i(n,1)}{2n}.$$

Then, it follows that

$$\begin{aligned}
\mathbb{E}[G_{n+1}^i(z)Z^i(n+1,1)] &= \mathbb{E}[(G_n^i(z)+K_n^i(z))(Z^i(n,1)+B_n^i)|\mathcal{F}_n] \\
&= \frac{2n+2+z}{2n}\mathbb{E}[G_n^i(z)Z^i(n,1)] + \frac{1}{2n}\mathbb{E}[Z^i(n,1)].
\end{aligned}$$

Since $Z^i(i-1,1) = 1$, this recursive formula yields to our result. $\square$

Next, we observe that for $1 \le i \le n+1$ the variable $Z^i(n,1)$ is the degree of the vertex i after the n-step, which first moment is given by (see [32])

$$\mathbb{E}[Z^1(n,1)] = \prod_{j=1}^{n-1}\frac{2j+1}{2j}, \quad \text{and} \quad \mathbb{E}[Z^i(n,1)] = \prod_{j=i-1}^{n-1}\frac{2j+1}{2j}, \quad \text{for } 2 \le i \le n+1 \tag{14}$$

with the convention that $\prod_{j=n}^{n-1} \frac{2j+1}{2j} = 1$.

We recall some technical results that will be useful later on. We have the following well-known inequality,

$$1 + x \le e^x, \qquad x \in \mathbb{R}. \tag{15}$$

Then, we can easily deduce that

$$\prod_{j=i-1}^{n-1} \frac{2j+2+z}{2j} \le e^{\frac{2+z}{2}} \left(\frac{n-1}{i-1}\right)^{\frac{2+z}{2}} \quad \text{and} \quad \prod_{j=i-1}^{n-1} \frac{2j+1}{2j} \le e^{\frac{1}{2}} \left(\frac{n-1}{i-1}\right), \tag{16}$$

for $2 \le i \le n$. We recall also that by the Euler-Maclaurin formula we have that

$$\sum_{j=1}^{n} \left(\frac{1}{j}\right)^s = \left(\frac{1}{n}\right)^{s-1} + s \int_1^n \frac{\lfloor x \rfloor}{x^{s+1}} \mathrm{d}x, \quad \text{with} \quad s \in \mathbb{R} \setminus \{1\},$$

for $n \ge 1$. Then,

$$\sum_{j=1}^{n} \left(\frac{1}{j}\right)^s \le \left(1 + \frac{s}{1-s}\right) n^{1-s}, \quad \text{for } s \in (0,1), \tag{17}$$

and

$$\sum_{j=1}^{n} \left(\frac{1}{j}\right)^s \le \frac{s}{s-1}, \quad \text{for } s > 1. \tag{18}$$

Lemma 13. *There exists $z_0 \in (0,1)$ such that*

$$\mathbb{E}[G_{i-1}^i(z_0)] \le (i-1)^{\frac{1+z_0}{2}}, \quad \textit{for } i \ge 2. \tag{19}$$

Proof. First, we focus on finding the correct z_0. For $i \ge 4$, let v_i be the parent of the vertex i which is distributed according to the law

$$\mathbb{P}\left(v_i = j | T_{i-2}^{(0)}\right) = \frac{Z^j(i-2,1)}{2(i-2)}, \quad j \in \{1, 2, \ldots, i-1\}.$$

Then, we have that

$$\begin{aligned}
\mathbb{E}[G^i_{i-1}(z)] &= 1 + z\mathbb{E}[G^{v_i}_{i-2}(z)] \\
&= 1 + z\sum_{j=1}^{i-1}\mathbb{E}[G^j_{i-2}(z)\mathbf{1}_{\{v_i=j\}}] \\
&= 1 + \frac{z}{2(i-2)}\sum_{j=1}^{i-1}\mathbb{E}[G^j_{i-2}(z)Z^j(i-2,1)]. \qquad (20)
\end{aligned}$$

We observe that Lemma 12, (14) and (16) imply after some computations that

$$\begin{aligned}
&\mathbb{E}[G^j_{n-2}(z)Z^j(n-2,1)] \\
&\quad \le e^{\frac{2+z}{2}}\left(\frac{n-3}{j-1}\right)^{\frac{2+z}{2}}\left(\mathbb{E}[G^j_{j-1}(z)] + \frac{e^{\frac{1}{2}}}{2}(j-1)^{-\frac{3+z}{2}}\sum_{k=j-1}^{n-4}\left(\frac{1}{k}\right)^{\frac{3+z}{2}}\right) + \frac{1}{2}e^{\frac{1}{2}}\left(\frac{1}{(n-3)(j-1)}\right)^{\frac{1}{2}}
\end{aligned}$$

