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COMPARISON OF SCORING RULES IN POISSON VOTING GAMES

by

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Abstract: Scoring rules are compared by their equilibria in simple voting games with Poisson population uncertainty, using new techniques for computing pivot probabilities. Best-rewarding rules like plurality voting can generate discriminatory equilibria where the voters disregard some candidate as not a serious contender, although he may be universally liked, or symmetric to other candidates as in the Condorcet cycle. Such discriminatory equilibria are eliminated by worst-punishing rules like negative voting, but then even a universally disliked candidate may have to be taken seriously. In simple bipolar elections, equilibria are always majoritarian and efficient under approval voting, but not other scoring rules.

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1. Introduction

When there are only two alternatives in an election, the meaning of majority rule seems clear and straightforwardly implementable. But when voters have to select among three or more alternatives, there are many different voting rules that can be used. The impossibility theorems of social choice theory show that no ideal voting rule can extend the definition of majority rule by a unique pure-strategy equilibrium in all social choice situations (Muller and Satterthwaite, 1977). To move beyond such impossibilities, we should now study how the sets of equilibrium outcomes in an election may be systematically affected by changes in the voting rule.

This paper considers simple examples of social choice situations that illustrate some of the basic ways that voting reform can change rational voting behavior. To keep things simple, most of this paper focuses on winner-take-all elections in which there are just three alternative candidates; but some simple elections with more candidates are considered in Section 8.

The voting rules considered in this paper are all scoring rules. In a scoring rule, each voter's ballot must be a vector that specifies the number of points that the voter gives to each candidate. The vote vectors of all voters are summed, and the winning candidate has the most points. In case of a tie for the most points, we assume that a winner is chosen randomly among those with the most points, each with equal probability.

In a model of three-candidate elections where the candidates are numbered {1,2,3}, a vote vector can be denoted by a triple $c = (c_1,c_2,c_3)$, where c_i denotes the number of points given to candidate i. Different scoring rules are characterized by the different sets of vote vectors that are permitted to the voters in the election. In this paper, we consider a family of scoring rules for three-candidate elections which are characterized by two parameters A and B such that

$$0 \leq A \leq B \leq 1.$$

In an (A,B)-scoring rule, each voter must choose a vote vector that is a permutation of either (1,B,0) or (1,A,0). That is, the voter must give a maximum of 1 point to one candidate, a minimum of 0 points to some other candidate, and A or B points to the remaining candidate.

The set of (A,B) pairs that satisfy $0 \leq A \leq B \leq 1$ can be represented as a triangle (as shown in Figure 1 below), and the extreme points in this triangle will get particular emphasis here. The corner (A,B)=(0,0) is plurality voting, the familiar system in which each voter gives 1 point to one candidate and 0 points to all the others. The corner (A,B)=(1,1) is negative voting, in which each voter votes against one of the candidates by giving him 0 points, while giving 1 point to every other candidate, and the winner will be the candidate with the fewest such votes against him. The corner (A,B)=(0,1) is approval voting, in which each voter can give 1 or 0 points to each candidate, regardless of how many points he has given the others. (Voting (0,0,0) or (1,1,1) would be equivalent to abstaining and can be ignored here, because we assume that voting is costless and each voter knows his or her utility for each candidate.) The (A,B)-scoring rule where (A,B) = (0.5,0.5) is Borda voting.

Following the terminology of Cox (1987, 1990), the scoring rules where (A,B) is near (0,0) may be called best-rewarding rules, because the most important aspect of any voter's ballot is which candidate has been ranked as "best" to get the maximum of 1 point, the other candidates both getting close to 0 points. In contrast, the scoring rules where (A,B) is near (1,1) may be called worst-punishing rules, because the most important aspect of any voter's ballot is which candidate has been ranked as "worst" to get the minimum of 0 points, the other candidates both getting close to 1 point. Borda and approval voting are balanced between these best-rewarding and worst-punishing extremes, but approval voting differs from Borda voting on a dimension of flexibility.

In Section 4, we show that best-rewarding rules tend to generate many discriminatory equilibria in which the voters disregard some candidate as not a serious contender, possibly even a candidate who would be widely preferred. In Section 5, we show that such discriminatory equilibria can be eliminated by worst-punishing rules, but then even a universally disliked candidate may have to be taken seriously in equilibrium. In Section 6, we characterize the voting rules that yield discriminatory equilibria in the Condorcet cycle, thus breaking the symmetry of this example which has been central in proofs of impossibility theorems. Section 7 considers symmetric equilibria of bipolar elections and shows that majoritarian outcomes can be guaranteed only by approval voting. Section 8 shows that this majoritarianism and efficiency of approval voting can be extended to all equilibria of more general bipolar elections.

The analysis of voting games in this paper is based on the assumption that voters are instrumentally motivated, that is, that each voter chooses his ballot to maximize the utility that he gets from the election, which is assumed to depend only on which candidate wins the election. This assumption seems natural and realistic, but it implies that each voter cares about his choice of ballot only in the event that his ballot could pivotally change the outcome of the election. So this theory of rational voting necessarily implies that voters' decisions may depend on the relative probabilities of various ways that one vote may be pivotal in the election, even though these pivot probabilities may be very small in a large election.

Under plurality voting, for example, voters often choose to vote for their second-favorite candidate rather than their favorite candidate. This phenomenon occurs when the probability of their favorite candidate being in a close race (where a vote could make a difference) is much smaller than the probability of their second-favorite candidate being in a close race with their worst candidate, in which case it would be a waste to not vote for their second-favorite candidate. Such wasted-vote effects can be very important in equilibria under best-rewarding voting rules like plurality voting.

So to characterize rational equilibria with instrumental voters, we need a formal procedure to identify which pairs of candidates are more likely to be in a close race where one vote could determine the winner. Myerson and Weber (1993) made some simple assumptions about how the serious races might be determined, given any pattern of anticipated voting strategies, but we will see an example (in Section 6) where the assumptions of that paper appear to be inconsistent with any reasonable probability model. To avoid the such pitfalls, the simple assumptions about serious races in Myerson and Weber (1993) should be replaced by calculations based on some formal probability model.

Unfortunately, it can be very difficult to calculate the probabilities of these close-race events, where two candidates' scores are within one vote of each other and are ahead of all the other candidates. I have argued elsewhere (Myerson 1998a, 2000) that the difficulty of these probability calculations can be minimized by assuming a Poisson model of population uncertainty, in which there is uncertainty about the numbers of each type of voter, and beliefs about these numbers can be characterized by independent Poisson random variables.

So Poisson models are applied here to quantify all probabilities in our analysis. The general Poisson model is described in Section 2, and the calculus of Poisson probability magnitudes is presented in Section 3. The advantage of the Poisson model is only that it gives us a precise and tractable framework for seeing how beliefs about outcome probabilities and rational voting behavior can fit together in a logically consistent way under a wide variety of voting rules. The most important conclusions of our analysis here are, not the specific quantitative probabilities that may be computed for any one equilibrium, but the more general qualitative ways that these equilibria may change when the voting rule is changed. (See Myerson, 2000, for an argument that other Multinomial models should yield qualitatively similar results.)

2. General definitions

In a general social choice situation, we may let K denote the set of candidates (or alternatives) in the election. One candidate in K must be chosen as winner in the outcome of any voting game. In models of three-candidate elections, we let $K = \{1,2,3\}$.

Each voter has a type that determines his (or her) preferences over the candidates. We let T denote the set of voters' possible types. We assume here that voters have independent private values for the candidates, and so the preferences of a type t voter can described by a utility vector $u(t) = (u_i(t))_{i \in K}$, where $u_i(t)$ is the utility payoff to each voter of type t if candidate i wins the election.

The expected distribution of voters in the electorate is denoted by a probability distribution $r = (r(t))_{t \in T}$, where r(t) denotes the probability that any randomly sampled voter will have type t. This given distribution r must satisfy

$$r(t) \geq 0 \;\; \forall t \in T, \;\text{ and }\; \sum_{s \in T} r(s) = 1.$$

The expected number of voters is denoted here by the parameter n. In our Poisson models of population uncertainty, we assume that the actual number of voters participating in the election will be a Poisson random variable with mean n, and each voters' type will be independently drawn from T according to the probability distribution r. A Poisson distribution has a standard deviation that is the square root of its mean. So if the expected number of voters is 100,000,000, then the Poisson assumption implies uncertainty about this population with a standard deviation of 10,000. (Extended Poisson models with more uncertainty are formulated by Myerson, 1998b.)

These parameters (K, T, u, r, n) then characterize a social choice situation with Poisson population uncertainty. Then to complete the definition of a voting game, we must specify the voting rule. In general, we may let C denote the set of ballot options from which each voter must choose. In the scoring rules that we study here, these ballot options are vote vectors of the form $c = (c_i)_{i \in K}$, where c_i denotes the number of points that a voter is giving to candidate i when the voter chooses ballot c in the election. These parameters (K, T, u, r, n, C) then completely characterize a voting game with expected population size n.

A strategy function for the voters in such a voting game is any mapping σ from T into the set of probability distributions over C. That is, a strategy function σ will specify, for each type t in T and each ballot option c in C, a number $\sigma(c|t)$ denoting the probability that a voter of type t would choose ballot c in the election. Any strategy function σ must satisfy

$$\sigma(c|t) \geq 0, \forall c \in C, \;\text{ and }\; \sum_{d \in C} \sigma(d|t) = 1, \;\; \forall t \in T.$$

When the voters behave according to the strategy function σ, the probability that any randomly sampled voter will cast the ballot c is

(1) $\tau(c) = \sum_{t \in T} r(t)\, \sigma(c|t)$.

