GRAPHICAL MODELS AND IMAGE PROCESSING
Vol. 59, No. 2, March, pp. 109–116, 1997
ARTICLE NO. IP960416

NOTE

An Efficient Class of Alternating Sequential Filters in Morphology*

SOO-CHANG PEI AND CHIN-LUN LAI

Department of Electrical Engineering, National Taiwan University, Taipei, Taiwan, Republic of China

AND

FRANK Y. SHIH

Department of Computer and Information Science, New Jersey Institute of Technology, Newark, New Jersey 07102

Received December 21, 1995; revised September 23, 1996; accepted December 17, 1996

In this note, an efficient class of alternating sequential filters (ASFs) in mathematical morphology is presented to reduce the computational complexity in the conventional ASFs about a half. The performance boundary curves of the new filters are provided. Experimental results from applying these new ASFs to texture classification and image filtering (grayscale and binary) show that comparable performance can be achieved while much of the computational complexity is reduced. © 1997 Academic Press

I. INTRODUCTION

Mathematical morphology, a set-theoretical method, provides an approach to digital image processing based on the geometrical shape of an object. It was first investigated by Matheron [1] and Serra [2]. Using an *a priori* determined structuring element, features in an image can be extracted, suppressed, or preserved by applying morphological operators. Morphological operations tend to simplify image data while preserving their essential shape characteristics. The morphological operations can be employed for many purposes including digitization, enhancement, compression, restoration, segmentation, and description.

An important application in mathematical morphology is to use a morphological filter to efficiently extract the crucial structures of binary images. Morphological filtering was developed by Matheron [1] with the inception of morphological openings and closings. The class of alternating filters (AFs), composed of morphological openings and closings, has been demonstrated to be useful in image analysis applications. Sternberg [3] introduced a new class of morphological filters called alternating sequential filters (ASFs), which consist of iterative operations of openings and closings with structuring elements of increasing sizes. Transformations that apply alternating sequences of openings and closings introduce less distortion than individual openings and closings with the largest structuring element [2]. Schonfeld and Goutsias [4] have shown that ASFs are the best in preserving crucial structures of binary images in the least difference sense. The ASFs have been successfully used in a variety of applications such as remote sensing and medical imaging [5, 6].

A noticeable problem of applying ASFs to image analysis is their high computational complexity. For a structuring element of size n, the computational complexity is the fourth power of n when the opening–closing operation is used. This leads to a huge number of operations. In this note, an efficient class of ASFs based on the adjunctional property of morphology is proposed to reduce the computational complexity. Moreover, the performance bound of these new ASFs is derived. To illustrate the analytical results, image filtering and texture classification applications are adopted.

This paper is organized as follows. In Section II, the definitions of alternating sequential filters are reviewed. Section III describes morphological adjunction and the new class of alternating sequential filters. Their properties and numerical bound are derived in Section IV. Experimental results and discussion of texture classification and

* This work was supported by the National Science Council in Taiwan, Republic of China, and by the New Jersey Institute of Technology, Newark, New Jersey.

1077-3169/97 $25.00

image filtering applications are given in Section V. Conclusions are drawn in Section VI.

II. PRELIMINARY: ALTERNATING SEQUENTIAL FILTERS

Alternating sequential filters (ASFs) in morphology [3] are a combination of iterative morphological filters with increasing sizes of structuring elements. They offer a hierarchical structure for extracting the geometrical characteristics of objects. It had been proved that AS filters have less distortion in feature extraction than those filters which directly process the images with the largest structuring element. The alternating filter is composed of morphological openings and closings whose primitive morphological operations are dilation and erosion. Detailed definitions of dilation, erosion, opening, and closing are given in [2, 3].

Let X denote a binary image and B a binary structuring element. The alternating filter is defined as an opening followed by a closing or a closing followed by an opening and is represented as

$$AF_B(X) = (X \circ B) \cdot B \tag{1}$$

or

$$AF_B(X) = (X \cdot B) \circ B. \tag{2}$$

Another type of AF is defined as

$$AF_B(X) = ((X \circ B) \cdot B) \circ B \tag{3}$$

or

$$AF_B(X) = ((X \cdot B) \circ B) \cdot B. \tag{4}$$

An alternating sequential filter (ASF) is an iterative application of $AF_B(X)$ with increasing sizes of structuring elements, denoted as

$$ASF(X) = AF_{B_N} AF_{B_{N-1}} \ldots AF_{B_1}(X), \tag{5}$$

where N is an integer and $B_N, B_{N-1}, \ldots, B_1$ are structuring elements with decreasing sizes. The B_N is constructed by

$$B_N = B_{N-1} \oplus B_1, \text{ for } N \geq 2. \tag{6}$$

The ASFs offer a method of extracting the features of images hierarchically. That is, these features can be divided into different layers according to their corresponding structuring element sizes. The features, such as size distribution [8, 9], of each layer can be used in many applications, e.g., feature classification and recognition.

