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THE EVOLUTION OF SOCIAL AND ECONOMIC NETWORKS

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SOCIAL SCIENCE WORKING PAPER 1044

September 1998

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Abstract

We examine the dynamic formation and stochastic evolution of networks connecting individuals whose payoffs from an economic or social activity depends on the network structure in place. Over time, individuals form and sever links connecting themselves to other individuals based on the improvement the resulting network offers them relative to the current network. Such a process creates a sequence of networks that we call an 'improving path'. The changes made along an improving path make the individuals, who added or deleted the relevant link(s) at each date, better off. Such sequences of networks can cycle, and we study conditions on underlying allocation rules that characterize cycles.

Building on an understanding of improving paths, we consider a stochastic evolutionary process where in addition to intended changes in the network there is a small probability of unintended changes or errors. Predictions can be made regarding the relative likelihood that the stochastic evolutionary process will lead to any given network at some time. The evolutionary process selects from among the statically stable networks and cycles. We show that in some cases, the evolutionary process selects inefficient networks even though efficient ones are statically stable. We apply these results to the matching literature to show that there are contexts in which the evolutionarily stable networks coincide with the core stable networks, and thus achieve efficiency.

JEL classification numbers: A14, D20, J00

Key words: networks, evolution

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1 Introduction

Network structure is important in determining the outcome of many important social and economic relationships. For example, networks play a fundamental role in determining how information is exchanged. Such information may be as simple as an invitation to a party, or as consequential as information about job opportunities (e.g., Boorman [1975], Montgomery [1991], and Topa [1996]), literacy (e.g., Basu and Foster [1996]), consumer products (e.g.,Ellison and Fudenberg [1995] and Iacobucci and Hopkins [1992]), or even information regarding the returns to crime (e.g., Glaeser, Sacerdote, and Scheinkman [1996]). Networks also play fundamental roles in the payoffs earned from bargaining with an organization (e.g., Wang and Wen [1998]) and in the exchange of goods and services. Exchange examples include trading networks and alliances (e.g., Bell [1996], Maxfield [1997], Kirman, Oddou, and Weber [1986], Tesfatsion [1997] and [1998]), and networks through which financial help or insurance is exchanged in developing countries (e.g., Lund and Fafchamps [1997]). Even standard matching problems (e.g., the marriage and college admissions problems studied by Gale and Shapley [1962] and Roth and Sotomayor [1989]) are special situations where network relationships are important.

Despite the fundamental importance of network structures in many social and economic settings, there is still a lack of foundational theoretical models which analyze how networks emerge and how the decisions of individuals contribute to network formation. In this paper, we examine the dynamic formation and stochastic evolution of networks, taking into account the incentives that individuals have to form (or sever) links with each other. Our aim is to develop a working model of the dynamic formation of networks, and to examine some of the features and predictions of that model.

1.1 An Overview of the Model and Results

Our approach is to model network formation as a dynamic process in which individuals form and sever links based on the improvement that the resulting network offers them relative to the current network. Networks are modeled as graphs, where nodes or vertices represent individuals and links or edges represent connections between the individuals. Links are non-directed and thus reciprocal. A link between two individuals can be formed only if both individuals agree to add the link, while a single individual can sever an existing link. Each individual receives a payoff or net benefit based on the network configuration that is in place. This payoff can be interpreted as the utility or production that an individual obtains from the social interaction that occurs through the network.

The primary tool that we introduce to analyze dynamic network formation is the concept of a sequence of networks that emerge when individuals form or sever links based on the improvement the resulting network offers relative to the current network. Such a sequence, called an 'improving path', has the properties that (i) each network in the sequence differs from the previous network by the addition or deletion of a single link, and (ii) the addition or deletion of the link benefits the individual(s) whose consent is necessary for the change. This myopic behavior is natural in the context of large networks where players may have limited information about the incentives of others, and generally provides a useful starting point for the study of the evolution of networks. Later in the paper, in the context of matching models, we show that this methodology is adaptable to allow for other sorts of behavior by players, such as the simultaneous change of a number of links.

The first part of the paper is devoted to analyzing improving paths. The improving paths emanating from any starting network must lead to either a pairwise stable network (where no two players want to form a link, and no individual player wants to sever a link) or a cycle (where a number of networks are repeatedly visited). We show that there always exists either a pairwise stable network or a cycle from which there is no exit. We give a simple trading network example to show that it is possible for cycles to exist while pairwise stable networks fail to exist. We also characterize the conditions, on primitives of the model, for which cycles do not exist.

The second part of the paper uses improving paths as the foundation for an evolutionary analysis, where in addition to intended changes in the network, unintended mutations or errors are introduced. Such unintended changes may be due to exogenous forces acting on the network, or simply miscalculations or errors on the part of an individual making an assessment or taking an action. Such a process can be described as a Markov chain and well-developed results concerning limiting behavior of Markov processes can be applied (Freidlin and Wentzell [1984], Kandori, Mailath, and Rob [1993] and Young [1993]). This stochastic process ultimately leads to specific predictions concerning the relative amounts of time that will be spent in various networks. The stochastic evolutionary results incorporate the concept of improving paths, as the dynamic process naturally moves along improving paths between mutations. Thus the process naturally gravitates to pairwise stable networks and cycles, but periodically is bumped away by mutation. The intuition for which networks are visited most often comes from the idea of resistance (based on that of Freidlin and Wentzell [1984]). In the network context, resistance keeps track of how many mutations are needed to get from some given network to an improving path leading to another network. Very roughly, networks that are harder to get away from and easier to get back to, in terms of resistance, are favored by the evolutionary process (although this favoritism depends on the full configuration of resistance among different networks). We apply these ideas to several examples to study the set of evolutionarily stable networks. For instance, Proposition 8 says that even in contexts where a unique (Pareto) efficient network is pairwise stable, it may not be evolutionarily stable. Thus the system may spend no measurable amount of time in an efficient network, even if that network is pairwise (statically) stable.

In the last section of the paper, we apply the stochastic evolutionary model to matching problems, such as the Gale-Shapley marriage problem and the college admissions (hospital-intern) problem (see Roth and Sotomayor [1989]). Such matching problems fit nicely into a network setting, and previous studies of such matching problems have concentrated on static notions of stability and on centralized procedures and algorithms. The methodology outlined above can be used to analyze which matchings one expects to arise endogenously, in the absence of some coordinating procedure. Theorem 10 shows that, in these problems, the set of evolutionarily stable networks coincides with the set of (static) core stable networks, which are necessarily Pareto efficient. Examples show how this relationship depends on the definition of improving path that is applied.

1.2 The Closely Related Literature

The papers most closely related to this one are Jackson and Wolinsky [1996], Dutta and Mutuswami [1997], and Watts [1997]. The model and the notion of pairwise stability that underlies the analysis conducted here is from Jackson and Wolinsky [1996]. Their focus was on developing a model for the study of (static) stability of networks and using this model to understand the relationship between stability and efficiency of networks. Dutta and Mutuswami [1997] looked at this relationship in further detail. As the Jackson and Wolinsky (and Dutta and Mutuswami) analyses are static, they leave open the question of which stable networks will form (if any, as they do not consider cycles). Watts [1997] analyzes the formation of networks in a dynamic framework. Watts [1997] extends the Jackson and Wolinsky [1996] model to a dynamic process, but limits attention to the specific context of the 'connections model' discussed by Jackson and Wolinsky and a particular deterministic dynamic. Thus, the new contributions here are in terms of both the network models that are admitted and the analysis of improving paths, cycles, mutation and the evolutionary process.

There are other papers that provide theoretical models of network formation in strategic contexts. Aumann and Myerson [1988] were the first to take an explicit look at network formation in a strategic context where individuals had discretion over their connections; these connections defined a communication structure that was applied to a cooperative game. Slikker and van den Nouweland [1997] have extended the Aumann and Myerson [1988] model to a one-stage model of link formation and payoff division. However, the analysis in those papers is devoted to issues in cooperative game theory such as the characterization of value allocations (see also, Myerson [1977]). Recent work by Bala and Goyal [1996] is closer in motivation to our paper, as they are also interested in network formation in a dynamic framework. However, their approach differs significantly from ours both in modeling and results. Bala and Goyal [1996] examine directed communication networks (similar to a directed version of the connections model with no deterioration of communication), in a repeated game with a focus on learning, and find that learning leads to efficiency.

Finally, recent work on evolution and learning in game theory studies how individuals play games when social structure is important in determining who interacts with whom (see for example, Ellison [1993], Ellison and Fudenberg [1995], and Young [1998]). This work concentrates on understanding the implication that social or network structure has on play in games, whereas the current paper concentrates on understanding the evolution of the network structure. Ultimately, these two literatures can be brought together, as the approach outlined above can be applied to endogenize the social structure in game theoretic interaction models.

The remainder of the paper is organized as follows. In Section 2 we provide the definitions comprising the basic model. In Section 3 we analyze improving paths, cycles, and conditions under which cycles do not exist. Section 4 contains the evolutionary analysis and dynamic stability results, including a discussion of dynamic stability and efficiency. In Section 5 we apply the evolutionary process to matching models.

2 A Model of Networks

The model of social and economic networks that we consider is based on that of Jackson and Wolinsky [1996], henceforth referred to as JW. The following definitions outline the model and a few examples.

2.1 Players

Let $N = \{1, \ldots, n\}$ be the finite set of *players*. Depending on the application, a player may be a single individual, a firm, a country, or some other autonomous unit.

2.2 Networks and Graphs

The network relations among the players are represented by graphs whose nodes or vertices represent the players and whose links (edges or arcs) capture the pairwise relations. We focus on non-directed networks where links are reciprocal. The complete graph, denoted g^N, is the set of all subsets of N of size 2. The set of all possible *networks or graphs* on N is $\{g | g \subset g^N\}$. Let ij denote the subset of N containing i and j and is referred to as the *link* ij. The interpretation is that if $ij \in g$, then nodes i and j are directly connected, while if $ij \notin g$, then nodes i and j are not directly connected.

Let $g+ij$ denote the network obtained by adding link ij to the existing network g and let $g-ij$ denote the network obtained by deleting link ij from the existing network g (i.e., $g+ij = g \cup \{ij\}$ and $g - ij = \frac{g}{\{ij\}}$).

Let $N(g) = \{i | \exists j \text{ s.t. } ij \in g\}$ be the set of players involved in at least one link and $n(g)$ be the cardinality of $N(g)$.

If $g' = g + ij$ or $g' = g - ij$, then we say that g and g' are *adjacent.*

2.3 Standard Architectures

There are several network configurations that we will refer to frequently.

A network $g \subset g^N$ is a *star* if $g \neq \phi$ and there exists $i \in N$ such that if $jk \in g$, then either $j = i$ or $k = i$. Individual i is the center of the star.

