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VOTING FOR VOTERS:
A MODEL OF ELECTORAL EVOLUTION[1]

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Abstract. We model the decision problems faced by the members of societies whose new members are determined by vote. We adopt a number of simplifying assumptions: the founders and the candidates are fixed; the society operates for k periods and holds elections at the beginning of each period; one vote is sufficient for admission, and voters can support as many candidates as they wish; voters assess the value of the streams of agents with whom they share the society, while they belong to it. In spite of these simplifications, we show that interesting strategic behavior is implied by the dynamic structure of the problem: the vote for friends may be postponed, and it may be advantageous to vote for enemies. We discuss the existence of different types of equilibria in pure strategies and point out interesting equilibria in mixed strategies.

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1. Introduction

Human societies evolve, grow and shrink, as the result of exit and entry. We are interested in the evolution of those societies where entry is regulated by the use of formal voting procedures: new members are admitted only if they receive enough support from inside, according to well specified rules.

Clubs and learned societies are examples of human groups that fit our description exactly. Others may only meet part of the features we require here. For example, parliaments are elected according to well specified rules, but their size is fixed, while our focus will be on the forces that determine the growth or the stagnation of groups. In other cases, entry and exit are the result of informal procedures, whose description as voting rules might be too simplistic even as an approximation. Our model, thus, only applies to a restricted set of societies.

Election rules are social constructs: they may come from an agreement among different founders, they may reflect the will of a unique founder or they may be the result of successive amendments, but they must be set purposely. Once the rules for election to a society are set, participants in the election are bound to engage in strategic considerations that involve non-myopic behavior. In particular, voters cannot overlook the fact that newly elected members will become voters in later elections: this may lead to postpone the election of individually attractive candidates who might vote in unattractive ways, or to accelerate the election of a poor candidate whose vote is needed. We are interested in the evolution of groups which results from considerations of this type being made by rational agents under well specified voting rules. The features we have emphasized should make it clear that electoral evolution is the result of nonmyopic behavior which is quite typical to human societies.

Since this paper is a first attempt at modeling such facts, we allow ourselves some strong simplifying assumptions. The founders and the rules of election of a society are fixed in advance (we don't explain why they join to create the society or why they agree on these rules). The candidates to enter the society are fixed as well (we don't explain why they don't try to create other societies, or any other process by which eligible candidates could change from election to election). We assume that nobody leaves the society once admitted (thus concentrating on entry and not on exit). We study finite horizon situations where members of the society know at all times when it will be dissolved and voting takes place at a finite number of periods (when in fact many societies operate under an uncertain horizon). We assume a specific voting method, whereby each member can vote for as many candidates as he wishes, and it is enough for a candidate to receive a vote in order to be admitted (this is the method of 'voting by quota one'; many others are worth considering). We postulate that agents' preferences are defined over streams of members in the society. Under these assumptions, we provide theorems on the existence and the characteristics of different types of equilibria of the games generated in such dynamic voting contexts. Although clearly restricted by our assumptions, these results bear witness to the abundance of possibilities within our model.

In addition to general theorems, we also provide many examples, some of which reflect quite unexpected phenomena. The simplicity of our model, when it comes to examples, becomes an asset: whatever counterintuitive results we exhibit are robust, since they happen even in simple situations. For instance, we shall prove that agents may want to vote for their enemies. This would not be surprising if they needed the votes of others in order to advance their friends to membership. But it is quite striking under our extreme assumption of vote by quota one, where each voter alone can assure his friends' admission! Also, many of our examples postulate a very simple structure of preferences: each voter is assumed to classify candidates as enemies or friends, and streams of elected members are valued as the sum of utilities derived from elected friends — one unit per period — plus the sum of disutilities derived from having enemies elected — essentially minus one per period. Revealing interesting strategic behavior under such simple preferences reinforces our points.

Our closest reference is “Voting by Committees”, by Barberà, Sonnenschein and Zhou (1991), where the question of electing members for a society is treated as a one period problem. That paper characterizes the set of all strategy-proof mechanisms respecting the sovereignty of voters when their preferences over sets of candidates satisfy one of two alternative restrictions, called additivity or separability: they are the methods of voting by committees. We shall not describe the general class, but simply say that they contain an interesting subclass of methods, which in addition to the preceding properties will also respect anonymity and neutrality; i.e., will treat all voters and all candidates alike. This subclass consists of the methods based on voting by quota: each agent can vote for as many candidates as he wishes, and all candidates who get at least q votes are elected, where q is fixed a priori. Our main interest in the present paper is on phenomena that only arise when the society’s horizon is greater than one period, and this is why we have chosen to work with multiperiod models whose one period version takes the form of voting by quota. Since these methods are strategy-proof in their one shot version, we can be sure that whatever strategic behavior arises when several periods are considered must have a dynamic source.

As already mentioned, our ambition is to study the evolution of societies who resort to voting as a means to include or to exclude members. It has both a normative and a positive viewpoint. Many interesting questions come to mind. Just to mention one topic on the descriptive side, we would like to understand why some societies maintain their defining features along their history, while others change so much that their own founders would not recognize them. However, our ambition must be tempered by the fact that the game theoretic analysis quickly becomes complex and presents several alternative routes. Accordingly, the paper contains examples, which point at the complexities of the analysis, as well as technical results on how to solve for equilibria and what types of equilibria to look for. It is structured as follows. In Section 2 we present the model, based on a gallery of assumptions. Section 3 contains examples. These examples show that the simplicity of the one period model is immediately lost if we have several periods. They also prove that some counterintuitive phenomena, like strategic voting for enemies, can occur if the number of periods is not too small. They also indicate that it will be worth analyzing not one but several solution concepts, because each one of them can provide some insight on the phenomena we try to model. One example shows that, although we concentrate on pure-strategy equilibria, the use of mixed strategies, or even correlated strategies, may be most reasonable in some cases. In Section 4 we analyze subgame-perfect equilibria and 'quasi-strong equilibria',[5] and we discuss the fact that the streams of members for a society can be attained in equilibrium, given the rules, through different distributions of the individual votes. In this section we also look for Pareto-undominated equilibria. Unfortunately, Pareto undominated equilibrium profiles are often not perfect equilibria. Thus, the members may wish to adopt less profitable outcomes in order to gain the stability that a perfect equilibrium yields. Section 5 is devoted to the existence of perfect equilibria in pure strategies: we provide a sufficient condition under which there will exist such equilibria, and examples showing that the condition is not necessary. We are able to show that if certain additivity assumptions are satisfied, every game that is generated by a generic 2-stage voting scheme has a pure-strategy perfect equilibrium.

[5]i.e., equilibria that have the additional property that no deviator can benefit if the set of deviators does not include the set of all voters at the start of a deviation.

2. The model

We want to analyze the results from imposing some electoral rules on the evolution of societies. The necessary elements to describe the rules, which we call *(finite horizon) voting schemes*, are the following:

(1) A nonempty set of *original founders*, denoted F^0, who belong to *society* at the initial stage and from stage to stage vote to bring in other members and/or to remove members. 'Society' may be an organization, a club, a foundation or similar enterprises.

(2) A set of *candidates* from whom new members can be chosen. This population may vary from stage to stage.

(3) A set of *voters* for each stage. Often, all elected members can vote at all stages following their election for as long as they belong to the society.

(4) A set of *rules* which specify under what conditions a person is admitted to the society, or is expelled, or resigns.

(5) A number of stages k during which the society operates. After k stages the society dissolves, having concluded its tasks, and *the play* is over.

An important part of the outcome of the voting scheme is the resulting *stream of members*, denoted $\mathcal{F} := \{F^1, F^2, \ldots, F^k\}$, where F^t represents the members at stage t, after the elections, expulsions and resignations at that stage. Another part may be information concerning who voted at each stage and for whom. Some of the above may be unknown to some, or all the agents. All of the information that is available to agent i until stage t constitutes his $(t-1)$-stage history.

The decision on how to vote at each stage, that every voter i faces, should take into consideration the priorities that each agent has over the various streams.[6]

[6] One can think of complicated priorities on events that may even be concealed. For example, a voter might not

As mentioned in the introduction, we make many simplifying assumptions in order to render the model simple and yet still capture some dynamic aspects of the workings of the voting scheme. In fact, we suppress many aspects in order not to 'blur' the purely dynamic issues. Obviously, other, more complicated and more realistic models should be studied. As we show, even the present simple model possesses enough intricacies to render the analysis interesting.

Some simplifying assumptions.

1. FIXED POPULATION. *We assume that the population is finite and fixed and includes the nonempty set of the original founders* F^0. Therefore, we can denote the set of *agents* by N. $N \setminus F^0$ is called *the set of the original candidates* and is denoted by C^0. Similarly, we write C^t for $N \setminus F^t$. Members of C^{t-1} are the candidates from whom the voters F^{t-1} can choose at stage t.
2. NO FIRING. *We assume that an elected candidate will stay in the society all the time.* There are no provisions to fire him.
3. NO RESIGNATION. NORMALIZATION. *Once an agent is admitted to the society, he will stay there throughout the performance of the society. Staying alone in the society has a zero utility.*[7]

   The no resignation requirement makes sense if staying in the society is highly prestigious. Nevertheless, even then it is a restriction. For example, it rules out strategies involving threats to resign, as punishments, if deviations occur.

   In this paper we take the position that an agent becomes a player only after he joins the

like an agent j, if he knew that agent p also voted for j, but otherwise he might have loved to have j in the society. Perhaps he does not even know who elected j. We shall not consider such complications in this paper.

[7]Sometimes we change the normalization, so that a zero utility corresponds to a situation where the agent stays in the society together with the original founders F^0. The reader will have no difficulty in deciding to which normalization we refer in each instance.

society. We shall rarely compare his utilities while in the society to his utilities before he joined the society.

4. 1-QUOTA VOTING. The rule for electing a candidate into the society is simple: *every voter can bring any number of candidates into the society at any stage, simply by casting a vote for them at the beginning of that stage*. This rule is known as *voting by quota 1.*
5. STREAMS OF MEMBERS ARE ALL THAT MATTER. We assume that *each agent cares only about the streams of members in the society* and does not care, for example, about who voted, or who did not vote for each member. This allows us to require that all his actions are based only on what he knows about the developing streams.
6. COMMON HISTORIES. We assume that *at each stage the elected candidates are known to everyone*. Thus, for every agent i the relevant $(t-1)$-stage histories are the same;[8] namely, subsequences of the streams terminating at F^{t-1}. These will be denoted h^t, $t = 1, 2, \ldots, k$. Thus, $h^t := \{F^0, F^1, \ldots, F^{t-1}\}$.

We now have all the ingredients to convert the above setup into a *game form*: The set of *players* is N, the *pure strategies* available to player i are choices of sets that specify at each stage t the candidates that he votes for at that stage only as a function of the history at that stage. Thus, we do not allow the strategies to depend on what agent voted for whom, in the past.