III. THE NEW CLASS OF ASFs

A. *Morphological Adjunction*

A pair of morphological operators (ε, δ) on Z^2 is called an adjunction [7] if

$$\delta(Y) \subseteq X \Leftrightarrow Y \subseteq \varepsilon(X), \text{ where } X, Y \subseteq Z^2. \tag{7}$$

That is, if (ε, δ) is an adjunction, then ε is an erosion and δ a dilation. For each erosion there exists a unique dilation such that this pair constitutes an adjunction.

Let a function f be dilated by a structuring element B n times, denoted as $f(\delta^n)B$. That is equivalent to $f \oplus nB$. This notation is also applied for erosions n times as ε^n. If (ε, δ) is an adjunction, it has the following property [10].

PROPOSITION 1. *Let a filter be of the form* $\psi = \varepsilon^{e_1}\delta^{d_1}\varepsilon^{e_2}$. *If* $e_1 = d_1 = e_2$, *this filter can be reduced to*

$$\psi = \varepsilon^{e_1}\delta^{e_1}\varepsilon^{e_1} = \varepsilon^{e_1}. \tag{8}$$

Similarly, a filter of the form $\phi = \delta^{d_1}\varepsilon^{d_1}\delta^{d_1}$ *can be reduced to*

$$\phi = \delta^{d_1}\varepsilon^{d_1}\delta^{d_1} = \delta^{d_1}. \tag{9}$$

Filters that are of the form

$$\psi = \varepsilon^{e_n}\delta^{d_n} \ldots \varepsilon^{e_2}\delta^{d_2}\varepsilon^{e_1}\delta^{d_1} \tag{10}$$

are called adjunctional filters [7]. There exists a large amount of redundancy in adjunctional filters which can be removed.

B. *Redundancy Removal in ASFs*

In order to simplify the notations of different ASFs, several definitions are given to represent different types of ASFs as follows.

DEFINITION 1. An ASF with $AF_B(X)$ having the form of Eq. (1) is called a TYPE-I ASF.

DEFINITION 2. An ASF with $AF_B(X)$ having the form of Eq. (2) is called a TYPE-II ASF.

DEFINITION 3. An ASF with $AF_B(X)$ having the form of Eq. (3) is called a TYPE-III ASF.

DEFINITION 4. An ASF with $AF_B(X)$ having the form of Eq. (4) is called a TYPE-IV ASF.

For convenience, a type-i ASF with a structuring element B is denoted as ASF_i^B. It is observed that the type-

III and type-IV ASFs have computational redundancy in their structure. If a type-III ASF has n levels, it can be rewritten as

$$\begin{aligned} ASF^{B}_{III}(X) &= (((((((((X \circ B) \cdot B) \circ B) \circ 2B) \cdot 2B) \\ &\quad \circ 2B) \cdots \circ nB) \cdot nB) \circ nB \\ &= X(\varepsilon\delta\delta\varepsilon\varepsilon\delta\varepsilon^{2}\delta^{2}\delta^{2}\varepsilon^{2}\varepsilon^{2}\delta^{2} \cdots \varepsilon^{n}\delta^{n}\delta^{n}\varepsilon^{n}\varepsilon^{n}\delta^{n})B \\ &= X(\varepsilon\delta^{2}\underline{\varepsilon^{2}\delta\varepsilon^{2}}\delta^{4}\varepsilon^{4}\delta^{2} \cdots \varepsilon^{n}\delta^{2n}\varepsilon^{2n}\delta^{n})B \\ &= X(\varepsilon\delta^{2}\underline{\varepsilon^{3}}\delta^{4}\varepsilon^{4}\delta^{2} \cdots \varepsilon^{n}\delta^{2n}\varepsilon^{2n}\delta^{n})B \\ &\quad \vdots \\ &= X(\varepsilon\delta^{2}\varepsilon^{3}\delta^{4} \cdots \delta^{2n}\varepsilon^{2n}\delta^{n})B \\ &= ((((((X \circ B) \cdot B) \circ 2B) \cdot 2B) \\ &\quad \cdots \circ nB) \cdot nB) \circ nB \\ &= ASF^{\mathrm{B}}_{I}(X) \circ nB. \end{aligned} \tag{11}$$