A network $g \subset g^N$ is a *circle* if there exists a sequence of players $i_0, \ldots, i_K$, with a player appearing at most once, such that $g = \{i_0 i_1, i_1 i_2, \ldots, i_K i_0\}$.

A network $g \subset g^N$ is a *line* if there exists a sequence of players $i_0, \ldots, i_K$, with a player appearing at most once, such that $g = \{i_0 i_1, i_1 i_2, \ldots, i_{K-1} i_K\}$.

2.4 Paths and Components

A *path in g connecting i_1 and i_n* is a set of distinct nodes $\{i_1, i_2, \ldots, i_n\} \subset N(g)$ such that $\{i_1 i_2, i_2 i_3, \ldots, i_{n-1} i_n\} \subset g$.

A nonempty graph $g' \subset g$ is a *component* of g, if for all $i \in N(g')$ and $j \in N(g')$, $i \neq j$, there exists a path in g' connecting i and j, and for any $i \in N(g')$ and $j \in N(g)$, $ij \in g$ implies $ij \in g'$.

2.5 Value Functions and Strong Efficiency

The *value of a network* is represented by $v : \{g | g \subset g^N\} \to I\!R$. So, $v(g)$ represents the total utility or production of the graph. The set of all such functions is V. In some applications the value will be an aggregate of individual utilities or productions, $v(g) = \sum_i u_i(g)$, where $u_i : \{g | g \subset g^N\} \to I\!R$.

A graph $g \subset g^N$ is *strongly efficient* if $v(g) \geq v(g')$ for all $g' \subset g^N$.

2.6 Allocation Rules and Pareto Efficiency

An *allocation rule* $Y : \{g | g \subset g^N\} \times V \to I\!R^N$ describes how the value associated with each network is distributed to the individual players. $Y_i(g, v)$ may be thought of as the payoff to player i from graph g under the value function v. For simplicity, if v is fixed, we will simply write $Y_i(g)$.

The allocation rule may represent several things. When considering a purely social network, the allocation rule may represent the utility that each individual receives from the network and this utility might not be transferable. When considering an exchange or production network, the allocation rule may represent either the trades or production accruing to each individual, the outcome of a bargaining process, or some exogenous redistribution.

2.7 Pairwise Stability

The following concept describes networks for which no player would benefit by severing an existing link, and no two players would benefit by forming a new link.

A network g is *pairwise stable* with respect to v and Y if

1. for all $ij \in g$, $Y_i(g, v) \geq Y_i(g - ij; v)$ and $Y_j(g, v) \geq Y_j(g - ij; v)$, and
2. for all $ij \notin g$, if $Y_i(g, v) < Y_i(g + ij; v)$ then $Y_j(g, v) > Y_j(g + ij; v)$.

When a network g is not pairwise stable it is said to be *defeated* by g' if either $g' = g + ij$ and (2) is violated for ij, or if $g' = g - ij$ and (1) is violated for ij.

There are variations on the notion of pairwise stability, discussed in JW and also in the appendix here. Dutta and Mutuswami [1997] discuss alternative approaches that capture coalitional deviations.

2.8 Examples

There are two examples of settings from JW that we will refer to frequently in illustrating definitions and results.

2.9 Connections Model

The symmetric connections model from JW is described as follows. Players form links with each other in order to exchange information. If player i is connected to player j, by a path of t links, then player i receives a payoff of δt from his indirect connection with player j. We assume $0 < \delta < 1$, and so the payoff δt decreases as the path connecting players i and j increases; thus information that travels a long distance becomes diluted and is less valuable than information obtained from a closer neighbor. Each direct link ij results in a cost c to both i and j. This cost can be interpreted as the time a player must spend with another player in order to maintain a direct link.

Formally, the payoff player i receives from network g is equal to $u_i(g) = \sum_{j_i} \delta^{t(ij)} - \sum_{j:ij \in g} c$ where $t(ij)$ is the number of links in the shortest path between i and j (setting $t(ij) = \infty$ if there is no path between i and j). Here the value of network g is $v(g) = \sum_{i \in N} u_i(g)$, and $Y_i(v, g) = u_i(g)$. The incentives in forming links come from the consideration of direct costs and benefit, as well as the benefits of indirect connections.

2.10 Co-Author Model

The co-author model of JW is described as follows. Each player is a researcher who spends time writing papers. If two players are connected, then they are working on a paper together. The amount of time researcher i spends on a given project is inversely related to the number of projects, n_i, that he is involved in. Formally, player i's payoff is represented as

$$U_i(g) = \sum_{j:ij \in g} \left[\frac{1}{n_i} + \frac{1}{n_j} + \frac{1}{n_i n_j} \right]$$

for $n_i > 0$. For $n_i = 0$, set $u_i(g) = 0$. Again, $v(g) = \sum_{i \in N} u_i(g)$, and $Y_i(v, g) = u_i(g)$. Here, the interesting tradeoffs from connection come from the benefit of gains from a co-author's time $(1/n_j)$, at the expense of diluting the synergy (interaction) term $1/(n_i n_k)$ with other co-authors.

3 Improving Paths and Cycles

Our focus in this paper is on the dynamic formation of networks. Before proceeding to study an evolutionary process, let us first focus on the paths that might be followed as a network evolves. To avoid confusion, we emphasize that below the idea of a path represents changes from one network to another, rather than a path along links within a given network.

3.1 Improving Paths

An improving path is a sequence of networks that can emerge when individuals form or sever links based on the improvement the resulting network offers relative to the current network. Each network in the sequence differs by one link from the previous one. If a link is added, then the two players involved must both agree to its addition, with at least one of the two strictly benefiting from the addition of the link. If a link is deleted, then it must be that at least one of the two players involved in the link strictly benefits from its deletion.

Formally, an *improving path* from a network g to a network g' is a finite sequence of graphs $g_1, \ldots, g_K$ with $g_1 = g$ and $g_K = g'$ such that for any $k \in \{1, \ldots, K-1\}$ either:

1. $g_{k+1} = g_k - ij$ for some ij such that $Y_i(g_k - ij) > Y_i(g_k)$, or
2. $g_{k+1} = g_k + ij$ for some ij such that $Y_i(g_k + ij) > Y_i(g_k)$ and $Y_j(g_k + ij) \geq Y_j(g_k)$.

Thus an improving path is a sequence of networks that might be observed in a dynamic process where players are myopically adding and deleting links. Let us say a few words about this myopic behavior, as it will also play a role in the evolutionary analysis that follows. It is possible that under myopic behavior a player deletes a link making him or herself better off, but then this leads another player to delete another link which in turn leaves the first player worse off relative to the starting position. If the first player foresaw this, he or she might choose not to sever the link to begin with. This sort of consideration is not taken into account in our analysis, and may be important when there are relatively small numbers of forward-looking players who are well-informed about the value of the network and the motivations of others. However, in larger networks and networks where players' information might be local and limited, or in networks where players significantly discount the future, myopic behavior is a more natural assumption, and a reasonable starting point for our analysis.

In addition to the assumption of myopic behavior, there are other assumptions, in the definition of improving path, which can be varied. For example, the definition can be adapted to allow for the simultaneous addition or deletion of several links at a time. We consider this possibility of simultaneous actions when we discuss matching problems in Section 5. We also discuss other definitions of improving path in the appendix; these other definitions provide specific restrictions on the order in which actions can be taken.

The improving paths emanating from any starting network lead either to a pairwise stable network or to a cycle (where a number of networks are repeatedly visited in some sequence). In fact, one can define pairwise stability by simply saying that a network is pairwise stable if there are no improving paths emanating from it.

3.2 Symmetric Connections Example

Consider the symmetric connections model with 4 players. The set of improving paths depend on the relative size of c and δ. If $\delta^2 < \delta - c$, then links are very cheap and players have an incentive to add every link and never to delete a link. Here, from any network that is not the fully connected network there exists an improving path leading to any larger network (i.e., network whose links are a superset of the given network). If $0 < \delta - c < \delta^2$, then players are willing to add links to a player with whom they are not already connected (directly or indirectly), but are not willing to add (and are willing to delete) a link with someone who is also indirectly connected to them by an indirect path of length 2. (Whether or not the same holds for indirect connections of length 3 depends on the comparison of c to $\delta - \delta^3$.) In this case, there are many improving paths leaving the empty network, some of which lead to the efficient network (a star; see JW Proposition 1), but others which lead to lines, and in some cases circles. If $c > \delta$, then there are no improving paths emanating from the empty network. Provided c is not too large, there are other pairwise stable networks, but none of which have any "loose ends" (players with just one link connecting them to the network). For some intermediate networks, for instance a line connecting all 4 players, there exist improving paths which lead back to the empty network, but also improving paths leading to a circle. (For more on pairwise stability, efficiency, and dynamic link formation in the connections model, see JW and Watts [1997].)

The multiplicity of improving paths emanating from some networks, and the variety of pairwise stable networks in this example sets the stage for the later evolutionary analysis. For instance, when $c >?$, it may be that although there are no improving paths emanating from the empty network, a mutation or two which introduce links could lead to a network from which there are improving paths leading to more efficient networks.

3.3 Stable States

A network g is a *stable state* if it is pairwise stable and there exists an improving path connecting the empty network to g.

The notion of a stable state is from Watts [1997] (although with a different process), and here is tied directly to the idea of an improving path. This notion identifies the networks that may be reached by a process where players act to improve their situation starting from the empty network.

3.4 Co-Author Example

In the co-author model, all pairwise stable networks are stable states. Here, a straightforward set of calculations show that there is at least one improving path leading from the empty network to any specific pairwise stable network.

3.5 Cycles

It is possible for a dynamic process, and improving paths in particular, to cycle among a set of networks. Let us examine this possibility in some detail.

A set of networks C form a *cycle* if for any $g \in C$ and $g' \in C$ there exists an improving path connecting g to g'.

A cycle C is a *maximal cycle* if it is not a proper subset of a cycle.

A cycle C is a *closed cycle* if no graph in C lies on an improving path leading to a graph which is not in C.

Note that a closed cycle is necessarily a maximal cycle.

3.6 Asymmetric Connections Example (Existence of a Cycle)

Consider an asymmetric variation of the connections model with 5 players, where δ is player specific, denoted δ_i. Assume that $\delta_1 < c < \delta_1 + \delta_1^2 - \delta_1^3 - \delta_1^4$; so player 1 is willing to add a link to make a circle if player 1 is at the end of a line involving all players. But player 1 does not want to be directly linked to a player who is not linked to anyone else. Assume that the reverse is true for player 3, so $\delta_3 > c > \delta_3 + \delta_3^2 - \delta_3^3 - \delta_3^4$, (for example, let $\delta_1 < (\sqrt{5} - 1)/2 < \delta_3$ and set $c = (\delta_1 + \delta_3)/2$). Here, player 3 prefers to delete a link if he is in a circle with everyone else. Assume that $\delta_i > c > \delta_i - \delta_i^2$ for all other players. These players are willing to link with any player they are not directly or indirectly connected to, but these players do not wish to shorten an indirect connection of distance 2 (but may or may not wish to shorten longer paths).