This vector $\tau = (\tau(c))_{c \in C}$ is the expected vote distribution corresponding to the strategy function σ in the voting game. Each candidate i's expected score in points per voter is then

$$S_i(\tau) = \sum_{c \in C} \tau(c)\, c_i.$$

The actual outcome of the election will depend on how many voters actually cast each of the ballot options in C. These numbers can be listed as a vote profile vector $x = (x(c))_{c \in C}$ in which each component x(c) denotes the number of voters who cast the ballot c in the election. The set of all possible vote profiles with ballot set C is denoted here by Z(C), where

$$Z(C) = \{x \in \mathbb{R}^C \mid x(c) \text{ is a nonnegative integer}, \forall c \in C\}.$$

With Poisson population uncertainty, for any expected vote distribution τ, the numbers of voters who choose each ballot option c are independent Poisson random variables with means $n\tau(c)$. This independence of the counts in the vote profile is a unique characteristic of the Poisson model (called independent actions by Myerson, 1998a). Thus, when the expected vote profile is $n\tau = (n\tau(c))_{c \in C}$, the probability that any x in Z(C) will be the actual vote profile in the election is

$$P(x \mid n\tau) = \prod_{c \in C} \left(e^{-n\tau(c)}\, (n\tau(c))^{x(c)} / x(c)!\right).$$

Another noteworthy property of the Poisson model is that any single voter in the election should assess this same probability distribution $P(\bullet \mid n\tau)$ for the vote profile that will be generated by all the other voters in the election, counting everyone's ballots except his own. (This property is called environmental equivalence by Myerson, 1998a.)

When the vote profile is x, the winner will be a candidate with the most points, in the set

$$W(x) = \text{argmax}_{i \in K} \sum_{c \in C} x(c)\, c_i.$$

Assuming random selection in ties, the probability of i winning given a vote profile x is

$$Q(i|x) = 1/\#W(x) \text{ if } i \in W(x), \; Q(i|x) = 0 \text{ if } i \notin W(x).$$

Given any expected vote profile $n\tau$, the corresponding probability distribution over the winner of the election may then be denoted by $q = (q(i))_{i \in K}$, where each

(2) $$q(i) = \sum_{x \in Z(C)} P(x|n\tau)\, Q(i|x).$$

For any vote profile x and any ballot option c in C, we let $x+[c]$ denote the vote profile which differs from x only in that the number of c-ballots is increased by one. That is, $y=x+[c]$ when $y(c)=x(c)+1$ and $y(d) = x(d)$ for all $d \neq c$. Thus, given any expected vote profile $n\tau$, a voter of type t should want to choose the ballot option c that maximizes his expected utility

$$\sum_{x \in Z(C)} P(x|n\tau) \sum_{i \in K} Q(i|x+[c])\, u_i(t).$$

So we may say that (σ,τ,q) is an <u>equilibrium</u> of the voting game with expected size n iff τ and q are the expected vote distribution and win probability distribution corresponding to σ (as in equations (1) and (2)) and, for each c in C and each t in T,

(3) $$\sigma(c|t) > 0 \text{ implies that } c \in \operatorname{argmax}_{d \in C} \sum_{x \in Z(C)} P(x|n\tau) \sum_{i \in K} Q(i|x+[d])\, u_i(t).$$

We consider here only equilibria in which weakly dominated actions have been eliminated for all types. That is, any type t should assign zero probability in equilibrium to a ballot option c if there exists some other ballot option d in C such that

$$\sum_i Q(i|x+[c])\, u_i(t) \le \sum_i Q(i|x+[d])\, u_i(t),\ \forall x \in Z(C), \text{ with strict inequality for some x.}$$

In an (A,B)-scoring rule, dominance implies that a voter with independent private values should give 1 or B points to his best candidate, and should give 0 or A points to his worst candidate.

The focus in this paper is on elections with large numbers of voters, and so we shall look at the limits of such equilibria as the expected number of voters n goes to infinity, holding fixed the other parameters of the Poisson voting game (K, T, u, r, C). Thus, we may say that a <u>large</u>

equilibrium sequence of this structure (K, T, u, r, C) is any sequence of equilibria $\{(\sigma_n,\tau_n,q_n)\}_{n\to\infty}$ of the finite voting games (K, T, u, r, n, C) such that the vectors (σ_n,τ_n,q_n) are convergent to some limit (σ,τ,q) as $n\to\infty$ in the sequence. We may also refer to this limit (σ,τ,q) as a large equilibrium of (K, T, u, r, C).

When the expected vote profile is $n\tau_n$ in a voting game of expected size n, any set M that is a subset of Z(C) can be interpreted as an event that has probability

$$P(M|n\tau_n) = \sum_{x\in M} P(x|n\tau_n).$$

We will be particularly interested in two kinds of events: the event that a particular candidate can win the election, and the event that there is a close race where one vote may make a pivotal difference between one candidate or another winning. So for each candidate i, let $\Omega(i)$ denote the event that candidate i is a winner or tied to win,

$$\Omega(i) = \{x\in Z(C)|\ Q(i|x) > 0\}.$$

For any pair of candidates i and j and any ballot option c, let $\Lambda(c,i,j)$ denote the event that adding one more ballot c could change the winner from i to j,

$$\Lambda(c,i,j) = \{x\in Z(C)|\ Q(i|x) > Q(i|x+[c]),\ Q(j|x) < Q(j|x+[c])\}$$

Let $\Lambda(i,j)$ denote the event that there is a close race between i and j such that one additional vote could pivotally change the winner from one to the other of these two candidates,

$$\Lambda(i,j) = \cup_{c\in C} (\Lambda(c,i,j) \cup \Lambda(c,j,i)).$$

Let D denote the set of pairs of candidates {i,j} who are distinguishable by the voters, in the sense that voters are not completely indifferent between them,

$$D = \{\{i,j\}|\ u_i(t) \neq u_j(t) \text{ for some t in T}\}.$$

Let Λ^* denote the event that a close race exists where one additional vote could be pivotal between some pair of distinguishable candidates,

$$\Lambda^* = \cup_{\{i,j\}\in D} \Lambda(i,j).$$

Notice that a rational voter cares about his vote only in the event that there is at least one close race among distinguishable candidates, so that his vote could make a difference. That is,

$$\operatorname{argmax}_{d\in C} \sum_{x\in Z(C)} P(x|n\tau_n) \sum_{i\in K} Q(i|x+[d])\, u_i(t)$$
$$= \operatorname{argmax}_{d\in C} \sum_{x\in\Lambda^*} P(x|n\tau_n) \sum_{i\in K} Q(i|x+[d])\, u_i(t).$$

So even though the probability of a close race may be quite small when n is large, rational voters would act the same if all probabilities were replaced by conditional probabilities given the event of a close race Λ^*. There must be some pairs of candidates for which the conditional probability of a close race given Λ^* has a positive limit (or limit supremum) in any large equilibrium sequence. So for any two candidates $\{i,j\}$, we may say that the $\{i,j\}$-race is <u>serious</u> in a large equilibrium sequence iff i and j are distinguishable and

$$\limsup_{n\to\infty} P(\Lambda(i,j)|n\tau_n)/P(\Lambda^*|n\tau_n) > 0.$$

That is, the race between i and j is serious if, in the event that a close race exists in the election, the conditional probability that i and j are in this close race has a positive limit as the expected population gets large. Any race that is not serious becomes of infinitesimal importance relative to the serious races, as $n\to\infty$, in the rational voters' expected-utility maximization problems.

In a large equilibrium sequence, we may say that a candidate i is <u>serious</u> iff there is some other candidate j such that the $\{i,j\}$ race is serious. We may also say that a candidate i is <u>out of contention</u> in a large equilibrium iff the candidate is not serious. We say that a large equilibrium is <u>discriminatory</u> iff there is a candidate in K who is not serious. So discriminatory equilibria represent situations in which the voters perceive great differences in the chances of different candidates, so that some candidates lose virtually all significance in the voters' decision-making.

Notice that a serious candidate is not necessarily likely to win. We may say that a candidate i is strong in a large equilibrium sequence $\{(\sigma_n,\tau_n,q_n)\}_{n\to\infty}$ if the probability of i winning has a positive limit $q(i) > 0$. In the winner-take-all voting games that are considered here, any strong candidate will be serious, but a serious candidate might not be a strong candidate.

3. Computing magnitudes and probability ratios of events in large voting games

The probability of any close race will generally tend to zero as the expected population n becomes large. But we can identify which races are serious in a large equilibrium by comparing the rates at which their probabilities go to zero. These rates can be usefully measured by a concept of magnitude defined as follows.

Given a large equilibrium sequence $\{(\sigma_n,\tau_n,q_n)\}_{n\to\infty}$, the magnitude of an event M is

$$\mu(M) = \lim_{n\to\infty} \log(P(M|n\tau_n))/n.$$

(Here, log denotes the natural logarithm, base e.) In particular, for any candidates i and j, we let μ_i denote the magnitude of candidate i winning,

$$\mu_i = \mu(\Omega(i)) = \lim_{n\to\infty} \log(P(\Omega(i)|n\tau_n))/n,$$

and we let $\mu_{i,j}$ denote the magnitude of a close race between i and j,

$$\mu_{i,j} = \mu(\Lambda(i,j)) = \lim_{n\to\infty} \log(P(\Lambda(i,j)|n\tau_n))/n.$$

So if we can show that a close race between one pair of distinguishable candidates has a magnitude that is strictly greater than the magnitude of a close race between another pair of candidates, then the latter race is not serious. This fact can give us a practical way to identify the serious races, once we have learned how to compute these magnitudes.

The key to computing magnitudes is given summarized by the magnitude theorem from Myerson (2000). To state this theorem, we need some notation. For any positive number θ, let

$$\psi(\theta) = \theta(1-\log(\theta))-1,$$

and let $\psi(0) = -1$. Then ψ is concave, is maximized at $\psi(1) = 0$, and has slope $\psi'(\theta) = -\log(\theta)$.

