

# How many random edges make a dense hypergraph non-2-colorable?

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## Abstract

We study a model of random uniform hypergraphs, where a random instance is obtained by adding random edges to a large hypergraph of a given density. The research on this model for graphs has been started by Bohman et al. in [7], and continued in [8] and [16]. Here we obtain a tight bound on the number of random edges required to ensure non-2-colorability. We prove that for any  $k$ -uniform hypergraph with  $\Omega(n^{k-\epsilon})$  edges, adding  $\omega(n^{k\epsilon/2})$  random edges makes the hypergraph almost surely non-2-colorable. This is essentially tight, since there is a 2-colorable hypergraph with  $\Omega(n^{k-\epsilon})$  edges which almost surely remains 2-colorable even after adding  $o(n^{k\epsilon/2})$  random edges.

## 1 Introduction

Research on random graphs and hypergraphs has a long history with thousands of papers and two monographs by Bollobás [9] and by Janson et al. [15] devoted to the subject and its diverse applications. In the classical Erdős-Rényi model [14], a random graph is generated by starting from an empty graph and then adding certain number of random edges. More recently, Bohman, Frieze and Martin [7] considered a generalized model where one starts with a fixed graph  $G = (V, E)$  and then inserts a collection  $R$  of additional random edges. We denote the resulting random graph by  $G + R$ . The initial graph  $G$  can be regarded as given by an adversary, while the random perturbation  $R$  represents noise or uncertainty, independent of the initial choice. This scenario is analogous to the *smoothed analysis* of algorithms proposed by Spielman and Teng [19], where an algorithm is assumed to run on the worst-case input, modified by a small random perturbation.

Usually, one investigates *monotone properties* of random graphs or hypergraphs; i.e., properties which cannot be destroyed by adding more edges, like the property of containing a certain fixed subgraph. Given a monotone property  $\mathcal{A}$  of graphs on  $n$  vertices, we can ask what are the parameters for which a random graph has property  $\mathcal{A}$  almost surely, i.e. with probability tending to 1 as the number of vertices  $n$  tends to infinity. In our setting, we start with a fixed hypergraph  $H$  and inquire how many random edges  $R$  we have to add so that  $H + R$  has property  $\mathcal{A}$  almost surely. This question is too general to get concrete and meaningful results, valid for all hypergraphs  $H$ . Therefore, rather than considering a completely arbitrary  $H$ , we start with a hypergraph from a certain natural class.

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One such class of graphs was considered in [7], where the authors analyze the question of how many random edges need to be added to a graph  $G$  of minimal degree at least  $dn$ ,  $0 < d < 1$ , so that the resulting graph  $G + R$  is almost surely Hamiltonian. Further properties of random graphs in this model are explored in [8].

In [16], Krivelevich et al. considered a slightly more general setting, in which one performs a small random perturbation of a graph  $G$  with at least  $dn^2$  edges. Observe that since  $G$  has at least  $dn^2$  edges, removing a small set of random edges would leave the total number of edges in  $G$  essentially unchanged. Therefore one only has to focus on the case of adding random edges. In [16], the authors obtained tight results for the appearance of a fixed subgraph and for certain Ramsey properties in this model. In the same paper, they also considered random formulas obtained by adding random  $k$ -clauses (disjunctions of  $k$  literals) to a fixed  $k$ -SAT formula. Krivelevich et al. proved that for any formula with at least  $n^{k-\epsilon}$   $k$ -clauses, adding  $\omega(n^{k\epsilon})$  random clauses of size  $k$  makes the formula almost surely unsatisfiable. This is tight, since there is a  $k$ -SAT formula with  $n^{k-\epsilon}$  clauses which almost surely remains satisfiable after adding  $o(n^{k\epsilon})$  random clauses. A related question, which was raised in [16], is to find a threshold for non-2-colorability of a random hypergraph obtained by adding random edges to a large hypergraph of a given density.

For an integer  $k \geq 2$ , a  $k$ -uniform hypergraph is an ordered pair  $H = (V, E)$ , where  $V$  is a finite non-empty set, called set of *vertices* and  $E$  is a family of distinct  $k$ -subsets of  $V$ , called the *edges* of  $H$ . A 2-coloring of a hypergraph  $H$  is a partition of its vertex set  $V$  into two color classes so that no edge in  $E$  is monochromatic. A hypergraph which admits a 2-coloring is called 2-colorable.

2-colorability is one of the fundamental properties of hypergraphs, which was first introduced and studied by Bernstein [6] in 1908 for infinite hypergraphs. 2-colorability in the finite setting, also known as “Property B” (a term coined by Erdős in reference to Bernstein), has been studied extensively in the last forty years (see, e.g., [10, 11, 13, 5, 18]). While 2-colorability of graphs is well understood being equivalent to non-existence of odd cycles, for  $k$ -uniform hypergraphs with  $k \geq 3$  it is already  $NP$ -complete to decide whether a 2-coloring exists [17]. Consequently, there is no efficient characterization of 2-colorable hypergraphs. The problem of 2-colorability of random  $k$ -uniform hypergraphs for  $k \geq 3$  was first studied by Alon and Spencer [4]. They proved that such hypergraphs with  $m = (c2^k/k^2)n$  edges are almost surely 2-colorable. This bound was improved later by Achlioptas et al. [1]. Recently, the threshold for 2-colorability has been determined very precisely. In [2] it was proved that the number of edges for which a random  $k$ -uniform hypergraph becomes almost surely non-2-colorable is  $(2^{k-1} \ln 2 - O(1))n$ .

Interestingly, the threshold for non-2-colorability is roughly one half of the threshold for  $k$ -SAT. It has been shown in [3] that a formula with  $m$  random  $k$ -clauses becomes almost surely unsatisfiable for  $m = (2^k \ln 2 - O(k))n$ . The two problems seem to be intimately related and it is natural to ask what is their relationship in the case of a random perturbation of a fixed instance. Recall that from [16] we know that for any  $k$ -SAT formula with  $n^{k-\epsilon}$  clauses, adding  $\omega(n^{k\epsilon})$  random clauses makes it almost surely unsatisfiable. In fact, the same proof yields that for any  $k$ -uniform hypergraph  $H$  with  $n^{k-\epsilon}$  edges, adding  $\omega(n^{k\epsilon})$  random edges destroys 2-colorability almost surely. Nonetheless, it

turns out that this is not the right answer. It is enough to use substantially fewer random edges to destroy 2-colorability: roughly a square root of the number of random clauses necessary to destroy satisfiability. The following is our main result.

**Theorem 1.1** *Let  $k, \ell \geq 2$ ,  $\epsilon \geq 0$  be fixed and let  $H$  be a 2-colorable  $k$ -uniform hypergraph with  $\Omega(n^{k-\epsilon})$  edges. Then the hypergraph  $H'$  obtained by adding to  $H$  a collection  $R$  of  $\omega(n^{\ell\epsilon/2})$  random  $\ell$ -tuples is almost surely non-2-colorable.*

Observe that for  $\epsilon \geq 2/\ell$ , the result is easy. Regardless of the hypergraph  $H$ , it is well known that a collection of  $\omega(n)$  random  $\ell$ -tuples on  $n$  vertices is almost surely non-2-colorable. So we will be only interested in the case when  $\epsilon < 2/\ell$ . For such  $\epsilon$  we obtain the following result, which shows that the assertion of Theorem 1.1 is essentially best possible.