In such a setting a cycle exists. Start with the circle $\{12, 23, 34, 45, 15\}$. Player 3 wants to delete the link 23, and no one else is interested in deleting a link. So, we move to $\{12, 34, 45, 15\}$. Player 1 now wants to delete the link 12, and no one else is interested in deleting a link. So, we move to $\{34, 45, 15\}$. Players 2 and 3 now want to add the link 23, and no one is interested in deleting a link. So, we move to $\{23, 34, 45, 15\}$. Now, players 1 and 2 want to add the link 12. So, we move back to circle $\{12, 23, 34, 45, 15\}$.

Note that this cycle is reachable from an improving path from the empty network. For instance first add link 15, then 45, then 34, then 23, then 12. However, this cycle is not closed, since player 3 could start by severing 34 instead.

Note also that there is an asymmetry in payoffs in this example that allows for the cycle. If a network is completely symmetric and payoffs are symmetric (such as the circle in the symmetric connections model, where each link is similar to every other link in value) then there cannot be a cycle containing the network that consists of entirely subgraphs or entirely supergraphs, since no one should want to delete a link that they just added.

The concept of improving path provides for an easy proof of the following existence result.

Theorem 1 *For any v and Y there exists at least one pairwise stable network or closed cycle of networks.*

Proof: Notice that a network is pairwise stable if and only if it does not lie on an improving path to any other network. So, start at any network. Either it is pairwise stable or it lies on an improving path to another network. In the first case the result is established so consider the second case. Follow the improving path. Given the finite number of possible networks, either the improving path ends at some network which has no improving paths leaving it, which then must be pairwise stable, or it can be continued through each network it hits. In the second case, the improving path must form a cycle. Thus, we have established that there always exists either a pairwise stable network or a cycle. So consider the case where there are no pairwise stable networks. We show that there must be a closed cycle. Since there must exist a cycle, given the finite number of networks there must exist a maximal cycle. Consider the collection of all maximal cycles. By the definition of maximal cycle, there must be at least one such cycle for which there is no improving path leaving the cycle. (There can be improving paths leaving some of the maximal cycles, but these must lead to another maximal cycle. If all maximal cycles had improving paths leaving them, then there would be a larger cycle, contradicting maximality.) Thus, there exists a closed cycle. ∎

It is necessary in Theorem 1 to allow for the existence of either pairwise stable networks or closed cycles. There are cases where only pairwise stable networks exist and no cycles exist (for instance the connections model with very low costs so that all networks are on an improving path to the complete network). Also, there are cases where only closed cycles exist and there are no pairwise stable networks. The following example illustrates this point.

3.7 Trading Example (Non-existence of a Pairwise Stable Network)

Consider a situation where players benefit from trading with other players with whom they are linked, and trade can only flow along links. In this example, players begin by forming a network. Subsequently, they receive random endowments and the players trade along paths of the network. Trade flows without friction along any path and each connected component trades to a Walrasian equilibrium.

There are two goods. All players have identical utility functions for the two goods which are symmetric Cobb-Douglas of the form $U(x, y) = xy$. Each player has a random endowment, which is independently and identically distributed. A player's endowment is either (1,0) or (0,1), each with probability 1/2. Links are formed before players' endowments are realized. For a given network, Walrasian equilibria occur on each connected component, regardless of the configuration of links. For instance, three players a line have the same trades as three players in a circle (triangle), but with a lower total cost of links. Let the cost of a link be equal to 5/96 (for each player).

Let us show that if n is a least 4, then there does not exist a pairwise stable network.

The utility of being alone is 0. Not accounting for the cost of links, the expected utility for a player of being connected to one other is 1/8. (There is a 1/2 probability that the realized endowments will differ, in which case the players will trade to an allocation of (1/2, 1/2) which results in a utility of 1/4 for each of the two players. There is also a 1/2 probability that the realized endowments will be identical in which case the utility will be 0 for each player.) Similar calculations show that, not accounting for the cost of links, the expected utility for a player of being connected (directly or indirectly) to two other players is 1/6; and of being connected to three other players is 3/16. Most importantly, the expected utility of a player is strictly concave in the number of other players that he is directly or indirectly connected to. Thus the marginal gain of being connected to an additional player is decreasing in the number of players that one is already connected to.

Accounting for the cost of a link, it becomes clear that if k players are in a component, then there must be exactly $k-1$ links. If there are more than $k-1$ links, then there is at least one link that could be severed without changing the component structure of the network. Thus, some player can sever a link thereby saving the cost of the link but not losing any expected utility from trading.

Note that if g is pairwise stable, then any component with 3 or more players cannot contain a player who has just one link. This result follows from the fact that a player connected to another player, who is not connected to anyone else, loses at most $1/6 - 1/8 = 1/24$ in expected utility by severing the link, but saves the cost of 5/96 and so should sever this link.

From these two observations it follows that if there were to exist a pairwise stable network, then it would have to consist of pairs of connected players (as two completely unconnected players benefit from forming a link), and one unconnected player if n is odd. If n is at least 4, then there must exist at least two pairs. However, such a network is not pairwise stable, since any two players in opposite pairs gain from forming a link. Thus, there is no pairwise stable network. From Theorem 1, we know that there exists a closed cycle.

3.8 Ruling out Cycles:

Let us explore conditions on Y and v which rule out the existence of cycles. If there are no cycles, then a dynamic process that follows improving paths will necessarily come to rest at a pairwise stable network.

Fix Y and v. If there exists an improving path from g to g', then let us use the symbol $g \to g'$. Given the transitivity of $\to$, there are no cycles if and only if $\to$ is asymmetric. Although this provides a direct characterization of the existence of cycles, Theorem 2 provides what turns out to be a more useful characterization.

The following definition is used in Theorem 2.

Y and v *exhibit no indifference* if for any g and g' that are adjacent either g defeats g' or g' defeats g.

Theorem 2 *Fix v and Y. If there exists a function, $w : \{g | g \subset g^N\} \rightarrow I\!R$, such that* [$g'$ defeats g] $\Leftrightarrow$ [$w(g') > w(g)$ and g' and g are adjacent]*, then there are no cycles. Conversely, if Y and v exhibit no indifference, then there are no cycles only if there exists a function, $w : \{g | g \subset g^N\} \rightarrow I\!R$, such that* [$g'$ defeats g] $\Leftrightarrow$ [$w(g') > w(g)$ and g' and g are adjacent].

Note that w is independent of the players involved in the link that is being added or deleted. Thus, in a rough sense, w is similar to a potential function. Theorem 2 shows that the existence of a cycle is tied to the existence of a single function (that is player independent) that represents the incentives of players with regards to any adjacent changes. Although this is a demanding condition, it is precisely what is needed to rule out cycles.

The proof of Theorem 2 appears in the appendix. The proof of the first part of the theorem, that the existence of such a w precludes the existence of cycles, is direct. The proof of the second part of the theorem, that the existence of such a w is necessary for the absence of cycles, is more involved. It starts from a result in decision theory, that a binary relation on a finite set (like $\rightarrow$) has a representation by such a w if and only if it is negatively transitive and asymmetric. Here, the binary relation of improving paths ($\rightarrow$) is transitive and asymmetric when there are no cycles, but may fail to be negatively transitive. We then show that the relation of improving paths may be extended to a more complete relation, that is negatively transitive, asymmetric and still agrees with ??on adjacent networks.

The supposition in the last part of Theorem 2, that Y and v exhibit no indifference is critical to the existence of such a w. To see this, consider the following example with $n = 3$. Suppose that $\{12, 23, 13\}$ defeats $\{12, 23\}$ defeats $\{12\}$ defeats $\{12, 13\}$, but that players 2 and 3 are both indifferent between $\{12, 23, 13\}$ and $\{12, 13\}$. Suppose also, that no other network defeats any other. Here there are no cycles, and yet the existence of such a w would require that $w(\{12, 23, 13\}) > w(\{12, 13\})$, while $\{12, 23, 13\}$ does not defeat $\{12, 13\}$.

3.9 Exact Pairwise Monotonicity

The existence of a function satisfying the role of w is sometimes difficult to check, but in some situations there is a natural candidate for w, which is simply v. This is captured in the following condition, which is a slight modification of the pairwise monotonicity condition of JW.

Y is exactly pairwise monotonic relative to v if g' defeats g if and only if $v(g') > v(g)$ (and g' is adjacent to g).

Exact pairwise monotonicity provides a nice alignment of individual incentives and overall value. It implies that strongly efficient networks are pairwise stable (but not necessarily that all pairwise stable networks are efficient). By Theorem 2, exact pairwise monotonicity rules out cycles.

Corollary 3 *If Y is exactly pairwise monotonic relative to v, then there are no cycles.*

While ruling out cycles implies that a dynamic process which follows improving paths will come to rest at a pairwise stable network, this condition does not imply that any pairwise stable network, and in particular that any strongly efficient network, is reachable if we start at the empty network. However, the addition of the following condition does lead to this conclusion.

3.10 Single Peakedness

A value function v is *single peaked* if $v(g) > v(g')$ implies $v(g') > v(g'')$, and if $v(g'') > v(g')$ implies $v(g') > v(g)$, for any $g \subset g' \subset g''$

The idea of a single peaked value function is as follows: consider growing a network by adding links one by one. Suppose that adding links adds value initially. The value function is 'single peaked' if once you add a link that lessens value, then continuing to add links will continue to lessen value.

A path is an *increasing path* if it only involves adding links.

Proposition 4 *Suppose that $v(g+ij) \neq v(g)$ for any g and $ij \notin g$. If Y is exactly pairwise monotonic relative to a v that is single peaked, then for every pairwise stable network there exists an increasing improving path leading from the empty network to that pairwise stable network. Therefore, in such a case the set of stable states coincides with the set of pairwise stable networks, and every strongly efficient network is a stable state.*

Proof: Consider any pairwise stable network g and some $g - ij$. By exact pairwise monotonicity it follows that $v(g-ij) < v(g)$. Thus by single peakedness, $v(g-ij-kl) < v(g-ij)$ and so on for any order of removal of links. By exact pairwise monotonicity any order of addition of links to get to $v(g)$ defines a increasing improving path (note that no two players want to delete a link at any point since the value of the resulting network must be lower and the severing player's payoff could be no higher by exact pairwise monotonicity). ∎

Proposition 4 rules out cases where adjacent networks have identical values. That case requires a significant complication of the pairwise monotonicity condition, with little gain in insight.