Equation (11) states that a type-III ASF can be reduced to a type-I ASF but with an additional opening operation padded onto the end. Similarly, an n level type-IV ASF can be rewritten as

$$\begin{aligned} ASF^{B}_{IV}(X) &= (((((((((X \cdot B) \circ B) \cdot B) \cdot 2B) \circ 2B) \\ &\quad \cdot 2B) \cdots \cdot nB) \circ nB) \cdot nB \\ &= ((((((X \cdot B) \circ B) \cdot 2B) \circ 2B) \\ &\quad \cdots \cdot nB) \circ nB) \cdot nB \\ &= ASF^{B}_{II}(X) \cdot nB. \end{aligned} \tag{12}$$

Thus, a type-IV ASF can be reduced to a type-II ASF but with an additional closing operation padded onto the end. That is, by using the adjunction property, a type-III (type-IV) ASF can be implemented by a type-I (type-II) ASF, adding an additional opening (closing) operation. Compared with the original implementation of type-III and type-IV ASFs, the computational complexity of this new method is reduced to about $\frac{2}{3}$ of the original defined operations.

C. *The New Class of ASFs*

The definitions of the new class of ASFs are given as follows.

Definition 5. An ASF of the following form is called a TYPE-V ASF:

$$\begin{aligned} ASF^{B}_{V}(X) = (((((((X \circ B) \cdot B) \cdot 2B) \circ 2B) \\ \circ 3B) \cdot 3B) \cdots. \end{aligned} \tag{13}$$

Definition 6. An ASF of the following form is called a TYPE-VI ASF.

$$\begin{aligned} ASF^{B}_{VI}(X) = (((((((X \cdot B) \circ B) \circ 2B) \cdot 2B) \\ \cdot 3B) \circ 3B) \cdots. \end{aligned} \tag{14}$$

Definition 7. An ASF of the following form is called a TYPE-VII ASF.

$$\begin{aligned} ASF^{B}_{VII}(X) = (((((((((X \circ B) \cdot B) \circ \mathrm{B}) \cdot 2B) \circ 2B) \\ \cdot 2B) \circ 3B) \cdot 3B) \circ 3\mathrm{B}) \cdots. \end{aligned} \tag{15}$$

Definition 8. An ASF of the following form is called a TYPE-VIII ASF.

$$\begin{aligned} ASF^{B}_{VIII}(X) = (((((((((X \cdot B) \circ B) \cdot \mathrm{B}) \circ 2B) \cdot 2B) \\ \circ 2B) \cdot 3B) \circ 3B) \cdot 3\mathrm{B}) \cdots. \end{aligned} \tag{16}$$

It is noted that ASF_V, ASF_{VI}, ASF_{VII}, and ASF_{VIII} are the modifications of ASF_I, ASF_{II}, ASF_{III}, and ASF_{IV}, respectively. Definitions of the eight different types of ASFs are listed in Table 1. To achieve the computational efficiency of the new ASFs, they are rewritten as follows.

Let n be an odd number throughout the following derivations. A type-V ASF can be expressed as

$$\begin{aligned} ASF^{B}_{V}(X) &= (((((X \circ B) \cdot B) \cdot 2B) \circ 2B) \\ &\quad \cdots \circ nB) \cdot nB \\ &= X(\varepsilon\delta\delta\varepsilon\delta^{2}\varepsilon^{2}\varepsilon^{2}\delta^{2} \cdots \varepsilon^{n}\delta^{n}\delta^{n}\varepsilon^{n})B \\ &= X(\varepsilon\delta^{2}\varepsilon\delta^{2}\varepsilon^{4}\delta^{2} \cdots \varepsilon^{n}\delta^{2n}\varepsilon^{n})B \\ &= X(\varepsilon\delta^{3}\varepsilon^{5} \cdots \delta^{2n-3}\varepsilon^{2n-1}\delta^{2n}\varepsilon^{n})B \\ &= ((((X \circ B) \cdot 2B) \circ 3B) \cdots \circ nB) \cdot nB. \end{aligned} \tag{17}$$