**Theorem 1.2** *For fixed  $k, \ell \geq 2$  and  $0 \leq \epsilon < 2/\ell$ , there exists a 2-colorable  $k$ -uniform hypergraph  $H$  with  $\Omega(n^{k-\epsilon})$  edges such that its union with a collection  $R$  of  $o(n^{\ell\epsilon/2})$  random  $\ell$ -tuples is almost surely 2-colorable.*

The rest of this paper is organized as follows. In the next section we present an example of the hypergraph which shows that our main result is essentially best possible. In Section 3 we discuss some natural difficulties in proving Theorem 1.1 and describe how to deal with them in the case of bipartite graphs. This result also serves as a basis for induction which we use in Section 4 to prove the general case of 2-colorable  $k$ -uniform hypergraphs.

**Remark 1.3** *We have two alternative ways of adding random edges. Either we can sample a random  $\ell$ -tuple  $|R|$  times, each time uniformly and independently from the set of all  $\binom{n}{\ell}$   $\ell$ -tuples. Or we can pick each  $\ell$ -tuple randomly and independently with probability  $p = |R|/\binom{n}{\ell}$ . Since 2-colorability is a monotone property, it follows, as in Bollobás [9], Theorem 2.2 and a similar remark in [16], that if the resulting hypergraph is almost surely non-2-colorable (2-colorable) in one model then this is true in the other model as well. This observation can sometimes simplify our calculations.*

**Notation.** Let  $H = (V, E)$  be a  $k$ -uniform hypergraph. In the following, we use the notions of *degree* and *neighborhood*, generalizing their usual meaning in graph theory. For a vertex  $v \in V$ , we define its degree  $d(v)$  to be the number of edges of  $H$  that contain  $v$ . More generally, for a subset of vertices  $A \subset V$ ,  $|A| < k$ , we define its degree  $d(A) = |\{e \in E : A \subset e\}|$ . For a  $(k-1)$ -tuple of vertices  $A$ , we define its *neighborhood* as  $N(A) = \{w \in V \setminus A : A \cup \{w\} \in E\}$ . Also, for a  $(k-2)$ -tuple of vertices  $A$ , we define its *link* as  $\Gamma(A) = \{\{u, v\} \in V \setminus A : A \cup \{u, v\} \in E\}$ .

Throughout the paper we will systematically omit floor and ceiling signs for the sake of clarity of presentation. Also, we use the notations  $a_n = \Theta(b_n)$ ,  $a_n = O(b_n)$  or  $a_n = \Omega(b_n)$  for  $a_n, b_n > 0$  and  $n \rightarrow \infty$  if there are absolute constants  $C_1$  and  $C_2$  such that  $C_1 b_n < a_n < C_2 b_n$ ,  $a_n < C_2 b_n$  or  $a_n > C_1 b_n$  respectively. The notation  $a_n = o(b_n)$  means that  $a_n/b_n \rightarrow 0$  as  $n \rightarrow \infty$ , and  $a_n = \omega(b_n)$  means  $a_n/b_n \rightarrow \infty$ . The parameters  $k, \ell, \epsilon$  are considered constant.

## 2 The lower bound

The following example proves Theorem 1.2 and shows that our main result is essentially best possible.

**Construction.** Partition the set of vertices  $[n]$  into three disjoint subsets  $X, Y, Z$  where  $|X| = |Y| = n^{1-\epsilon/2}$ . Let  $H$  be a  $k$ -uniform hypergraph whose edge set consists of all  $k$ -tuples which have exactly one vertex in  $X$ , one vertex in  $Y$  and  $k-2$  vertices in  $Z$ . By definition the number of edges in  $H$  is  $\Theta(n^{k-\epsilon})$ .

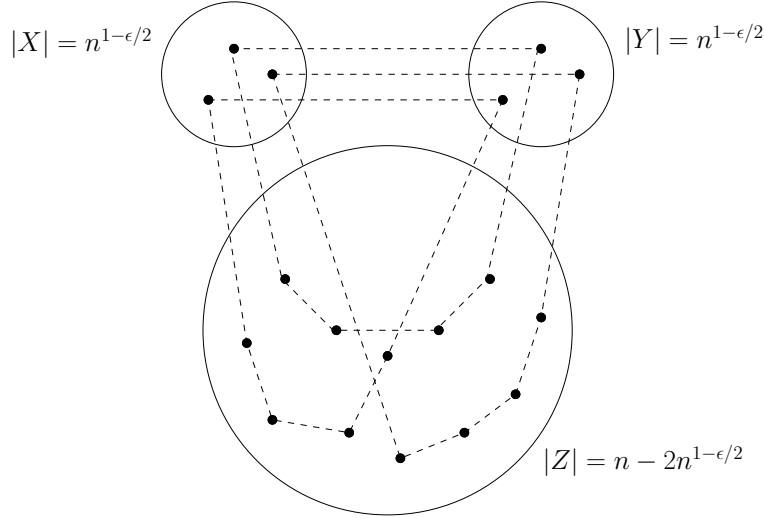


Figure 1: Construction of the hypergraph  $H$ .

**Claim.** Color all the vertices in  $X$  by color 1 and vertices in  $Y$  by color 2. Note that no matter how we assign colors to the remaining vertices, this gives a proper 2-coloring of  $H$ . Let  $R$  be a set of  $o(n^{\ell\epsilon/2})$  random  $\ell$ -tuples. Then almost surely we can 2-color  $Z$  so that none of the  $\ell$ -tuples in  $R$  is monochromatic, i.e., there exists a proper 2-coloring of  $H + R$ .

To prove this claim we transform  $R$  into another random instance  $R'$  that contains only single vertices with a fixed *prescribed color* and edges of size two which must not be monochromatic. Following Remark 1.3 we can assume that  $R$  was obtained by choosing every  $\ell$ -tuple in  $[n]$  randomly and independently with probability  $p = o(n^{\ell\epsilon/2-\ell})$ . First note that almost surely there is no  $\ell$ -tuple in  $R$  whose vertices are all in  $X$  or in  $Y$ . Indeed, since  $|X| = |Y| = n^{1-\epsilon/2}$ , the probability that there is such an  $\ell$ -tuple is at most  $2\binom{n^{1-\epsilon/2}}{\ell}p = o(1)$ . Also, every  $\ell$ -tuple in  $R$  which has vertices in both  $X$  and  $Y$  is already 2-colored so we discard it.

For every  $v \in Z$  we add it to  $R'$  with prescribed color 1 if there is a subset  $A$  of  $Y$  of size  $\ell-1$  such that  $A \cup \{v\} \in R$ . Since  $\epsilon < 2/\ell \leq 1$ , the probability of this event is

$$p_1 = \binom{|Y|}{\ell-1} p = \binom{n^{1-\epsilon/2}}{\ell-1} p \leq n^{(\ell-1)(1-\epsilon/2)} p = o(n^{-1+\epsilon/2}) = o(n^{-1/2}).$$

Similarly, if there is a subset  $B$  of  $X$  of size  $\ell - 1$  such that  $B \cup \{v\} \in R$  then we add  $v$  to  $R'$  with prescribed color 2. The probability  $p_2$  of this event is also  $o(n^{-1/2})$ .