3.11 Examples

For some values of δ and c the connections model satisfies exact pairwise monotonicity and single peakedness (for instance if c is very small or very large). For some intermediate values of c when δ is close to 1, for instance where $\delta > c > n(\delta - \delta^{n-1})$, the connections model satisfies exact pairwise monotonicity but not single peakedness. Here?a player only wants to add a link if it is to a player who is not already in his or her component, and a player wants to sever any links whose deletion would not change the component; both of which are value increasing operations. Yet for other choices of c and δ, for example when $c > \delta$ and c is low enough for a star to be efficient, the connections model fails to satisfy both exact pairwise monotonicity and single peakedness.

The co-author model does not satisfy the exact pairwise monotonicity condition since the strongly efficient network, where all players are arranged in pairs, is not pairwise stable as players in different pairs want to form links even though these extra links decrease the value of the network. It fails to satisfy single peakedness, since adding a new connection to an existing component can sometimes lower value while adding a link between two completely unconnected agents always increases value, regardless of the other components' configurations.

4 A Stochastic Process and the Evolution of Networks

Based on the notion of improving path, and the implicit associated dynamic, networks evolve as players myopically form or sever links based on the improvement the resulting network offers relative to the current network. In this section, we explicitly describe a dynamic process which allows for additional stochastic changes to a network. Occasionally two players will add a link that they normally would not add, or a single player will sever a link that he normally would not sever. This random element in the process will allow the dynamic formation process to deviate from an improving path. As a network evolves, the formation process will occasionally jump from one improving path to another. We will examine which networks the stochastic process will spend a positive amount of time in as the probability of mutations goes to zero.

These stochastic mutations in the formation process have several different interpretations or justifications. They may be thought of as errors made by the players. They might also represent a lack of knowledge on the part of the players and be a form of experimentation. Such mutations might also be due to exogenous factors that are beyond the players' control. More fundamentally, what follows may be thought of as a check on the robustness, or stochastic stability (in the language of Foster and Young [1991]), of networks. Although a number of networks may be pairwise stable, it can turn out that they differ in how they respond to random perturbations. For instance it may be relatively easy to leave and hard to get back to some networks, and vice versa for others.

Fix Y and v. The discussion that follows will be with respect to such a fixed Y and v, although we omit notation indicating this dependence.

4.1 A Stochastic Dynamic Process

At each time a pair of players ij is randomly identified with probability $p(ij) > 0$. The (potential) link between these two players is the only link that can be altered at that date. (One may think of a random matching process where players randomly bump into each other, and time is identified with the bumping times.) If the link is already in the network, then the decision is whether to sever it, and otherwise, the decision is whether to add the link. The players involved act myopically, adding the link if it makes each at least as well off and one strictly better off, and severing the link if its deletion makes either player better off. After the action is taken, there is some small probability ε that a tremble (or mutation) occurs and the link is deleted if it is present, and added if it is absent. (In the appendix, we discuss alternative definitions of this process.)

The above process naturally defines a Markov chain with different states corresponding to the network obtained at the end of a given period. The Markov chain is irreducible, given non-zero trembles, as it is possible for the process to eventually transition from any state to any other. The Markov chain is also aperiodic, given non-zero trembles, since it is possible for the network to return to itself at the end of any period. A finite state irreducible, aperiodic Markov chain has a unique corresponding stationary distribution. Since the transition matrix is continuous in ε, the stationary distribution is also continuous in ε. Thus, as ε goes to zero, the stationary distribution will converge to a unique limiting stationary distribution. As ε goes to zero, we can examine the limiting stationary distribution to see what percentage of the time in the long run the network is of any given form.

Note that we assume that the probability of a tremble, ε, is independent of both the network, g, that the process is currently starting from, and the link, ij, that has been randomly identified. However, since adding a link requires both players' cooperation, one could argue that the probability of a tremble that adds a link should be smaller than the probability of a tremble that severs a link. Also, one might argue that players' incentives to experiment are decreasing in their current payoff, and so the probability of a tremble should decrease as the relative payoff to players i and j increases. One could allow the probability of a tremble to equal $k_{g,ij}\varepsilon > 0$, for some constant $k_{g,ij}$ which depends on both g and ij. However, from the analysis of Young [1993] (and Bergin and Lipman [1996]) we know that the set of stochastically stable states, of such a Markov chain, is independent of $k_{g,ij}$. Thus allowing the probability of a tremble to equal $k_{g,ij}\varepsilon$ will not affect our the set of stochastically stable states; so for simplicity we let the probability of a tremble equal ε.

Nevertheless, Bergin and Lipman [1996] show that if mutation rates for different states are allowed to go to zero at different rates, then the set of stochastically stable states of a Markov chain may change; and thus the evolutionary process does matter. This is an important observation, and suggests that one should pay attention to appropriate orders of magnitude of mutation rates for various networks. For instance, if, instead of considering a rescaling of mutation rates, as above, one lets an exponent on the mutation rate be dependent on the current network and link in question, then the results can be affected. With this noted, in what follows, we concentrate on the case where mutation rates are of the same order.

The following definitions will be important in the discussion of the stochastic evolutionary process.

4.2 Evolutionary Stability

A network that is in the support of the limiting stationary distribution of the above-described Markov process (as ε goes to 0) is *evolutionarily stable.*

For a given network, g, let $im(g) = \{g' \text{ such that } g' \to g\}$. This is the set of networks that live on the improving paths leading back to g.

4.3 Resistance of a Path

The *resistance of a path* $p = \{g_1, \ldots, g_K\}$ from g' to g, denoted $r(p)$, is computed by $r(p) = \sum_1^{K-1} I(g_i, g_{i+1})$, where $I(g_i, g_{i+1}) = 0$ if $g_I \in \text{im}(g_{i+1})$ and $I(g_i, g_{i+1}) = 1$ otherwise.

Resistance keeps track of how many trembles must occur along a path, with the idea that a tremble is necessary to move from one network to an adjacent one whenever it is not in the players' interests to sever or add the link in question.

4.4 g-Trees

Given a network g, a g-tree is a directed graph which has as vertices all networks and has a unique directed path leading from each g' to g. Let $T(g)$ denote all the g-trees, and represent a t in $T(g)$ as a collection of ordered pairs of networks, so that $g'g'' \in t$ if and only if there is a directed edge connecting g' to g'' in the g-tree t.

4.5 Example of a g-Tree

Assume there are four possible networks: $g1, g2, g3, g4$. Then there are 13 possible $g4$-trees. To see this consider Figure 1. There are six possible $g4$-trees with the same shape as tree A. (To find the remaining five trees, let graphs $g1$, $g2$, and $g3$ switch places.)

There are six possible $g4$-trees with the same shape as tree B, and one possible tree of shape C.

Let $r(g', g)$ denote the minimum $r(p)$ over all paths p from $g'tog$, and set $r(g, g) = 0$. Note that $r(g', g) = 0$ if and only if g' is in $im(g)$ or $g' = g$. Thus, if g' and g are in the same cycle, then $r(g', g) = 0$.

4.6 Resistance of a Network

The *resistance* of a network g is computed as

$$r(g) = \min_{t \in T(g)} \sum_{g'g'' \in t} r(g', g'').$$

Theorem 5 *The set of evolutionarily stable networks is the set* $\{g | r(g) \leq r(g') \forall g'\}$.

The proof of Theorem 5 is in the appendix, and is based on results of Freidlin and Wentzell [1984] and Young [1993] concerning limiting distributions of aperiodic, irreducible Markov processes.

Remark: The set of evolutionarily stable networks is always nonempty as we are taking a minimum over a finite set.

Remark: Theorem 5, and also Theorem 9 which follows, hold for any definition of improving path (such as the definition of ordered improving path found in the appendix). As illustrated in Section 5, these results are easily adapted to variations on the definition of improving path.

Before illustrating the implications of Theorem 5 in examples, we provide two auxiliary results. The first notes that the only networks that are evolutionarily stable are either pairwise stable or part of a closed cycle. The second provides a simplified method of calculating resistance.

Lemma 6 *If $g' \in \text{im}(g)$ and $g \notin \text{im}(g')$, then $r(g) \leq r(g')$, with strict inequality if g is pairwise stable or in a closed cycle. Thus, if g is evolutionarily stable, then either g is pairwise stable or part of a closed cycle. Furthermore, if one network in a closed cycle is evolutionarily stable then all networks in the closed cycle are evolutionarily stable.*

Proof: Let us start by showing that if $g' \in im(g)$ and $g \notin im(g')$, then $r(g) \leq r(g')$. Consider a g'-tree relative to which $r(g')$ is obtained. Construct a g-tree by starting with the g'-tree and directing an edge from g' to g, and erasing the edge that led away from g. Since $g' \in im(g)$, it follows that the added edge has 0 resistance, so this g-tree has a resistance of no more than $r(g')$. Thus, $r(g) \leq r(g')$.

Moving out of the order of the statement of Lemma 6, let us next verify that if one network, g, in a closed cycle is evolutionarily stable then any other network, g', in the same closed cycle is also evolutionarily stable. To see this simply start with a g-tree relative to which $r(g)$ is obtained, and switch the places of g' and g. Since g and g' are both connected to each other by improving paths, $r(g'', g) = r(g'', g')$ and $r(g, g'') = r(g', g'')$ for any g''. Thus, the resistance will be unchanged, and so g' must also be evolutionarily stable.

Next, let us show that if $g' \in im(g)$ and $g \notin im(g')$ and g is pairwise stable, then $r(g) < r(g')$. Again, construct a g-tree by starting with a g'-tree relative to which $r(g')$ is obtained, directing an edge from g' to g, and erasing the edge that led away from g. Note that if g is pairwise stable, then erasing the edge that led away from g saved at least 1 unit of resistance. Since the added edge has 0 resistance, it follows that $r(g) < r(g')$. This argument is extended to the case where g is in a closed cycle, as follows. If the edge that led away from g in the original g'-tree had a positive resistance, then the same argument as above works. If not, then it must be that the edge leading away from g in the original g'-tree pointed to some network in the closed cycle containing g. In that case, there must have been some g'' in the closed cycle containing g that had an edge exiting the closed cycle. Construct the g'' tree from the g' tree as described above. By the definition of closed cycle it must be that $g \in im(g'')$ and $g'' \notin im(g')$. Thus the above argument holds, establishing that g'' is evolutionarily stable. So, it follows that g must also be evolutionarily stable as it is part of the same closed cycle. ∎

Remark: From the results in section 3, it follows that if Y is exactly pairwise monotonic, then the set of evolutionarily stable networks contains only pairwise stable networks (and no cycles).

