EDGE-COLORING LINEAR HYPERGRAPHS WITH MEDIUM-SIZED EDGES

VANCE FABER[1] AND DAVID G. HARRIS[2]

Abstract. Motivated by the Erdős-Faber-Lovász (EFL) conjecture for hypergraphs, we consider the list edge coloring of linear hypergraphs. We show that if the hyper-edge sizes are bounded between i and $C_{i,\epsilon}\sqrt{n}$ inclusive, then there is a list edge coloring using $(1+\epsilon)\frac{n}{i-1}$ colors. The dependence on n in the upper bound is optimal (up to the value of $C_{i,\epsilon}$).

1. Introduction

Let $H = (V, E)$ be a hypergraph with $n = |V|$ vertices. Each edge $e \in E$ can be regarded as a subset of V. We say that H is *linear* if $|e \cap e'| \leq 1$. We define the *minimum rank* of H as $\rho = \min_{e \in E} |e|$ and similarly the *maximum rank* of H as $P = \max_{e \in E} |e|$. A graph is a special case with $\rho = P = 2$.

For any hypergraph H, one may define the *line graph* of H, to be an (ordinary) graph $L(H)$ on vertex set E, with an edge $\{e_1, e_2\}$ in $L(H)$ iff $e_1 \cap e_2 \neq \emptyset$. The edge chromatic number $q(H)$ (respectively list edge chromatic number $q_{\text{list}}(H)$) is the chromatic number χ (respectively list chromatic number χ_{list}) of $L(H)$.

A long-standing conjecture, known now as the Erdős-Faber-Lovász conjecture, can be stated as

Conjecture 1.1 (EFL). *Let H be a linear hypergraph with n vertices and no rank-1 edges. Then $q(H) \leq n$.*

There has been partial progress to proving this result; in [3] Kahn showed that $q(H) \leq n + o(n)$. See [5] for a more recent review of results.

In this paper, we will show a related result for hypergraphs in which the edges all have medium size. More specifically, we show the following:

Theorem 1.2. *For any integer $i \geq 3$ and any $\epsilon > 0$, there exists some value $C_{i,\epsilon} > 0$ with the following property. For any linear hypergraph H on n vertices, with minimum rank $\rho \geq i$ and maximum rank $P \leq C_{i,\epsilon}\sqrt{n}$, it holds that*

$$q_{list}(H) \leq (1+\epsilon)\frac{n}{i-1}$$

In particular, the EFL conjecture holds for hypergraphs of minimum rank $\rho \geq 3$ and maximum rank $P \leq C\sqrt{n}$, for some universal constant $C = C_{3,1}$. By way of comparison, [6] showed that the EFL conjecture holds for hypergraphs of minimum rank $\rho \geq \sqrt{n}$. However, Theorem 1.2 can be significantly stronger than EFL in cases where ρ is large. We also note that the dependence of P on n is optimal, up to the value of the constant $C_{i,\epsilon}$; see Section 4 for further details.

2. Preliminaries

2.1. Notation.

To simplify some notation, we define the truncated logarithm by

$$\log(x) = \begin{cases} \ln(x) & \text{if } x \geq e \\ 1 & \text{otherwise} \end{cases}$$

[1]IDA/Center for Computing Sciences, Bowie MD 20707. Email: vance.faber@gmail.com

[2]Department of Computer Science, University of Maryland, College Park, MD 20742. Email: davidgharris29@gmail.com.

Thus, $\log(x) \geq 1$ for all $x \in \mathbf{R}$.

If $H = (V, E)$ is a hypergraph and $E' \subseteq E$, then $H(E')$ is the hypergraph (V, E'). If $V' \subseteq V$, then $H[V']$ is the induced hypergraph (V', E'), where E' is the set of edges involving only vertices of V'. If the hypergraph H is understood, then we sometimes write $q_{\text{list}}(E')$ as shorthand for $q_{\text{list}}(H(E'))$.

2.2. **Background facts.** Our proof is based upon two powerful theorems for list-coloring hypergraphs. The first is due to Kahn [4], restated in terms of linear hypergraphs.