With this notation, a pure strategy for agent i, can be expressed as $\sigma_i := (\sigma_i^1, \sigma_i^2, \ldots, \sigma_i^k)$, where $\sigma_i^t(h^t)$ denotes the set of agents chosen by agent i, given[9] the history h^t.

[8] Actually, if ballots are not secret, histories may be more complicated than simply the past stream of members. They may include information such as who voted for whom, and when. In this paper we shall not employ such histories.

[9] Note that we allow a strategy of an agent to depend on the part of the stream that existed before he entered the coalition. Usually, however, this may not be the case. Note also that there is a redundancy in this notation: What agent i "votes for" in stages before he was admitted into the society has no effect on the resulting stream of members. We use this notation for the sake of brevity.

From this description one can realize that we formally allow a player at each stage to vote even for agents that were already elected (including himself) and we allow an agent to vote even if he is not elected. This is done merely for mathematical convenience. Of course such votes will have no effect on the stream of members. Given a strategy profile $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$, the stream of members is given by

$$F^t = F^t(\sigma) = F^{t-1} \cup \big(\cup_{i \in F^{t-1}} \sigma_i^t(h^t) \big), \qquad (t = 1, 2, \dots, k). \tag{2.1}$$

Most of this paper will deal with pure strategies. Since the game is of perfect recall, by Kuhn's (1953) theorem (see also Selten 1975), even when we do employ mixed strategies we can restrict ourselves to behavioral strategies, in which case it is sufficient to consider the probability distribution on the various histories.

To convert our game form into a game we now introduce priorities and utilities.

7. Known utilities. We assume that *the priorities of agent i are given by complete and transitive binary relations on the set of outcomes* and therefore they can be represented by a *utility function u_i*. Later, when we deal with mixed strategies, we shall assume that *these utilities are, in fact, von Neumann Morgenstern utilities.*[10]

Once an agent is in the society, every stream that is better for him than staying alone is assigned a positive utility. Every stream that is worse for him is assigned a negative utility (still larger than the utility of not being in the society).

We now present several *possible simplifying assumptions on the utilities*, ranging from simple to more complicated considerations. Some of them will be employed in the examples of the next section, to illustrate some of the issues. Others will be needed for the proofs.

[10] This, of course involves more assumptions on the binary priority relations.

The simplest model in this paper assumes that for every pair of distinct agents i, j, either i *likes* j, or i *dislikes* j. Expressing it differently, we say that either j is a *friend of* i or he is an *enemy* of i, where friendship and enmity merely mean that he wants or does not want the person in the society. This does *not* imply that a voter will always vote for his friend. He may be reluctant to do so if, for example, he thinks that his friend may bring enemies to the society.

We do not assume that the "friendship" relation is either symmetric, or transitive: Agent j can be a friend of i, yet i is regarded as an enemy by j. Also, a friend of a friend need not be a friend.

'A friend' may be interpreted in several ways, such as: 'the voter enjoys his company', 'the voter thinks he will be useful for the workings of the society', 'that his opinion should be heard, because it is relevant', etc. Likewise 'an enemy' can have opposite interpretations.

We then assume that *each agent wishes to spend as much time as possible with friends and as little time as possible with enemies and that this is all he cares for*. If the stages are equally spaced in time, it then makes sense to denote by 1 the utility of having a friend in the committee for one stage and by $(-1-\varepsilon)$ — the utility of having an enemy for one stage, where ε is a small positive number, added to break ties.[11]

If the voters are not sophisticated and only durations of time spent with 'friends' and 'enemies' matter, it makes sense to choose additive utilities. We summarize the above formally:

8a. Pure friendship and enmity. *The utility for a stream of members, given by (2.1), for an agent who succeeds in entering the society is given by*

$$u_i(\mathcal{F}) = \sum_{\{t\geq 1:\ i\in F^t\}} |F^t \cap \mathrm{fr}\,(i)| - (1+\varepsilon) \sum_{\{t\geq 1:\ i\in F^t\}} |F^t \cap \mathrm{en}\,(i)|, \tag{2.2}$$

[11] We decided to require a positive ε in order to express the fact that, other things being equal, the members would like to have a society with as few conflicts as possible: it is worse to have a friend and an enemy for a certain period of time than to have neither of them for that period.

where $|S|$ *denotes the cardinality of* S, $\mathrm{fr}\,(i)$ *denotes the set of friends of* i *and* $\mathrm{en}\,(i)$ *denotes the set of enemies of* i. Here, $\mathrm{fr}\,(i) \cup \mathrm{en}\,(i) = N \setminus \{i\}$ for each agent i.

In a more sophisticated model we can still assume that whether or not to vote for a person is decided on purely personal grounds; namely, only on the merits of the person and not, e.g., on who is already in the society, but now we let agents also take into consideration *how much* they like/dislike each person.

Individual considerations may be quite complicated: a voter may like one person and dislike another. He may want a person in the society, because he thinks that his views should be heard. He may want a person in to balance an extreme stand of a founder, etc. Here we make the strong assumption that whatever these considerations are, they can be summed up by *each agent providing each individual with a time-independent and society-independent "weight function", so that the sum of the weights reflects the utility of the voter for one stage.*

Naturally the weights still allow us to distinguish between friends and enemies. Friends will be agents with positive weight and enemies — with negative weights. If the weight is zero, we can call him neutral for the voter.

We couple the above assumption with the idea that *a voter wants to spend as much time as possible with friends and as little time as possible with enemies.* Together, the above brings about the next simplifying assumption:

8b. FRIENDS AND ENEMIES. ADDITIVITY WITHIN EACH STAGE AND ACROSS STAGES. *Every agent* i *has a weight function* $w_i : N \to \Re$. *His utility* $u_i(\mathcal{F})$ *for a stream of members* $\mathcal{F}$ *serving in the society is given by*:

$$u_i(\mathcal{F}) = \sum_{\{t \geq 1:\ i \in F^t\}} \sum_{a \in F^t} w_i(a). \tag{2.3}$$

Thus, $w_i(a)$ can be interpreted as the utility that i accumulates from spending one stage in the society together with a.

How a weight function w_i is determined in real life is hard to tell. Presumably it reflects player i's opinion on the importance that the agent belongs to the society. As indicated previously, a friend may carry a high weight and yet not be invited to join.

On a higher level of sophistication we can consider a model in which not only individuals but also groups matter. Thus, we now assume only that agents have priorities over the various groups that may compose the society and these priorities need not be sums of weights for individual members. We still assume additivity across stages. Formally:

8c. Additivity across stages. *Each member i of the population has a 'utility-per-stage' function $v_i : \{1, 2, \ldots, k\} \times 2^N \to \Re$, that depends only on t, and on the set of members that stay with him in the society*[12] *so that his utility for a stream $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ is*

$$u_i(\mathcal{F}) = \sum_{\{t \geq 1:\ i \in F^t\}} v_i(t, F^t). \tag{2.4}$$

Again, additivity across stages makes sense if the stages are equally spaced in time. Note that now we no longer assume 'time independence': We allow that the same set of members adds a different utility per stage to a player if it appears at a different stage. This may be the case, e.g., if some of the agents are experts, whose services are important only at a late stage in the life of the society.

A next level is the most general one still compatible with our assumptions:

8d. General stream dependence. *The utility of an agent who became a member of the*

[12] Assuming that indeed he is already in the society at that stage. We do not discuss the utility of an agent who is not a member of a society, because, as we shall see subsequently, we prefer not to regard him as a player.

society is only a function of the stream of members that occurred:[13]

$$u_i(\mathcal{F}) = u_i(F^0, F^1, \ldots, F^k). \tag{2.5}$$

To complete the descriptions above we make a last assumption:

9. Common knowledge. *All utilities as well as all the descriptions above are common knowledge.*

Who are the players? We have set up the society protocol and we have converted it into a game. Clearly, the way we formulated it, the set of players is N. Yet, we can also regard the situation as *a sequence of several games*, one starting at each stage, with different players, where the players at each stage t are the set of voters F^{t-1} and the other agents are considered extraneous entities. Indeed, agents do not really become players until they enter the society. The only votes that count are those of agents who are members by that stage. They create the continuation and it is their interest that matters.[14] Thus, if we want to talk about refinements of equilibria, we sometimes prefer to make them relative to the set of voters at each stage. Accordingly, we shall employ the following definition:

Definition 2.1. An equilibrium strategy profile σ is called *sequentially-Pareto-undominated*, if for every $t \in \{1, \ldots, k\}$ there does not exist another equilibrium strategy profile which coincides with σ up to stage $t-1$, whose outcome is weakly preferred by all voters in F^{t-1} and strongly preferred by at least one of them. The payoff that such a strategy yields is called a *sequentially-Pareto-undominated outcome*.

[13] We even allow his utility to depend on events that occurred before he entered the society.

[14] There are two ways of looking at it. On the one hand, the voters at a stage make their own decisions. They can even dictate to the elected candidates how to vote in the future, threatening not to bring them into the society if no agreement is reached. On the other hand they also have to take into account that the people who are going to participate are pursuing their own interests and will not abide by the agreement if they can benefit by violating it.

The concept of ‘strong equilibrium’ was introduced in Aumann (1959). We shall encounter in the next section games for which strong equilibria do not exist. Nevertheless, we shall show in Section 4 that it is often possible to achieve ‘quasi-strong equilibria’ as defined below:

Definition 2.2. An equilibrium strategy profile σ is called *quasi-strong*, if at no stage can any voter of that stage benefit by a deviation that involves a proper subset of the voters.

This concept is in a sense weaker than Aumann’s, because it does not allow for deviations involving all the voters. In another sense it is stronger, because it tells us that no voter can gain even if others lose.

3. Some interesting simple examples

A universal equilibrium profile. One equilibrium profile always exists in pure strategies:[15]

If there is more than one founder, each founder votes at stage 1 for every candidate — friends and enemies and (off the equilibrium path) every voter votes always for every candidate. This is certainly an equilibrium point, as nobody can change the outcome.

If there is only one founder he chooses that stream that maximizes his utility given that as soon as there are at least two voters, each will vote for every candidate. For example, under pure friendship and enmity (Assumption 8a),[16] he will vote for all his friends in the first stage, if he has more friends than enemies (and every candidate will be brought in at the second stage) and if the number of friends does not exceed the number of enemies he will vote for nobody until the last stage, whereupon he will bring all his friends.

A transitive friendship relation. Here we assume additivity within each stage and across stages (Assumption 8b). If friendship is transitive, then the following is an equilibrium profile: *Each founder votes for all his friends at the first stage* and (off the equilibrium path) each voter votes for all his friends. Indeed, under this strategy, a founder need not be afraid that any of his candidates will bring anybody later and no voter can gain by deviation, neither by voting for fewer friends nor by bringing in enemies.