Noting that only pairwise stable networks and closed cycles matter in the dynamic process, and that resistance along improving paths is 0, we can simplify the calculations of resistance as follows. Given a closed cycle C and a network g, let $r(C, g) = r(g', g)$ where g' is any network in C, and similarly $r(g, C) = r(g, g')$ where $g' \in C$. These are well defined since $r(g', g) = r(g'', g)$ (and similarly $r(g, g') = r(g, g'')$) for any g' and g'' in C since there is a path of zero resistance between g'' and g'.

Given a network g, a *restricted g-tree* is a directed graph which has as its root g, and as other vertices g the pairwise stable graphs and closed cycles, and has a unique directed path leading from each other vertex to g. Denote the set of restricted g-trees by $RT(g)$. In the following lemma, we let x denote a generic vertex which could be a graph g' or a cycle c. The following lemma follows from the proof of Theorem 5 (from a result by Young (1993)).

Lemma 7

$$r(g) = \min_{\tau \in RT(g)} \sum_{x'x'' \in t} r(x', x'').$$

The implication of Lemma 7 is that the resistance of a network may be calculated by restricting attention to a simpler problem. The intuition behind the result is straightforward: paths that can be made via improving paths add nothing to resistance. Thus, the resistance comes in only from transitions for which there are no improving paths. Since any network that is not pairwise stable or in a cycle lies on an improving path to a pairwise stable network or cycle, their additional consideration does not add to the resistance, while a transition from some pairwise stable network or closed cycle to another network will involve some resistance.

Now let us illustrate the results above through the following examples.

4.7 Symmetric Connections Example

In this example, we use the above results to find the set of evolutionary stable networks. Consider the connections model with 4 identical players. For each player assume that $\delta < 1, 0 < \delta - c < \delta^2$ and $\delta - c > \delta^3$. The two pairwise stable network structures are a circle and a star, and there are no cycles. There are three different circles possible that may be catalogued by the player who is across from player 1, and there are four different stars possible that may be catalogued by the player at the center. Given the symmetry of this setting, the resistance of one star is the same as the resistance of any other star, and similarly for the circles. So let us calculate the resistance of the circle $\{12, 14, 23, 34\}$ and the star $\{12, 13, 14\}$.

First let us calculate the resistance of the star $g = \{12, 13, 14\}$. Note that $\{12, 13, 14, 23, 34\}$ is in $im(g)$ (2 severs 23 and 4 severs 34). Thus, the distance from the circle $\{12, 14, 23, 34\}$ to $im(g)$ is 1, and the same is true for each of the other circles. The distance from another star, say $\{12, 23, 24\}$ directly to g is 2 since $\{12\}$ is in $im(g)$ and no network within distance 1 of $\{12, 23, 24\}$ is in $im(g)$. However, such a star has a resistance of just 1 to some circle ($\{12, 23\}$ is on an improving path to $\{12, 14, 23, 34\}$) and so on a restricted g-tree, the other stars can be directed to a circle. Thus, the total resistance of g is 6. (Note that each pairwise stable network must contribute at least 1 unit to the resistance, and so this must be the minimum.)

Next let us calculate the resistance of the circle $g' = \{12, 14, 23, 34\}$. The distance from the star g to $im(g')$ is 1 since $\{12, 14\}$ is in $im(g')$. Similarly for each other star. The distance from another circle, say $\{13, 14, 23, 24\}$ to g' is 1 since $\{12, 13, 14, 23, 24\}$ lies in $im(g')$ (3 severs 13, 2 severs 24, 3 and 4 add 34). Thus, the resistance of the circle is also 6. Following Lemmas 6 and 7, the evolutionarily stable networks are the stars and circles.

4.8 Insurance Example

Consider an example where financial help or mutual insurance is exchanged in a developing country. (For an empirical study see Lund and Fafchamps [1997]).

Each player can be thought of as a household that receives its income from crops. Every period there is a probability that someone in the household will need serious medical attention, which will cause the household to fall below a subsistence level, which is normalized to a level of 0. In particular, assume that in each period a player (household) has a probability of p of not having any medical needs and thus having an income of a\$, and a probability of $1-p$ of having a medical need and thus having an income of 1\$, where $0 < p < 1$. Each player I has a utility function for income U_I, that is increasing, concave and satisfies $U_i(0) = 0$. For simplicity, we assume that income is consumed each period, and abstract away from additional smoothing that may come from savings.

The players (households) can form networks through which they provide each other with mutual insurance. Each player decides to form or sever links based on his expected utility. For the purposes of this example, mutual insurance is assumed to pass through the network in the following manner. If player j has a\$ this period and he is directly connected to k players who each have medical needs, then j brings all k players up to the subsistence level, as long as doing so does not bring j below subsistence level; thus player j gives each of the k players $\min\{a/k, 1\}$. In situations where several players are connected to the same player who has medical needs, then they each pay an even share of the amount needed to bring the player up to subsistence. For instance, if players i and j each have a\$ and are both directly connected to player l, who has $\hat{u}1$\$, then players i and j each give player l $\min\{a/2, 1/2\}$.) If one of the players is constrained, then the other continues until he hits his constraint or player one reaches subsistence. After the k players directly connected to player j are brought to the subsistence level, player j helps players who are directly connected to one of these k players, who are not already at their subsistence level, as long as doing so does not bring j below subsistence level. Next player j helps anyone who is two links away, etc.; and this process continues until player j either runs out of money or has helped everyone who needs his help.

Consider the specific example where there are 5 players, a=4 and $|U_i(-1)|$ is large enough, compared to $U_i(4)$, so that any two players, who are not directly or indirectly connected, are always willing to form a link. Thus any PS network will consist of one component. Since $a = (n-1)$, every player in a single-component network will have income greater than or equal to subsistence level as long as at least one player in the network receives a\$. Thus there is no benefit (but rather only a liability) to having a direct connection instead of an indirect connection.

The three PS network structures are the star, line, and half-star (example, $\{12, 23, 35, 34\}$ is a half-star with player 3 in the center). There are 5 possible stars, 60 possible lines and 60 possible half-stars.

Next we find the set of evolutionary stable networks. There are no cycles. Let us consider the resistance of the line $g = \{23, 31, 14, 45\}$. The half-star $\{23, 31, 14, 15\}$ has a distance of 1 from $\text{im}(g)$ as link 15 can be severed to get a subgraph of g. The star $\{13, 12, 15, 14\}$ has a distance of 1 from the half-star $\{23, 31, 14, 15\}$. The remaining 11 half-stars, with player 1 in the center, each have a distance of 1 from the star $\{12, 13, 14, 15\}$. The 12 half-stars of the form $\{i1, 1j, jk, jn\}$ each have a distance of 1 from a half-star with 1 in the center. The remaining 36 half-stars each have a distance of 1 from a half-star of the form $\{i1, 1j, jk, jn\}$. Each of the remaining 59 lines is a distance of 1 from a half-star. The remaining 4 stars are each a distance of 1 from a half-star. Thus the total resistance of g is 124. Similarly it can be shown that every star, half-star and line has a resistance equal to 124. Thus, by Theorem 2, all PS networks are evolutionary stable.

In the examples above, all of the pairwise stable networks are evolutionary stable. Next we give an example where only a subset of the pairwise stable networks are evolutionarily stable.

4.9 Co-Author Example (Selection from Pairwise Stable Networks)

Consider the co-author model from JW with $n = 7$. In this case (see proposition 3 in JW), the pairwise stable networks are the complete network and the networks where five players are completely interconnected and the two remaining players are connected only to each other. There are no cycles.

Consider a restricted g-tree for the complete network. Each of the other pairwise stable networks has distance 1 from an improving path to the complete network. By severing the link between the two paired players, one obtains a network on an improving path to the complete network (either player will link with a member of the group of 5 if they have that opportunity, and then would link with each of the others, and so forth). Thus, the complete network has a resistance of 7!/(5!2!)=21, which is the minimum possible given the number of pairwise stable networks (22).

However, consider a restricted g-tree for one of the other pairwise stable networks. The complete network lies more than a distance of 1 away from an improving path leading to some other pairwise stable network, since severing only one link leads to a network only on an improving path back to the complete network. Thus, for any of these pairwise stable networks, the resistance will be greater than 21.

Thus, by Theorem 5 and Lemma 7, the unique evolutionarily stable network is the complete network.

The following observation was used in the above example. Let PSC denote the pairwise stable networks and the closed cycles, with a closed cycle treated as a single object. Denote a generic object in PSC by x.

Observation: For x in PSC, if $\min_{\{x' \in PSC, x' \neq x\}} r(x, x') \geq \max_{\{x'' \in PSC, x'' \neq x\}} r(x'', x)$, then x is evolutionarily stable, and it is uniquely so if the inequality is strict.

Intuitively, it is relatively difficult to get away from x and easy to get back to x. In the case where PSC has only two elements, PSC=$\{x, x'\}$, this implies that x is evolutionarily stable if and only if $r(x, x') \geq r(x', x)$. This observation is applied in the example below.

In the next example, the uniquely evolutionarily stable network is not a stable state. This emphasizes the importance of mutations in the evolutionary process. In this example, the empty network is pairwise stable, and thus the only stable state as any two agents are worse off by forming a link. Thus, without mutations, the process would never advance. However, with mutations, some links eventually form allowing the process to reach non-degenerate networks, which are, in fact, efficient.

4.10 A Connections Example with Increasing Returns (Unique Evolutionarily Stable Network is not a Stable State)

Consider a variation on the connections model where the payoff to any individual is scaled by δ times n, where n is the number of direct links that the given individual has. The value of a link increases as connectedness increases; so this model exhibits increasing returns. (For example, substituting $n\delta$ where δ was before, the middle person in a three-player line receives a payoff of $4\delta - 2c$, while the end players receive $\delta + \delta^2 < c$.)

Suppose that $c > \delta$ so that starting from the empty network initial inertia exists, as in the symmetric connections model; and so the unique stable state is the empty network. However, if $c < \delta + \delta^2$, then once two links in a row form additional links will be added. If $c < 5\delta - 4\delta^2$, then with four individuals, the dynamic process will lead to the complete network (which is efficient). The complete network is the unique evolutionarily stable network as one has to break four links in the complete network to get to an improving path to the empty network, and one needs only to form two links to get from the empty network to an improving path leading to the complete network.

4.11 Evolutionary Stability and Efficiency

There is no guarantee that the evolutionary process will lead to an efficient network. Theorem 1 in JW shows that there are situations where no strongly efficient network is pairwise stable (and where there are no cycles), for a large class of allocation rules Y. Thus there are examples where no strongly efficient network is evolutionarily stable, for the same class of Y's (i.e., those which are anonymous and component balanced, as defined below).