Similarly, a type-VI ASF can be expressed as

$$\begin{aligned} ASF^{B}_{VI}(X) &= (((((X \cdot B) \circ B) \circ 2B) \cdot 2B) \\ &\quad \cdots \cdot nB) \circ nB \\ &= ((((X \cdot B) \circ 2B) \cdot 3B) \cdots \cdot nB) \circ nB. \end{aligned} \tag{18}$$

TABLE 1
Definitions of Eight Different Types of ASF

ASF type	Definition
ASF_I	$X \circ B \cdot B \circ 2B \cdot 2B \cdots$
ASF_{II}	$X \cdot B \circ B \cdot 2B \circ 2B \cdots$
ASF_{III}	$X \circ B \cdot B \circ B \circ 2B \cdot 2B \circ 2B \cdots$
ASF_{IV}	$X \cdot B \circ B \cdot B \cdot 2B \circ 2B \cdot 2B \cdots$
ASF_V	$X \circ B \cdot B \cdot 2B \circ 2B \cdots$
ASF_{VI}	$X \cdot B \circ B \circ 2B \cdot 2B \cdots$
ASF_{VII}	$X \circ B \cdot B \circ B \cdot 2B \circ 2B \cdot 2B \cdots$
ASF_{VIII}	$X \cdot B \circ B \cdot B \circ 2B \cdot 2B \circ 2B \cdots$

It is clear that, for each $AF_B(X) = (X \circ B) \cdot B$ layer, since either a closing or an opening operation can be saved in ASF_V and $ASFVI$, their complexities are nearly one half the amount of those in ASF_I and ASF_{II}. On the other hand, rewriting the type-VII and type-VIII ASFs by the same method produces similar results as follows:

$$\begin{aligned} ASF^B_{VII}(X) &= (((((((((X \circ B) \cdot B) \circ B) \cdot 2B) \circ 2B) \\ &\quad \cdot 2B) \cdots \circ nB) \cdot nB) \circ nB \\ &= ((((((X \circ B) \cdot 2B) \circ 3B) \\ &\quad \cdots \cdot (n-1)B) \circ nB) \cdot nB) \circ nB \\ &= ASF^B_V \circ nB \end{aligned} \tag{19}$$

and

$$\begin{aligned} ASF^B_{VIII}(X) &= (((((((((X \cdot B) \circ B) \cdot B) \circ 2B) \cdot 2B) \\ &\quad \circ 2B) \cdots \cdot nB) \circ nB) \cdot nB \\ &= ((((((X \cdot B) \circ 2B) \cdot 3B) \\ &\quad \cdots \circ (n-1)B) \cdot nB) \circ nB) \cdot nB \\ &= ASF^B_{VI} \cdot nB. \end{aligned} \tag{20}$$

Therefore, an additional advantage in reducing complexity can be achieved if we rewrite the four new defined ASFs by using the adjunction property. However, although additional advantages are obtained, there are some important questions to be addressed. What are the properties of the new ASFs? Do they have a performance boundary? These questions will be answered in the next section.

IV. PROPERTIES OF THE NEW CLASS OF ASFs

PROPERTY 1. *The four newly defined operations are increasing.*

PROPERTY 2. *The four newly defined operations are idempotent.*

The proofs of Properties 1 and 2 are trivial since they are composed of increasing and idempotent operators. These two properties are used to ensure that the proposed new operators are morphological filters. In the following properties the upper and lower bounds of these new ASFs are derived. To ease the analysis, the padded operations of types- V–VIII are ignored. Thus, an odd number n level type-V ASF can be written as $X \circ B \cdot 2B \cdots \circ nB$.

PROPERTY 3. *A type-V ASF is nearly bounded by a type-II ASF (upper bound) and exactly bounded by a type-I ASF (upper bound) when n is odd, and is nearly bounded by a type-I ASF (lower bound) and exactly bounded by a type-II ASF (upper bound) when n is even.*