This equilibrium profile is perfect (see Selten (1975)), because the strategy for each player remains a best reply against any possible trembles of the others. Surprisingly, it is not necessarily a sequentially-Pareto-undominated equilibrium profile (See Example 3.2 below).

[15] This was first observed by Hans Reijnierse (private communication).
[16] Assuming that ε is small enough.

The case $\mathbf{k = 1}$. This case is quite clear under additivity within a stage (Assumption 8b): Having each founder voting for his friends is certainly an equilibrium profile. It is perfect and Pareto-undominated, but it is *not necessarily strong.* For example, under pure friendship and enmity (Assumption 8a), if there are several founders, each having one and a different friend then the set of all founders can all benefit by all voting for nobody. *This example, which can easily be extended to any number of stages, demonstrates that one cannot always obtain a strong equilibrium profile.*

We remark that under friends and enemies and additivity within each stage (Assumption 8b), every voting profile that produces the set of all friends of all the original founders as an outcome and in which each founder votes at least for his friends, constitutes also an equilibrium profile. These profiles produce the same outcome, so they are all Pareto-undominated but they need not be perfect: voting for one's friends only is a best reply against any tremble.

Complications can occur if additivity does not prevail, as the following example shows:[17]

Example 3.1. $F^0 = \{1, 2\}$, $k = 1$, $C^0 = \{a, b\}$,

$$u_1(\emptyset) = 2, \quad u_1(a) = 3, \quad u_1(b) = 1, \quad u_1(ab) = 0,$$

$$u_2(\emptyset) = 3, \quad u_2(a) = 0, \quad u_2(b) = 2, \quad u_2(ab) = 1.$$

Here, $u_i(S)$ stands for the utility of Founder i for $S \cup \{1, 2\}$. A similar convention will be used throughout.

Possible scenario: Founder 1 likes to stay alone. He thinks it is a good idea to bring a to the society and it is a bad idea to bring b. It is a disaster to bring both, because the two will fight all

[17]Here, and in the sequel, we sometimes omit curly brackets and commas. We write, for example, $u_1(ab)$ instead of $u_1(\{a, b\})$.

the time. Founder 2 does not like a's views. He somewhat prefers b, but would above all like to stay alone. Bringing both is a 'compromise' between the previous two undesirable events.

The pure-strategy equilibrium points are (b, b), (a, ab) and (ab, ab). None of them is perfect — they are all eliminated by weak domination. The only perfect equilibrium is mixed, in which Founder 1 votes for $\emptyset$ and a with equal probabilities and Founder 2 votes for $\emptyset$ and b with equal probabilities.

This example demonstrates that sometimes one has to resort to mixed strategies if one wants a perfect equilibrium profile. We shall return to this issue in Section 5.

The case k = 2. This case carries other types of complications as is manifested by the following two examples. These complications appear already under pure friendship and enmity (Assumption 8a). This assumption will prevail for the rest of this section.

Example 3.2. $N = \{a, b, c, d, e, f\}$; $F^0 = \{a, b\}$; $\mathrm{fr}\,(a) = \{c\}$; $\mathrm{fr}\,(b) = \{d\}$, $\mathrm{fr}\,(c) = \mathrm{fr}\,(d) = \mathrm{fr}\,(e) = \mathrm{fr}\,(f) = \emptyset$. $k = 2$. (It does not matter who the candidates e and f have as friends.)

Since friendship here is vacuously transitive, the following is a perfect equilibrium profile: *a votes for c at both stages and b votes for d at both stages, regardless of the histories*. Nevertheless, there is another equilibrium profile that is preferred by *both* players: *players a and b bring their friends only in the second stage and if anyone deviates in the first stage, both a and b invite all the remaining candidates in the second stage*. In this strategy each founder ties the hands of the other founder: "If you do not abide, we shall punish you by bringing in all the enemies." This is even a subgame-perfect equilibrium and sequentially-Pareto-undominated,[18] but it is *not* perfect:

[18] Another variant, in which the deviator is punished only by the other person, in case of deviation, is *not* subgame-perfect but is more convincing: why should the deviator agree, and abide by punishing himself? This is another

Whatever the action of the other person, voting only for one's friend in the last stage is never worse and in some cases better than the prescribed action.

We see already in this simple example the dilemma: Which equilibrium to recommend? A perfect equilibrium which yields small but 'safe' profits or an equilibrium which maximizes profits, but uses threats whose credibility is questionable?

Example 3.3. $N = \{1,2,3,a,b,c,d,e,f,g,p,q,r,s\}$; $k = 2$; $F^0 = \{1,2,3\}$; $\mathrm{fr}\,(1) = \{g\}$; $\mathrm{fr}\,(2) = \{e,f\}$; $\mathrm{fr}\,(3) = \{a,b,c,d\}$; $\mathrm{fr}\,(a) = \{p,q\}$; $\mathrm{fr}\,(b) = \{q,r\}$; $\mathrm{fr}\,(c) = \{p,r\}$; $\mathrm{fr}\,(d) = \{p,q,r\}$; $\mathrm{fr}\,(e) = \{s,p\}$; $\mathrm{fr}\,(f) = \{s,q\}$; $\mathrm{fr}\,(g) = \{s,p,q\}$; $\mathrm{fr}\,(p) = \mathrm{fr}\,(q) = \mathrm{fr}\,(r) = \mathrm{fr}\,(s) = \emptyset$.

We reach a conclusion by the following heuristic arguments: At first one thinks that 1 should not invite g at stage 1, because inviting him would bring about three enemies of 1 in the second stage. Similarly, 2 should apparently not invite any of his friends, because that would bring him more enemies in the last stage. Player 3, however, should invite *all* his four friends (not less!) in the first stage, because that will bring him only three enemies in the next stage, with a net profit of $1 - 3\varepsilon$, compared to not inviting any friend in the first stage.

Realizing that p, q are going to be in the society in the last stage anyhow, player 2 should not hesitate to vote for his friends in the first stage: He gets two friends at that stage but suffers from only one additional enemy next stage.

Realizing that also s will be present in the last stage anyhow, it now follows that 1 can only gain by bringing his friend in stage 1.

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manifestation of the known dilemma: Why should one trust a promise of a person, who already proved that he does not keep his promises, because he deviated in the first stage. Note that formally the strategies in this variant are not functions of the stream only. They also depend on knowing who deviated in the first stage. Of course, this knowledge can be *deduced* if one remembers for whom he himself voted and what was the outcome of the first stage.

Thus, the following is an equilibrium profile: *Every voter brings all his friends as soon as he is allowed to vote.*

The utilities (not including utilities for time spent with the original founders and ignoring multiples of ε) are: $u_1 = -14, u_2 = -10, u_3 = -2, u_g = -10, u_e = u_f = u_a = u_b = u_c = -12, u_d = -10, u_p = u_q = u_r = u_s = -10$.

It can be checked that this is indeed an equilibrium profile and, moreover, it is perfect.[19]

This is *not* a sequentially-Pareto-undominated equilibrium. Like in the previous example, there is a sequentially-Pareto-undominated, subgame-perfect but not perfect equilibrium that will be strictly preferred by all original founders, and in fact, by everyone who will find himself eventually in the society; namely, to invite nobody in the first stage, invite one's friends in the second stage and punish deviations by each voter inviting everyone in the second stage.

To sum up: We exhibited here a "safe" equilibrium outcome that does not yield much to the founders and another "not so safe" that brings about higher utilities to the founders, and moreover brings about a society with much fewer frictions in it. Which one (if any) should be chosen has to be decided by the members. Do they trust their co-founders to honor the "agreement" in the second case? Do they believe that the "punishment" will be carried out in case of a breach? The answer to such questions, we feel, is beyond the scope of the theory.

When many common enemies exist. We have seen in the previous example how a punishment can force an equilibrium. In fact, if there are enough common enemies, then *any agreement* between the current founders, at any stage other than the last, can be enforced by a strategy that stipulates

[19] Any "tremble" can be observed only in the last stage when it is still to one's advantage to bring all his friends. The considerations of this example will be employed in Section 5 to produce classes of 2-stage voting schemes for which pure-strategy perfect profiles always exist.

that out of the agreement all voters will vote for all common enemies as soon as they recognize that they are off the equilibrium path. This is even subgame-perfect.

The question then becomes: Which agreements are the players likely to sign? Realizing that almost all agreements can be made binding as explained above, this case should be handled with the tools of cooperative game theory and this is outside the scope of the present paper.

We keep the above in mind but we wish to make the following two observations: (1) In real life one can usually extend the set of candidates so as to include as many common enemies as one 'wishes'. (2) Nevertheless, a threat to bring these common enemies is often not credible as a general procedure. It often would be considered unthinkable, because it would undermine the very foundations upon which the society rests. Thus, although such threats may be feasible, often they are not viable, which brings us again to the recognition that a model does not usually capture all the intricacies of a real situation.

The helpful enemy. We have seen how voting for an enemy may be beneficial off the equilibrium path. The following example will show that voting for an enemy may be beneficial also along the equilibrium path.

Example 3.4. $N = \{a, b_1, b_2, \ldots, b_5, c_1, c_2, \ldots, c_5, d, e\}$; $F^0 = \{a\}$; $\mathrm{fr}\,(a) = \{b_1, \ldots, b_5\}$; $\mathrm{fr}\,(b_i) = \{c_i\}, i = 1, \ldots, 5$; $\mathrm{fr}\,(c_i) = \{d\}, i = 1, \ldots, 5$; $\mathrm{fr}\,(d) = \{e\}$; $\mathrm{fr}\,(e) = \emptyset$; $k = 4$.

The founder would like to bring all his friends, but if he simply does so at the first stage then each b_i will bring c_i in the next stage. This is because the b_i's will not fear[20] that c_i will bring d before the last stage, knowing that if c_i does so, d will bring e. To prevent this from happening,

[20]We are using the fact that, because ε is positive (Assumption 8a), a voter will prefer to postpone a vote for a friend if this friend will bring an enemy at the next stage. He will gain an ε by postponing one stage.

the founder can vote for e in the first stage. A complete strategy profile is this:

$$\sigma_e^t = \emptyset, \qquad (t \in \{2,3,4\}), \quad \forall F^{t-1};$$

$$\sigma_d^t = \{e\}, \qquad (t \in \{2,3,4\}), \quad \forall F^{t-1};$$

$$\sigma_{c_i}^4 = \{d\}, \qquad i \in \{1,\dots,5\};$$

$$\sigma_{c_i}^t = \begin{cases} \{d\}, & \text{if } e \in F^{t-1}, \\ \emptyset, & \text{otherwise,} \end{cases} \qquad (i \in \{1,\dots,5\}), \quad (t \in \{2,3\});$$

$$\sigma_{b_i}^4 = \{c_i\}, \qquad (i \in \{1,\dots,5\});$$

$$\sigma_{b_i}^3 = \begin{cases} \{c_i\}, & \text{if } d \in F^2, \\ \emptyset, & \text{otherwise,} \end{cases} \qquad (i \in \{1,\dots,5\});$$

$$\sigma_{b_i}^2 = \begin{cases} \{c_i\}, & \text{if } d \in F^1, \\ \{c_i\}, & \text{if } e \notin F^1, \\ \emptyset, & \text{otherwise,} \end{cases} \qquad (i \in \{1,\dots,5\});$$

$$\sigma_a^t = \emptyset, \quad (t \in \{2,3,4\});$$

$$\sigma_a^1 = \{b_1,\dots,b_5,e\}.$$

One can verify that this is indeed an equilibrium profile.