Nevertheless, one would hope that if a strongly efficient network is pairwise stable, then the evolutionary process would select this network as one of the evolutionarily stable networks. Proposition 8, below, however, shows that this is not the case. Even if there is a unique efficient network and it is pairwise stable, it can fail to be evolutionarily stable for a wide class of allocation rules.

The following definitions will be used in Proposition 2.

Given a permutation $\pi : N \to N$, let $g^\pi = \{ij | i = \pi(k), j = \pi(l), kl \in g\}$ and v^π be defined by $v^\pi(g^\pi) = v(g)$. An allocation rule Y is *anonymous* if, for any permutation π, $Y_{\pi(i)}(g^\pi, v^\pi) = Y_i(g, v)$.

A value function v is *component additive* if $v(g) = \sum_{h \in C(g)} v(h)$. (Here, implicitly $v(\emptyset) = 0$, although the definitions can be extended to avoid this normalization.)

An allocation rule Y is *component balanced* if $\sum_{I \in N(h)} Y_i(g, v) = v(h)$ for every g and $h \in C(g)$ and component additive v.

Proposition 8 *If $N \geq 3$, then there is no Y which is anonymous and component balanced and such that for each v at least one strongly efficient graph is evolutionarily stable, even if there exists a strongly efficient graph that is pairwise stable.*

Proof: Let $n = 3$ and consider the component additive v such that for all i, j, and k, $v(\{ij\}) = -c$, $v(\{ij, jk\}) = -3c/2$, and $v(\{ij, jk, ik\}) = v > 0$. So, links are initially costly and then valuable. First, note that $Yj(\{ij\}) = -\frac{c}{2}$ for each ij by anonymity and component balance, and similarly $Y_j(\{ij, jk, ik\}) = \frac{v}{3}$. Second, note that $\{ij\}$ is only on an improving path to $\{\}$. This follows since at least one of j and k will not want to add the link jk (if it is improving for k, then it must be that $Y_j(\{ij, jk\}) < -c/2$). Both i and j are better off by severing ij. Lastly, note that $\{ij, jk\}$ lies on an improving path to $\{ij\}$ as at least one of j and k will benefit by severing jk, regardless of the choice of $Y(\{ij, jk\})$. Thus $\{ij, jk\}$ may or may not lie on an improving path to $\{ij, jk, ik\}$.

From these observations, it follows that the resistance of $\{\}$ is 1, and the resistance of $\{ij, jk, ik\}$ is at least 2, and that these are the only two pairwise stable networks (and there are no cycles). Thus, $\{\}$ is the unique evolutionarily stable network. This is easily extended to contexts where $n > 3$. ∎

4.12 Relative Probabilities of Evolutionarily Stable Networks

In many of the previous examples, there were several networks, of different forms, that were all evolutionarily stable. Thus each of these networks received positive weight in the limiting distribution of the evolutionary process. One might ask how relatively likely each evolutionary stable network is, in the limiting distribution. In fact, there exists a closed form expression for the stationary distribution of the evolutionary process, for any epsilon. To present this expression, the following notation is necessary.

Let $Q(g, \varepsilon) = \sum_{t \in T(g)} \prod_{g'g'' \in t} p(g', g'', \varepsilon)$, where $p(g', g, \varepsilon)$ is the probability of transitioning from g' to g in a given period when the probability of a tremble is ε.

Let $MT(g) = \{t \in T(g) | r(g, t) \leq r(g, t') \text{ for all } t' \text{ in } T(g)\}$, where $r(g, t)$ represents the resistance of a graph g along a specific g-tree t; $r(g, t) = \sum_{g'g'' \in t} r(g', g'')$. Thus $MT(g)$ is the set of resistance minimizing trees.

Let $\bar{Q}(g, \varepsilon) = \sum_{t \in MT(g)} \prod_{g'g'' \in t} p(g, g', \varepsilon)$.

Let ES denote the set of evolutionarily stable graphs.

Theorem 9 *The unique stationary distribution of the dynamic process given ε is described by*

$$m(g, \varepsilon) = \frac{Q(g, \varepsilon)}{\sum_{g'} Q(g', \varepsilon)}.$$

The limiting stationary distribution of network g, $m(g) = \lim_{\varepsilon \to 0} m(g, \varepsilon)$, is strictly positive if and only if g is evolutionarily stable, and then

$$m(g) = \lim_{\varepsilon} \frac{\bar{Q}(g, \varepsilon)}{\sum_{g' \in ES} \bar{Q}(g', \varepsilon)}.$$

Proof: The expression for $m(g, \varepsilon)$ follows from Lemma 6.3.1 in Freidlin and Wentzell [1984] (see also Kandori, Mailath, and Rob [1993]). The expression for $m(g)$ follows from Theorem 5 and the expression for $m(g, \varepsilon)$, since $\prod_{g'g'' \in t} p(g, g', \varepsilon)$ is on the order of ε raised to the power $r(g, t)$. ∎

4.13 Asymmetric Connections Example (Application of Theorem 9)

Let $n = 3$ and consider a variation of the connections model, where all players receive the same payoff from the same connection (i.e., all players have the same δ), but players have different costs of connection. Let c_1 represent player 1's cost of directly connecting to player 2 or 3. Let c_{21} represent player 2's cost of directly connecting to player 1 and c_{23} represent player 2's cost of connecting to player 3. Let $c3$ represent player 3's cost of connecting to either player 1 or 2. Assume that $\delta < c_1 < \delta + \delta^2$, $\delta < c_{23} < \delta + \delta^2$, $\delta + \delta^2 < c_{21}$, and $c_3 < \delta$.

It is easy to check that the set of pairwise stable networks coincides with the set of evolutionarily stable networks, which are $\{13, 23\}$ and $\{\}$.

Consider a version of the evolutionary process where each of the three possible links has an equal probability of being identified at any given time. Thus, the network $\{12, 13\}$ has equal probabilities $(1 - \varepsilon)/3$ of changing to $\{12\}$ or $\{13\}$, and a probability of $\varepsilon/3$ of changing to $\{12, 13, 23\}$. In Figure 2, the solid arrows represent moves between networks which have a resistance of 0 (so the arrow from $\{12, 13\}$ to $\{12\}$ indicates that $\{12\}$ defeats $\{12, 13\}$). Generally, from any network in Figure 2, there is a probability of $(1-\varepsilon)/3$ of changing to another given adjacent network if there is a solid arrow to that given adjacent network, and there is a probability of $\varepsilon/3$, otherwise. From these observations it is easy to find the resistance minimizing trees, MT($\cdot$), of $\{\}$ and $\{13, 23\}$. Each one of these trees has the same resistance, 1, and correspondingly $\prod_{g'g''\in t} p(g, g', \varepsilon)$ for such a tree is $\varepsilon(1-\varepsilon)^6/3^7$. Thus, by Theorem 9, the relative probabilities of $\{\}$ and $\{23, 13\}$ can be found by identifying the number of resistance minimizing trees for $\{\}$ and $\{13, 23\}$.

This set of resistance minimizing trees can be generated from Figure 2. By definition, the two pairwise stable networks $\{\}$ and $\{13, 23\}$ do not have solid arrows exiting them. The dashed arrows exiting the two pairwise stable networks represent the moves of resistance 1 from one pairwise stable network to the other. For instance, the dashed line from the network $\{\}$ to $\{23\}$, then connects to $\{13, 23\}$ via a solid line. Notice that the network $\{\}$ does not have a dashed arrow going to $\{12\}$ since the only solid arrow exiting $\{12\}$ goes back to $\{\}$. To find all the resistance minimizing trees for the network $\{\}$, we must delete arrows so that every other network has only one arrow exiting it and so that every other network has a path leading to $\{\}$. Straightforward calculations show that there are 80 such trees for $\{\}$, and there are 48 resistance minimizing trees for $\{13, 23\}$. Thus, from Theorem 9 it follows that $m(\{\}) = 80/128 = .625$ and $m(\{13, 23\}) = .375$.

5 Matching Models

We have mentioned several times that the results on evolutionary stability can easily be adapted to alternative notions of improving path. We now illustrate this fact in the context of matching problems, such as the Gale and Shapley [1962] marriage problem and the hospital-intern (and college admissions) problem. (See Roth and Sotomayor [1989] for a detailed background on these problems.) This section is also of independent interest as both an application of the model, and as an analysis of evolutionary dynamics in the Gale and Shapley matching world.

In matching problems, there are restrictions on the set of admissible networks so that only some subset G of all possible networks are feasible. We provide definitions for two of the most extensively studied of these problems.

5.1 Marriage Problems

For the marriage problem, the set of players N is divided into a set of men, $M = \{m_1, \ldots, m_j\}$, and a set of women, $W = \{w_1, \ldots, w_k\}$. A network, g, is feasible if each woman is linked to at most one man, and each man is linked to at most one woman. Let G denote the set of such feasible networks.

Let $m_i(g) = \{j | ij \in g\}$ denote the match of player i in the network g.

In a marriage problem, $v = \sum_i u_i(g)$, and $Y_i(v, g) = u_i(g)$, where for each i, $u_i : G \to I\!R$ depends only on the match of I. That is, for each i u_i is such that $u_i(g) = u_i(g')$ whenever $m_i(g) = m_i(g')$.

5.2 Hospital-Intern and College Admissions Problems

For the hospital-intern (or college admissions) problem, the set of players N is divided into a set of hospitals, $H = \{h_1, \ldots, h_j\}$ and a set of interns, $I = \{i_1, \ldots, i_k\}$. A network, g, is feasible if each intern is linked to at most one hospital, and each hospital, h, is linked to at most q_h interns, where $q_h > 0$ is the quota for the hospital; thus each hospital has a fixed number of slots.

Again, $v = \sum_i u_i(g)$, and $Y_i(v, g) = u_i(g)$, where for each i, $u_i : G \to I\!R$ depends only on the match of I. That is, for each i u_i is such that $u_i(g) = u_i(g')$ whenever $m_i(g) = m_i(g')$.

Additionally, we work under the assumption of responsive preferences (see Roth and Sotomayor [1989]). This is the condition that a hospital (college) has a ranking over interns (students) and an empty slot, such that preferences over subsets are consistent with the hospital's (college's) ranking.

Preferences are *responsive* if for each $h \in H$ there exists $y_h : N \to I\!R$ such that

(i) if $m_h(g) = m_h(g') \cup i$, then $u_h(g) > u_h(g')$ if and only if $y_h(i) > y_h(\emptyset)$,

(ii) if $m_h(g) = m_h(g')/i$, then $u_h(g) > u_h(g')$ if and only if $y_h(i) < y_h(\emptyset)$, and

(iii) if $m_h(g) = m_h(g') \cup i/j$, then $u_h(g) > u_h(g')$ if and only if $y_h(i) > y_h(j)$.