Fix an ordering  $v_1 < v_2 < \dots$  of all vertices in  $Z$ . For every pair of vertices  $u, w \in Z$  we add an edge  $\{u, w\}$  to  $R'$  if there is an  $\ell$ -tuple  $L \in R$  such that the two smallest vertices in  $L \cap Z$  are  $u$  and  $w$ . Since the number of such possible  $\ell$ -tuples is at most  $\binom{n}{\ell-2}$ , and  $\epsilon < 2/\ell$ , the probability of this event is

$$p_3 \leq \binom{n}{\ell-2} p = O(n^{\ell-2} p) = o(n^{\ell\epsilon/2-2}) = o(n^{-1}).$$

Also note that by definition all the above events are independent since they depend on disjoint sets of  $\ell$ -tuples. By our construction, any 2-coloring of  $Z$  in which singletons in  $R'$  get prescribed colors and no 2-edge is monochromatic gives a proper 2-coloring of  $R$ . Therefore, to complete the proof of Theorem 1.2, it is enough to prove the following simple statement.

**Lemma 2.1** *Let  $R'$  be a random instance which is obtained as follows. For  $i = 1, 2$  we choose every vertex in  $[n]$  with probability  $p_i = o(n^{-1/2})$  (independently for  $i = 1, 2$ ) and prescribe to it color  $i$ . In addition we choose every pair of vertices to be an edge in  $R'$  with probability  $p_3 = o(n^{-1})$ . Then almost surely there exists a 2-coloring of  $[n]$  in which all singletons in  $R'$  get prescribed colors and no edge is monochromatic.*

**Proof.** Let  $G$  be the graph formed by edges from  $R'$ . The probability that there is a vertex with conflicting prescribed colors is  $np_1p_2 = o(1)$ . The probability that  $G$  contains a cycle is at most  $\sum_{s=3}^n n^s p_3^s = O(n^3 p_3^3) = o(1)$ . Finally the probability that there exists a path between two vertices with any prescribed color is also bounded by

$$\sum_{s=1}^n \binom{n}{2} (p_1 + p_2)^2 n^{s-1} p_3^s = o(n(p_1 + p_2)^2) = o(1).$$

Therefore almost surely no vertex gets prescribed conflicting colors, every connected component of  $G$  is a tree and contains at most one vertex with prescribed color. This immediately implies the assertion of the lemma, since every tree can be 2-colored, starting from the vertex with prescribed color (if any).  $\square$

### 3 Bipartite graphs

Now let's turn to Theorem 1.1. First, consider the case of  $k = \ell = 2$ . Here, we claim that for any bipartite graph  $G$  with  $\Omega(n^{2-\epsilon})$  edges, adding  $\omega(n^\epsilon)$  random edges makes the graph almost surely non-bipartite. This will follow quite easily, since it turns out that almost surely we will insert an edge inside one part of a bipartite connected component of  $G$ , creating an odd cycle (see the proof of Proposition 3.1).

However, with the more general hypergraph case in mind, we are also interested in a scenario where random  $\ell$ -tuples are added to a bipartite graph, and  $\ell > 2$ . Then we ask what is the probability that the resulting hypergraph is 2-colorable (i.e., no 2-edge and no  $\ell$ -edge should be monochromatic). We prove the following special case of Theorem 1.1.

**Proposition 3.1** *Let  $\ell \geq 2$ ,  $0 \leq \epsilon < 2/\ell$  and let  $G$  be a bipartite graph with  $\Omega(n^{2-\epsilon})$  edges. Then the hypergraph  $H$  obtained by adding to  $G$  a collection  $R$  of  $\omega(n^{\ell\epsilon/2})$  random  $\ell$ -tuples is almost surely non-2-colorable.*

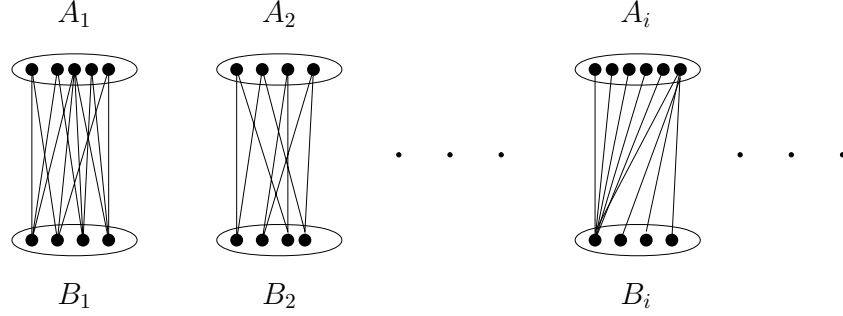


Figure 2: Components of the bipartite graph  $G$ .

**Proof.** Consider the connected components of  $G$  which are bipartite graphs on disjoint vertex sets  $(A_1, B_1), (A_2, B_2), \dots$  (see Figure 2). Denote  $a_i = |A_i|$ ,  $b_i = |B_i|$  and assume  $a_i \geq b_i$ . The number of edges in each component is at most  $a_i b_i$ . Since the total number of edges is at least  $cn^{2-\epsilon}$  for some constant  $c > 0$ , we have

$$\sum a_i^2 \geq \sum a_i b_i \geq cn^{2-\epsilon}.$$

Observe that for  $\ell = 2$ , the number of pairs of vertices inside the sets  $\{A_i\}$  is  $\sum \binom{a_i}{2} \geq \frac{1}{2}(cn^{2-\epsilon} - n) \geq c'n^{2-\epsilon}$ , so a random edge lands inside one of these sets with probability at least  $c'n^{-\epsilon}$ . Consequently, the probability that none of the  $\omega(n^\epsilon)$  random edges ends up inside some  $A_i$  is at most  $(1 - c'n^{-\epsilon})^{\omega(n^\epsilon)} = o(1)$ . Thus almost surely,  $G + R$  contains an odd cycle.

On the other hand, in the general case we are adding  $\omega(n^{\ell\epsilon/2})$  random  $\ell$ -tuples, which might never end up inside any vertex set  $A_i$ . The probability of hitting a specific  $A_i$  is  $\binom{a_i}{\ell} / \binom{n}{\ell} = O(a_i^\ell / n^\ell)$ . For example, if  $G$  has  $n^\epsilon$  components with  $a_i = b_i = n^{1-\epsilon}$ , then this probability is at most  $O(\sum a_i^\ell / n^\ell) = O(n^{-(\ell-1)\epsilon})$ . Hence we need  $\omega(n^{(\ell-1)\epsilon})$  random  $\ell$ -tuples to hit almost surely some  $A_i$ . This suggests a difficulty with the attempt to place a random  $\ell$ -tuple in a set which is forced to be monochromatic by the original graph. We have to allow ourselves more freedom and consider sets which are monochromatic only under certain colorings.

More specifically, each of the sets  $A_i, B_i$  must be monochromatic under any coloring, and at least half of them must share the same color. We do not know a priori which sets will share the same color, yet we can estimate the probability that *any* of these configurations allows a feasible coloring together with the random  $\ell$ -tuples. First, it is convenient to assume that the sets have roughly equal size, in which case we have the following claim.