We say that $\alpha = (\alpha(c))_{c\in C}$ is the offset-ratio vector of a vote profile x, relative to the expected vote profile $n\tau_n$, iff

$$\alpha(c) = x(c)/(n\tau_n(c))$$

Then let $M/(n\tau_n)$ denote the set of all offset-ratio vectors of vote profiles in M relative to $n\tau_n$

$$M/(n\tau_n) = \{(x(c)/(n\tau_n(c)))_{c\in C} \mid x \in M\}.$$

Writing $n\tau_n\alpha = (n\tau_n(c)\alpha(c))_{c\in C}$, we have $\alpha \in M/(n\tau_n)$ iff $n\tau_n\alpha \in M$.

These definitions are useful because, for any sequence of vote profiles $\{x_n\}_{n\to\infty}$ such that

$$\lim_{n\to\infty} x_n(c)/(n\tau_n(c)) = \alpha(c) \text{ and } \lim_{n\to\infty} \tau_n(c) = \tau(c),\ \forall c\in C,$$

the magnitude of this sequence is

$$\lim_{n\to\infty} \log(P(x_n|n\tau_n))/n$$

$$= \lim_{n\to\infty} \sum_{c\in C} (-n\tau(c) + x_n(c)\log(n\tau_n(c)) - \log(x_n(c)!))/n$$

$$= \lim_{n\to\infty} \sum_{c\in C} (-n\tau(c) + x_n(c)\log(n\tau_n(c)) - x_n(c)(\log(x_n(c))-1))/n$$

$$= \lim_{n\to\infty} \sum_{c\in C} \tau_n(c)\ \psi(x_n(c)/(n\tau_n(c)))$$

$$= \sum_{c\in C} \tau(c)\ \psi(\alpha(c)).$$

(Here the second equality uses Stirling's formula; see Abramowitz and Stegun, 1965.) Then the magnitude theorem of Myerson (2000) then tells us that the magnitude of any event is determined by the magnitude of the most likely points in the event.

Magnitude Theorem. Given $M \subseteq Z(C)$, the magnitude of M is

$$\mu(M) = \lim_{n\to\infty} \log(P(M|n\tau_n))/n = \lim_{n\to\infty} \max_{y_n \in M} \log(P(y_n|n\tau_n))/n$$

$$= \lim_{n\to\infty} \max_{\alpha_n \in M(n\tau_n)} \sum_{c \in C} \tau_n(c)\,\psi(\alpha_n(c)).$$

For any event M and any ballot c, if $\alpha(c)$ is the unique limit as $n\to\infty$ of the $\alpha_n(c)$ values in the optimal solutions of the (last) maximization problem in the magnitude theorem then, as $n\to\infty$, all probability in the event M becomes concentrated in the set of vote profiles x where the offset ratio $x(c)/(n\tau_n(c))$ is close to $\alpha(c)$. (See Myerson, 2000.)

The magnitude theorem gives us an optimization problem with nice mathematical structure, because the objective function is smooth and concave. Furthermore, the events that concern us in the analysis of large voting games generally have the simple geometrical structure of a cone defined by a finite collection of linear inequalities. For example, the event of candidate 1 winning $\Omega(1)$ is a cone defined by the linear inequalities that candidate 1's total point score (a linear function of the vote profile) should be greater than or equal to each other candidate's total point score. Under an (A,B)-scoring rule where $0<A<B<1$, this cone is in a twelve-dimensional space, because a vote profile must count the numbers of each of the 12 ballot options in C (3! permutations of (1,B,0) and (1,A,0)). But the following dual magnitude theorem gives us a way to reduce the dimensionality of this magnitude problem down to the number of constraints that define the cone.

Dual Magnitude Theorem. Let M be a cone defined by

$$M = \{x \in \mathbb{R}_+^C \mid \sum_{c \in C} b_k(c)\,x(c) \geq 0 \;\forall k \in J\},$$

where J is a finite set, and the numbers $b_k(c)$ are given for each k in J and c in C. Given a vote distribution τ, suppose that λ in $\mathbb{R}_+^J$ is an optimal solution to the dual problem

$$\text{minimize}_\lambda \ \textstyle\sum_{c\in C} \tau(c)\big(\exp(\sum_k \lambda_k\, b_k(c)) - 1\big) \ \text{ subject to } \ \lambda_k \geq 0\ \forall k\in J.$$

Then letting

$$\alpha(c) = \exp(\textstyle\sum_k \lambda_k\, b_k(c)),\ \forall c\in C,$$

yields the optimal solution to the magnitude problem

$$\text{maximize}_{\alpha\in M/\tau} \textstyle\sum_{c\in C} \tau(c)\ \psi(\alpha(c)),$$

and the optimal values of the objectives in these two problems are equal. (Notice that $M/(n\tau) = M/\tau$, because M is a cone.)

Proof. By the first-order conditions at the dual optimum λ, we must have, for each j in J,

$$\textstyle\sum_{c\in C} \tau(c) \exp(\sum_k \lambda_k\, b_k(c))\, b_j(c) = 0 \ \text{ if } \ \lambda_j > 0,$$

$$\textstyle\sum_{c\in C} \tau(c) \exp(\sum_k \lambda_k\, b_k(c))\, b_j(c) \geq 0 \ \text{ if } \ \lambda_j = 0.$$

So the letting

$$\alpha(c) = \exp(\textstyle\sum_{k\in J} \lambda_k\, b_k(c)) \ \text{ and } \ x(c) = n\tau(c)\alpha(c),\ \forall c\in C,$$

yields a vote profile x that satisfies the cone inequalities. But now consider changing the offset ratios from α in any direction δ that keeps us in the cone. The objective function of the magnitude problem is concave, and its derivative at α in the direction δ is

$$\textstyle\sum_{c\in C} \delta(c)\ \tau(c)\ \psi'(\alpha(c)) = -\sum_{c\in C} \delta(c)\ \tau(c) \log(\alpha(c))$$

$$= -\textstyle\sum_{c\in C} \delta(c)\ \tau(c) \sum_{k\in J} \lambda_k\, b_k(c) = -\sum_k \lambda_k \sum_{c\in C} b_k(c)\ \tau(c)\ \delta(c) \leq 0,$$

because staying in the cone when the offset ratios change from α to $\alpha+\delta$ implies that

$$\textstyle\sum_{c\in C} b_k(c)\ \tau(c)\ \delta(c) \geq 0$$

for every binding constraint k where $\lambda_k > 0$. Q.E.D.

As an application of this result, consider the event M where the number of votes in some set G is greater than or equal to the number in some other set H, where $G \subseteq C$, $H \subseteq C$, and $G \cap H = \emptyset$,

$$M = \{x; \sum_{c\in G} x(c) \geq \sum_{d\in H} x(d)\}$$

Let γ and η denote the expected fractions of voters choosing ballots in G and H,

$$\gamma = \sum_{c\in G} \tau(c), \quad \eta = \sum_{d\in H} \tau(d).$$

So we get the dual problem

$$\text{minimize}_{\lambda>0} \sum_{c\in G} \tau(c)(e^{\lambda} - 1) + \sum_{d\in H} \tau(d)(e^{-\lambda} - 1) + \sum_{b\in C\setminus(G\cup H)} \tau(b)(e^{0} - 1)$$

$$= \gamma(e^{\lambda} - 1) + \eta(e^{-\lambda} - 1).$$

If $\gamma \geq \eta$ then the optimal solution has $\lambda=0$, and the event M has magnitude 0, achieved at the expected vote profile where all offset ratios $\alpha(c)$ are 1. But when $\gamma < \eta$, the optimal solution has

$$\alpha(c) = e^{\lambda} = \sqrt{\eta/\gamma}, \quad \forall c\in G,$$

$$\alpha(d) = e^{-\lambda} = \sqrt{\gamma/\eta}, \quad \forall d\in H,$$

$$\alpha(b) = e^{0} = 1, \quad \forall b\in C\setminus(G\cup H),$$

and the magnitude is

$$\mu(M) = \gamma\sqrt{\eta/\gamma} - \gamma + \eta\sqrt{\gamma/\eta} - \eta = 2\sqrt{\gamma\eta} - \gamma - \eta$$

$$= -(\sqrt{\gamma} - \sqrt{\eta})^2.$$

This is also the magnitude of having equal numbers of votes in G and H. For large n, almost all probability in this event M is concentrated near the vote profiles that have these magnitude-maximizing offset ratios α, so that the numbers of votes in G and H are near

$$n\gamma\sqrt{\eta/\gamma} = n\eta\sqrt{\gamma/\eta} = n\sqrt{\gamma\eta},$$

which is the geometric mean of the expected numbers of votes in G and H.

The event that no voters choose ballots in some set $G \subseteq C$ can be viewed as a the special case of this analysis, where $H = \emptyset$. So if γ is the expected fraction of votes in G, then the event that no voters actually choose ballots in G has magnitude $-\gamma$. In particular, the event of zero turnout in the election (that is, the vote profile x such that all $x(c) = 0$) has magnitude -1. (Indeed, the probability of 0 is e^{-n} for a Poisson random variable with mean n.) But the cone events considered here all include the zero vector in Z(C), and so their magnitudes cannot be less than -1. Also, the magnitudes of these events cannot be greater than 0, because the natural logarithm of a probability is never positive.

For any vote profile x, any expected vote profile $n\tau_n$, and any ballot option c, we have

$$P(x-[c]|n\tau_n)/P(x|n\tau_n) = x(c)/(n\tau_n(c)).$$

So the ratio of probabilities of two vote profiles that differ by a single c ballot is equal to the c-offset ratio. Thus, if the offset ratios $\alpha = (\alpha(c))_{c\in C}$ uniquely achieve the maximal magnitude of an event M, then

$$\lim_{n\to\infty} P(M-k[c]|n\tau_n)/P(M|n\tau_n) = \alpha(c)^k,$$

where $M-k[c]$ is the event that adding k more c ballots would make M occur. This result is called the <u>offset theorem</u> in Myerson (2000).