Next we give the definition of a core stable network. A network g is core stable if there is no group of players who prefer network g' to g and who can change the network from g to g' without the cooperation of the remaining players. In the marriage problem, it turns out that a network is core stable if and only if no player wants to sever his/her current link and no two players want to simultaneously sever their existing links and link with each other. This notion of core stability has been explored in great detail in matching models, beginning with Gale and Shapley (and again, see Roth and Sotomayor [1989] for more detail).

5.3 Core Stability

A network g is *core stable* if there does not exist any set of players A and $g' \in G$ such that

(i) $Y_i(g') \geq Y_i(g)$for each i in A (with at least one strict inequality),

(ii) if $ij \in g'$ but $ij \notin g$, then $i \in A$ and $j \in A$, and

(iii) if $ij \notin g'$ but $ij \in g$, then either $i \in A$ or $j \in A$.

Example (Contrast of core stability and pairwise stability[1])

Preferences are

$$\begin{aligned} m_1 &: \quad w_1, \quad w_2 \\ m_2 &: \quad w_2, \quad w_1 \\ w_1 &: \quad m_1, \quad m_2 \\ w_2 &: \quad m_2, \quad m_1. \end{aligned}$$

The above table can be read as follows: m_1's first choice for a spouse is w1 and his second choice is w_2. The remaining preferences can be read in a similar fashion. The unique core stable matching is $\{m_1w_1, m_2w_2\}$. However, both $\{m_1w_1, m_2w_2\}$ and $\{m_1w_2, m_2w_1\}$ are pairwise stable. Also, both networks are evolutionarily stable (using the definition given in section 4). From either network, two links need to be severed to get to an improving path to the other network.

5.4 Simultaneous Improving Paths

A *simultaneous improving path*, is a sequence of networks $g_0, \ldots, g_K$ in G such that if g' follows g in the sequence then either

(i) $g' = g - ij$ and either $Y_i(g') > Y_i(g)$ or $Y_j(g') > Y_j(g)$, or

(ii) $g' \in \{g+ij-ik, g+ij-ik-jm, g+ij, g+ij-jm\}$ where $ij \notin g$ and $Y_i(g') \geq Yi(g)$ and $Y_j(g') \geq Y_j(g)$ (with one inequality holding strictly).

Note that improving paths are a subset of simultaneous improving paths. Here the simultaneity refers to the fact that a player may make several changes at once: a player may both sever an existing link and add a new one. Note that the above definition can be altered so that the players, when adding a new link, sever the minimum necessary number of links in order to add the given link. In the context of the marriage and hospital-intern problems, a core stable network g is any network from which there is no simultaneous improving path leaving g.

[1]Gale and Shapley have a notion that they call pairwise stability that is core stability when A is restricted to have no more than two members. Here, we mean pairwise stability in the sense of JW, as indicating the lack of improving paths.

As illustrated in the following example, cycles can exist with the notion of simultaneous improving path.

Example (Existence of a cycle in a marriage problem)

Consider a marriage problem with two men and two women, where preferences are as follows:

$$\begin{aligned} m_1 &: \quad w_1, \quad w_2 \\ m_2 &: \quad w_2, \quad w_1 \\ w_1 &: \quad m_2, \quad m_1 \\ w_2 &: \quad m_1, \quad m_2. \end{aligned}$$

There exists a cycle under the definition of simultaneous improving path: $\{m_1w_1\}$ to $\{m_2w_1\}$ to $\{m_2w_2\}$ to $\{m_1w_2\}$ to $\{m_1w_1\}$. Interestingly, there are no cycles in the marriage problem under the notion of improving path, as a cycle in this setting requires some simultaneous changes to be made.

As we shall see as a key step in the proof of Theorem 10, however, there are no closed cycles when considering simultaneous improving paths in the marriage problem.

Let us now discuss an evolutionary process that corresponds to the notion of simultaneous improving path.

5.5 A Simultaneous Evolutionary Process

At each time a pair of players is randomly identified. If the link is already in the network, then the decision is whether to sever it; otherwise the two players are allowed to form a link and at the same time sever up to one existing link each. (Their actions are constrained to lead to a feasible g in G, so in some cases they must sever an existing link in order to add the new link). The players involved act myopically, adding the link (with corresponding severances) if it makes each at least as well off and one strictly better off, and severing the link if its deletion makes either player better off. After the action is taken, there is some small probability that a tremble occurs and the link is deleted if it is present.

Here, we consider only trembles that delete the given link; thus one does not have to worry about the constraints imposed by feasibility, which might bind in the case of adding a link. This turns out to be irrelevant due to the natural tendency towards the addition of beneficial links, the restrictions on numbers of links, and the absence of externality effects.[2]

[2]What is important is that it is impossible to have a situation where someone hesitates to add a

5.6 S-Evolutionarily Stable Networks

The set of networks that is the support of the limiting stationary distribution (of the simultaneous improving process) is the set of *S-evolutionarily stable networks.*

Next we explore the set of S-evolutionarily stable networks. In particular, we show in Theorem 10 that the set of S-evolutionarily stable networks is equal to the set of core stable networks. This result is somewhat surprising for two reasons. First, as we have just seen in the example above, there can exist cycles with simultaneous improving paths. Thus, one has to show that no networks in these cycles are evolutionarily stable. Second, the example below shows that the set of pairwise stable networks does not coincide with the set of evolutionarily stable networks, and as we are making parallel changes in these definitions one might not expect the set of core stable and S-evolutionarily stable networks to always coincide either.

Example (Contrast between Pairwise Stable and Evolutionarily Stable Networks)

Consider a marriage problem with two men and two women. Preferences are:

$$\begin{aligned} m_1 &: \ w_1, \ \ w_2 \\ m_2 &: \ w_2, \ \ \text{alone} \\ w_1 &: \ m_1, \ \ \text{alone} \\ w_2 &: \ m_2, \ \ m_1. \end{aligned}$$

Thus m_2 prefers being alone to being matched with w_1. Here $\{m_1w_1, m_2w_2\}$ and $\{m_1w_2\}$ are both pairwise stable. However, only $\{m_1w_1, m_2w_2\}$ is evolutionarily stable. This result follows since it takes two mutations to get from $\{m_1w_1, m_2w_2\}$ to $\{m_1w_2\}$, but only one mutation to go the other way.

Theorem 10 *Consider the marriage problem where players' preferences are strict (and players are allowed to prefer staying alone to being in some matches). The set of S-evolutionarily stable networks coincides with the set of core stable networks.*

The proof of Theorem 10 is in the appendix. We first show that although there exist cycles under the definition of simultaneous improving path, there are no closed cycles. The proof builds on this observation, and utilizes the lattice structure of the marriage problem to build restricted g-trees for any core stable network that has a resistance of $K-1$, where K is the number of core stable networks.

Theorem 10 extends to the college admissions (hospital-intern) problem.

link, that once added would lead to changes in incentives for the subsequent networks (such as in the increasing returns connections example). In the marriage model, and the college admissions model with responsive preferences, if a link were added that did not benefit the players involved, it would simply be deleted at the next opportunity.

Corollary 11 *Consider the college admissions (hospital-intern) problem with strict and responsive preferences. The set of S-evolutionarily stable networks coincides the set of core-stable networks.*

Proof: This follows from the proof Theorem 10, by considering a related marriage market where each college is replicated qh times. (For more on the relationship between the college admissions and marriage problems see Chapter 5 in Roth and Sotomayor [1989].) ∎

Example (Role of responsiveness)

Consider a college admissions problem with 1 college and 3 students. The set of feasible networks allows only for links that involve the college and a student. Here the term 'full network' refers to the network with all three feasible links. Assume that each student prefers being in the college to being out. The college prefers to have all three students to none, but prefers none to having any proper subset of students. Thus the college's preferences fail to satisfy responsiveness.

There are two pairwise stable networks: one that has each student linked to the college and another where none are linked to the college. However, the only network that is 'core-stable' is the full network.

Under evolutionary stability the resistance of the full network is 2 (to get from the null network to the full network, at least two links must exist for the third to be added). However, the resistance of the null network is 1 (delete one link from the full network and it lies on an improving path back to the null network). Thus, only the null network (which is Pareto inefficient and out of the core) is evolutionarily stable. Similarly, only the null network is S-evolutionarily stable.

6 Conclusion

We have developed a model for the study of the dynamic formation and evolution of networks based on a tool of improving paths. We have studied cycles under improving paths, a stochastic evolutionary process, and variations of that process, as well as examples and applications of the model and results. As we have emphasized, there are many alternatives in modeling choices, especially with respect to the dynamic process and to the particulars of the definition of improving path.

One modeling decision that deserves further attention is the assumption that players are myopic. It would be natural to have forward-looking players in situations with a small number of players who are well-informed about all the other players, the allocations and valuations, and who care about the future. A interesting problem for future research, is to develop an appropriate definition of improving path for forward looking players, deal with existence issues, and find the set of evolutionary stable networks for these forward-looking improving paths.

To get an idea of what types of differences might emerge from the myopic model, consider the following three-player example. Let players have payoffs which meet the following inequalities: $Y_1(12) < 0 < Y_1(12,13) < Y_1(13) < Y_1(12,13,23)$; $Y_2(23) < 0 < Y_2(12,23) < Y_2(12) < Y_2(12,13,23)$; $Y_3(13) < 0 < Y_3(13,23) < Y_3(23) < Y_3(12,13,23)$; $Y_i(\{\}) = 0$ and $Y_i(ij,jk) < Y_i(12,13,23)$, for all i. Thus $\{12,13,23\}$ is the unique efficient network.

If players are myopic, then there are two pairwise stable networks, $\{12,13,23\}$ and $\{\}$. From the network $\{12,13,23\}$ one link must be severed to get to an improving path to $\{\}$. From the network $\{\}$ one link must be added to get to an improving path to $\{12,13,23\}$. Thus, by Theorem 5, both networks are evolutionary stable and it is easy to show (using Theorem 9) that the evolutionary process will split its time equally between the two networks.

Now assume that players are non-myopic and care about future payoffs. Assume also that each player knows every other player's payoff function. Suppose the players are currently in the network $\{12\}$ and suppose that the link $\{12\}$ is identified and can thus be severed by either player 1 or 2. If player 1 were myopic then he would sever the link. However, if player 1 values the future enough, then he will decide not to sever the link as long as he believes that the other players have incentive to do the same thing when faced with a similar decision. Thus if all players value the future enough, the unique stable network will be {12,13,23}, and so {12,13,23} will be the unique evolutionary stable network as well. In this example, if players care enough about the future and are not myopic, they will end up at the efficient network.