**Lemma 3.2** *Suppose we have  $t$  disjoint subsets  $A_1, \dots, A_t$  of  $[n]$  of size  $\Theta(n^{1-\alpha})$ . Let  $\alpha \geq \epsilon/2$ ,  $t = \Omega(n^{\frac{\ell}{\ell-1}(\alpha-\epsilon/2)})$  and let  $R$  be a collection of  $\omega(n^{\ell\epsilon/2})$  random  $\ell$ -tuples on  $[n]$ . Then the probability that  $R$  can be 2-colored in such a way that each  $A_i$  is monochromatic is at most  $e^{-\omega(t)}$ .*

**Proof.** Consider the  $2^t$  possible colorings in which all  $A_i$  are monochromatic. For each such coloring there is a set of indices  $I, |I| \geq t/2$  such that the sets  $A_i, i \in I$  share the same color. Since  $A_i$  are disjoint we have  $|\cup_{i \in I} A_i| \geq c_1 t n^{1-\alpha}$  for some  $c_1 > 0$ . The probability that one random  $\ell$ -tuple falls inside this set is at least  $\binom{c_1 t n^{1-\alpha}}{\ell} / \binom{n}{\ell} \geq c_2 (t n^{-\alpha})^\ell$  for some  $c_2 > 0$ . Since  $t^{\ell-1} = \Omega(n^{\ell(\alpha-\epsilon/2)})$ , it implies that

$$\Pr \left[ \cup_{i \in I} A_i \text{ contains no } \ell\text{-tuple from } R \right] \leq \left( 1 - (c_2 t n^{-\alpha})^\ell \right)^{\omega(n^{\ell\epsilon/2})} \leq e^{-\omega(t^\ell n^{-\ell(\alpha-\epsilon/2)})} = e^{-\omega(t)}.$$

Therefore, by the union bound over all choices of  $I$ , we get

$$\Pr \left[ \exists I \text{ such that } \cup_{i \in I} A_i \text{ contains no } \ell\text{-tuple from } R \right] \leq 2^t e^{-\omega(t)} = e^{-\omega(t)}.$$

In particular, almost surely there is no 2-coloring of  $R$  in which all  $A_i$  are monochromatic.  $\square$

Now we can finish the proof of Proposition 3.1 for  $\ell \geq 3$ . Partition the components of  $G$  according to their size and let  $G_s$  contain all the components with  $|A_i| \in [2^{s-1}, 2^s)$ . If there is any  $A_i$  of size at least  $n^{1-\epsilon/2}$ , we are done immediately because one of the  $\omega(n^{\ell\epsilon/2})$  random  $\ell$ -tuples a.s. ends up in  $A_i$  and this destroys the 2-colorability. So we can assume that  $s \leq \lfloor (1 - \epsilon/2) \log_2 n \rfloor$ . Recall that  $\ell \geq 3$  and consider the following sum

$$\sum_{s=1}^{\lfloor (1-\epsilon/2) \log_2 n \rfloor} 2^{\frac{\ell-2}{\ell-1}s} n^{\frac{\ell}{\ell-1}(1-\epsilon/2)} \leq \frac{n^{\frac{\ell-2}{\ell-1}(1-\epsilon/2)}}{1 - 2^{-\frac{\ell-2}{\ell-1}}} \cdot n^{\frac{\ell}{\ell-1}(1-\epsilon/2)} \leq 4n^{2-\epsilon}.$$

Since  $G$  has at least  $cn^{2-\epsilon}$  edges, there is a subgraph  $G_s$  containing at least  $\frac{\epsilon}{4} 2^{\frac{\ell-2}{\ell-1}s} n^{\frac{\ell}{\ell-1}(1-\epsilon/2)}$  edges. As each component of  $G_s$  has at most  $2^{2s}$  edges, the number of components of  $G_s$  is  $t = \Omega(2^{-\frac{\ell}{\ell-1}s} n^{\frac{\ell}{\ell-1}(1-\epsilon/2)})$ . We set  $2^s = n^{1-\alpha}$ ,  $\alpha \geq \epsilon/2$  which means that  $t = \Omega(n^{\frac{\ell}{\ell-1}(\alpha-\epsilon/2)})$ . To summarize, we have  $t$  disjoint sets  $A_i$  of size  $\Theta(n^{1-\alpha})$ , each of which must be monochromatic under any feasible coloring. Thus we can apply Lemma 3.2 to conclude that for  $H = G + R$ , almost surely there is no feasible 2-coloring.  $\square$

## 4 General hypergraphs

In this section we deal with the general case of a 2-colorable  $k$ -uniform hypergraph  $H$ , to which we add a collection of random  $\ell$ -tuples  $R$ . Our goal is to prove Theorem 1.1 which asserts that if  $H$  has  $\Omega(n^{k-\epsilon})$  edges then adding to it  $\omega(n^{\ell\epsilon/2})$  random  $\ell$ -tuples makes it almost surely non-2-colorable. The proof will proceed by induction on  $k$ . The base case when  $k = 2$  follows from Proposition 3.1, so we can assume that  $k > 2$  and that the result holds for  $k - 1$ .

We start with a series of lemmas which allow us to make simplifying assumptions. Depending on the hypergraph  $H$ , we either reduce the problem to the  $(k - 1)$ -uniform case or prove directly that  $H + R$  is not 2-colorable.

Since we have  $\omega(n^{\ell\epsilon/2})$  random  $\ell$ -tuples available, we can divide them into a constant number of batches, where each batch still has  $\omega(n^{\ell\epsilon/2})$   $\ell$ -tuples. We will use a separate batch for each step of the induction. We write  $R = R_1 \cup R_2 \cup \dots \cup R_k$  where  $|R_i| = \omega(n^{\ell\epsilon/2})$  for each  $i$ .

**Lemma 4.1** *Let  $H_k$  be a  $k$ -uniform hypergraph on  $n$  vertices with  $c_1 n^{k-\epsilon}$  edges. Consider all  $(k-1)$ -tuples  $A \subset V(H_k)$  with degree greater than  $n^{1-\epsilon/2}$ . If there are at least  $\frac{c_1}{4} n^{k-1-\epsilon}$  such  $(k-1)$ -tuples in  $H_k$  then  $H_k + R$  is almost surely non-2-colorable.*

**Proof.** For each  $(k-1)$ -tuple  $A$  of degree  $> n^{1-\epsilon/2}$ , the neighborhood  $N(A)$  contains  $\Omega(n^{\ell-\ell\epsilon/2})$  distinct  $\ell$ -tuples. Therefore a random  $\ell$ -tuple lands inside  $N(A)$  with probability  $\Omega(n^{-\ell\epsilon/2})$ . Consequently, the probability that none of  $\omega(n^{\ell\epsilon/2})$  random  $\ell$ -tuples from  $R_k$  ends up inside  $N(A)$  is at most  $(1 - \Omega(n^{-\ell\epsilon/2}))^{\omega(n^{\ell\epsilon/2})} = o(1)$ . If we have  $t \geq \frac{c_1}{4} n^{k-1-\epsilon}$  such  $(k-1)$ -tuples, then the expected number of them, whose neighborhood does *not* contain any  $\ell$ -tuple in  $R_k$ , is  $o(t)$ . Therefore, by Markov's inequality, we get almost surely at least  $\frac{t}{2} \geq \frac{c_1}{8} n^{k-1-\epsilon}$   $(k-1)$ -tuples with an  $\ell$ -edge in their neighborhood. Denote by  $H_{k-1}$  the  $(k-1)$ -uniform hypergraph formed by these  $(k-1)$ -tuples.