Proof. By using the anti-extensive and extensive properties of the opening and closing filters, it is observed that

$$X \circ B \subset (X \cdot B) \circ B \subset X \cdot B. \tag{21}$$

Since closing is increasing, if a closing operator is applied at each side, Eq. (21) can be rewritten as

$$\begin{aligned} (X \circ B) \cdot 2B &\subset ((X \cdot B) \circ B) \cdot 2B \\ &\subset (X \cdot B) \cdot 2B \quad [\cdot 2B], \end{aligned}$$

where the present equation is obtained by applying the operation inside the brackets to the previous equation. By continuously utilizing the same method, the above equation implies

$$\begin{aligned} &X \circ B \cdot B \circ 2B \cdot 2B \circ 3B \cdot 3B \\ &\quad \subset X \circ B \cdot 2B \circ 3B \cdot 3B \\ &\quad \subset X \cdot B \circ B \cdot 2B \circ 2B \cdot 3B \circ 3B \cdot 3B \end{aligned}$$

when n is odd. It is noted that the left-hand side of the above equation is exactly a type-I ASF while the right-hand side is a type-II ASF with an additional closing padded to its end. On the other hand, if n is even, Eq. (21) implies

$$\begin{aligned} &X \circ B \cdot B \circ 2B \cdot 2B \circ 2B \subset X \circ B \cdot 2B \circ 2B \\ &\quad \subset X \cdot B \circ B \cdot 2B \circ 2B. \end{aligned}$$

It is observed that the left-hand side of the above equation is a type-I ASF with an additional opening padded to its end while the right-hand side is an exact type-II ASF. That is, the type-V ASFs are bounded by the type-I and type-II ASFs. ■

PROPERTY 4. *A type-VI ASF is nearly bounded by a type-I ASF (lower bound) and exactly bounded by a type-II ASF (upper bound) when n is odd, and is nearly bounded by a type-II ASF (upper bound) and exactly bounded by a type-I ASF (lower bound) when n is even.*

Proof. By using the same method in the proof of Property 3, the equation

$$X \circ B \subset (X \circ B) \cdot B \subset X \cdot B \tag{22}$$

implies

$$\begin{aligned} &X \circ B \cdot B \circ 2B \cdot 2B \circ 3B \cdot 3B \circ 3B \\ &\quad \subset X \cdot B \circ 2B \cdot 3B \circ 3B \\ &\quad \subset X \cdot B \circ B \cdot 2B \circ 2B \cdot 3B \circ 3B \end{aligned}$$

in case n is an odd number. It is noted that the left-hand side and the right-hand side are nearly the type-I and exactly the type-II ASF, respectively. If n is an even number, Eq. (22) implies

$$X \circ B \cdot B \circ 2B \cdot 2B \subset X \cdot B \circ 2B \cdot 2B$$
$$\subset X \cdot B \circ B \cdot 2B \circ 2B \cdot 2B.$$

Thus, the left-hand side and right-hand side are exactly the type-I and nearly the type-II ASF, respectively. ■

PROPERTY 5. *The type-VII and type-VIII ASFs are bounded by the type-I and type-II ASFs if the padded operations are ignored.*

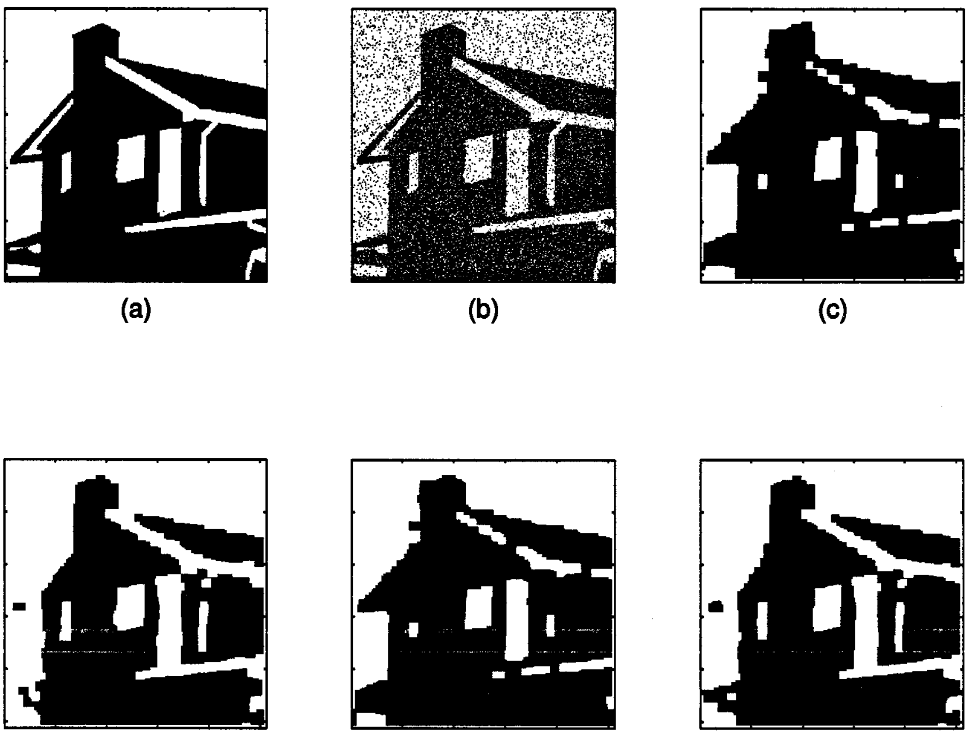
Proof. It is trivial that this property can be obtained by observing Eqs. (17), (18), (19), and (20). That is, the type-VII ASF is equivalent to the type-V ASF, and the type-VIII ASF is equivalent to the type-VI ASF if the padded operations are ignored. ■