Denoting the expected vote profile by $\omega = (\omega(c))_{c\in C}$ (instead of $n\tau_n$), we can express the sensitivity of $P(x|\omega)$ to the expected vote profile ω by the formula

$$\begin{aligned} \partial\log(P(x|\omega))/\partial\omega(c) &= \partial(-\omega(c) + x(c)\log(\omega(c)) - \log(x(c)!))/\partial\omega(c) \\ &= x(c)/\omega(c) - 1 \\ &= \alpha(c) - 1, \text{ when } \alpha(c) = x(c)/\omega(c). \end{aligned} \tag{4}$$

The highest possible magnitude of 0 holds for an event that includes points x_n such that the offset ratios $x_n(c)/(n\tau_n(c))$ converge to 1 as $n\to\infty$ for all c. To estimate probabilities in this region where all offset ratios are close to 1, we can apply the Normal approximation to the Poisson distribution. (See Theorem 3 of Myerson 2000.) In this Normal approximation for the game of expected size n, the components of the realized vote profile $(x_n(c))_{c \in C}$, are approximated as the integer roundings of independent Normal random variables, where each $x_n(c)$ has the mean $n\tau_n(c)$ and standard deviation $\sqrt{n\tau_n(c)}$. Assuming that these means are large, this Normal approximation can be applied to estimate probabilities in a 0-magnitude event where all offset ratios are close to 1.

4. Problems of too many discriminatory equilibria: Above the Fray

We can now apply these techniques to compare equilibria under different voting rules. We may begin with a general proposition that best-rewarding voting rules like plurality voting (where A=B=0) tend to have many discriminatory equilibria. Recall that B is the upper bound on the number of points that a voter can give to the middle-ranked candidate on his ballot.

Proposition 1. For any (A,B)-scoring rule with $B < 0.5$, for any pair of candidates $\{i,j\}$, if all voters have strict preferences on $\{i,j\}$ and neither i nor j is expected to be unanimously preferred over the other, then a discriminatory large equilibrium sequence exists in which $\{i,j\}$ is the only serious race.

Proof. It suffices to consider $\{i,j\} = \{1,2\}$. In a discriminatory equilibrium, each voter will want to maximize his probability of making an impact on the serious race, by giving one point to the candidate in $\{1,2\}$ whom he prefers, and giving zero points to the candidate in $\{1,2\}$ whom he does not prefer. So all voters must be expected to choose among the four ballots

$$(1,0,A),\ (1,0,B),\ (0,1,A),\ (0,1,B).$$

When there are m voters, the total points of candidates 1 and 2 always sum to m with this strategy, and so the high scorer among 1 and 2 never has less than m/2 total points. But candidate 3 always has less than m/2 total points because $A \leq B < 1/2$. So candidate 3 cannot win or be in a close race with any positive fraction of the large expected turnout. Thus the pivot magnitudes $\mu_{1,3}$ and $\mu_{2,3}$ are both -1. Now let $\gamma = \tau(1,0,A)+\tau(1,0,B)$. Because the voters are not expected to unanimously prefer either 1 or 2, we have $0 < \gamma < 1$. So the magnitude of a {1,2} close race satisfies

$$\mu_{1,2} = -(\sqrt{\gamma} - \sqrt{1-\gamma})^2 > -1 = \mu_{1,3} = \mu_{2,3}.$$

This condition confirms the existence of a discriminatory equilibrium in which candidate 3 is not serious. Q.E.D.

To illustrate this proposition, let us consider as Example 1 a simple voting game where existence of discriminatory equilibria seems very undesirable from a social-choice perspective. In this game there are three candidates, K = {1,2,3}, and there are two types of voters T = {**1**,**2**}. (We use boldface here for type values, to make them easier to distinguish from candidates' names.) Any randomly sampled voter is equally likely to be type **1** or **2**, so

$$r(\mathbf{1}) = 0.5 = r(\mathbf{2}).$$

The utility values are

$$u(\mathbf{1}) = (6,0,9), \quad u(\mathbf{2}) = (0,6,9).$$

So type **1** voters prefer candidate 1 over candidate 2, and type **2** voters prefer candidate 2 over candidate 1, but all voters prefer candidate 3 over both candidates 1 and 2. We may call this game "Above the Fray", to indicate something of candidate 3's superior position in the contest.

Under plurality voting or any (A,B)-scoring rule such that $B < 0.5$, we can find a discriminatory equilibrium in which candidate 3 is not serious. In this equilibrium, each voter wants to maximize his impact for the serious candidate that he prefers against the other serious candidate in $\{1,2\}$, but still wants to give the admired nonserious candidate 3 as many points as possible (B) subject to the constraint of achieving this maximal impact on the serious race. So the equilibrium strategy σ and expected vote distribution τ satisfy

$$\sigma(1,0,B|\mathbf{1}) = 1 = \sigma(0,1,B|\mathbf{2}) \text{ and } \tau(1,0,B) = 0.5 = \tau(0,1,B).$$

Given $B < 0.5$, a close race between candidates 1 and 2 occurs near the expected vote distribution $n\tau$, and so it is an event of the highest magnitude $\mu_{1,2} = 0$. But with everyone expected to vote $(1,0,B)$ or $(0,1,B)$, the score of candidate 3 (B points per voter) is always less than the average of 1's and 2's scores (0.5 points per voter) with any positive turnout, and so $\mu_{1,3} = -1 = \mu_{2,3}$. These magnitudes confirm the perception that only the $\{1,2\}$ race is serious. The candidates' expected scores (points per voter) are then $S(\tau) = (0.5, 0.5, B)$, and (given $B < 0.5$) the distribution of candidates' win probabilities is $q = (0.5, 0.5, 0)$ in this equilibrium. That is, the good candidate 3 has almost no chance of winning in this discriminatory equilibrium.

This discriminatory equilibrium vanishes when $B \geq 0.5$. If everyone were expected to vote $(1,0,B)$ or $(0,1,B)$, then candidates 1 and 2 could be in a close race only when their scores were both 0.5 points per voter. So with $B \geq 0.5$, a close $\{1,2\}$ race with only $(1,0,B)$ and $(0,1,B)$ ballots could occur only when candidate 3 was also involved in the close race, which would contradict the assumption that candidate 3 was not serious.

Under any (A,B) scoring rule, this example also has a good equilibrium in which 3 is serious, everyone votes $(A,0,1)$ or $(0,A,1)$, and the good candidate 3 wins with probability one. We can show that this good equilibrium is unique under approval voting and negative voting.

Under approval voting (where A=0 and B=1), dominance implies that everyone gives an approval point to his favorite candidate, and no one gives an approval point to his worst candidate. So type **1** voters must vote (1,0,1) or (0,0,1), and type **2** voters must vote (0,1,1) or (0,0,1), and so the expected vote distribution in a large equilibrium must satisfy

$$\tau(1,0,1) + \tau(0,1,1) + \tau(0,0,1) = 1.$$

With only these ballots, the magnitude of a close {1,2} race is $\mu_{1,2} = -1$, because a close {1,2} race could not occur with any positive turnout. The magnitude of a close {1,3} race is $\mu_{1,3} = -(1-\tau(1,0,1))$, because a close {1,3} race occurs when the only votes are (1,0,1) ballots. So if $\tau(1,0,1)$ were positive, then a close {1,3} race would have strictly higher magnitude than a close {1,2} race, but then the type **1** voters (who prefer 3 over 1) should not give a second approval point to candidate 1. Thus, we must have $\tau(1,0,1) = 0$, and all type **1** voters must vote (0,0,1) in a large equilibrium. A similar argument shows that type **2** voters must also vote (0,0,1) with probability 1, and so the unique large equilibrium of this example under approval voting has

$$\sigma(0,0,1|\mathbf{1}) = 1 = \sigma(0,0,1|\mathbf{2}) \text{ and } \tau(0,0,1) = 1.$$

This unique approval-voting equilibrium uses only single-point ballots that are also feasible in plurality voting, but the equilibrium set under approval voting is significantly different from the equilibrium set under plurality voting, which also includes the bad discriminatory equilibrium.

It is even easier to show that this good equilibrium is unique under negative voting, where A=B=1. Under negative voting, casting a (1,1,0) ballot against the most-preferred candidate 3 would be dominated for any voter in this example, and so $\tau(0,1,1)+\tau(1,0,1) = 1$ in any equilibrium. The limiting expected vote share $\tau(0,1,1)$ against candidate 1 cannot be strictly less than the limiting expected vote share $\tau(1,0,1)$ against candidate 2, because then the only serious race would be {1,3}, which would make all voters would want to vote (0,1,1) against candidate 1. Similarly $\tau(0,1,1)$ cannot be strictly greater than $\tau(1,0,1)$, because then the only serious race would then be {2,3}, which would make all voters want to vote (1,0,1) against candidate 2. Thus in the limit of any large equilibrium sequence, we must have $\tau(0,1,1) = \tau(1,0,1) = 0.5$, which is achieved by the strategy function with $\sigma(1,0,1|\mathbf{1}) = 1 = \sigma(0,1,1|\mathbf{2})$.