Theorem 2.1 ([4]). *For every $\epsilon > 0$ and integer $P > 1$ there exists a constant $c = c(\epsilon, P)$, such that any linear hypergraph H with maximum rank P and maximum degree $\Delta \geq c$ satisfies*

$$q_{list}(H) \leq \Delta(1 + \epsilon)$$

Observe that if $P \leq O(1)$, then Theorem 2.1 immediately shows Theorem 1.2, namely that $q_{\text{list}}(H) \leq \frac{n}{\rho - 1}(1+\epsilon)$. Indeed, Kahn's asymptotic proof of EFL [3] uses this proof strategy. However, Theorem 2.1 does not give useful information when P is increasing with n.

In this second case, we follow a strategy of [2] and use another graph property, *local sparsity*, based on the triangle counts in the line graph.

Definition 2.2. *A* triangle *in a graph G is a set of three vertices v_1, v_2, v_3 such that $(v_1, v_2) \in E, (v_1, v_3) \in E, (v_2, v_3) \in E$.*

Theorem 2.3 ([7]). *Suppose that every vertex of G has degree at most d, and every vertex participates in at most f triangles. Then*

$$\chi_{list}(G) \leq O\Big(\frac{d}{\log(d^2/f)}\Big)$$

We note that [1] had earlier showed a similar bound $\chi(G) \leq O(\frac{d}{\log(d^2/f)})$ for the ordinary chromatic number. However, our proofs really need list-coloring as a subroutine: even if our goal was only to bound the ordinary chromatic index $q(H)$, we would still need the list-coloring provided by Theorem 2.3.

3. Main result

In this section, we prove Theorem 1.2. We let L denote the line graph of H. Also, we let $\epsilon > 0$ be an arbitrary fixed quantity which we view as a *constant* throughout the proof.

Observation 3.1. *Every vertex $v \in H$ is in at most $\frac{n-1}{\rho-1}$ edges of H.*

Proof. Since H is linear, the edges containing v do not share any other vertices. ☐

Proposition 3.2. *Suppose we are given a linear hypergraph $H = (V, E)$ and a partition $E = E_1 \sqcup E_2$. Let $P_1 = \max_{e_1 \in E_1} |e_1|$ and let $\rho_2 = \min_{e_2 \in E_2} |e_2|$.*

Then

$$q_{list}(H) \leq \max\Big(q_{list}(E_2), q_{list}(E_1) + \frac{(n-1)P_1}{\rho_2 - 1}\Big)$$

Proof. Suppose each edge has a palette of size $Q \geq q_{\text{list}}(E_2)$. Select an arbitrary list-coloring of $H(E_2)$. Consider the residual palette for each edge $e \in E_1$; that is, the palette available to e after removing the colors selected by edges $f \in E_2$ with $f \cap e \neq \emptyset$. By Observation 3.1, each vertex v is in at most $\frac{n-1}{\rho_2 - 1}$ edges in E_2, and thus each edge $e \in E_1$ touches at most $\frac{(n-1)P_1}{\rho_2-1}$ edges in E_2. Thus, each edge in E_1 has a residual palette of size at least $Q - \frac{(n-1)P_1}{\rho_2-1}$.

As long as $Q - \frac{(n-1)P_1}{\rho_2-1} \geq q_{\text{list}}(E_1)$, we can list-color $H(E_1)$ with the residual palette, thus giving a full list-coloring of H. Thus, our coloring procedure succeeds as long as

$$Q \geq q_{\text{list}}(E_2) \qquad \text{and} \qquad Q \geq \frac{(n-1)P_1}{\rho_2 - 1} + q_{\text{list}}(E_1)$$

□

Given a hypergraph H, we let A_i denote the set of edges $e \in E$ such that $2^i \leq |e| < 2^{i+1}$. Note that each edge $e \in E$ corresponds to a vertex of L, and for any $U \subseteq E$ we denote by $L[U]$ the induced subgraph of L on the edges of U.

Proposition 3.3. *For any integer $i \geq 1$, we have*

$$q_{list}(A_i) \leq O(\frac{n}{i} + \frac{n}{\log(n/2^{2i}))})$$

Proof. By Observation 3.1, every vertex in H touches $O(n/2^i)$ edges in A_i. As each edge in A_i contains at most 2^{i+1} vertices, it follows that $L[A_i]$ has maximum degree $O(n)$.