Example 3.5. The game of chicken. In this example, $F^0 = \{1,2\}$, $C^0 = \{x_1,x_2,y_1,y_2\}$, $k = 3$. Founder 1 likes only x_1, who likes only y_1. Founder 2 likes only x_2, who likes only y_2. Agents y_1 and y_2 like only each other.

Skipping formalities, each founder can essentially either choose his friend in the first stage, or refrain from doing so. (He strictly loses by voting for an enemy at this stage.) Unfortunately, if player 1 votes for his friend at the first stage, player 2 will lose if he too votes for his own friend. The reason is that in this case it is clear that both y_1 and y_2 will be present in stage 3, so there will be no reason for both x_1 and x_2 to refrain[21] from voting for their friends in stage 2. These friends

[21] If, say y_2 were not present at stage 3, x_1 would not have invited y_1 at stage 2, as ε is positive and x_1 knows that y_1 will bring y_2 (an enemy of x_1) at the last stage.

are enemies of the founders. Putting together the relevant information and ignoring ε, we get the following payoff as functions of the choices in the first stage:

	$\emptyset$	x_2
$\emptyset$	0, 0	-3, 1
x_1	1, -3	-4, -4

This is the famous game 'chicken'. It has two pure-strategy equilibrium points: $(x_1, \emptyset)$ and $(\emptyset, x_2)$, yielding payoffs $(1, -3)$ and $(-3, 1)$, respectively. In addition, the players can each use a mixed strategy (1/2,1/2) that yields a more sensible payoff $(-1.5, -1.5)$. All these are undominated and therefore perfect (see Kohlberg and Mertens (1986), Appendix D).

Even more sensible for the players is to employ a correlated strategy under which a mechanism chooses one of $(\emptyset, \emptyset)$, $(\emptyset, x_2)$, $(x_1, \emptyset)$ with equal probabilities, informing founder 1 what row was chosen and informing founder 2 what column was chosen. The founders then choose whatever row/column they were told, thereby reaching an equilibrium expected payoff equal to $(-2/3, -2/3)$. *This example comes to show that mixed and correlated strategies should not be ignored.*

4. Common voting and partial common voting

At the beginning of this section we study *common voting* profiles; namely, profiles under which, at each stage, all voters vote for the same set of candidates. We show that every equilibrium outcome that can be reached by a pure-strategy profile can also be reached by a common-voting profile that generates the same stream of members. These profiles have the additional advantage that they are quasi-strong equilibria (Definition 2.2). A quasi-strong equilibrium gives each voter the assurance that, without his participation, no subgroup of the other players will agree to deviate, because none of them will gain, and some may even lose.

We then proceed to characterize and, at least theoretically, construct all the equilibrium streams, and therefore all equilibrium outcomes that can be achieved by pure strategies. We also indicate where to look when we want to get all the sequentially-Pareto-undominated equilibrium streams, as well as all the subgame-perfect streams.

In the last part of this section we provide interesting procedures that produce equilibrium profiles that only 'partially' employ common voting, or even some in which the voters vote for distinct sets.

A key role in reaching some of these results is expressed in the following:

Remark 4.1. *Quota one implies that whoever the voters bring in can also be brought by one voter. Consequently, if a set S of candidates is chosen in an equilibrium profile of a 1-stage game, this set has the property that, if elected, no voter would have preferred that more members were added to it.*

All strategies in this section are pure and we shall rarely repeat this fact. To avoid trivialities we assume henceforth that $C^0 \neq \emptyset$.

Proposition 4.2. *Let σ be a pure-strategy equilibrium profile for a 1-stage game Γ. The strategy profile $\bar{\sigma}$, generated from σ by common voting, is a quasi-strong equilibrium profile for Γ.*

Proof. If $|F^0| = 1$, then $\bar{\sigma} = \sigma$, it is an equilibrium point and vacuously a quasi-strong one. Let $|F^0| > 1$. The set S of players that was elected under $\bar{\sigma}$ is the same set that was elected under σ; therefore, it yields the same payments. Any deviation from $\bar{\sigma}$, made by a nonempty *proper* subset of the founders, can only yield a set that contains S, because the remaining founders still vote for S. Therefore, if such a deviation from $\bar{\sigma}$ resulted with some members gaining, then, in σ each of them could have forced the same better payment, by alone adding the same additional candidates, contrary to the fact that σ is an equilibrium profile for Γ. ■

One should be careful when one tries to generalize Proposition 4.2 to multi-stage games: At future stages 'new' players may enter the game and one is inclined to take into account possible agreements involving them, as a condition to be elected. Consider the following:

Example 4.3. Let $F^0 = \{1, 2\}$, $C^0 = \{a, b\}$. Under pure friendship and enmity (Assumption 8a), agents 1 and 2 like agent a. Agent a likes agent b. For all other pairs (i, j), j is an enemy of i. $k = 2$. Assume also that agent a prefers to be in the society to not being there, no matter who else is with him.

The following strategy profile is subgame-perfect:

$$\sigma_1^1 = \emptyset, \quad \sigma_1^2 = \{a\}, \quad \sigma_2^1 = \emptyset, \quad \sigma_2^2 = \{a\}, \quad \sigma_a^2 = \{b\},$$

where these actions are taken on and off the equilibrium path.

This profile is already in common voting for the original founders, who vote the same way throughout the play, on and off the equilibrium path. Nevertheless, this profile is not immune to deviation involving a proper subset of the founders: Agents 1 and a can deviate by 1 voting for a already in the first stage and a promising to vote for $\emptyset$ at the second stage. By this deviation agent 1 gains and agent a also gains, because he becomes elected.[22]

The point is that the strategy profile above is *not* in common voting with agents that may later be admitted to the society!

Indeed, augmenting the above example and requesting that both 1 and 2 vote also for b at the second stage, if a is elected in the first stage, then no profitable deviation can take place by a proper subset of the founders. For example, it will do no good that a will refrain from voting b, because founder 2 will still vote for b.

With this understanding we can generalize Proposition 4.2 as follows:

Theorem 4.4. *Let Γ be a game representing a voting scheme obeying general stream dependence (Assumption 8d). Let σ be a Nash equilibrium of Γ. Let $\bar{\sigma}$ be the profile derived from σ by common voting at each stage, on and off the equilibrium path.*[23] *Then, $\bar{\sigma}$ is a quasi-strong equilibrium of Γ, giving the same stream as σ. If σ is a subgame perfect profile then $\bar{\sigma}$ is also subgame-perfect.*

Proof. Since actions in σ may depend only on the history of membership (and not on who voted for whom), common voting preserves the set of candidates voted into the society at every stage, both on and off the equilibrium path. Therefore, the outcome stream $\mathcal{F} := \{F^0, F^1, \ldots, F^k\}$ of σ coincides with that of $\bar{\sigma}$.

The profile $\bar{\sigma}$ is also an equilibrium point. Indeed, if a deviation of an agent i from $\bar{\sigma}$ profits him, then he could profit the same way by deviating alone from σ, voting at each stage for those members

[22] One can question how safe is this agreement between 1 and a. Obviously, a will desire not to honor the agreement. This, however, is irrelevant to the claim that 1 and a can both gain if they follow this agreement.

[23] The requirement of common voting refers only to the profile σ. Of course, if a player, or several players, decide to deviate, they need not adhere to the common voting stipulation.

who were elected due to $\bar{\sigma}_{-i}$ together with those elected by him in his deviation. A contradiction. By the same token, $\bar{\sigma}$ is subgame perfect, if σ was subgame perfect.

Suppose that $\bar{\sigma}$ is not quasi strong. Then, there exists a profile τ that coincides with $\bar{\sigma}$ up to a certain stage $t^\star$ and deviates from that stage, where the deviation is done by a proper subset of $F^{t^\star-1}$, together with agents who are admitted later to the society. Let $\mathcal{G} := (F^0, F^1, \ldots, F^{t^\star-1}, G^{t^\star}, \ldots, G^k)$ be the stream of members that result from τ. Suppose that a member i of $F^{t^\star-1}$, who is not necessarily a deviator, prefers $\mathcal{G}$ to $\mathcal{F}$. We claim that he alone could generate $\mathcal{G}$, if all other agents obey $\bar{\sigma}$. Indeed, consider an arbitrary subgame starting at an arbitrary stage t, $t \geq t^\star$, on, or off the equilibrium path of $\bar{\sigma}$. Let H^t be the set of members elected at this stage due to $\bar{\sigma}$, then it is contained in the set of members elected at this stage due to τ, because there are members of $F^{t^\star-1}$ who still vote as in $\bar{\sigma}$. This very same set can be voted into the society by agent i alone. It follows that i can benefit by a deviation from $\bar{\sigma}$, contrary to the fact that $\bar{\sigma}$ is an equilibrium profile. ■

We have shown that all pure-strategy equilibrium outcomes can be generated by common voting. The natural question that now comes to mind is how to characterize all streams that constitute such outcomes. Proposition 4.5, Theorem 4.6 and Corollary 4.7 provide an answer.

Proposition 4.5. *Assume that there are at least two founders in a 1-stage game Γ. A set S of candidates chosen can result from a pure-strategy equilibrium profile iff S has the property that no founder would prefer to add members to S.*[24]

Proof. The ‘only if’ part is explained in Remark 4.1. Conversely, suppose S has this property and is voted, say, by common voting. Then no player can benefit by deviating alone: He cannot delete members from S and he does not want to add members to S. ■

[24] Such an S always exists, for example $S = C^0$.

Thus, for a multi-person set of founders, to generate all equilibrium outcomes for a 1-stage game one has to examine all subsets S of C^0 and select those that have the property that no founder would like to augment them. This task is manageable by a computer if $|N|$ is reasonably small and $k = 1$. It becomes less so when the number of stages increases.

To extend Proposition 4.5 to a k-stage game, for $k > 1$, we employ a process that we call *collation*, explained subsequently.