It may be interesting to see how this argument for uniqueness under negative voting still applies when we modify this example by changing the expected fractions of types 1 and 2 to

$$r(\mathbf{1}) = 0.6,\ \ r(\mathbf{2}) = 0.4.$$

In this modified example, the limiting expected vote shares against candidates 1 and 2 must still be $\tau(0,1,1) = \tau(1,0,1) = 0.5$, because otherwise the candidate in {1,2} who was expected to get fewer negative votes would be the only serious challenger to candidate 3, which would make everyone want to vote against him. So the expected vote distributions must have the form

$$\tau_n(1,0,1) = 0.5 + \varepsilon_n,\ \ \tau(0,1,1) = 0.5 - \varepsilon_n,$$

for all n, where $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$. With $r(\mathbf{1}) = 0.6$, this expected vote distribution τ_n can be achieved only if type **1** voters randomize between voting (1,0,1) and (0,1,1). In equilibrium, ε_n must be just large enough to make type **1** voters indifferent between voting (0,1,1) against 1 and voting (1,0,1) against 2, even though they actually prefer 1 over 2. With $u_1 = (6,0,9)$, type **1** voters would be willing to so randomize only if

$$P(\Lambda((0,1,1),1,3)|n\tau_n)/P(\Lambda((1,0,1),2,3)|n\tau_n) = (9-0)/(9-6) = 3.$$

That is, 1's expected lead over 2 must make the probability of candidate 1 winning in a close {1,3} race three times larger than the probability of candidate 2 winning in a close {2,3} race. In the limit of these equilibria τ_n, all probability in $\Lambda((0,1,1),1,3)$ becomes concentrated where the offset ratios are $\alpha(1,0,1) = 1$ and $\alpha(0,1,1) = 0$, because the votes against 1 must disappear to make a close {1,3} race. The partial-derivative formula (4) in Section 3 then implies

$$\partial\log(P(\Lambda((0,1,1),1,3)|n\tau_n))/\partial\varepsilon_n = n(\alpha(1,0,1)-1) + (-n)(\alpha(0,1,1)-1) = n.$$

Similarly, all probability in $\Lambda((1,0,1),2,3)$ becomes concentrated where the offset ratios are $\alpha(1,0,1) = 0$ and $\alpha(0,1,1) = 1$, because (1,0,1) votes must disappear in a close {2,3} race, and so

$$\partial\log(P(\Lambda((1,0,1),2,3)|n\tau_n))/\partial\varepsilon_n = n(0 - 1) + (-n)(1 - 1) = -n.$$

When ε_n is 0, $\log(P(\Lambda((0,1,1),1,3)|n\tau_n))-\log(P(\Lambda((1,0,1),2,3)|n\tau_n))$ is 0, by symmetry. So to make $\log(P(\Lambda(1,3)\ n\tau_n))-\log(P(\Lambda(2,3)|n\tau_n))$ equal to log(3), we need $(n - -n)\varepsilon_n = \log(3)$. Thus, ε_n must be $0.55/n$ in a large equilibrium sequence for this modified example.

5. Problems of too few discriminatory equilibria: One Bad Apple

We saw in the preceding section that best-rewarding rules like plurality voting can easily generate discriminatory equilibria that sometimes seem problematic or undesirable. Our next proposition shows how worst-punishing rules like negative voting frequently do not yield any discriminatory equilibria, which can also be problematic. (Recall that A is the lower bound on the number of points that a voter can give to the middle-ranked candidate on his ballot.)

Proposition 2. For any (A,B)-scoring rule with $A \geq 0.5$, if all voters have strict preferences over the three candidates then discriminatory equilibrium sequences do not exist.

Proof. Suppose, contrary to the proposition, that we have a discriminatory large equilibrium sequence in which {1,2} is the only serious race. In such equilibria, for all large n, each voter would want to maximize his probability of making an impact on the serious race, and so all voters must be expected to choose among the four ballots

$(1,0,A), (1,0,B), (0,1,A), (0,1,B).$

So when candidates 1 and 2 have equal scores, they must each get an average of 0.5 points per voter. But candidate 3 in such a scenario would have a score of at least $A \geq 0.5$ points per voter, and so candidate 3 could never get fewer points than the low scorer among 1 and 2. Thus candidate 3 could never be more than one vote away from winning in a close {1,2} race. So the probability of a close race involving 3 cannot be less than the probability of a close {1,2} race, which contradicts the assumption that candidate 3 is not serious. Q.E.D.

To illustrate the implications of Proposition 2, let us consider now as Example 2 a simple voting game where absence of discriminatory equilibria seems undesirable from a social choice perspective. In this game, there are again three candidates $K = \{1,2,3\}$, and two types of voters $T = \{\mathbf{1},\mathbf{2}\}$, and any randomly sampled voter is equally likely to be type **1** or **2**,

$$r(\mathbf{1}) = 0.5 = r(\mathbf{2}).$$

The utility values in this example are

$$u(\mathbf{1}) = (9,6,0), \quad u(\mathbf{2}) = (6,9,0).$$

So again, type **1** voters prefer candidate 1 over candidate 2, and type **2** voters prefer candidate 2 over candidate 1, but now all voters prefer both candidates 1 and 2 over candidate 3. We may call this game "One Bad Apple", because our concern is that the presence of one universally undesirable candidate may in some way spoil the whole election, as one rotten apple can spoil a whole barrel of apples.

In this example, the majority-preferred outcome can be guaranteed only if there exists a discriminatory equilibrium where candidate 3 is not serious, and Proposition 2 tells us that this equilibrium cannot exist unless $A < 0.5$. Indeed, when $A < 0.5$, this example has a discriminatory large equilibrium where

$$\sigma(1,0,A|\mathbf{1}) = 1 = \sigma(0,1,A|\mathbf{2}), \text{ and } \tau(1,0,A) = \tau(0,1,A) = 0.5.$$

Here the pivot magnitudes are $\mu_{1,2} = 0$ and $\mu_{1,3} = \mu_{2,3} = -1$, confirming that only the {1,2} race is serious. For example, under plurality or approval voting, everyone votes (1,0,0) or (0,1,0) in this discriminatory equilibrium, and so the bad apple 3 cannot win with any positive turnout.

But now consider the case where $A \geq 0.5$. If candidate 3 was not serious then the type **1** voters would vote (1,0,A), and the type **2** voters would vote (0,1,A), and so any close race involving candidates 1 and 2 would necessarily involve candidate 3 as well. But each voter prefers his second-favorite candidate over a lottery where all three candidates have equal probability, and so each voter would then want to deviate to (1,A,0) or (A,1,0).

Under negative voting, the unique large equilibrium of this example has limiting strategy

$$\sigma(1,0,1|\mathbf{1}) = 2/3,\ \sigma(1,1,0|\mathbf{1}) = 1/3,\ \sigma(0,1,1|\mathbf{2}) = 2/3,\ \sigma(1,1,0|\mathbf{1}) = 1/3,$$

and the limiting vote distribution is

$$\tau(1,1,0) = \tau(1,0,1) = \tau(0,1,1) = 1/3.$$

So each candidate's expected score in the limit is 2/3 points per voter, and all three races have magnitude 0 and are serious. Thus, even though all voters dislike candidate 3 in this example, candidate 3 must be a serious candidate in a large equilibrium under negative voting.

We may now ask how likely the bad candidate 3 is to actually win the election in this large equilibrium under negative voting. Just because all candidates have equal expected scores per voter in the limit does not imply that they have equal chance of winning in large equilibria, because their expected scores can converge differently to 2/3, from above or below.

If all three close races were equally likely in this example then, under negative voting, the voters would all vote sincerely against candidate 3. To induce some voters of each type to vote against their second-favorite candidate in {1,2}, the probability of a close {1,2} race must be somewhat greater than either race involving 3, which can happen if, for large finite n, the expected fraction of votes against 3 is slightly larger than 1/3 while the expected fraction of votes against 1 and 2 are each slightly less than 1/3. Because the possible pivot events $\Lambda(c,i,j)$ all occur here with offset ratios approaching 1, we can use the Normal approximation to estimate their probabilities, and the probability of any possible pivot event $\Lambda(c,i,j)$ is essentially the same as the probability of candidates i and j being tied for first place (in the sense that the ratio of these probabilities goes to 1 as $n\to\infty$, by the offset theorem). So, given any large n, let p_{ij} denote the probability of i and j being tied for first place. To make type **1** voters indifferent between voting (1,1,0) against 3 and voting (1,0,1) against 2, these probabilities must satisfy

$$(9-0)p_{13} + (6-0)p_{23} = (9-6)p_{12} + (0-6)p_{23}.$$

To make type **2** voters indifferent between voting (1,1,0) against 3 and voting (0,1,1) against 1, we similarly need

$$(6-0)p_{13} + (9-0)p_{23} = (9-6)p_{12} + (0-6)p_{13}.$$

These equations imply

$$p_{13} = p_{23} = p_{12}/7.$$

That is, the probability of a close {1,3} race and the probability of a close {2,3} race must each be 1/7 of the probability of a close {1,2} race. So for any large n (noting the symmetry among candidates 1 and 2 in this game), we may look for an equilibrium of the form

$$\sigma_n(1,0,1|\mathbf{1}) = 2/3-\varepsilon_n = \sigma_n(0,1,1|\mathbf{2}),\ \ \sigma_n(1,1,0|\mathbf{1}) = 1/3+\varepsilon_n = \sigma_n(1,1,0|\mathbf{2})$$

which gives us the expected vote distribution

$$\tau_n(1,0,1) = 1/3 - 0.5\varepsilon_n = \tau_n(0,1,1), \ \ \tau_n(1,1,0) = 1/3 + \varepsilon_n.$$

Near the expected vote profile, the numbers x(c) for each ballot option c can be approximated as the integer-roundings of independent Normal random variables with mean and variance both equal to $n\tau_n(c)$. From this random vector x, we define

$$z_2 = (x(1,0,1) - x(0,1,1))/\sqrt{n}, \ \ z_3 = (x(1,1,0) - x(0,1,1))/\sqrt{n}.$$