for $2 \le j \le n-3$. Then the inequalities (17) and (18) imply that

$$\begin{aligned}
&\sum_{j=2}^{n-3}\mathbb{E}[G^j_{n-2}(z)Z^j(n-2,1)] \\
&\quad \le e^{\frac{2+z}{2}}(n-3)^{\frac{2+z}{2}}\left(\sum_{j=2}^{n-3}\left(\frac{1}{j-1}\right)^{\frac{2+z}{2}}\mathbb{E}[G^j_{j-1}(z)] + e^{\frac{1}{2}}\frac{3+z}{1+z}(n-3)^{\frac{1}{2}}\right) + \frac{e^{\frac{1}{2}}}{(n-3)^{\frac{1}{2}}}, \qquad (21)
\end{aligned}$$

for $n \ge 5$. Similarly, one gets that

$$\mathbb{E}[G^1_{n-2}(z)Z^1(n-2,1)] \le e^{\frac{2+z}{2}}(n-3)^{\frac{2+z}{2}} + \frac{e^{\frac{3+z}{2}}}{2}\frac{3+z}{1+z}(n-3)^{\frac{2+z}{2}} \qquad (22)$$

and

$$\mathbb{E}[G^{n-2}_{n-2}(z)Z^{n-2}(n-2,1)] \le e^{\frac{2+z}{2}}\mathbb{E}[G^{n-2}_{n-3}(z)] + \frac{1}{2}(n-3)^{-1}, \qquad (23)$$

for $n \ge 4$. Next, we define the functions

$$A^1_n(z) = \left(e^{\frac{2+z}{2}} + \frac{e^{\frac{1+z}{2}}}{2}\frac{3+z}{1+z}\right)(n-3)^{-\frac{1}{2}}, \qquad A^2_n(z) = \left(e^{\frac{2+z}{2}} + \frac{1}{2}(n-3)^{-\frac{3+z}{2}}\right)(n-3)^{-1}$$

and

$$A^3_n(z) = 2e^{\frac{2+z}{2}} + e^{\frac{1}{2}}(n-3)^{-\frac{4+z}{2}} + e^{\frac{3+z}{2}}\frac{3+z}{1+z},$$

for $n \geq 4$ and $z \in (0,1)$. Then one can find $z_0 \in (0,1)$ such that

$$3^{-\frac{1+z_0}{2}} + \frac{z_0}{2}\left(A_4^1(z_0) + A_4^2(z_0) + A_4^3(z_0) + \frac{1}{2}\right) \leq 1.$$

Now, we proceed to prove by induction (19) with $z_0 \in (0,1)$ such that the previous inequality is satisfied. For $i = 2, 3$, it must be clear since

$$\mathbb{E}[G_1^2(z_0)] = 1 \quad \text{and} \quad \mathbb{E}[G_2^3(z_0)] = 1 + z_0.$$

Suppose that it is true for $i = n-1 \geq 2$. We observe from (20) and the inequalities (21), (22) and (23) that

$$\begin{aligned}
\mathbb{E}[G_{n-1}^n(z_0)] &\leq 1 + (n-1)^{\frac{1+z_0}{2}} \frac{z_0}{2}\left(A_n^1(z_0) + A_n^2(z_0) + \frac{1}{2} + A_n^3(z_0)\right) \\
&\leq (n-1)^{\frac{1+z_0}{2}}\left(3^{-\frac{1+z_0}{2}} + \frac{z_0}{2}\left(\frac{1}{2} + A_4^1(z_0) + A_4^2(z_0) + A_4^3(z_0)\right)\right) \\
&\leq (n-1)^{\frac{1+z_0}{2}},
\end{aligned}$$

the second inequality is because the functions $A_n^1(\cdot)$, $A_n^2(\cdot)$ and $A_n^1(\cdot)$ are decreasing with respect to n and the last one is by our choice of z_0. $\square$

Finally, we have all the ingredients to prove Lemma 9.

Proof of Lemma 9. We deduce from the inequality (15) that for $n \geq 2$ we have

$$\prod_{j=i-1}^{n-1} \frac{2j+1+z}{2j} \leq e^{\frac{1+z}{2}}\left(\frac{n-1}{i-1}\right)^{\frac{1+z}{2}} \qquad \text{for } i \geq 2.$$

We consider $z_0 \in (0,1)$ such that equation (19) in Lemma 13 is satisfied. Then from (12) and (13) we have that

$$\mathbb{E}[G_n^1(z_0)] \leq e^{\frac{1+z_0}{2}} n^{\frac{1+z_0}{2}} \qquad \text{for } i \geq 1 \text{ and } n \geq 1,$$

which is our claim. $\square$

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