Consider a tree game Γ, representing a k-stage voting scheme. Consider an arbitrary subgame starting at the last stage. This is a tree form for a 1-stage voting scheme, connected to the root of Γ by a unique path. Its endpoints represent the payoff vectors that would be obtained if the players proceeded along this path and continued along the subgame. Thus, if we fix an action[25] for each voter of the subgame (and remember it), we can delete the subgame and connect the resulting payoff vector to the new endpoint, at the root of the subgame. Fixing actions at each last-stage subgame allows us to delete them, thus converting the tree to a $(k-1)$-stage game. By this collation we can obtain backward induction results, by considering only 1-stage games, even though Γ is not a perfect-information game. Note that strategies constructed in this way usually do not depend only on the streams. To force a strategy that depends only on the stream, we have to require that actions taken at paths that correspond to the same stream are the same actions. Since this paper allows only stream-dependent strategies, we assume in this paper that this requirement is imposed during collation.

Note that two 1-stage games, belonging to the same stage, may have the same set of voters and yet differ in the resulting payoff vectors. This can happen because the streams leading to these voters differ. However, if the voting scheme obeys additivity across stages (Assumption 8c), two

[25] This action can also be a mixed strategies at each stage, which together form a behavioral strategy. We shall use mixed strategy collation in Section 5.

such 1-stage games are strategically equivalent — their payoff vectors differ by a constant, which is the difference between the payoff vectors accumulated until these stage games were reached. This results in a great saving when attempting to construct equilibrium strategies which are *markovian*; namely, depend at each stage only on the set of voters and on the number of stages left, and not on the paths reaching these 1-stage games.

To sum up: *collation is a protocol, during which one assigns fixed moves*[26] *to all last stage games and then truncates these games, assigning the appropriate payoff vectors to the new endpoints. This is continued until the root is assigned a payoff vector. If one takes care to use the same moves at vertices corresponding to the same stream, up to that stage, then the resulting strategy profile will be only a function of the stream.*

Theorem 4.6. *Let* Γ *be a game representing a* k*-stage voting scheme obeying general stream dependence (Assumption 8d). If, during collation, we always choose an equilibrium profile for each 1-stage game, the resulting profile is a subgame-perfect equilibrium profile for* Γ*. Conversely, every subgame-perfect equilibrium profile can result in this fashion. If the 1-stage profiles are quasi-strong, then the resulting profile is quasi-strong (Definition 2.2).*

Proof. A. Suppose that during collation, we always choose an equilibrium profile for a one-stage game. Let σ be the resulting profile for Γ. Let (σ_{-i}, τ_i) be an arbitrary profile resulting from a deviation by player i. We show that this deviation does not yield this player any benefit. Indeed, switching to σ_i at all last-stage subgames does not decrease his payoff, because σ_i is a best reply to σ_{-i} at all stage k games. After the switches, collate on the last-stage games, and continue in the same fashion. Performing this procedure k times, we observe that player i's payment never decreases. Finally, we arrive at his original payment due to σ_i.

[26] Pure, or mixed.

B. Suppose that at each one-stage game the chosen profile was a quasi-strong equilibrium. Suppose now that a deviation τ occurred, subject to the restriction that at the start of every subgame, there was at least one voter who adhered to σ for that stage. Let i be an arbitrary agent, who was a member of the society when the deviation started, and we show that he cannot benefit from τ. Indeed, consider all k-stage subgames, and instruct all the players to revert to σ. This will not harm agent i, because the profiles restricted to the last stage were quasi-strong. Collate on this stage and continue in the same fashion $k-1$ times. One winds up with player i not harmed, and getting the payment as in σ. Thus, τ does not benefit agent i.

C. Let σ be a subgame-perfect profile. Its restriction to any last-stage subgame is an equilibrium profile. Collate on all the subgames of this stage and look at the games of stage $k-1$. Again, σ restricted to this subgame (after collation) is an equilibrium, because σ was subgame perfect. Continuing in this fashion, we see that σ was indeed obtained by the process of collation. ∎

Corollary 4.7. *The following collation protocols yield all possible pure-strategy subgame-perfect equilibrium streams:*

Starting with the last stage and continuing backwards, as long as there are at least two voters, select a set of candidates that has the property that, if elected, no agent prefers to add candidates to this stage. If there is one voter, select for him a move that maximizes his payment. Having done that for a stage, perform collation and continue in the same fashion until all stages are exhausted.

Proof. Proposition 4.5 and Theorem 4.6. ∎

Interestingly, subgame perfect outcomes do not yield strict refinements to Nash equilibrium outcomes, as the following theorem shows.

Theorem 4.8. *Let σ be a Nash equilibrium profile for a voting game Γ. There exists a subgame perfect equilibrium profile $\hat{\sigma}$ yielding the same stream of members.*

Proof. Denote by τ the universal equilibrium profile for Γ as defined at the beginning of Section 3. Note that τ is a subgame perfect profile. Let $\hat{\sigma}$ be equal to σ along the equilibrium path of σ, and equal to τ otherwise. Both σ and $\hat{\sigma}$ have the same equilibrium path. Off the equilibrium path, $\hat{\sigma}$ is subgame perfect, due to τ. If an agent has a profitable deviation from $\hat{\sigma}$ starting on the equilibrium path, he could have achieved it alone against σ_{-i} by switching to τ_i after the starting point of the deviation. ■

Consider again a one-stage multi-founder game. It may well happen that several sets S have the property that no founder would have preferred to add more candidates, given that they were elected. If such a set S_1 is contained in another such a set S_2, then the payment to each of the founders under S_2 is not greater than the payment under S_1, since otherwise a founder who would have preferred to vote for S_2, rather than for S_1 could have forced this outcome. Consequently, all sequentially-Pareto-undominated equilibrium outcomes in a one stage game can be found through the common-voting procedure described in Proposition 4.5 but choosing only sets S that are minimal under inclusion. Similarly, we can obtain all subgame perfect sequentially-Pareto-undominated equilibrium payoffs in a multi-stage game by performing the construction of Theorem 4.6, but restricting ourselves at each stage to sets S that are minimal under inclusion. (Of course some equilibria reached by this construction may not be sequentially-Pareto-undominated.)

If we were only interested in equilibrium outcomes we could stop here. But we are also interested in other equilibrium profiles that lead to such outcomes, in particular those obtained by pure strategies. We shall close this section by producing a wider class of equilibrium profiles. These extend the common-voting class in that they involve only partial common voting, or even no common voting at all. These profiles will play an important role in Section 5, when we deal with perfect equilibria. In view of Theorem 4.6, it is sufficient to consider 1-stage voting games.

Proposition 4.9. *Let Γ be a 1-stage voting game having at least two founders. Let S be a set of candidates from C^0, having the property that, if elected, no original founder will prefer to add players to S. For each founder i, choose a set P_i, contained in S, that is a best response to*[27] *$S \setminus P_i$. Let $C = S \setminus \cup_{j \in F^0} P_j$. Finally, let $V_i = P_i \cup C$. Under these conditions, $\{V_i \colon i \in F^0\}$ is an equilibrium profile for Γ.*

The proof requires two lemmas:

Lemma 4.10. *Let P_i be a best response of founder i against $S \setminus P_i$, where S is an arbitrary given set of candidates from C^0 containing P_i. If $Q \subseteq S \setminus P_i$, then $P_i \cup Q$ is also a best response of i to $S \setminus P_i$*

Proof. Q is covered anyhow by $S \setminus P_i$, so it makes no difference whether i includes Q in his vote, or not. ∎

Lemma 4.11. *Let P_i be a best response of founder i against $S \setminus P_i$, where S is an arbitrary set of candidates containing P_i. If $R \subseteq P_i$ then $P_i \setminus R$ is a best response of i to $(S \setminus P_i) \cup R$.*

Proof. Voting $P_i \setminus R$ against $(S \setminus P_i) \cup R$, would yield player i the utility gained from S being elected. If voting for another set, Q, would yield him a higher utility, then voting $Q \cup R$ would be a better response to $S \setminus P_i$ than voting P_i, because $(Q \cup R) \cup (S \setminus P_i) = Q \cup (R \cup (S \setminus P_i))$. ∎

[27] Such a set always exists; for example $\emptyset$.

Proof of Proposition 4.9. P_i is a best response of i against $S \setminus P_i$; therefore, V_i is a best response of i against $S \setminus P_i = (S \setminus P_i) \cap (\cup_{j \in F^0 \setminus \{i\}} V_j)$ (Lemma 4.10). By Lemma 4.11, $(C \cup P_i) \setminus \cup_{j \in F \setminus \{i\}} P_j$ is a best response of i against $\cup_{j \in F^0 \setminus \{i\}} V_j$. Invoking Lemma 4.10 once more, we find that V_i is a best response of i against $\cup_{j \in F^0 \setminus \{i\}} V_j$. ∎

5. Perfect equilibria in pure strategies

Common-voting equilibria are usually not perfect. A voter may be tempted to deviate, figuring that the others will continue to vote in the same way with high probability, in order to extract some profit in case of 'trembles'. In this section we provide a sufficient condition for the existence of perfect equilibria in pure strategies and show how one can construct them (Proposition 5.2 and Theorem 5.12). We then show by examples that this condition is not necessary, as there are other cases in which pure-strategy perfect equilibria exist. Nevertheless, we show that for 2-stage games *with additive preferences across stages and within a stage (Assumption 8b)*, pure-strategy perfect equilibria always exist (Theorem 5.7). Whether this result can be extended to games with more stages is still an open problem.

We are able to prove the main theorems of this section under the assumption that the voting scheme is *generic*; in the sense that *different streams yield different utilities for each player.* Example 5.6 shows that this assumption is necessary for the results.

Definition:. For a set $S \subseteq C^0$ we say that i *supports* x with respect to S if $S \succ_i S \setminus \{x\}$. Here, $\succ_i$ means: 'Preferred by i'.

The following lemma is easily proved by induction.

Lemma 5.1. *For all $n \geq 1$, for all $0 < \varepsilon < 1$, it is true that $1 - (1-\varepsilon)^n \leq n\varepsilon$.*

Proposition 5.2. *Let Γ be a generic 1-stage multi-founder voting game. If S is a set of candidates, V_i is the set of candidates supported by founder i in S, and the strategy profile $\{V_i\}_{i \in F^0}$ is a Nash equilibrium with $S = \cup_{i \in F^0} V_i$, then $\{V_i\}_{i \in F^0}$ is a perfect equilibrium of Γ.*

Proof. Denote $c = |C^0|$, $f = |F^0|$. Denote by d the minimum payoff difference for any two sets of candidates and any founder. Similarly, denote by M the maximal payoff difference for any two sets of candidates and any founder; i.e.,

$$d = \min_{\substack{i \in F^0 \\ T_1, T_2 \subseteq C^0}} |u_i(T_1) - u_i(T_2)|, \qquad M = \max_{\substack{i \in F^0 \\ T_1, T_2 \subseteq C^0}} |u_i(T_1) - u_i(T_2)|. \tag{5.1}$$

The voting scheme is generic, and $c \geq 1$, therefore $d > 0$ and $M > 0$.

Assume fixed positive ε_1 and ε_2. Assume initially that they are each less than $\dfrac{1}{4c}$ and that $\varepsilon_2 \leq \varepsilon_1$. Additional conditions will be provided later.