By induction, we know that  $H_{k-1} + R_1 + \dots + R_{k-1}$  is almost surely non-2-colorable. Therefore for every 2-coloring respecting  $R_1 \cup \dots \cup R_{k-1}$ , there is a monochromatic  $(k-1)$ -tuple  $A$  in  $H_{k-1}$ . Without loss of generality assume that all vertices in  $A$  are colored by 1. By definition, the neighborhood  $N(A)$  contains an  $\ell$ -edge  $L \in R_k$ . Either  $L$  is monochromatic, or one of its vertices  $x$  is colored by 1 as well. But then  $A \cup \{x\}$  is a monochromatic edge of  $H_k$ . This implies that there is no feasible 2-coloring for  $H_k + R_1 + \dots + R_k$ .  $\square$

Thus we only need to treat the case where there are at most  $\frac{c_1}{4} n^{k-1-\epsilon}$   $(k-1)$ -tuples with degree greater than  $n^{1-\epsilon/2}$ , therefore at most  $\frac{c_1}{4} n^{k-\epsilon}$  edges through such  $(k-1)$ -tuples. We will get rid of these high degrees by removing a constant fraction of edges and making all degrees of  $(k-1)$ -tuples at most  $n^{1-\epsilon/2}$ . This would also imply a bound of  $n^{2-\epsilon/2}$  on the degrees of  $(k-2)$ -tuples, etc. However, in the following we show that for  $(k-2)$ -tuples we can assume an even stronger bound. More specifically, we prove that if we have many edges through  $(k-2)$ -tuples of degrees  $n^{2-\delta}$  with  $\delta \leq \frac{\ell}{2(\ell-1)}\epsilon$ , then we can proceed by induction. For this purpose, we first show the following.

**Lemma 4.2** *Let  $\ell \geq 2$  and let  $G$  be a graph on  $n$  vertices with  $n^{2-\delta}$  edges. Then  $G$  contains  $\frac{1}{2} n^{1-\delta}$  disjoint subsets of vertices  $F_1, F_2, \dots$  such that the vertices in each  $F_j$  have disjoint neighborhoods of sizes  $d_1, d_2, \dots$ , satisfying  $d_i \geq \frac{1}{2} n^{1-\delta}$  and*

$$\sum d_i^\ell \geq \frac{n^{\ell-(\ell-1)\delta}}{2^\ell}.$$

**Proof.** We iterate the following construction for  $j = 1, 2, \dots, \frac{1}{2} n^{1-\delta}$ .

- Take the vertex  $v_1$  of maximum degree  $d_1$  and remove all the edges incident to its neighbors. Note that by maximality of  $d_1$ , at most  $d_1^2$  edges are removed.
- In step  $i$ , take the vertex  $v_i$  of maximum degree  $d_i$  in the remaining graph and remove the edges incident to its neighbors (again, at most  $d_i^2$  edges). Repeat these steps, as long as  $\sum d_i^2 < \frac{1}{4} n^{2-\delta}$ .
- When the procedure terminates, define  $F_j = \{v_1, v_2, \dots\}$ . Then return to the original graph, but remove the vertices in  $F_j$  and all their edges permanently.



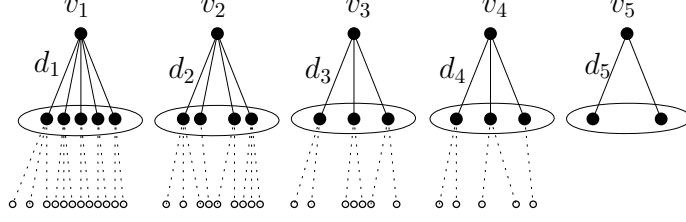


Figure 3: Construction of  $F_j = \{v_1, v_2, \dots\}$ . The neighborhood of  $v_i$  is incident with at most  $d_i^2$  edges.

By construction, the neighborhoods of the vertices in every  $F_j$  are disjoint and hence with each  $F_j$ , we remove  $\sum d_i \leq n$  edges from the graph. The sets  $F_j$  are also disjoint (although the neighborhoods of vertices from different  $F_j$ 's are not necessarily disjoint). Since we constructed  $\frac{1}{2}n^{1-\delta}$  sets  $F_j$ , there are at least  $n^{2-\delta} - \frac{1}{2}n^{1-\delta} \cdot n = \frac{1}{2}n^{2-\delta}$  edges available at the beginning of every construction.

Inside the construction of  $F_j$ , we repeat as long as  $\sum d_i^2 < \frac{1}{4}n^{2-\delta}$  and therefore we remove at most  $\frac{1}{4}n^{2-\delta}$  edges from the graph we started with. Hence, at every step the remaining graph still has at least  $\frac{1}{4}n^{2-\delta}$  edges and so its maximum degree is at least  $\frac{1}{2}n^{1-\delta}$ . When we terminate we have  $\sum d_i^2 \geq \frac{1}{4}n^{2-\delta}$ . This, together with the fact that  $d_i \geq \frac{1}{2}n^{1-\delta}$ , implies that for every  $F_j$  we have

$$\sum d_i^\ell \geq \left(\frac{1}{2}n^{1-\delta}\right)^{\ell-2} \sum d_i^2 \geq \frac{n^{\ell-(\ell-1)\delta}}{2^\ell}. \quad \square$$

**Lemma 4.3** *Let  $H_k$  be a  $k$ -uniform hypergraph on  $n$  vertices with  $c_1 n^{k-\epsilon}$  edges. Consider  $(k-2)$ -tuples of degree  $n^{2-\delta}$  where  $\delta \leq \frac{\ell}{2(\ell-1)}\epsilon$ . If there are at least  $\frac{c_1}{4}n^{k-\epsilon}$  edges through such  $(k-2)$ -tuples then  $H_k + R$  is almost surely non-2-colorable.*

**Proof.** Consider a  $(k-2)$ -tuple  $A$  of degree  $n^{2-\delta}$ . The link of  $A$  in  $H_k$  is a graph  $\Gamma(A)$  with  $n^{2-\delta}$  edges. By Lemma 4.2, we find  $\frac{1}{2}n^{1-\delta}$  subsets  $F_j$  such that vertices in  $F_j$  have disjoint neighborhoods in  $\Gamma(A)$  with sizes satisfying  $\sum d_i^\ell \geq 2^{-\ell}n^{\ell-(\ell-1)\delta}$ . We repeat this construction for each  $(k-2)$ -tuple of degree  $n^{2-\delta}$  with  $\delta \leq \frac{\ell}{2(\ell-1)}\epsilon$ . For each of them, we construct  $\frac{1}{2}n^{1-\delta}$  sets as above. Assuming that the total number of edges through such  $(k-2)$ -tuples is at least  $\frac{c_1}{4}n^{k-\epsilon}$ , we get  $\frac{c_1}{8}n^{k-1-\epsilon}$  sets  $F_j$  in total.

Now fix a set  $F_j$ . Call it *good* if after adding random  $\ell$ -tuples from  $R_k$  there is at least one vertex in  $F_j$  whose neighborhood in  $\Gamma(A)$  contains a random  $\ell$ -tuple. If this is not the case, call it *bad*. We estimate the probability that  $F_j$  is bad. By Lemma 4.2, the total number of  $\ell$ -tuples in the neighborhoods of vertices in  $F_j$  is

$$\sum \binom{d_i}{\ell} = \Omega\left(\sum \frac{d_i^\ell}{\ell!}\right) = \Omega\left(\frac{n^{\ell-(\ell-1)\delta}}{2^\ell \ell!}\right) = \Omega(n^{\ell-\ell\epsilon/2}).$$

Thus the probability that a random  $\ell$ -tuple falls inside some neighborhood of  $F_j$  is  $\sum \binom{d_i}{\ell} / \binom{n}{\ell} = \Omega(n^{-\ell\epsilon/2})$ . After adding the entire batch of random  $\ell$ -tuples  $R_k$ ,

$$\Pr[F_j \text{ is bad}] = \left(1 - \Omega(n^{-\ell\epsilon/2})\right)^{-\omega(n^{\ell\epsilon/2})} = o(1).$$

Consequently, the expected fraction of bad  $F_j$ 's is  $o(1)$ . By Markov's inequality, this fraction is almost surely at most one half, which means that at least  $\frac{c_1}{16}n^{k-1-\epsilon}$  sets  $F_j$  have a vertex  $v \in F_j$  whose neighborhood contains some  $\ell$ -tuple from  $R_k$ . For each such  $F_j$ , we have a set  $A$  of size  $k-2$  which together with  $v$  forms a  $(k-1)$ -tuple whose neighborhood in  $H_k$  contains an  $\ell$ -tuple from  $R_k$ . We could get the same  $(k-1)$ -tuple in  $k-1$  different ways, but in any case we have at least  $\frac{c_1}{16k}n^{k-1-\epsilon}$  such  $(k-1)$ -tuples which form an edge set of a  $(k-1)$ -uniform hypergraph  $H_{k-1}$ .