Now, consider some edge $e \in A_i$; we want to count how many triangles of L it participates in. There are two types of triangles. The first involves three edges around a single vertex; for each $v \in e$ there are at most $n/2^i$ choices for the other two edges, giving a total triangle count of at most $2^{i+1} \times (n/2^i)^2 = O(n^2/2^i)$.

The second type of triangle involves three edges, each intersecting at a distinct point. There are $O(n)$ choices for the second edge e'. By linearity, for any given choice of vertices $v \in e, v' \in e'$ there is at most one possible choice for the third edge e'' intersecting e at v and e' at v'. Hence, the total number of triangles of this second type is at most $n \times (2^{i+1})^2 \leq O(n2^{2i})$.

Now, applying Theorem 2.3, we see that

$$q_{\text{list}}(A_i) \leq O(\frac{n}{\log(\frac{n^2}{O(n^2/2^i+n2^{2i})})}) \leq O(\frac{n}{i} + \frac{n}{\log(n/2^{2i})})$$

□

The remainder of this proof will be expressed in terms of an integer parameter k (which does not depend on n), and which we will set later in the construction.

Observation 3.4. *For each integer $k \geq 1$, there is some integer N_k, such that whenever $n \geq N_k$ it holds that*

$$q_{list}(A_1 \cup A_2 \cup \cdots \cup A_k) \leq (1+\epsilon)\frac{n}{\rho - 1}$$

Proof. By Observation 3.1, $L[A_1 \cup A_2 \cup \cdots \cup A_k]$ has maximum degree $\frac{n-1}{\rho-1}$ and maximum rank $2^{k+1} - 1$. Now apply Theorem 2.1. □

Corollary 3.5. *If $i \geq k$ and $P \leq \sqrt{ne^{-k}}$, then $q_{list}(A_i) \leq O(n/k)$.*

Proof. This is vacuously true for $i \geq \log_2 P$, since then A_i is empty. Otherwise, Proposition 3.3 gives $q_{\text{list}}(A_i) \leq O(\frac{n}{i} + \frac{n}{\log(n2^{-2i})}) \leq O(\frac{n}{k} + \frac{n}{\log(nP^{-2})}) \leq O(n/k)$. □

Proposition 3.6. *Suppose $i \geq k$ and $P \leq \sqrt{ne^{-k}}$. Let $x = \lceil \log_2 k \rceil$. Then*

$$q_{list}(A_i \cup A_{i+x} \cup A_{i+2x} \cup A_{i+3x} \dots) \leq O(n/k)$$

Proof. Let $s = \lceil \log_2 n \rceil$. For each integer $j \geq 0$, define

$$B_j = A_{i+sx} \cup A_{i+(s-1)x} \cup A_{i+(s-2)x} \cup \cdots \cup A_{i+(j+1)x} \cup A_{i+jx}$$

We will prove that $q_{\text{list}}(B_j) \leq cn/k$ for some sufficiently large constant c and $j \geq 0$, by induction downward on j.

When $j = s$, this is vacuously true as $B_j = \emptyset$. For the induction step, we use Proposition 3.2 using the decomposition $E_1 = A_{i+jx}$ and $E_2 = A_{i+(j+1)x} \cup A_{i+(j+2)x} \cup \cdots \cup A_{i+sx} = B_{j+1}$.

Note that $P_1 \leq 2^{i+jx+1}-1$ and $\rho_2 \geq 2^{i+(j+1)x}$. By induction hypothesis, $q_{\text{list}}(E_2) = q_{\text{list}}(B_{j+1}) \leq cn/k$. By Corollary 3.5, $q_{\text{list}}(E_1) = q_{\text{list}}(A_{i+jx}) \leq c'n/k$, for some constant $c' \geq 0$. This gives

$$\begin{aligned} q_{\text{list}}(B_j) &\leq \max(q_{\text{list}}(E_2), q_{\text{list}}(E_1) + \frac{(n-1)P_1}{\rho_2 - 1}) \\ &\leq \max(cn/k, c'n/k + \frac{(n-1)(2^{i+jx+1}-1)}{2^{i+(j+1)x}-1}) \\ &\leq \max(cn/k, c'n/k + 4n/k) \end{aligned}$$

which is at most cn/k when $c \geq c' + 4$. □

Proposition 3.7. *For each integer $k \geq 1$, there is some integer N'_k such that whenever $n > N'_k$ and $P \leq \sqrt{ne^{-k}}$ it holds that*

$$q_{list}(A_k \cup A_{k+1} \cup A_{k+2} \cup \dots) \leq O(\frac{n \log k}{k})$$