Properties 3, 4, and 5 suggest the performance boundary of the new class of ASFs. If they are applied to one of the image processing applications, it is interesting to note that we can predict the performance in advance. Thus, the proposed ASFs have much better computational efficiency, while their filtered results remain comparable to the traditional ASFs.

V. EXPERIMENTAL RESULTS AND DISCUSSION

In this section, two kinds of applications are adopted as computer simulations to illustrate our analysis. One is for image filtering and the other for texture classification. Both new and conventional types of ASFs are used for comparison of their advantages and disadvantages.

Figures 1a and 1b respectively show a binary image and a corrupted version produced by adding pepper–salt noise with 0.1 occurrence probability. The structuring elements are square blocks with sizes 3×3, 5×5, and 7×7. Figures 1c–1f show the filtered results of type-I, -II, -V, and -VI ASFs, respectively. Similarly, the grayscale image "Lena," its distorted version, and its filtered results are shown in Figs. 2a–2h. From the experimental results, it is observed that, in contrast to the present ASFs, the newly defined ASFs yield comparable results. Phenomenon of the boundary properties can be easily observed in Figs. 1a–1d. That is, the dark part of the filtering result, brought by the type-V and type-VI ASFs, is contained in that of the type-I filtering result while the bright part of the filtering result, given by the type-V and VI ASFs, is contained in



(a) (b) (c)

(d) (e) (f)

FIG. 1. (a) A binary image, (b) a version corrupted by pepper–salt noise (occurrence probability $P_0 = 0.1$), and filtered results from (c) type-I, (d) type-II, (e) type-V, and (f) type-VI ASFs.