The joint distribution of z_2 and z_3 is approximately Multivariate-Normal, with means $E(z_2) = 0$ and $E(z_3) = 1.5\varepsilon_n\sqrt{n}$, variances both close to 2/3, and correlation 1/2. Candidates 1 and 2 are tied for first place when z_2 is between $-0.5/\sqrt{n}$ and $0.5/\sqrt{n}$ and z_3 is positive. But when z_2 is close to 0, the conditional distribution of z_3 is approximately Normal with mean $1.5\varepsilon_n\sqrt{n}$ and variance 1/2. Let $\Phi(x,m,v)$ and $\Phi'(x,m,v)$ denote respectively the cumulative and density at x for a Normal with mean m and variance v. So p_{12} is approximately

$$p_{12} \approx (1/\sqrt{n})\,\Phi'(0,0,2/3)\,(1 - \Phi(0,\ 1.5\varepsilon_n\sqrt{n},\ 1/2))$$

Similarly, 1 and 3 are tied for first place when z_3 is between $-0.5/\sqrt{n}$ and $0.5/\sqrt{n}$ and z_2 is positive. But when z_3 is close to 0, the conditional distribution of z_2 is approximately Normal with mean $-0.75\varepsilon_n\sqrt{n}$ and variance 1/2. So

$$p_{13} \approx (1/\sqrt{n})\,\Phi'(0,\ 1.5\varepsilon_n\sqrt{n},\ 2/3)\,(1 - \Phi(0,\ -0.75\varepsilon_n\sqrt{n},\ 1/2)).$$

This approximation yields $p_{12}/p_{13} = 7$ when $\varepsilon_n = 0.628/\sqrt{n}$. (Thus, for example, if $n =$ 9,000,000 then we get the equilibrium expected vote profile $n\tau_n(1,1,0) = 3{,}001{,}884$ and $n\tau_n(1,0,1) = n\tau_n(0,1,1) = 2{,}999{,}058$.) Simulation analysis shows that the probability of the bad candidate 3 winning is 0.044 in these negative-voting equilibria with large n.

6. Breaking symmetry of cyclic majorities in the Condorcet Cycle

We have considered above (and will consider in other sections below) simple examples that have just two types of voters. In such two-type examples, the meaning of majority rule seems clear: It means that there should never be another candidate whom a majority of the voters would strictly prefer over the winner of the election. When we consider such examples, we can evaluate voting rules by whether their equilibrium outcomes are consistent with majority rule, because majority-rule outcomes always exist.

But the great impossibility theorems of social choice theory tell us that majority-rule outcomes cannot be defined for all social choice situations. The Condorcet cycle example is the simplest and best-known of these situations where majority-rule outcomes do not exist. In this section, we consider a version of this Condorcet cycle, to show that, even when "majority rule" is not well-defined, we can still find systematic differences among voting rules in terms of their tendency to admit discriminatory equilibria.

So let us consider as Example 3 a replicated version of the Condorcet cycle where there are three candidates K = {1,2,3}, three types of voters T = {**1**,**2**,**3**}, any randomly sampled voter is equally likely to be of any type

$$r(\mathbf{1}) = r(\mathbf{2}) = r(\mathbf{3}) = 1/3,$$

and the utility values of the candidates for each type are

$$u(\mathbf{1}) = (9,6,0),\ \ u(\mathbf{2}) = (0,9,6),\ \ u(\mathbf{3}) = (6,0,9).$$

So type **1** voters have the preference ordering $1 \succ 2 \succ 3$, type **2** voters have the preference ordering $2 \succ 3 \succ 1$, and type **3** voters have the preference ordering $3 \succ 1 \succ 2$.

The symmetries of the candidates and types in this example imply that the voting game must always have a symmetric equilibrium in which each candidate has the same 1/3 probability of winning, and each pair of candidates is equally likely to be in a close race. In such a symmetric equilibrium, each voter should vote sincerely, giving 1 point to his most-preferred candidate, 0 points to his worst candidate, and B points to his middle candidate. (The choice is B rather than A for the middle candidate, because we have assumed that each voter would prefer the middle candidate when the other choice is equally likely to be the best or worst candidate.)

The main question of this section is to characterize the scoring rules such that this Condorcet cycle also has discriminatory equilibria which break the symmetry of the candidates. So let us look for discriminatory equilibria in which, say, candidate 3 is not serious. If {1,2} were the only serious race, then type **1** voters would all vote (1,0,A) (because they prefer 1 over 2 but think 3 is worst), type **3** voters would all vote (1,0,B) (because they also prefer 1 over 2 but think 3 is best), and type **2** voters would all vote (0,1,A) or (0,1,B) (because they prefer 2 over 1). So for some ρ in [0,1], the expected vote distribution must be

$$\tau(1,0,A) = 1/3 = \tau(1,0,B),\ \ \tau(0,1,A) = (1-\rho)/3,\ \ \tau(0,1,B) = \rho/3$$

in the limit of any discriminatory equilibrium sequence where 3 is not serious.

The decision of the type **2** voters will depend on whether a close race involving candidate 3 is more likely to be with 1 or with 2. We have assumed that $u_3(\mathbf{2})-u_1(\mathbf{2})$ is more than $u_2(\mathbf{2})-u_3(\mathbf{2})$, and so the type **2** voters should be more concerned about influencing a {1,3} race (where they would prefer to vote (0,1,B)) than about influencing a {2,3} race (where they would prefer to vote (0,1,A)) unless a close {2,3} race is much more likely than a close {1,3} race.

But we now claim that a close {2,3} race cannot be more likely than a close {1,3} race in a discriminatory equilibrium where 3 is not serious, so that the type **2** voters should all vote (0,1,B), yielding ρ=1. To verify this claim, consider the dual optimization problem for the event

$\Omega(3)$ where candidate 3 wins. For each i in {1,2}, let λ_i denote the Lagrange multiplier for the constraint that candidate i should not have a higher score than candidate 3. Then the dual magnitude problem for $\Omega(3)$ is to find values of $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ that minimize the dual objective

$$\begin{aligned}
&(1/3)\exp((A-1)\lambda_1+A\lambda_2) + (1/3)\exp((B-1)\lambda_1+B\lambda_2) \\
&\qquad + ((1-\rho)/3)\exp(A\lambda_1+(A-1)\lambda_2) + (\rho/3)\exp(B\lambda_1+(B-1)\lambda_2) - 1 \\
&= \exp(A(\lambda_1+\lambda_2))\,(\exp(-\lambda_1) + (1-\rho)\exp(-\lambda_2))/3 \\
&\qquad + \exp(B(\lambda_1+\lambda_2))\,(\exp(-\lambda_1) + \rho\exp(-\lambda_2))/3 - 1
\end{aligned}$$

If $\lambda_1 < \lambda_2$ then switching the values of λ_1 and λ_2 would strictly reduce this convex objective function. Thus, an optimal solution must have $\lambda_1 \geq \lambda_2 \geq 0$. The λ_i cannot both be zero, or else candidate 3 would be winning at the expected vote profile (contradicting the assumption that 3 is not serious). So we must have $\lambda_1 > 0$, which implies that the magnitude maximizing region in $\Omega(3)$ must be near a close {1,3} race and $\mu_3 = \mu_{1,3}$. If there is not also a close {2,3} race in this region, then $\mu_3 = \mu_{1,3} > \mu_{2,3}$, in which case the claim is verified. So it remains to consider the case where $\mu_3 = \mu_{1,3} = \mu_{2,3}$. By concavity of the magnitude problem, the magnitude-maximizing points in $\Omega(3)$ must be in the region where both linear constraints are binding, and so the most likely way of candidate 3 winning or being in any close race is in the region near a three-way tie among all the candidates, where the offset ratios α are as given by the dual magnitude theorem. So by the offset theorem, changing a vote profile in this region by adding k (1,0,A) votes and subtracting k (0,1,A) votes would change the probability by a multiplicative factor of

$$\begin{aligned}
&(\alpha(0,1,A)/\alpha(1,0,A))^k = (\exp(\lambda_1 A+\lambda_2(A-1))/\exp(\lambda_1(A-1)+\lambda_2 A))^k \\
&\qquad = \exp(k(\lambda_1-\lambda_2)) \geq 1,
\end{aligned}$$

because $\lambda_1 \geq \lambda_2$. So for any given point x in this region where a close {2,3} race exists, we can find a corresponding point x+k[1,0,A]−k[0,1,A] where a close {1,3} race exists and which is at least as likely as x. (Let k be the difference of candidate 2's points minus 1's points at x+[0,1,A].) Thus, as claimed, a close {1,3} race cannot be less likely than a close {2,3} race in a discriminatory equilibrium where 3 is not serious.

Now with $\rho=1$, a discriminatory equilibrium must have the expected vote distribution

$$\tau(1,0,A) = \tau(1,0,B) = \tau(0,1,B) = 1/3,$$

where 2/3 of the voters are expected to vote for 1 and only 1/3 for 2, and so the magnitude of a tie between candidates 1 and 2 is

$$\mu_{1,2} = -(\sqrt{2/3} - \sqrt{1/3})^2 = -0.05719\,.$$

By the dual magnitude theorem, the magnitude of candidate 3 winning (which is also the magnitude of 3 being in a close race) is the minimal value of the dual objective

$$(\exp((A-1)\lambda_1+A\lambda_2) + \exp((B-1)\lambda_1+B\lambda_2) + \exp(B\lambda_1+(B-1)\lambda_2))/3 - 1$$

subject to $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$. Solving this problem numerically, we find that its optimal value is less than −0.05719 as long as (A,B) is below a curve that goes through the points

(0, 0.649), (0.1, 0.621), (0.2, 0.593), (0.3, 0.563), (0.4, 0.532), (0.5, 0.500).

This curve is shown in Figure 1. Thus, for (A,B)-scoring rules below this curve, we can find discriminatory equilibria of this Condorcet cycle example such that any single candidate is out of contention. These scoring rules include the best-rewarding rules like plurality voting.