Proof. Suppose every edge begins with a palette of size $\frac{cn \log k}{k}$ for some constant c. Let $x = \lceil \log_2 k \rceil$. We randomly partition the colors into x classes. For any class i and edge e, let $Q_{i,e}$ denote the number of class-i colors in the palette of edge e.

We have $\mathbf{E}[Q_{i,e}] \geq \frac{cn \log k}{kx} \geq \frac{cn}{2k}$. Furthermore, $Q_{i,e}$ is the sum of independent random variables (whether each color in the palette of e goes into $Q_{i,e}$), and so by Chernoff's bound we have

$$\Pr(Q_{i,e} \leq \frac{cn}{4k}) \leq e^{-\Theta(n/k)}$$

For fixed k, this is smaller than $n^2k/2$ for sufficiently large n. Since there are at most n^2 edges and k classes, by the union bound there is a positive probability that every edge has at least $\frac{cn}{4k}$ colors of each class in its palette.

Next, for each i in the range $i = 1, \dots, x$, we use the class-i colors to color $L[A_{k+i} \cup A_{k+i+x} \cup A_{k+i+2x} \cup \dots]$. By Proposition 3.6 this succeeds for c sufficiently large. □

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. We will prove that $q_{\text{list}}(H) \leq \frac{n}{i-1}(1 + O(\epsilon))$; the result follows easily by rescaling.

By Proposition 3.7, there is some constant c such that for any integer $k > 1$ and $n > N'_k$ we have $q_{\text{list}}(A_k \cup A_{k+1} \cup A_{k+2} \cup \dots) \leq \frac{cn \log k}{k}$. We select k sufficiently large so that

$$\frac{c \log k}{k} \leq \frac{\epsilon}{i-1}$$

By Observation 3.4, for $n > N_k$ we have

$$q_{\text{list}}(A_1 \cup A_2 \cup \dots \cup A_{k-1}) \leq \frac{n(1+\epsilon)}{i-1}$$

Now, suppose every edge has a palette of size $Q = \frac{(1+3\epsilon)n}{i-1}$. Randomly partition the colors into two classes, where a color goes into class I with probability $p = \frac{1.5n\epsilon}{Q}$ and goes into class II with probability $1-p$. A Chernoff bound argument similar to Proposition 3.7 shows that for $n > N''$, with positive probability every edge has at least $n\epsilon$ class-I colors and $\frac{n}{i-1}(1+\epsilon)$ class-II colors. In such a case, we can color $L[A_k \cup A_{k+1} \cup A_{k+2} \cup \dots]$ using class-I colors and $L[A_1 \cup A_2 \cup \dots \cup A_{k-1}]$ using class-II colors.

So far, we have shown that there is an integer k and integer $M = \max(N_k, N'_k, N''_k)$ such that whenever $P \leq \sqrt{ne^{-k}}$ and $n > M$ that $q_{\text{list}}(H) \leq \frac{n}{i-1}(1 + O(\epsilon))$. Set $C_{i,\epsilon} = \min(e^{-k/2}, \frac{1}{\sqrt{M}})$.

When $P \leq C_{i,\epsilon}\sqrt{n}$, then $P \leq \sqrt{ne^{-k}}$. Also, when $P \leq C_{i,\epsilon}\sqrt{n}$, we have $P \leq \sqrt{\frac{n}{M}}$. Since $P \geq i \geq 2$, this implies $n \geq 4M$. So $q_{\text{list}}(H) \leq \frac{n}{i-1}(1 + O(\epsilon))$. □

4. Dependence of n on P

In this section, we show (roughly speaking) that Theorem 1.2 must include a condition of the form $P \leq C_{i,\epsilon}\sqrt{n}$ where $C_{i,\epsilon} \leq \sqrt{\frac{1+\epsilon}{i-1}}$.