Define the following completely mixed strategy for each founder i:

(1) For each $x \in V_i$, vote for $V_i \setminus \{x\}$ with probability ε_1.

(2) For any other set of candidates, except V_i, vote for this set with probability $\dfrac{\varepsilon_2}{2^c}$.

(3) Vote for V_i with the residual probability. This probability is greater than $1 - c\varepsilon_1 - \varepsilon_2$ as $|V_i| \leq c$, and from the restrictions already imposed on the epsilons it is greater than $\frac{1}{2}$.

As ε_1 and ε_2 tend to zero, this completely mixed strategy tends to V_i for every founder i.

Let i be an arbitrary fixed founder. The proof will conclude if we show that V_i is his best reply against the others using these strategies, provided the epsilons are small enough.

Consider two possible types of deviation by agent i. The first is a deviation that makes a difference when all others vote for the candidates they support, and the second is a deviation that makes no difference when all others vote for the candidates they support.

The first type of deviation causes a loss of at least d whenever all others vote V_j, and a gain of at most M in other cases. The loss occurs with a probability of at least $\dfrac{1}{2^{f-1}}$ (as this number is less than the probability of every other founder j voting V_j) and the gain can occur with a probability of at most

$$1-(1-c\varepsilon_1-\varepsilon_2)^{f-1} \leq (f-1)(c\varepsilon_1+\varepsilon_2) \leq cf(\varepsilon_1+\varepsilon_2), \tag{5.2}$$

as at least one founder $j \neq i$ must vote for a set different from V_j. The first inequality is implied by Lemma 5.1.

A sufficient condition for the expected loss from such a deviation to exceed the expected gain is therefore

$$\frac{d}{2^{f-1}} \geq Mcf(\varepsilon_1+\varepsilon_2), \tag{5.3}$$

and this always holds if $\varepsilon_1 < \dfrac{d}{Mcf2^f}$ as $\varepsilon_2 \leq \varepsilon_1$.

We now investigate the other type of deviation, and find restrictions on the epsilons to ensure that it too will not be profitable.

Consider a deviation by agent i to $(V_i \setminus R) \cup A$, where $R \cup A \subseteq V_{-i}$, $R \subseteq V_i$ and $A \cap V_i = \emptyset$. Thus, player i removes members of R from his bid and adds members of A, and each of these candidates is supported by at least one other founder. Denote $Q = A \cup R \neq \emptyset$ and $V_i' = (V_i \setminus R) \cup A$.

There are three cases of bids of the other founders we now consider. The first, where V_i' gives a sure loss of at least d relative to V_i, the second, where a gain of up to M is possible, and the third, where the payoff to i from V_i and V_i' is the same.

The first case is when the others vote for $V_{-i} \setminus \{x\}$ for some $x \in Q$. Regardless of whether i supports x and does not vote for him ($x \in R$), or whether i does not support x and does vote for him ($x \in A$), the deviation to V_i' gives a loss of at least d compared to voting V_i. For each $x \in Q$ denote the probability of this subcase by $\eta_1(x)$.

The second case (possibility of gain) is when the vote of the others does not contain x for some $x \in Q$, but it is not $V_{-i} \setminus \{x\}$. Denote the probability of these subcases by $\eta_2(x)$ for each $x \in Q$. Note that the $|Q|$ such possibilities are not mutually exclusive. Note also that these two cases cover all situations where any member of Q is missing from V_{-i}.

Any other set voted for by the others (such a set must contain Q) gives the same payoff to i from both V_i and V_i'.

A sufficient condition for V_i' not to be a profitable deviation is that the expected loss is greater than the expected gain. A sufficient condition for this is

$$d \sum_{x \in Q} \eta_1(x) > M \sum_{x \in Q} \eta_2(x). \tag{5.4}$$

Let $m(x)$ be the number of supporters of x with respect to S, not including agent i. For all $x \in Q$ it is true that $m(x) \geq 1$.

The following bounds hold, as we explain:

$$\eta_1(x) \geq \varepsilon_1^{m(x)}(1 - c\varepsilon_1 - \varepsilon_2)^{f-m(x)-1} \geq \frac{\varepsilon_1^{m(x)}}{2^{f-m(x)-1}} \geq \frac{\varepsilon_1^{m(x)}}{2^f}. \tag{5.5}$$

The first inequality holds, as the event $\eta_1(x)$ includes the event that each supporter of x votes $V_j \setminus \{x\}$ and all others vote V_j. The second inequality is implied by $1 - c\varepsilon_1 - \varepsilon_2 > \frac{1}{2}$.

$$\begin{aligned} \eta_2(x) &\leq 1 - (1 - \varepsilon_2)^{f-1} + \varepsilon_1^{m(x)}(1 - (1 - c\varepsilon_1 - \varepsilon_2)^{f-m(x)-1}) \\ &\leq (f-1)\varepsilon_2 + \varepsilon_1^{m(x)}(f - m(x) - 1)(c\varepsilon_1 + \varepsilon_2). \end{aligned} \tag{5.6}$$

The first inequality holds, as for this case to occur, at least one of the events [at least one founder j votes for neither V_j nor $V_j \setminus \{y\}$ for any candidate y] which has probability no greater than $1 - (1 - \varepsilon_2)^{f-1}$, or [all the supporters j of x vote for $V_j \setminus \{x\}$ and at least one of the other founders

j' votes for a set different from $V_{j'}$] must occur. The second inequality holds from two applications of Lemma 5.1.

If we now assume that $\varepsilon_2 \le \dfrac{\varepsilon_1^{m(x)+1}}{f-1}$ then (5.6) implies

$$\eta_2(x) \le \varepsilon_1^{m(x)+1} + \varepsilon_1^{m(x)}(f-1)(c+1)\varepsilon_1 \le 2cf\varepsilon_1^{m(x)+1}. \tag{5.7}$$

Inequalities (5.5) and (5.7) together imply

$$\eta_2(x) \le \eta_1(x)\frac{2cf\varepsilon_1^{m(x)+1}2^f}{\varepsilon_1^{m(x)}} = \eta_1(x)2^{f+1}cf\varepsilon_1. \tag{5.8}$$

Now, using inequality (5.8), inequality (5.4) is implied by

$$d\sum_{x\in Q}\eta_1(x) > M2^{f+1}cf\varepsilon_1\sum_{x\in Q}\eta_1(x). \tag{5.9}$$

This is equivalent (since $\sum_{x\in Q}\eta_1(x) > 0$), to

$$\varepsilon_1 < \frac{d}{Mcf2^{f+1}}. \tag{5.10}$$

Taking all the restrictions together, and using the fact that $m(x) \le f-1$, for all $x \in Q$, we have that

$$\varepsilon_1 < \frac{d}{Mcf2^{f+1}}, \qquad \varepsilon_2 \le \frac{\varepsilon_1^f}{f-1}, \tag{5.11}$$

imply that V_i is a best response to the mixed strategies of the others.

Since we can take a sequence of epsilons that tend to zero while keeping all the restrictions, the proof is complete, as we have a sequence of completely mixed strategy equilibrium profiles tending to $\{V_i\}_{i\in F^0}$. ∎

Guessing a set of candidates S, for a single game Γ, that generates $\{V_i\}_{i\in F^0}$ that covers S, and furthermore, constitutes an equilibrium profile, might be a difficult task. Searching for all such S's makes it even more difficult. Even then, as we shall see (Example 5.5), we may not construct all pure-strategy perfect-equilibrium profiles. Sometimes, we are not interested in a specific voting game, but rather in a large class of games, and we wish to prove that every game in this class has a pure-strategy perfect-equilibrium profile. We may even want to characterize such a profile. In such cases, the following two corollaries might be useful. In fact, one of them will be employed subsequently.

Corollary 5.3. *Let Γ be a generic one-stage voting game. If there exists a set of votes $\mathcal{P} = \{P_i\}_{i\in F^0}$ where $P_i \subseteq C^0$, satisfying*

(1) *$\mathcal{P}$ is an equilibrium profile for Γ,*

(2) *$P_i \cap P_j = \emptyset$, whenever $i \neq j$,*

then Γ has a pure-strategy perfect-equilibrium profile.

Terminology: Profile $\mathcal{P}$, satisfying (1) and (2) above will henceforth be called *a generalized-partition equilibrium profile.*

Proof. Denote by S the union $\cup_{i\in F^0} P_i$. Let $V_i := \{x \in S\colon x \text{ is supported by } i \text{ with respect to } S\}$. It follows that $V_i \supseteq P_i$ because $\mathcal{P}$ is an equilibrium profile. For the same reason, $\{V_i\}_{i\in F^0}$ is an equilibrium profile. Therefore, $\{V_i\}_{i\in F^0}$ satisfies the conditions of Theorem 5.2 and constitutes a pure-strategy perfect-equilibrium profile. ∎

This corollary is a special case of the following:

Corollary 5.4. *Let Γ be a generic one-stage voting game. Consider an arbitrary equilibrium profile*

$\{P_i \cup C\}_{i \in F^0}$, employing partial common voting as in Proposition 4.9. If $P_i \cap P_j = \emptyset$, whenever $i \neq j$, and every agent in C is supported by every voter, with respect to $S = C \cup (\cup_{j \in F^0} P_j)$ then Γ has a pure-strategy perfect-equilibrium profile.

The proof is similar to the previous one and will be omitted.

Proposition 5.2 raises the question whether the conditions are also necessary for the existence of pure-strategy perfect equilibrium. We answer the question negatively, by the following example:

Example 5.5. The population consists of:

$$F^0 = \{1, 2\}, \quad C^0 = \{a, b\}.$$

There is only one period; $k = 1$. The utilities of the founders are:

$$u_1(\emptyset) = 2, \quad u_1(\{a\}) = 3, \quad u_1(\{b\}) = 4, \quad u_1(\{a, b\}) = 1,$$

$$u_2(\emptyset) = 4, \quad u_2(\{a\}) = 2, \quad u_2(\{b\}) = 1, \quad u_2(\{a, b\}) = 3.$$

The payoff matrix is given by[28]

	$\emptyset$	a	b	ab
$\emptyset$	2, 4	3, 2	4, 1	1, 3
a	3, 2	3, 2	1, 3	1, 3
b	4, 1	1, 3	4, 1	1, 3
ab	1, 3	1, 3	1, 3	1, 3

In this example the pure equilibrium profiles are $(\{a\}, \{a, b\})$, $(\{b\}, \{a, b\})$ and $(\{a, b\}, \{a, b\})$. None of them satisfies the conditions of Proposition 5.2. Nevertheless, $(\{a\}, \{a, b\})$ and $(\{b\}, \{a, b\})$

[28]For simplicity we omit the curly brackets that denote sets.

are perfect equilibrium profiles.[29] This shows that Proposition 5.2 does not yield necessary conditions. On the other hand, Example 3.1 shows that voting schemes exist that do not have any pure-strategy perfect equilibrium. Providing a necessary and sufficient condition for the existence of pure-strategy perfect equilibrium in a 1-stage game remains an open question.