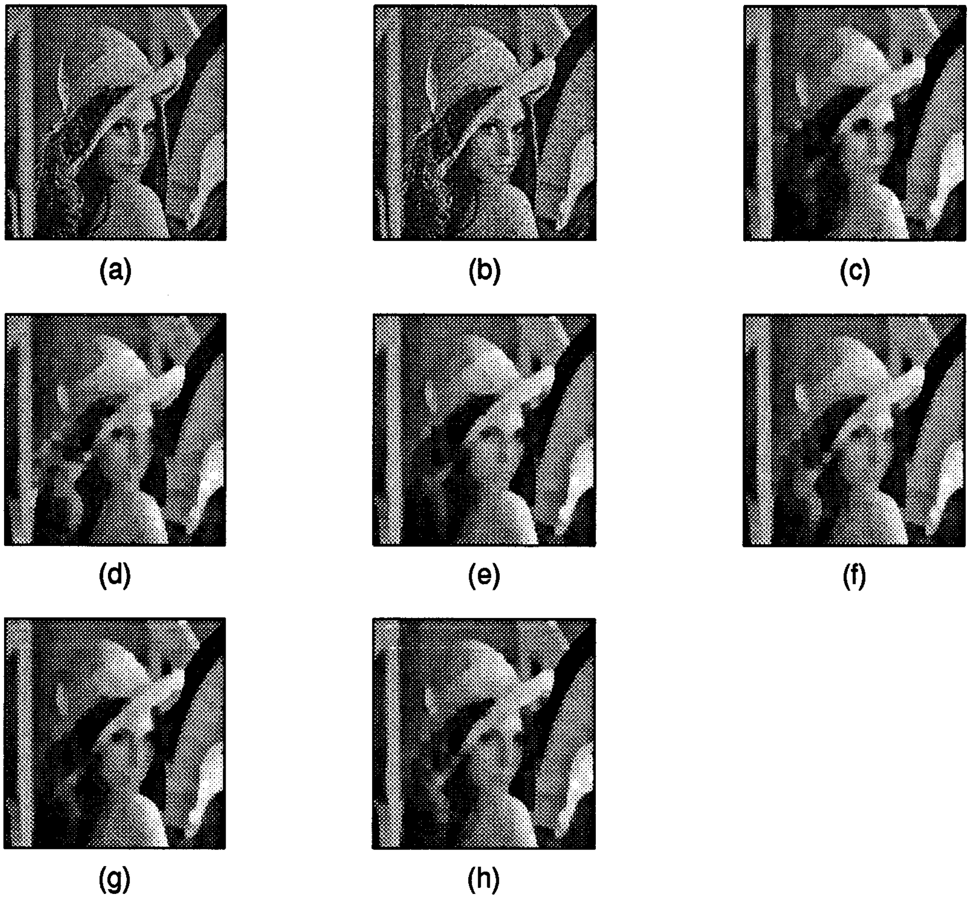


FIG. 2. (a) Grayscale "Lena" image, (b) a version corrupted by Gaussian noise ($\sigma = 10$), and filtered results from (c) opening–closing by 7×7 structuring element, (d) closing–opening by 7×7 structuring element, (e) type-I, (f) type-II, (g) type-V, and (h) type-VI ASFs.

that of the type-II filtering result. The boundary properties are useful in image filtering because we can predict the performance of a filter in advance from its boundary curves. However, it is a little hard to notice these properties in Fig. 2 since grayscale is used. From Figs. 2c and 2d, it is observed that ASFs are better than those morphological filters which are produced by directly applying the largest structuring element.

For performance comparisons, an application of texture classification is adopted. We use 20 texture images of size 512×512 from Brodatz in classification. Each texture image is decomposed into 5 component images by choosing squared structuring elements with increasing sizes of 1, 9, 25, 49, and 81 pixels. The decomposition method is similar to [11] and is described briefly as follows. Let B_i denote a flat structuring element with size $i \times i$ and $B_0 = (0, 0)$ in E^2. The texture image $f(x, y)$ is processed first by an AF with the largest structuring element B_9. By sequentially applying this procedure through decreasing the size of the structuring element, the image primitives of different sizes can be decomposed into different image layers s_i. This decomposition procedure can be described by the following iterative formulas:

$$\begin{cases} f_0(x, y) = & f(x, y), \\ s_i(x, y) = & (AF)f_i(x, y), \quad j = 0, 1, \ldots n, \\ f_{i+1}(x, y) = & f_i(x, y) - s_i(x, y). \end{cases}$$

For each component image, it is first divided into 16 blocks; then the mean value of the training blocks (diagonal blocks) is calculated and stored. The test images are randomly selected from these 20 textures and corrupted by the zero mean Gaussian noise whose variance equals 100. They are also decomposed into 5 components and the remaining 12 blocks, not training blocks, of each component are used to calculate their mean value. The five-dimensional mean vector is then used as the difference measurement. For each type of ASF, 300 trials are used.