By the induction hypothesis,  $H_{k-1} + R_1 + \dots + R_{k-1}$  is almost surely non-2-colorable. Therefore, for any 2-coloring which respects the  $\ell$ -edges from  $R_1 + \dots + R_{k-1}$ , there must be a monochromatic  $(k-1)$ -edge  $B$  in  $H_{k-1}$ . However, since there is an  $\ell$ -edge from  $R_k$  in the neighborhood of  $B$ , one of its vertices should have the same color as  $B$ . This would form a monochromatic edge in  $H_k$  so there is no feasible 2-coloring for  $H_k + R_1 + \dots + R_k$ .  $\square$

Thus we can also assume that at most  $\frac{c_1}{4}n^{k-\epsilon}$  edges go through  $(k-2)$ -tuples of degree  $n^{2-\delta}$ ,  $\delta \leq \frac{\ell}{2(\ell-1)}\epsilon$ . Before the last part of the proof, we make further restrictions on the degree bounds and structure of our hypergraph, by finding a subhypergraph  $H_\alpha$  described in the following lemma.

**Lemma 4.4** *Let  $H_k = (V, E)$  be a  $k$ -uniform hypergraph with  $c_1n^{k-\epsilon}$  edges, such that at most  $\frac{c_1}{4}n^{k-\epsilon}$  edges go through  $(k-1)$ -tuples of degree  $\geq n^{1-\epsilon/2}$  and at most  $\frac{c_1}{4}n^{k-\epsilon}$  edges go through  $(k-2)$ -tuples of degree  $n^{2-\delta}$ ,  $\delta \leq \frac{\ell}{2(\ell-1)}\epsilon$ . Then for some constant  $\alpha \geq \epsilon/2$ ,  $H_k$  contains a subhypergraph  $H_\alpha$  with the following properties*

1.  $H_\alpha$  is  $k$ -partite, i.e.  $V$  can be partitioned into  $V_1 \cup V_2 \cup \dots \cup V_k$  so that every edge of  $H_\alpha$  intersects each  $V_i$  in one vertex.
2. Every vertex has degree at most  $n^{k-1-\frac{\ell}{2(\ell-1)}\epsilon}$ .
3. The degree of every  $(k-1)$ -tuple in  $V_1 \times V_2 \times \dots \times V_{k-1}$  is either 0 or between  $n^{1-\alpha}$  and  $2n^{1-\alpha}$ .
4. The number of edges in  $H_\alpha$  is at least

$$c_5 \left( n^{k-\epsilon-\frac{\epsilon-\alpha}{\ell-1}} + n^{k-\epsilon-\frac{\ell-2}{\ell-1}(\alpha-\epsilon/2)} \right),$$

for some constant  $c_5 = c_5(k, \ell, c_1)$ .

**Proof.** First, remove all edges through  $(k-1)$ -tuples of degree  $\geq n^{1-\epsilon/2}$  and through  $(k-2)$ -tuples of degree  $n^{2-\delta}$ ,  $\delta \leq \frac{\ell}{2(\ell-1)}\epsilon$ . We get a hypergraph  $H'$  such that the degrees of all  $(k-1)$ -tuples are at most  $n^{1-\epsilon/2}$ , the degrees of all  $(k-2)$ -tuples are at most  $n^{2-\frac{\ell}{2(\ell-1)}\epsilon}$ , and the number of edges is at least  $c_2n^{k-\epsilon}$  edges,  $c_2 = c_1/2$ . Consequently, the degree of every vertex in  $H'$  is at most  $n^{k-3} \cdot n^{2-\frac{\ell}{2(\ell-1)}\epsilon} = n^{k-1-\frac{\ell}{2(\ell-1)}\epsilon}$ .

Next, we use a well known fact, proved by Erdős and Kleitman [12] that every  $k$ -uniform hypergraph  $H'$  with  $c_2n^{k-\epsilon}$  edges contains a  $k$ -partite subhypergraph with at least  $c_3n^{k-\epsilon}$  edges where  $c_3 = \frac{k!}{k^k}c_2$ . This can be achieved for example by taking a random partition of the vertex set into  $k$  parts and computing the expected number of edges which intersect all of them. Let  $(V_1, V_2, \dots, V_k)$

be a partition, so that at least  $c_3 n^{k-\epsilon}$  edges of  $H'$  have one vertex in every  $V_i$ . Discard all other edges and denote this  $k$ -partite hypergraph by  $H''$ .

Consider all  $(k-1)$ -tuples in  $V_1 \times V_2 \times \dots \times V_{k-1}$  whose degree in  $H''$  is less than  $\frac{c_3}{2} n^{1-\epsilon}$ . Delete all their edges, which is at most  $\binom{n}{k-1} \frac{c_3}{2} n^{1-\epsilon} \leq \frac{c_3}{2} n^{k-\epsilon}$  edges in total. We still have at least  $c_4 n^{k-\epsilon}$  edges, where  $c_4 = c_3/2$ . Now the degree of every  $(k-1)$ -tuple in  $V_1 \times V_2 \times \dots \times V_{k-1}$  is either 0 or between  $c_4 n^{1-\epsilon}$  and  $n^{1-\epsilon/2}$ . Finally, we are going to find a subhypergraph in which all the non-zero degrees of  $(k-1)$ -tuples are  $\Theta(n^{1-\alpha})$  and the number of edges is at least

$$c_5 \left( n^{k-\epsilon-\frac{\epsilon-\alpha}{\ell-1}} + n^{k-\epsilon-\frac{\ell-2}{\ell-1}(\alpha-\epsilon/2)} \right).$$

The existence of such a subhypergraph can be proved by an elementary counting argument. Let  $n^{1-\alpha} = 2^i$  and partition  $V_1 \times V_2 \times \dots \times V_{k-1}$  into groups of  $(k-1)$ -tuples with degrees in intervals  $[2^i, 2^{i+1})$ , where  $i$  ranging between  $i_1 = \log_2(c_4 n^{1-\epsilon})$  and  $i_2 = \log_2(n^{1-\epsilon/2})$ . Consider the following two expressions:

$$\sum_{i=i_1}^{i_2} 2^{-i/(\ell-1)} \leq \frac{(c_4 n^{1-\epsilon})^{-\frac{1}{\ell-1}}}{1 - 2^{-\frac{1}{\ell-1}}} \leq 2(\ell-1) c_4^{-1} n^{-\frac{1-\epsilon}{\ell-1}}$$

and

$$\sum_{i=i_1}^{i_2} 2^{\frac{\ell-2}{\ell-1}i} \leq \frac{n^{\frac{\ell-2}{\ell-1}(1-\epsilon/2)}}{1 - 2^{-\frac{\ell-2}{\ell-1}}} \leq 4n^{\frac{\ell-2}{\ell-1}(1-\epsilon/2)}.$$