The next example will show that the requirement that the game is generic is needed for Proposition 5.2 to hold.

Example 5.6. The population is:

$$F^0 = \{1, 2\}, \quad C^0 = \{a, b\}.$$

There is only one period; $k = 1$. The utilities of the founders are:

$$u_1(\emptyset) = 2; \quad u_1(\{a\}) = 1; \quad u_1(\{b\}) = 3; \quad u_1(\{a, b\}) = 1.$$

$$u_2(\emptyset) = 0; \quad u_2(\{a\}) = 1; \quad u_2(\{b\}) = 1; \quad u_2(\{a, b\}) = 1.$$

The game is not generic, as for example $u_1(a) = u_1(\{a, b\})$. For $S = \{a\}$, founder 1 supports the empty set and founder 2 supports $\{a\}$. This voting profile is a Nash equilibrium. However, it is not a perfect equilibrium, as founder 1's strategy of $\emptyset$ is weakly dominated by voting $\{b\}$. This shows that requiring genericity is needed in Proposition 5.2. Note that (b, b) is a perfect Nash equilibrium which does support the conditions of Proposition 5.2.

An interesting application of Corollary 5.3 is the following:

Theorem 5.7. *Let Γ be a game representing a 2-stage generic voting scheme, whose utilities obey additivity across stages and additivity within each stage (Assumption 8b). Under these conditions, Γ has a perfect equilibrium in pure strategies.*

[29] Note that $(\{a\}, \{a, b\})$ can be eliminated by successive weak domination.

Proof. Any perfect equilibrium profile for Γ must specify for each subgame of the second stage a profile under which each voter votes precisely for the set of his friends (who are not already in the society). This is a perfect equilibrium of the subgame (Section 2, case $k = 1$) and unique, by genericity. With this understanding, we can construct a 1-stage game Γ^1 by collation. The proof will be concluded if we show that Γ^1 has a pure-strategy perfect equilibrium, as the combination of this strategy with the continuation is a perfect strategy[30] for Γ. To achieve that, it is sufficient, by Corollary 5.3, to exhibit a generalized-partition equilibrium profile for Γ^1. This we are about to do by a construction under which voters add candidates to the society piecewise: There will be a variable set of candidates, called a *current set*, that grows, or stays put, as the voters add to it during the construction, until it eventually becomes the outcome for Stage 1, as well as an outcome of Γ^1. We introduce the following terminology: Let A be a current set of candidates. We say that a, possibly empty, set of candidates taken from $C^0 \setminus A$, is *optimal for voter i w.r.t. A*, and denoted $X_i(A)$, if it is the best set of candidates that i could add to A, so as to increase his utility from the two stages. Note that $X_i(A)$ cannot contain enemies of i, since such candidates are enemies, and can only contribute more enemies at Stage 2. (The friends of i will be brought in anyhow by i at Stage 2.) In symbols, $X_i(A)$ is characterized by

$$(5.12)\qquad \begin{aligned} &w_i(A \cup X_i(A)) + w_i(\mathrm{en}_i\,(F^0 \cup A \cup \mathrm{fr}\,(F^0 \cup A \cup X_i(A)))) \geq \\ &w_i(A \cup B) + w_i(\mathrm{en}_i\,(F^0 \cup A \cup \mathrm{fr}\,(F^0 \cup A \cup B))), \qquad \text{all } B \subseteq \mathrm{fr}_i\,(C^0 \setminus A). \end{aligned}$$

(In this calculation friends of i at the second stage are omitted from both sides of each inequality.) Here, $w_i(T) := \sum_{t \in T} w_i(t)$, $\mathrm{fr}_i\,(S) := \{j \colon j \in \mathrm{fr}\,(i) \cap S\}$, $\mathrm{en}_i\,(S) := \{j \colon j \in \mathrm{en}\,(i) \cap S\}$ and $\mathrm{fr}\,(B) := \{\ell \colon \ell \in \mathrm{fr}\,(j) \text{ for some } j \text{ in } B\}$. Sums over the empty set are considered equal to zero. By genericity, the set $X_i(A)$ is unique.

[30] A proof for any game representing a k-stage voting scheme is given in Theorem 5.12.

The construction:

Starting with a current set $A = \emptyset$, a referee approaches the voters repeatedly, one by one, and suggests to them to add candidates to the current set. Each approached voter i adds $X_i(A)$ and the set $A \cup X_i(A)$ becomes a new 'current set' A. The referee continues to approach the voters, perhaps approaching a voter several times, taking care not to ignore voters whose optimal addition is not empty. This assures that after a finite number of approaches, there comes a situation when all optimal sets w.r.t. the current A are empty for all voters. At this the construction ends. This determines a pure-strategy profile $\{P_j\}_{j\in F^0}$, where P_j is the set consisting of all the members that voter j added along the construction.

It follows from the construction, that $\{P_j\}_{j\in F^0}$ is a generalized partition of $S := \cup_{j\in F^0} P_j$. It remains to show that it is an equilibrium profile for Γ^1. To this end we require a lemma, which unfortunately is not true if $k > 2$:

Lemma 5.8. *Assume the conditions and notations of Theorem 5.7. Let A and B be two sets of candidates, $A \subseteq B$. Let C be a set of friends of a voter i satisfying $C \cap B = \emptyset$. If $A \cup C \succ_i A$ then $B \cup C \succ_i B$.*

Proof. From the data it follows that the total weight of i from C exceeds the absolute value of the total weight of the new enemies that C brings at Stage 2.[31] When C is added to B he brings the same number of friends, namely $|C|$, and no new enemies. Perhaps even less — the previous ones that happen to be in $B \setminus A$. ∎

(Continuation of the proof of Theorem 5.7). If $(P_j)_{j\in F^0}$ is not an equilibrium profile, then a voter i can benefit from a deviation. A deviation means that he deletes a set T of candidates from

[31] Namely, $w_i(C) + w_i(en_i(\mathrm{fr}\,(C \setminus \mathrm{fr}\,(F^0 \cup A)))) > 0$.

his vote P_i and adds a set Q of candidates not in S.[32] At least one of these sets is not empty. The set T, if not empty, is a union of nonempty sets $T_1, T_2, \dots T_r$, which are, respectively, subsets of his votes $P_i^1, P_i^2, \dots, P_i^r$ taken when i was approached at times that we enumerate chronologically $1, 2 \dots, r$. Denote by $S_1, S_2, \dots, S_r$ the current sets at these times after his addition. Consider a hypothetical sequence when all founders vote as in the construction except that agent i votes $P_i^1 \setminus T_1$ at time 1, $P_i^2 \setminus (T_1 \cup T_2)$ at time 2, $\dots$, $P_i^r \setminus T$ at time r and at the first time he also adds the candidates of Q. The end of this sequence is the deviation, which, as we assumed, benefited player i. We now modify this sequence in such a way that player i will continue to benefit and at least as much. To this end, add T_1 to the hypothetical vote of voter i at all times, starting from time 1. This will benefit him at time 1. Indeed, he would benefit if the current set were $S_1 \setminus T_1$ because $X_i(S_1 \setminus P_i^1) = S_1$ is the unique optimal response and so, by Lemma 5.8, he would benefit by adding T_1 to $(S_1 \setminus T_1) \cup Q$. For the same reason i would benefit by adding T_1 at every part of the hypothetical sequence, since $S_1 \setminus T_1 \subseteq Q \cup (S_t \setminus (T_1 \cup T_2 \cdots \cup T_t))$ and $T_1 \cap (Q \cup (S_t \setminus (T_1 \cup T_2 \cdots \cup T_t)) = \emptyset$, $t \in \{1, 2, \dots, r\}$. After adding T_1 we are in an improved deviation that starts at time 2. We make a similar modification and continue for r times. Eventually, we arrive at an improved deviation at which only Q is added. But this is impossible, since the original construction ended when no voter could beneficially add members outside the current set. The contradiction shows that we are indeed at equilibrium. ∎

The construction in the above proof is not specific about the order in which the referee approaches the voters. We are going to show that although different orders yield different equilibrium profiles, the outcome S remains the same. Therefore, the *perfect equilibrium profile* that is generated as described in Proposition 5.2 is the same, regardless of the order of approach.

Lemma 5.9. *If $A \subseteq B \subseteq C^0$, then $A \cup X_i(A) \subseteq B \cup X_i(B)$ for every agent i in F^0.*

[32] It is irrelevant if he also votes for agents in $S \setminus P_i$, so we assume that he does not.

Proof. Assume negatively, that for some i in F^0, $D := (A \cup X_i(A)) \setminus (B \cup X_i(B)) \neq \emptyset$. By optimality of $X_i(A)$ and genericity of Γ, it follows from (5.12), replacing B by $X_i(A) \setminus D$, and noting that $D \cap A = \emptyset$, that

(5.13)

$$w_i(D) + w_i(\mathrm{en}\,_i(F^0 \cup A \cup \mathrm{fr}\,(F^0 \cup A \cup X_i(A)))) - w_i(\mathrm{en}\,_i(F^0 \cup A \cup \mathrm{fr}\,(F^0 \cup A \cup (X_i(A) \setminus D)))) =$$

$$w_i(D) + w_i(\mathrm{en}\,_i(\mathrm{fr}\,(D) \setminus \mathrm{fr}\,(F^0 \cup A \cup (X_i(A) \setminus D)))) > 0.$$

Using (5.12) once more, replacing $A, X_i(A), B$ by $B, X_i(B), X_i(B) \cup D$, respectively, we obtain:

$$w_i(D) + w_i(\mathrm{en}\,_i(\mathrm{fr}\,(D) \setminus \mathrm{fr}\,(F^0 \cup B \cup X_i(B)))) < 0. \tag{5.14}$$

However, $(A \cup X_i(A)) \setminus D \subseteq B \cup X_i(B)$, and enemies of i carry negative utilities; therefore, the left side of (5.14) is not smaller than the left side of (5.13) — a contradiction. ∎

Corollary 5.10. *Changing the order of the referee's approaches leads to the same final set S, although the actual votes of the players may be different.*

Proof. Let $\emptyset = T^0, T^1, \ldots, T^r = T$ be the sequence of 'current sets' generated by a different order of approaches. We shall show that $T^m \subseteq S$ for every m and therefore $T \subseteq S$. Reversing the roles of S and T one gets $S \subseteq T$ and this concludes the proof. Proceed by induction: Certainly $T^0 \subseteq S$. Suppose $T^{m-1} \subseteq S$ and $T^m \not\subseteq S$. Then, some i in F^0 has $X_i(T^{m-1}) \not\subseteq S$. Thus, a candidate a exists in $X_i(T^{m-1})$, $a \notin S$. From Lemma 5.5, $a \in X_i(T^{m-1}) \subseteq X_i(S)$, which contradicts the fact that the construction terminates when $X_i(S) = \emptyset$ for all i. ∎

One may now ask whether a perfect equilibrium profile is always unique under the conditions of Theorem 5.7. The following example settles this question negatively.