[INSERT FIGURE 1 ABOUT HERE]

So when voters in the Condorcet cycle use plurality voting or any of the best-rewarding rules below this curve, the symmetry of the Condorcet cycle can be broken in equilibrium. A perception that one candidate is not serious can become a self-fulfilling prophecy, so that the next candidate in the cycle 1>2>3>1 will be almost sure to win the election. Thus, a candidate i's success in the election may depend on manipulation of the voters' perceptions, to get them to focus on the equilibrium in which the candidate who can beat i is not taken seriously.

On the other hand, when voters in the Condorcet cycle use a rule above this curve, such as approval voting or negative voting, all three candidates must always be taken seriously as contenders to win the election.

According to the simpler theory of Myerson and Weber (1993), an (A,B)-scoring rule would have discriminatory equilibria for this Condorcet cycle example if and only if it satisfies the inequality $A+2B \leq 1$, which holds somewhat below the curve in Figure 1. To see the shortcomings of the Myerson-Weber theory, consider the case of A=B=0.4, which is below the curve in our Poisson analysis but would be above the line in Myerson-Weber analysis. With (A,B) = (0.4,0.4), if 3 were considered out of contention then the type **1** and type **3** voters would vote (1, 0, 0.4), while the type **2** voters would vote (0, 1, 0.4), and so the expected scores for the three candidates would be 2/3 points per voter for candidate 1, 1/3 for candidate 2, and 0.4 for candidate 3. From these expected scores, the Myerson-Weber analysis would conclude that candidate 3 should be considered as a more serious contender than candidate 2, because 3's expected score is greater than 2's, and so the conditions for a discriminatory equilibrium would not be recognized. But in our Poisson analysis, the universally middle-ranked candidate 3 gets the lowest pivot magnitudes $\mu_{1,3} = \mu_{2,3} = -1$, because 3 always get fewer points (with A=B=0.4) than the average of candidates 1 and 2, and so candidate 3 could not be in a close race when there is any positive turnout. The magnitude of a close {1,2} race is $\mu_{1,2} = -(\sqrt{2/3} - \sqrt{1/3})^2 =$

≈ 0.05719 in Poisson analysis, because it is the event that the (1,0,0.4) and (0,1,0.4) ballots get equal vote shares, instead of their expected 2/3 and 1/3 vote shares. Thus, the Poisson analysis appropriately finds discriminatory equilibria for this Condorcet cycle with A=B=0.4.

7. Majoritarian outcomes in symmetric equilibria of simple bipolar elections

In the preceding sections, we compared voting rules in terms of whether they admit discriminatory equilibria in which voters perceive candidates as having very different chances of contending to win the election. In this section, we show that voting rules may differ substantially even when we focus only on equilibria where the voters view similar candidates symmetrically. In particular, we consider now the question of whether it may be an advantage or disadvantage for a bloc of voters to have more than one serious candidate in the election. We simplify matters here, in this section and the next, by again considering examples in which there are only two types of voters. For such examples, we can then ask whether the equilibrium outcomes under different voting rules coincide with what the majority of voters would most prefer.

So let us consider as Example 4 a simple bipolar game where there are two types of voters $T = \{\mathbf{1},\mathbf{2}\}$, three candidates $K = \{1,2,3\}$, and the utility values of candidates to voters are

$$u(\mathbf{1}) = (1,0,0),\ \ u(\mathbf{2}) = (0,1,1).$$

Candidates 2 and 3 here are indistinguishable duplicates advocating type **2** interests. So type **2** voters prefer candidates 2 and 3, while type **1** voters prefer candidate 1. The expected fraction of type **1** voters $r(\mathbf{1})$ is left as a parameter to be specified later in the analysis, with $r(\mathbf{2}) = 1-r(\mathbf{1})$.

We may say that an outcome of this example is <u>majoritarian</u> iff the winner is a candidate who is preferred by at least half of the voters, that is, candidate 1 if there are more type **1** voters, but candidate 2 or 3 if there are more type **2** voters.

Under any (A,B)-scoring rule, this simple bipolar example always has a symmetric equilibrium in which the voters treat candidates 2 and 3 symmetrically and use the strategy

$$\sigma(1,A,0|\mathbf{1}) = \sigma(1,0,A|\mathbf{1}) = 1/2, \ \ \sigma(0,1,B|\mathbf{2}) = \sigma(0,B,1|\mathbf{2}) = 1/2.$$

In this equilibrium, each voter gives as many points as possible to the candidate(s) he prefers, and as few points as possible to the candidate(s) whom he does not prefer, splitting randomly in their treatment of the similar candidates 2 and 3. Unfortunately, these symmetric equilibria allow failures of majority rule under all (A,B)-scoring rules except approval voting.

Proposition 3. For this simple bipolar example, the equilibrium outcome is always majoritarian under approval voting, but a non-majoritarian outcome can occur in the symmetric equilibrium under any (A,B)-scoring rule other than approval voting.

Proof. In the case where $A > 0$, it can happen that the type **1** voters have a slight majority, but the type **1** voters all vote (1,A,0) and the type **2** voters all vote (0,1,B), making candidate 2 the winner. In the case where $A = 0$ and $B < 1$, it can happen that the type **2** voters have a slight majority, but the type **1** voters all vote (1,0,0) and the type **2** voters split equally among (0,1,B) and (0,B,1), making candidate 1 the winner. But in the equilibrium under approval voting, with $A=0$ and $B=1$, each candidate gets as many points as there are voters who prefer him, and so the set of voters who prefer the winner cannot be a strict minority. Q.E.D.

In this symmetric equilibrium, the expected score for the candidate 1 is $r(\mathbf{1})$ points per voter, and the expected score for candidates 2 and 3 is $(r(\mathbf{1})A+(1-r(\mathbf{1}))(1+B))/2$. So 1's expected score is largest, and the probability of candidate 1 winning goes to one as $n\to\infty$, when

$$r(\mathbf{1}) > (1+B)/(3+B-A).$$

Conversely, 1's expected score is lowest, and the probability of candidate 1 winning goes to 0 as $n\to\infty$, when $r(\mathbf{1}) < (1+B)/(3+B-A)$. This quantity $(1+B)/(3+B-A)$ is Cox's threshold of diversity for (A,B)-scoring rules with 3 candidates (see Cox, 1987, 1990, and Myerson, 1993b).

Consider now the (A,B)-scoring rules where $A + B < 1$. (These include the best-rewarding rules like plurality voting.) Under such voting rules, the expected fraction of type **1** voters can satisfy $1/2 > r(\mathbf{1}) > (1+B)/(3+B-A)$, and then the probability of a majority with two candidates both losing the election approaches one as $n\to\infty$. Thus, a majority bloc of voters may be weakened by having duplicate candidates under best-rewarding rules like plurality voting.

Consider now the (A,B)-scoring rules where $A + B > 1$. (These include the worst-punishing rules like negative voting.) Under such voting rules, the expected fraction of type **1** voters can satisfy $1/2 < r(\mathbf{1}) < (1+B)/(3+B-A)$, and then the probability that a minority with two candidates has a winner of the election approaches one as $n\to\infty$. So a minority bloc of voters may be strengthened by having duplicate candidates under worst-punishing rules like negative voting.

Under (A,B)-scoring rules where $A + B = 1$, the probability of nonmajoritarian outcomes in the symmetric equilibria of Proposition 3 goes to zero as $n\to\infty$. So the possible failure of majority rule for symmetric equilibria of this simple bipolar example does not seem very problematic under such rules that are well balanced between best-rewarding and worst-punishing.

8. Efficient majoritarian outcomes in more general bipolar elections with corruption

In the simple model of the previous section, we could apply the criterion of majoritarianism but not Pareto-efficiency because, among any two distinguishable candidates, the better candidate for one type of voter was always worse for the other type. We now consider a more general bipolar model in which both efficiency and majoritarianism can be tested, and we show that approval voting passes both tests. In this analysis, we extend the results of Myerson (1993a) to the Poisson framework (and the proof here is easier than in the original framework).

As before, suppose that there are two types of voters $T = \{\mathbf{1},\mathbf{2}\}$. We assume now that the type **1** voters are expected to form a strict majority, with $0.5 < r(\mathbf{1}) < 1$ and $r(\mathbf{2}) = 1 - r(\mathbf{1})$. For any type t, we may let ~t denote the other type in $\{1,2\}$.

We now allow any finite set of three or more candidates, but we assume that each candidate is associated with one type or the other. That is, set of candidates K is partitioned into sets K_1 and K_2, where K_t denotes the set of candidates of type t. As in the model of Myerson (1993a), we assume that each candidate k has a given known corruption level which we denote here by $f(k) \geq 0$. We may say that candidate k is <u>clean</u> if $f(k) = 0$, but is <u>corrupt</u> if $f(k) > 0$. For any type t voter, the utility from candidate k winning is

$$u_k(t) = 1 - f(k) \text{ if } k \in K_t, \; u_k(t) = -f(k) \text{ if } k \notin K_t.$$

That is, each voter gains one unit of utility from having the winner be a candidate of his own type, but also loses an amount of utility equal to the winner's corruption level regardless of type. We consider the generic case where no voter is indifferent between two candidates of different types (which can be guaranteed if no two candidates' corruption levels differ by exactly one). Finally, we assume that there exists at least one clean candidate in K_1, so that the election of this clean type 1 candidate could fulfill both the criteria of Pareto-efficiency and majority rule in the event that the type 1 voters have a majority (which has probability approaching one as $n \to \infty$).

Proposition 4 Under approval voting, in all large equilibria of a bipolar election as described above, the winner at the expected vote distribution will be a clean candidate in K_1, and so the probability of a clean majority-type winner goes to one as $n \to \infty$.