Normalizing by the right-hand side and taking the average, we get

$$\sum_{i=i_1}^{i_2} \left( \frac{2^{-\frac{i}{\ell-1}}}{4(\ell-1) c_4^{-1} n^{-\frac{1-\epsilon}{\ell-1}}} + \frac{2^{\frac{\ell-2}{\ell-1}i}}{8n^{\frac{\ell-2}{\ell-1}(1-\epsilon/2)}} \right) \leq 1$$

By the pigeonhole principle, there is an  $i$  such that the fraction of edges through  $(k-1)$ -tuples with degree between  $2^i = n^{1-\alpha}$  and  $2^{i+1} = 2n^{1-\alpha}$  is at least

$$\frac{2^{-\frac{i}{\ell-1}}}{4(\ell-1) c_4^{-1} n^{-\frac{1-\epsilon}{\ell-1}}} + \frac{2^{\frac{\ell-2}{\ell-1}i}}{8n^{\frac{\ell-2}{\ell-1}(1-\epsilon/2)}} = \frac{c_4}{4(\ell-1)} n^{-\frac{\epsilon-\alpha}{\ell-1}} + \frac{1}{8} n^{-\frac{\ell-2}{\ell-1}(\alpha-\epsilon/2)}$$

so the lemma holds with  $c_5 = c_4 \cdot \min \left\{ \frac{c_4}{4(\ell-1)}, \frac{1}{8} \right\}$ .  $\square$

Note that in this lemma, we lose more than a constant fraction of the edges. However, from now on, we do not use induction anymore and will prove directly that  $H_\alpha + R$  is almost surely non-2-colorable. We will proceed in  $t = c_5 \ell^{-k} n^{\frac{\ell}{\ell-1}(\alpha-\epsilon/2)}$  stages. For each stage, we allocate a certain number of random  $\ell$ -tuples. Namely, we set again  $R = R_1 \cup R_2 \cup \dots \cup R_k$ ,  $|R_i| = \omega(n^{\ell\epsilon/2})$ . Furthermore, we divide each  $R_j$  for  $j \leq k-1$  into  $t$  parts  $R_{1,j}, \dots, R_{t,j}$  so that

$$|R_{i,j}| = \omega \left( \frac{n^{\ell\epsilon/2}}{t} \right) = \omega \left( n^{\ell\epsilon/2 - \frac{\ell}{\ell-1}(\alpha-\epsilon/2)} \right).$$

The random set  $R_{i,j}$  will be used for the  $j$ -th “level” of the  $i$ -th stage. The following lemma describes one stage of the construction. Finally,  $R_k$  will be used in the last step of the proof.

**Lemma 4.5** *Let  $H_\alpha$  be a  $k$ -uniform  $k$ -partite hypergraph where the degree of every  $(k-1)$ -tuple in  $V_1 \times V_2 \times \dots \times V_{k-1}$  is either zero or is in the interval  $[n^{1-\alpha}, 2n^{1-\alpha}]$ , and the number of edges in  $H_\alpha$  is at least*

$$c_5 n^{k-\epsilon-\frac{\ell-2}{\ell-1}(\alpha-\epsilon/2)}.$$

*Then almost surely, there exists a family of  $q = \ell^{k-2}$  sets  $S_1, \dots, S_q$ ,  $n^{1-\alpha} \leq |S_i| \leq 2n^{1-\alpha}$ , such that for every feasible 2-coloring of  $H_\alpha + R_{i,1} + \dots + R_{i,k-1}$  at least one  $S_i$  is monochromatic.*

**Proof.** We are going to construct an  $\ell$ -ary tree  $T$  of depth  $k-1$ . We denote vertices on the  $j$ -th level by  $v_{a_1 a_2 \dots a_{j-1}}$  where  $a_i \in \{1, 2, \dots, \ell\}$ .  $T$  is rooted at a vertex in  $V_1$  and the  $j$ -th level is contained in  $V_j$ . We construct  $T$  in such a way that the vertices along every path which starts at the root and has length  $k-1$  form a  $(k-1)$ -tuple with degree  $\Theta(n^{1-\alpha})$  in  $H_\alpha$ . The neighborhoods of all branches of length  $k-1$  will be our sets  $S_i$  (not necessarily disjoint). In addition, the set of  $\ell$  children of every node on each level  $j \leq k-2$ , like  $\{v_{a_1 a_2 \dots a_{j-1} 1}, v_{a_1 a_2 \dots a_{j-1} 2}, \dots, v_{a_1 a_2 \dots a_{j-1} \ell}\}$ , will form an edge of  $R_{i,j}$ .

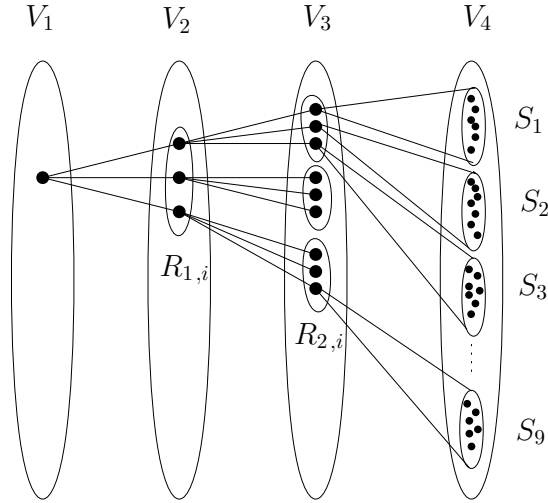


Figure 4: Construction of the tree  $T$ , for  $k = 4$  and  $\ell = 3$ . Branches of the tree form active  $(k-1)$ -tuples, with neighborhoods  $S_i$ . Each set of children on level  $j+1$  forms an edge of  $R_{i,j}$ .

Assuming the existence of such a tree, consider any 2-coloring of  $H_\alpha + R_{i,1} + \dots + R_{i,k-1}$ . Since the children of each vertex on level  $j < k-1$  form an  $\ell$ -edge in  $R_{i,j}$ , every vertex has children of both colors. In particular, there is always one child with the same color as its parent. Therefore, starting from the root, we can always find a monochromatic branch  $A$  of length  $k-1$ . Since all the extensions of this branch to edges of  $H_\alpha$  must be 2-colored, all the vertices in  $S_i = N(A)$  must have the same color.

We grow the tree level by level, maintaining the property that all branches have sufficiently many extensions to edges of  $H_\alpha$ . More precisely, we call an  $r$ -tuple in  $V_1 \times \dots \times V_r$  *active* if its degree is at least

$$\Delta_r = \frac{c_5}{2^r} n^{k-r-\epsilon-\frac{\ell-2}{\ell-1}(\alpha-\epsilon/2)}.$$

**Claim.** Every active  $r$ -tuple  $A$ ,  $r \leq k-2$ , can be extended to at least

$$d_r = \frac{\Delta_r}{4n^{k-r-1-\alpha}} = \frac{c_5}{2^{r+2}} n^{1-\epsilon/2 + \frac{1}{\ell-1}(\alpha-\epsilon/2)}$$

active  $(r+1)$ -tuples  $A \cup \{x\}$ ,  $x \in V_{r+1}$ .