Proposition 4.1. *For every real number x in the range $(0,1)$ and every $\delta > 0$, there is some integer N such that, for all $n \geq N$, there are linear hypergraphs H on n vertices which satisfy the following properties:*

(1) $q_{list}(H) > xn$
(2) $P = \rho = r$
(3) $\sqrt{nx} \leq r \leq (1+\delta)\sqrt{nx}$

Proof. Consider a finite projective plane with parameter q; this gives a linear hypergraph H' with q^2+q+1 vertices and every edge of rank $r = q+1$, and $q_{\text{list}}(H') = q^2+q+1$. Assuming that $q^2+q+1 \leq n$, we can form H from H' by adding $n-(q^2+q+1)$ isolated vertices.

Let $u = \sqrt{nx}$ and let q be the smallest prime with $q \geq u$. For n (and hence u sufficiently large), this q satisfies $q \leq u + u^\theta$ for some constant $\theta < 1$. In particular, $q^2+q+1 \leq nx + o(n) \leq n$ for n sufficiently large. Our choice of q ensures that that $q_{\text{list}}(H') = q^2+q+1 \geq u^2+u+1 > xn$. Also, $r > \sqrt{nx}$ and $r/\sqrt{n} \to \sqrt{x}$ as $n \to \infty$. □

Proposition 4.2. *For any integer $i \geq 3$ and $\epsilon \in (0,1)$, the term "$C_{i,\epsilon}\sqrt{n}$" in Theorem 1.2 cannot be replaced by any expression of the form $f(i,\epsilon,n)$, where for any fixed i,ϵ,δ there are infinitely many n with*

$$f(i,\epsilon,n) > (1+\delta)\sqrt{\frac{(1+\epsilon)n}{i-1}}$$

Proof. Apply Proposition 4.1 with $x = (1+\epsilon)/(i-1)$; note that since $i \geq 3$ and $\epsilon < 1$ we have $x \in (0,1)$. This ensures that for all $n > N$, there is a hypergraph H_n with $q_{\text{list}}(H_n) > xn$ and rank $\rho = P = r$ for $\sqrt{nx} \leq r \leq (1+\delta)\sqrt{nx}$.

Thus, for n sufficiently large, we have $\rho \geq i$. Also, for sufficiently large n, we have $P \leq (1+\delta)\sqrt{nx} = (1+\delta)\sqrt{\frac{(1+\epsilon)n}{i-1}}$. Thus, if $f(i,\epsilon,n) > (1+\delta)\sqrt{\frac{(1+\epsilon)n}{i-1}}$ for infinitely many n, then for infinitely many n we would have $q_{\text{list}}(H_n) \leq \frac{(1+\epsilon)n}{i-1} = xn$, a contradiction. □

5. Acknowledgments

Thanks to the anonymous reviewers for helpful suggestions and corrections.

References

[1] Alon, N., Krivelevich, M., Sudakov, B.: Coloring graphs with sparse neighborhoods. Journal of Combinatorial Theory, Series B 77, pp. 77-82 (1999)
[2] Faber, V.: Linear hypergraph edge-coloring — generalizations of the EFL conjecture. Bulletin of Mathematical Sciences and Applications 17, pp. 1-9 (2016)
[3] Kahn, J.: Coloring nearly-disjoint hypergraphs with $n + o(n)$ colors. Journal of Combinatorial Theory, Series A 59, pp. 31-39 (1992)
[4] Kahn, J.: Asymptotically good list-colorings. Journal of Combinatorial Theory Series A 73, pp. 1 - 59 (1996)
[5] Romero, D., Sánchez-Arroyo, A.: Advances on the Erdős-Faber-Lovász conjecture. Oxford Lecture Series in Mathematics and its Applications 34, pp. 272 - 283 (2007)
[6] Sánchez-Arroyo, A.: The Erdő-Faber-Lovász conjecture for dense hypergraphs. Discrete Mathematics 308, pp. 991-992 (2008)
[7] Vu, V.: A general upper bound on the list chromatic number of locally sparse graphs. Combinatorics, Probability and Computing 11, pp. 103-111 (2002)