Proof: In any given large equilibrium sequence, each voter's decision about whether to add an approval for any candidate k depends on a comparison of the probability that k might be in a close race with a worse candidate (where approving k would be preferred) or with a better candidate (where not approving k would be preferred). In particular, a voter should not approve a serious candidate whose serious races are all with better candidates. By dominance, all type **1** voters should approve the clean candidate in K_1, because they have no better candidates.

If some voters are willing to cross over and approve a candidate of the other type, then all voters of that candidate's type must strictly prefer to approve him also, because of the monotone increasing differences property of the utility function.

If the proposition failed, then the magnitude of a close race involving the clean candidate in K_1 could not be less than $-r(\mathbf{2})$, because his score would be maximal when the type **2** voters disappear. So we can rule out the existence of any serious candidate whom nobody is expected to approve, because the magnitude of a close race involving such a candidate would be -1.

If all serious candidates were of one type, then the most corrupt serious candidate would only have serious races with other candidates who are preferred by all voters, and so nobody would approve him. Thus, there are serious candidates both in K_1 and in K_2.

We now claim that no serious candidate can expect a positive rate of approval from both types of voter. To prove this claim by contradiction, let g be most corrupt such candidate. Let t denote g's type. So g expects approval from all type t and some type ~t voters.

If there are any serious candidates in K_t who are more corrupt than g, then let h be the most corrupt among these candidates. So h gets approval only from type t voters (by definition of g), and so the voters who are expected to approve h in equilibrium must be a strict subset of the voters who are expected to approve g. There must also be some candidate j in $K_{\sim t}$ such that the {h,j} race is serious, because otherwise h would only have serious races with other less corrupt candidates in K_t (and then nobody would approve h). But in the event of such a {h,j} close race, g must be in the close race also, and there must be zero turnout from the expected bloc of voters who would approve g but not h. Now we have to consider two cases. As case 1, suppose that there exists a candidate j in $K_{\sim t}$ such that the {h,j} race is serious and a positive fraction of type ~t voters are expected to approve both j and g. In the event of a close {h,j} race, as $n \to \infty$, there would be a zero offset ratio for these ~t voters who approve g and j but not h, and so there would be an arbitrarily larger probability for the event that differs by adding two votes that approve j and g but not h. But adding these two votes would put us in the event where a close race exists involving j and g but not h. Thus, the event of a {g,j} close race would be infinitely more likely than a {h,j} close race, as $n \to \infty$, contradicting the assumption that the {h,j} race is serious. As case 2, now suppose that every candidate j in $K_{\sim t}$ who has a serious race with h does not have a positive expected share of supporters in common with g. In the event of a close {h,j} race, as $n \to \infty$, there would be a zero offset ratio for these ~t voters who approve g but not h or j. So there would be an arbitrarily larger probability for the event that differs by adding two votes that approve g but not h or j, and subtracting two votes that approve h and g but not j (which could not have a zero offset ratio when h is in a close race at a positive score). Adding and subtracting these votes would put us in the event where a close race exists involving j and g but not h. So the event of a {g,j} close race would be infinitely more likely than a {h,j} close race, as $n \to \infty$, again contradicting the assumption that the {h,j} race is serious. Getting a contradiction in both cases, we conclude that there does not exist any serious candidate h in K_t who is more corrupt than g.

So g is the most corrupt serious candidate in K_t. Now let i denote the most corrupt serious candidate in $K_{\sim t}$. Voters of type ~t must prefer i over g, because otherwise i would be the worst serious candidate for both types of voters (and so nobody would approve i). So there does not exist any serious candidate in $K_{\sim t}$ who is worse than g for voters of type ~t.

So type ~t voters must consider g worst among all serious candidates. So type ~t voters should not approve g in equilibrium, contradicting the definition of g. Thus, as claimed above, no serious candidate g can expect a positive rate of approval from both voter types.

Recall that the clean candidate in K_1 gets approval from all type **1** voters. If some corrupt candidate in K_1 also expected a 100% approval rate from type **1** voters in the limit, then the event of a close race between that candidate and the clean candidate in K_1 would have magnitude 0. On the other hand, the event of type **1** voters not being a strict majority has strictly negative magnitude $-(\sqrt{r(\mathbf{1})} - \sqrt{r(\mathbf{2})})^2$, and any close races involving candidates in K_2 (who expect no approvals from type **1** voters) could not have a magnitude greater than this. So candidates in K_2 could not be in any serious races, if there were a corrupt candidate in K_1 who expected a 100% approval rate from type **1** voters in the limit. But we have seen that there must be some serious candidate in K_2. So only a clean candidate in K_1 can expect votes from all type **1** voters in the limit, and no other candidates can win at the expected outcome. Q.E.D.

REFERENCES

M. Abramowitz and I. Stegun, "Handbook of Mathematical Tables." New York: Dover (1965).

G. Cox, "Electoral equilibrium under alternative voting institutions." American Journal of Political Science 31 (1987), 82-108.

G. Cox, "Centripetal and centrifugal incentives in electoral systems." American Journal of Political Science 34 (1990), 903-935.

E. Muller and M. Satterthwaite, "The equivalence of strong positive association and strategy-proofness," Journal of Economic Theory 14 (1977), 412-418.

R. Myerson, "Effectiveness of Electoral Systems for Reducing Government Corruption: A Game Theoretic Analysis," Games and Economic Behavior 5 (1993a), 118-132.

R. Myerson, "Incentives to cultivate favored minorities under alternative electoral systems." American Political Science Review 87 (1993b), 856-869.

R. Myerson, "Population uncertainty and Poisson games," International Journal of Game Theory 27 (1998a), 375-392.

R. Myerson, "Extended Poisson games and the Condorcet jury theorem," Games and Economic Behavior 25 (1998b), 111-131.

R. Myerson, "Large Poisson games," Northwestern University Math Center discussion paper #1189*. To appear in Journal of Economic Theory (2000).

R. Myerson and R. Weber. 1993. "A theory of voting equilibria." American Political Science Review 87 (1993), 102-114.

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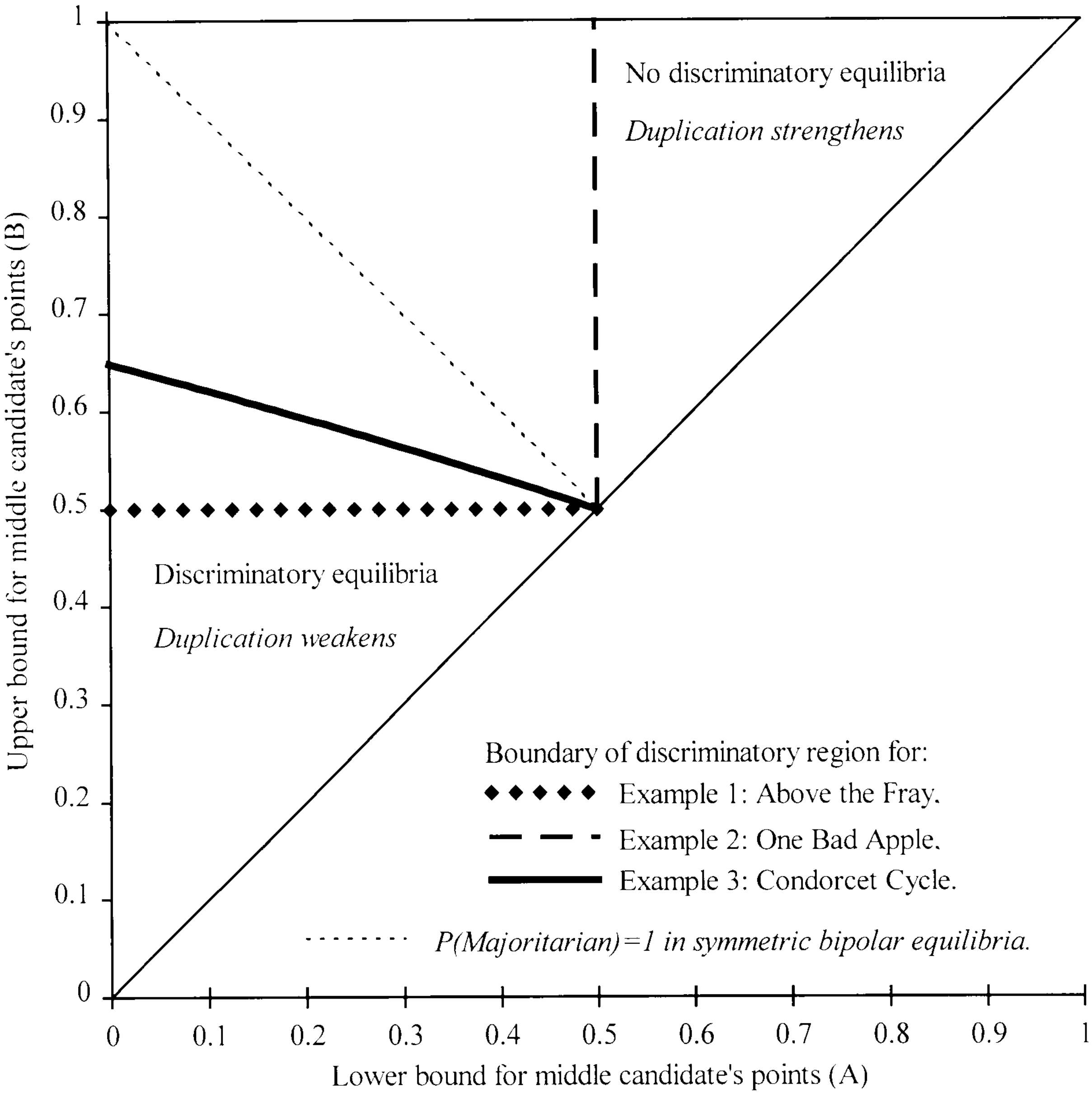

Figure 1. Characterizing Equilibria of (A,B)-Scoring Rules.