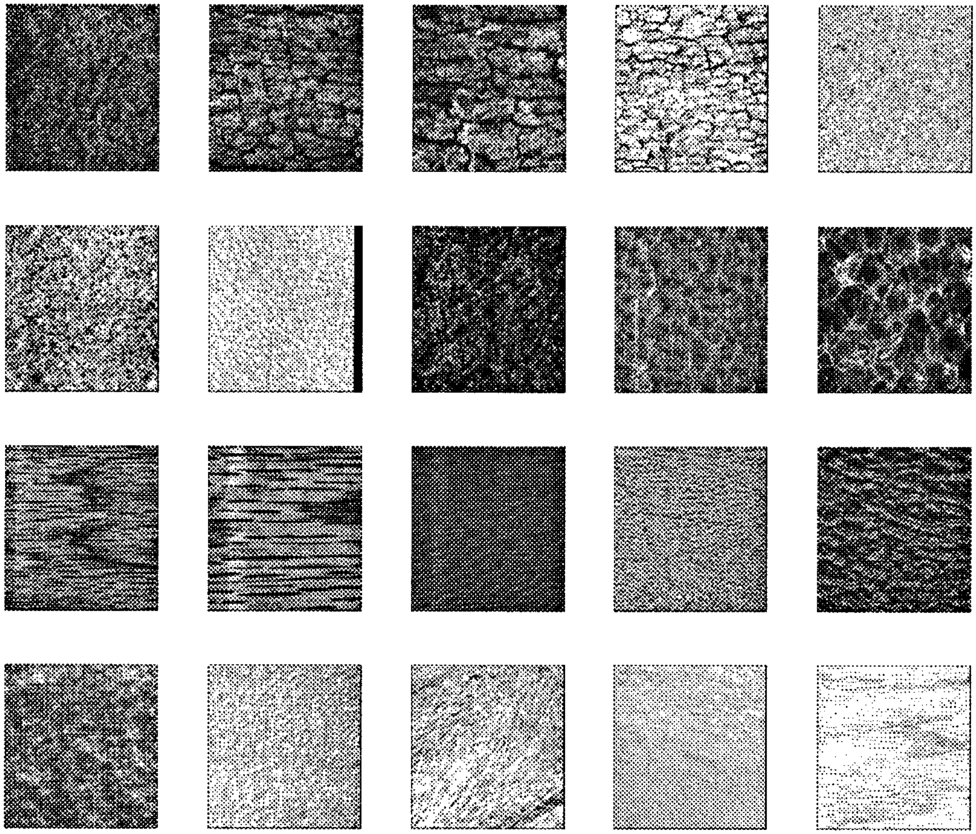


FIG. 3. The natural texture images used in classification.

Figure 3 shows the 20 texture images. The classification results from using type-I, -II, -V, and -VI ASFs are shown in Table 2. It is important to note that properties 3, 4, and 5 are not available here since the decomposition method of this application is not exactly the forms of type-V and type-VI ASFs. That is, we cannot ensure that the correct classification rates of ASF_V and ASF_{VI} are in the range made by the rates of ASF_I and ASF_{II}. Indeed, it is observed in Table 2 that the correct classification rates of type-V and type-VI ASFs are somewhat lower than those of type-I and type-II ASFs. However, only half of the computational complexity is needed for new ASFs.

From the experimental results it is concluded that the new types of ASFs not only offer efficient implementation structures in ASFs but also maintain comparable results in image decomposition and filtering. Moreover, the bounding properties are also illustrated in the experimental results.

TABLE 2
Classification Results of Type-I, -II, -V, and -VI ASFs

	Type-I	Type-II	Type-V	Type-VI
Correct classification rate	69%	76%	64%	67%

VI. CONCLUSIONS

A significantly efficient class of ASFs is introduced in this paper to outperform typical ASFs. By utilizing the adjunctional property in morphology, new types of ASFs are developed. As a result, the computational complexity can be reduced to nearly half of the original. Another advantage of new types of ASFs is that since their performance boundary exists, it serves as a performance curve with which we can predict their filtering results in advance. Experimental results also validate our analysis and show the excellent performance of the new types of ASF.

REFERENCES

1. G. Matheron, *Random Sets and Integral Geometry,* New York, Wiley, 1975.

2. J. Serra, *Image Analysis and Mathematical Morphology,* New York, Academic Press, 1982.
3. S. R. Sternberg, Grayscale morphology, *Comput. Vision, Graphics, Image Process.* **35,** 1986, 333–355.
4. D. Schonfeld and J. Goutsias, ''Optical morphological pattern restoration from noisy binary image,'' *IEEE Trans. Pattern Anal. Mach. Intelligence* **13**(1), 1991, 14–29.
5. I. Destival, Mathematical morphology applied to remote sensing, *Acta Astronaut.* **13,** 1986, 371–385.
6. F. Preteux, A. M. Laval-Jeantet, B. Roger, and M. H. Laval-Jeantet, New prospects in C.T. image processing via mathematical morphology, *Eur. J. Radiol.* **5,** 1985, 313–317.
7. H. J. A. M. Heijmans, A new class of alternating sequential filters, in *1995 IEEE Workshop on Nonlinear Signal and Image Processing,* Vol. 2, pp. 30–33.
8. C. R. Giardina and E. R. Dougherty, *Morphological Methods in Image and Signal Processing,* New Jersey, 1988.
9. E. R. Dougherty and J. B. Pelz, Morphological granulometric analysis of electrophotographic images-size distribution statistics for process control, *Opt. Engrg.* **30**(4), 1991, 438–445.
10. H. J. A. M. Heijmans, The algebraic basis of mathematical morphology. I. Dilations and erosions, *Comput. Vision, Graphics, Image Process.* **50,** 1990, 245–295.
11. W. Li and V. H. Coat, Composite morphological filters in multiresolution morphological decomposition, in *1995 IEE Workshop on Image Processing and Its Applications,* No. 410, pp. 752–756.