Example 5.11. The set of founders is $F^0 = \{1, 2\}$. The set of candidates is $C^0 = \{a, b, c\}$. $k = 2$ and we assume pure friendship and enmity (Assumption 8a). $\mathrm{fr}\,(1) = \{a\}, \mathrm{fr}\,(2) = \{b\}$,

$\mathrm{fr}(a) = \mathrm{fr}(b) = \{c\}$. The construction in Theorem 5.7 leads to $S = \emptyset$. However, it can be checked that 1 and 2 voting for their friends at all stages and a and b vote for their friend at Stage 2 is also a perfect equilibrium profile.

We conclude this section by extending Proposition 5.2 to several-stage voting schemes.

Theorem 5.12. *Let Γ be a game representing a k-stage generic voting scheme, obeying general stream dependence (Assumption 8d). If, during collation, we always manage to choose a pure-strategy perfect-equilibrium profile at each one-stage game,*[33] *the resulting strategy is perfect for Γ. Conversely, every pure-strategy perfect profile for Γ can be obtained by collation in this fashion.*

Proof. The game Γ is a game of perfect recall, therefore, by Kuhn's (1953) theorem (see also Selten (1975)), we can work only with behavioral strategies.

We regard Γ as given in extensive form. Denote by $\Gamma_{t,r}$ the 1-stage tree that corresponds to the r-th tree[34] of stage t. Denote by $\hat{\Gamma}_{t,r}$ the subgame of Γ that starts with $\Gamma_{t,r}$.

Collation with respect to a strategy τ converts the 1-stage tree $\Gamma_{t,r}$ to a 1-stage game $\Gamma_{t,r}(\tau)$, where, even if τ happens to be defined on all of Γ, we mean here its restriction *to the subgame $\hat{\Gamma}_{t,r}$ excluding the first stage of this subgame.* $\Gamma_{t,r}(\tau)$ is a 1-stage game, so, if its voters employ a strategy profile[35] $(\rho)^1$, we denote by $h_{t,r}(\tau)(\rho)^1$ the payoff vector that results.

Note that

$$h_{t,r}(\tau)(\rho)^1 = \hat{h}_{t,r}((\rho)^1(\tau)), \tag{5.15}$$

[33] Proposition 5.2 and Corollaries 5.3 and 5.4 might be useful here.

[34] Counted, e.g., from left to right.

[35] The superscript 1 comes merely to remind us that ρ is a 1-stage profile, or a restriction to a 1-stage profile, if ρ happens to be defined in a larger game.

where the right side is the payoff vector that results when the agents play $\hat{\Gamma}_{t,r}$, using ρ at the first stage and then continue with τ. Note that for $(t,r)=(1,1)$, $h_{1,1}(\tau)(\rho)^1$ is the expected payoff in Γ if $(\rho)^1$, followed by τ, is played.

A. Assume that σ is constructed backwards, by collation, such that at each 1-stage game, a pure-strategy perfect profile is chosen. We require that identical 1-stage moves are chosen at $\Gamma_{t,r}$'s with identical histories; i.e., identical streams of members until stage t (see Assumptions 5 and 6).[36] Then, for each (t,r), there exists a test sequence $(\sigma^m_{t,r})^\infty_{m=1}$ of completely mixed 1-stage strategy profiles converging to the restriction $\sigma_{t,r}$ of σ to $\Gamma_{t,r}$, such that for every agent i,

$$h_{t,r;i}(\sigma)(\sigma^m_{t,r;-i},\sigma_{t,r;i})^1 \geq h_{t,r;i}(\sigma)(\sigma^m_{t,r;-i},\sigma'_i)^1, \tag{5.16}$$

whenever σ'_i is a pure 1-stage move different from $\sigma_{t,r;i}$. Again, to ensure that eventually strategies depend only on histories, identical $\sigma^m_{t,r}$ should be chosen at $\Gamma_{t,r}$'s that result from the same stream up to stage t. This is always possible, because of the way σ was constructed.

Let us examine the payments at endpoints of $\Gamma_{t,r}(\tau)$, where τ is an arbitrary pure strategy that depends only on histories. Since Γ is assumed to be generic, payments to an agent i at two endpoints of $\Gamma_{t,r}(\tau)$ are different unless, and only unless, one of the following cases occurs:

i. Agent i is not a voter in $\Gamma_{t,r}$. Observe that in this case, whatever agent i "does" against whatever the other agents are doing in this game, or in a different continuation, is always a best reply.

or

ii. The two endpoints result from the same 1-stage "stream" in $\Gamma_{t,r}$. In this case τ, and any pure, or mixed (behavioral) strategy, that depends only on histories, specify the same

[36] At the expense of somewhat more technicality, a similar theorem can be proved even if the strategies are more complicated than those used in this paper.

continuation following these points. Indeed, to every path that follows one endpoint, there corresponds a path following the other endpoint that specifies the same stream of members.

It follows that if $\Gamma_{t,r}(\sigma)$ has the same payment to agent i at two endpoints then $\Gamma_{t,r}(\sigma^m_{-i}, \sigma_i)$ has the same payments to agent i, for every $m \in \{1, 2 \dots\}$, where σ^m denotes the aggregate of the $\sigma^m_{t,r}$'s.

Now, as m gets larger, the payoff vectors at the endpoints of $\Gamma_{t,r}(\sigma^m_{-i}, \sigma_i)$ approach those of $\Gamma_{t,r}(\sigma_{-i}, \sigma_i)$, and are equal whenever the latter are equal. Therefore, if $(\sigma_{t,r;i})^1$ is a best response to $(\sigma_{t,r;-i})^1$ in $\Gamma_{t,r}(\sigma_{-i}, \sigma_i)$, then it remains a best response in $\Gamma_{t,r}(\sigma^m_{-i}, \sigma_i)$, if m is large enough. Thus,

$$h_{t,r;i}(\sigma^m_{-i}, \sigma_i)(\sigma^m_{t,r;-i}, \sigma_{t,r;i})^1 \geq h_{t,r;i}(\sigma^m_{-i}, \sigma_i)(\sigma^m_{t,r;-i}, \sigma'_i)^1, \tag{5.17}$$

for all strategies σ'_i.

Now, $(\sigma^m)_{m=1}^\infty$ is a sequence of completely mixed (behavioral) strategies. Our proof will conclude, if we prove that for every agent i and every large enough m, σ_i is a best reply to σ^m_{-i}. Indeed, let σ'_i be an arbitrary pure strategy for agent i in Γ. By (5.15), it yields agent i the payoff $h_{1,1;i}(\sigma^m_{-i}, \sigma'_i)(\sigma^m_{1,1;-i}, \sigma'_{1,1;i})^1$ against σ^m_{-i}

Working backwards, we instruct agent i to switch to $\sigma_{t,r;i}$ at each stage. By (5.17), this will not decrease his payment at each 1-stage game, that sequentially becomes $\Gamma_{t,r}(\sigma^m_{-i}, \sigma_i)$. The final payoff to agent i eventually becomes $h_{1,1;i}(\sigma^m_{-i}, \sigma_i)(\sigma^m_{1,1;-i}, \sigma_{1,1;i})^1$. We have proved that

$$h_{1,1;i}(\sigma^m_{-i}, \sigma_i)(\sigma^m_{1,1;-i}, \sigma_{1,1;i})^1 \geq h_{1,1;i}(\sigma^m_{-i}, \sigma'_i)(\sigma^m_{1,1;-i}, \sigma'_{1,1;i})^1. \tag{5.18}$$

So, by (5.15), σ_i is indeed a best reply in Γ, to σ^m_{-i}, whenever m is large enough, and σ is a pure-strategy perfect profile in Γ.

B. Conversely, suppose that σ is a pure-strategy perfect profile for Γ, then there exists a sequence $(\sigma^m)_{m=1}^{\infty}$, of completely mixed behavioral strategies, converging to σ, such that for each agent i and for each m, σ_i is a best reply to σ^m_{-i}. This is true also in every subgame $\hat{\Gamma}_{t,r}$. Invoking (5.15), we find that

$$h_{t,r;i}(\sigma^m_{-i}, \sigma_i)(\sigma^m_{t,r;-i}, \sigma_{t,r;i})^1 \geq h_{t,r;i}(\sigma^m_{-i}, \sigma'_i)(\sigma^m_{t,r;-i}, \sigma'_{t,r;i})^1, \tag{5.19}$$

for every pure-strategy σ'_i of agent i, where, $\sigma^m_{t,r}$ is the restriction of σ^m to the tree $\Gamma_{t,r}$. In particular, this is true if σ'_i differs from σ_i only at $\Gamma_{t,r}$. Thus,

$$h_{t,r;i}(\sigma^m_{-i}, \sigma_i)(\sigma^m_{t,r;-i}, \sigma_{t,r;i})^1 \geq h_{t,r;i}(\sigma^m_{-i}, \sigma_i)(\sigma^m_{t,r;-i}, \sigma'_{t,r;i})^1, \tag{5.20}$$

for every pure-strategy 1-stage deviation $\sigma'_{t,r;i}$.

Now, the voting scheme is generic, so the payments corresponding to distinct endpoints of $\Gamma_{t,r}(\sigma)$ are different for any agent i, who is a voter, whenever these endpoints represent different 1-stage streams. As explained in the previous part, those endpoints that represent the same 1-stage stream have the same payoff vectors both in $\Gamma_{t,r}(\sigma)$ and in the games $\Gamma_{t,r}(\sigma^m_{-i}, \sigma_i)$, $m = 1, 2, \ldots,$ because all strategies depend only on streams. Thus, there exists a d, such that for all $m \geq d$, the relative order among the payments to each voter at $\Gamma_{t,r}(\sigma)$ and at $\Gamma_{t,r}(\sigma^m_{-i}, \sigma_i)$, are identical. We conclude that $\sigma_{t,r;i}$ is a best reply against $\sigma^m_{t,r;-i}$ in both games. In particular,

$$h_{t,r;i}(\sigma)(\sigma^m_{t,r;-i}, \sigma_{t,r;i})^1 \geq h_{t,r;i}(\sigma)(\sigma^m_{t,r;-i}, \sigma'_i)^1, \tag{5.21}$$

for every 1-stage deviation σ'_i of agent i. This means that the restriction of $(\sigma^m)_{m=d}^{\infty}$ to $\Gamma_{t,r}(\sigma)$, is an appropriate test sequence and $\sigma_{t,r}$ is therefore prefect for the 1-stage game $\Gamma_{t,r}(\sigma)$, obtained by collation with respect to σ. ■

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