*Proof.* Suppose that fewer than  $d_r$  extensions of  $A$  are active. Since the degrees of  $(k-1)$ -tuples are at most  $2n^{1-\alpha}$ , we get that any  $(r+1)$ -tuple has degree at most  $2n^{k-r-1-\alpha}$ . Therefore the number of edges through all active extensions of  $A$  is smaller than  $d_r \cdot 2n^{k-r-1-\alpha} = \frac{1}{2}\Delta_r$ . We also have inactive extensions of  $A$  which have degrees less than  $\Delta_{r+1}$ . The total number of edges through these extensions of  $A$  is smaller than  $n\Delta_{r+1} = \frac{1}{2}\Delta_r$ . But the total number of edges through  $A$  is at least  $\Delta_r$ . This contradiction proves the claim.  $\square$

We start our construction from an active vertex  $v \in V_1$ . Since  $H_\alpha$  has at least  $n\Delta_1$  edges, such a vertex must exist. By our claim,  $v$  can be extended to at least  $d_1$  active pairs  $\{v, x\}$ ,  $x \in W_2 \subset V_2$ . Consider this set of  $d_1$  vertices  $W_2$ . The probability that a random  $\ell$ -tuple falls inside  $W_2$  is  $\binom{d_1}{\ell} / \binom{n}{\ell} = \Omega(n^{-\ell\epsilon/2 + \frac{\ell}{\ell-1}(\alpha-\epsilon/2)})$ . Now we use  $\omega(n^{\ell\epsilon/2 - \frac{\ell}{\ell-1}(\alpha-\epsilon/2)})$  random  $\ell$ -tuples from  $R_{i,1}$  that we allocated for the first level of this construction. This means that almost surely, we get an  $\ell$ -edge  $\{v_1, \dots, v_\ell\} \in R_{i,1}$  such that  $\{v, v_i\}$  is an active pair for each  $i = 1, 2, \dots, \ell$ .

We continue growing the tree, using the random  $\ell$ -tuples of  $R_{i,j}$  on level  $j$ . Since we have ensured that each path from the root to the level  $j$  from an active  $j$ -tuple, it has at least  $d_j$  extensions to an active  $(j+1)$ -tuple. Again, the probability that a random  $\ell$ -tuple hits the extension vertices  $W_{j+1} \subset V_{j+1}$  for a given path is  $\binom{d_j}{\ell} / \binom{n}{\ell} = \Omega(n^{-\ell\epsilon/2 + \frac{\ell}{\ell-1}(\alpha-\epsilon/2)})$ . Almost surely, one of the  $\ell$ -tuples in  $R_{i,j}$  will hit these extension vertices and we can extend this path to  $\ell$  children on level  $j+1$ . The number of paths from the root to level  $j$  is bounded by  $\ell^{j-1}$  which is a constant, so in fact we will almost surely succeed to build the entire level.

In this way, we a.s. build the tree all the way to level  $k-1$ . Every path from the root to one of the leaves forms an active  $(k-1)$ -tuple and has degree  $\in [n^{1-\alpha}, 2n^{1-\alpha}]$ . Define  $S_1, S_2, \dots, S_q$  to be the neighborhoods of all these  $q = \ell^{k-2}$  paths. By construction, for any feasible 2-coloring of  $H_\alpha + R_{i,1} + \dots + R_{i,k-1}$ , one of these paths is monochromatic which implies that the corresponding set  $S_i$  is monochromatic as well.  $\square$

**Lemma 4.6** *Let  $H_\alpha$  be a  $k$ -uniform  $k$ -partite hypergraph where the degree of every vertex is at most  $n^{k-1 - \frac{\ell}{2(\ell-1)}\epsilon}$ , the degree of every  $(k-1)$ -tuple in  $V_1 \times V_2 \times \dots \times V_{k-1}$  is either zero or is in the interval  $[n^{1-\alpha}, 2n^{1-\alpha}]$ , and the number of edges in  $H_\alpha$  is at least*

$$c_5 n^{k-\epsilon - \frac{\epsilon-\alpha}{\ell-1}} + c_5 n^{k-\epsilon - \frac{\ell-2}{\ell-1}(\alpha-\epsilon/2)}.$$

*Then almost surely,  $H_\alpha + R$  is not 2-colorable.*

**Proof.** We apply Lemma 4.5 repeatedly in  $t = c_5 \ell^{-k} n^{\frac{\ell}{\ell-1}(\alpha-\epsilon/2)}$  stages. In each stage  $i$ , we almost surely obtain  $q = \ell^{k-2}$  sets  $S_{i,1}, \dots, S_{i,q}$ ,  $n^{1-\alpha} \leq |S_{i,j}| \leq 2n^{1-\alpha}$  such that for any 2-coloring of the hypergraph  $H_\alpha + \sum R_{i,j}$ , one of these sets must be monochromatic. If this happens, we call such a stage “successful”. After each successful stage, we remove all edges of  $H_\alpha$  incident with any of

the sets  $S_{i,1}, \dots, S_{i,q}$ . Since degrees are bounded by  $n^{k-1-\frac{\ell}{2(\ell-1)}\epsilon}$  and we repeat  $t = c_5 \ell^{-k} n^{\frac{\ell}{\ell-1}(\alpha-\epsilon/2)}$  times, the total number of edges we remove is at most

$$\sum_{i=1}^t \sum_{j=1}^q |S_{i,j}| n^{k-1-\frac{\ell}{2(\ell-1)}\epsilon} \leq tq \cdot 2n^{1-\alpha} \cdot n^{k-1-\frac{\ell}{2(\ell-1)}\epsilon} = 2c_5 \ell^{-2} n^{k-\epsilon-\frac{\epsilon-\alpha}{\ell-1}} \leq c_5 n^{k-\epsilon-\frac{\epsilon-\alpha}{\ell-1}}.$$

In particular, before every stage we still have at least  $c_5 n^{k-\epsilon-\frac{\ell-2}{\ell-1}(\alpha-\epsilon/2)}$  edges available, so we can use Lemma 4.5. Since the expected number of stages that are not successful is  $o(t)$ , by Markov's inequality, we almost surely get at least  $t/2$  successful stages. Eventually, we obtain sets  $S_{i,j}$  for  $1 \leq i \leq t/2$  and  $1 \leq j \leq q$  such that

- For  $i_1 \neq i_2$  and any  $j_1, j_2$ ,  $S_{i_1, j_1} \cap S_{i_2, j_2} = \emptyset$ .
- For any 2-coloring of  $H_\alpha + \sum R_{i,j}$  and any  $i$ , there is  $j_i$  such that  $S_{i, j_i}$  is monochromatic.

Finally, we add once again a collection  $R_k$  of  $\omega(n^{\ell\epsilon/2})$  random  $\ell$ -tuples. We do not know a priori which selection of sets  $S_{i,j}$  will be monochromatic but there is only exponential number of choices  $q^{t/2} = e^{O(t)}$ . For any specific choice of sets to be monochromatic, Lemma 3.2 says that the probability that after adding  $\omega(n^{\ell\epsilon/2})$  random  $\ell$ -tuples, there is a feasible 2-coloring keeping these sets monochromatic, is  $e^{-\omega(t)}$ . By the union bound, the probability that there exist a proper 2-coloring of  $H_\alpha + \sum R_{i,j} + R_k$  is at most  $q^{t/2} e^{-\omega(t)} = o(1)$ . This completes the proof of this lemma together with the proof of Theorem 1.1.  $\square$

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