Appendix

Proof of Theorem 2: First, suppose that there exists a cycle, so that there is some g such that $g \to g$. We show that there cannot exist such a w. Suppose, to the contrary, that there exists such a w. By transitivity of $>$, w satisfies $w(g) > w(g)$, which is impossible. So, if there is a cycle there cannot exist such a w, and so the existence of such a w precludes any cycles.

Next, assume there are no cycles and that for g and g' (that are adjacent) either g defeats g' or g' defeats g. We show that there exists such a w. The following Lemma (see Kreps [1988], Proposition 3.2) is helpful.

Lemma: If X is a finite set and b is a binary relation, then there exists $w : X \to I\!R$ such that $[w(x) > w(y)] \Leftrightarrow [xby]$, if and only if b is asymmetric and negatively transitive.

Since there are no cycles, our binary relation $\to$ is acyclic, and thus asymmetric. Also, $\to$ is transitive by the definition of improving path. However $\to$ is not necessarily negatively transitive. (For an easy example, consider the connections model with $n = 3$, $c = .1$, and $\delta = .9$.) Thus, we construct a binary relation b over the set of networks such that (i) $g \to g'$ implies $g'bg$, (ii) if g and g' are adjacent, then $g \to g'$ iff $g'bg$, and (iii) b is asymmetric and negatively transitive. Then, by (iii) we can apply the Lemma to obtain w, and Theorem 2 follows from (ii).

Construct b as follows.

Case 1. For every distinct g and g' at least one of the following holds: $g \to g'$ or $g' \to g$.

Set b by $g'bg$ iff $g \to g'$. We show that b is negatively transitive. Write $g''nbg'''$ if it is not the case that $g''bg'''$. Suppose that $gnbg'$ and $g'nbg''$. This implies that $g'bg$ and $g''bg'$. Thus, by transitivity, it follows that $g''bg$, and so by asymmetry $gnbg''$. Thus negative transitivity is satisfied.

Case 2. There exist distinct g and g' (which are not adjacent) such that g not$\to$ g' and g' not$\to$ g.

Define the binary relation b_1 as follows. Let $g''b_1g'''$ iff $g''' \to g''$, except on g and g' where we arbitrarily set $g'b_1g$. Note that by construction, (i) and (ii) are true of b_1. Note also that b_1 is acyclic (and hence asymmetric). To see the acyclicity of b_1, note that if there were a cycle that it would have to include g and g', as this is the only place that b_1 and $\to$ disagree. However, the existence of such a cycle would imply that $g' \to g$, which is a contradiction. Next, define b_2 by taking all of the transitive implications of b_1. Again, (i) and (ii) are true of b_2. By construction b_2 is transitive. We argue that b_2 is acyclic. Let us show this by constructing b_2; we will add one implication from b_1 and transitivity at a time, and we will verify acyclicity at each step. Consider the first new implication that is added and suppose that there exists a cycle. Let g''' and g'' be the networks in question. So $g''b_1g'''$ and $g'''nb_1g''$, but $g'''b_2g''$, and there exist a sequence of networks $\{g_0, g_1, \ldots, g_r\}$ such that $g'''b_1g_0b_1g_1 \ldots b_1g_rb_1g''$. This implies that there is a cycle under b_1, which is a contradiction. Iterating this logic implies that b_2 is acyclic.

Now, reconsider Cases 1 and 2 when b_2 is substituted for $\rightarrow$. Iterating on this process, we will eventually come to a case where we have constructed b_k, and relative to b_k we are in Case 1. Iterating on the argument under Case 2, it follows that (i) and (ii) will be true of b_k, and b_k will be transitive and asymmetric. Then, by the argument under Case 1, b_k will be negatively transitive. Let $b = b_k$ and the proof is complete.

Proof of Theorem 5 The proof is an application of a theorem from Young [1993]. To state Young's theorem, the following definitions are necessary.

Consider a stationary Markov process on a finite state space X with transition matrix P.

A set of *perturbations of* P is a range $(0, a]$ and a stationary Markov process on X with transition matrix $P(\varepsilon)$ for each ε in $(0, a]$, such that (i) $P(\varepsilon)$ is aperiodic and irreducible for each ε in $(0, a]$, (ii) $P(\varepsilon) \rightarrow P$, and (iii) $P(\varepsilon)_{xy} > 0$ implies that there exists $r \geq 0$ such that $0 < \varepsilon^r P(\varepsilon)_{xy} < \infty$.

The number r in (iii) above is the *resistance* of the transition from state x to y. There is a path from x to z of *zero resistance* if there is a sequence of states starting with x and ending with z such that the transition from each state to the next state in the sequence is of zero resistance. Note that from (ii) and (iii), this implies that if there is a path from x to z of zero resistance, then the n-th order transition probability associated with P of x to z is positive for some n.

The *recurrent communication classes* of P, denoted $X_1, \ldots, X_J$, are disjoint subsets of states such that (i) from each state there exists a path of zero resistance leading to a state in at least one recurrent communication class, (ii) any two states in the same recurrent communication class are connected by a path of zero resistance (in both directions), and (iii) for any recurrent communication class X_j and states x in X_j and y not in X_j such that $P(\varepsilon)_{xy} > 0$, the resistance of the transition from x to y is positive.

For two communication classes X_i and X_j, since each $P(\varepsilon)$ is irreducible, it follows that there is a sequence of states $x_1, \ldots, x_k$ with x_1 in X_i and x_k in X_j such that the resistance of transition from x_k to x_{k+1} is defined by (iii) and finite. Denote this by $r(x_k, x_{k+1})$. Let the resistance of transition from X_i to X_j be the minimum over all such sequences of $\sum_1^{K-1} r(x_k, x_{k+1})$, and denote it by $r(X_i, X_j)$.

Given a recurrent communication class X_i, an *i-tree* is a directed graph with a vertex for each communication class and a unique directed path leading from each class $j (\neq i)$ to i. The *stochastic potential* of a recurrent communication class X_j is then defined by finding an i-tree that minimizes the summed resistance over directed edges, and setting the stochastic potential equal to that summed resistance.

Also, given any state x, an *x-tree* is a directed graph with a vertex for each state and a unique directed path leading from each state $y(\neq x)$ to x. The resistance of x is then defined by finding an x-tree that minimizes the summed resistance over directed edges.

The following theorem is a combination of Theorem 9 and Lemmas 1 and 2 in Young:

Theorem (Young [1993]): Let P be the transition matrix associated with a stationary Markov process on a finite state space with a set of perturbations $\{P(\varepsilon)\}$ with corresponding (unique) stationary distributions $\{m(\varepsilon)\}$ be the corresponding (unique) stationary distribution. Then $m(\varepsilon)$ converges to a stationary distribution m of P, and a state x has $m_x > 0$ if and only if is in a recurrent communication class of P which has a minimal stochastic potential. This is equivalent to x having minimum resistance.

To apply this to our setting, note that under version 1 (or 2 or 5) of the dynamic process, each pairwise stable graph and closed cycle is a recurrent communication class of the corresponding process P (and these are exactly the recurrent communication classes of P). Next, the transition from any graph g to an adjacent one g' has probability on the order of ε if g is not in $im(g')$ (and thus has resistance 1), and is of the order of 1 otherwise (and thus has resistance 0).

Proof of Theorem 10: Consider the following results about the core stable networks (see Roth and Sotomayor [1989]):

Theorem A (Gale and Shapley, Knuth): There exists a man optimal stable matching which all men (weakly) prefer to any other stable matching and similarly there exists a woman optimal stable matching. Moreover, all men (weakly) prefer any matching to the woman optimal stable matching, and a similar statement holds for the women and the man optimal stable matching.

Theorem B (2.22 in Roth and Sotomayor): The set of players with no links is the same in all core stable matchings.

We use these in proving the following series of claims.

Claim 1: From any g that is not core stable, there exists a simultaneous improving path that leads to a core stable network g' that is weakly preferred by each man to g.

Proof of Claim 1: Consider any initial graph g that is not core stable. The following algorithm constructs a simultaneous improving path from g to a core stable graph. This algorithm is similar to the deferred-acceptance algorithm where the man proposes (see Theorem 2.8 in Roth and Sotomayor).

Throughout the algorithm, we assume a man (woman) is acceptable to a woman (man) only if he (she) is preferred to both her (his) current spouse and to being single.

Algorithm:

Step 1: Start at graph g. Let anyone who prefers being single to their current mate, sever their current tie.

Step 2: Let each man, who is not married to his first choice, propose to his first choice, as long as she is acceptable. Each woman rejects all offers that are not acceptable. Of the remaining offers she receives, she accepts the one she likes best and rejects all others. If a woman accepts an offer, she and the man sever all existing ties and marry.

Step 3: Any man who was rejected in step 2, or who is now single, proposes to his first choice, as long as she is acceptable. Repeat this step until there are no more acceptances.

Step 4: Each man who is not married to his first or second choice, proposes to his second choice, as long as she is acceptable. The woman rejects offers as in Step 2. If a woman accepts an offer she and the man sever all existing ties and marry.

Step 5: Any man who was rejected in Step 4 or who is now single, proposes to his second choice, as long as she is acceptable. Repeat this step until there are no more acceptances.

Repeat Steps 4 and 5 for each man's third choice, as long as she is acceptable. Keep repeating Steps 4 and 5 for each man's next choice, until every man has proposed to his last acceptable choice.

Go back to Step 1. Repeat the entire sequence, until there are no more acceptances.

In order to make the algorithm into a simultaneous improving path, no marriages can take place simultaneously. Thus, if at any step multiple marriages take place, we need to randomly order them so that the marriages occur sequentially.

Such an algorithm cannot cycle, since the deferred acceptance algorithm cannot cycle (see proof of Theorem 2.8 in Roth and Sotomayor). Thus the algorithm must stop at a graph g', which is by construction core stable. Since no man ever proposes to a woman who is worse than his current mate, we know that g' is weakly preferred by each man to g.

Claim 2: There exist no closed cycles.

Claim 2 follows directly from Claim 1.

Claim 3: If there is a singleton core stable network, then it is the unique S-evolutionarily stable network.

Claim 3 follows from Claim 2.

Claim 4: Any core stable network, g, that is not man-optimal is one link away from an improving path that leads to a core stable network, g', that all men weakly prefer and some men strictly prefer to g.

Proof of Claim 4: Find a man in g who is not linked to his man-optimal mate and sever that link. (By theorem B, that man must be linked under g.) Call the resulting network g''. Alter this man's preferences to be the same except that his previous mate and all those less preferred than her are now unacceptable. Note that g'' cannot be stable under the new preferences, as the woman who had her link severed must be the man-optimal mate of some currently matched man (again by Theorem B). So, by claim 1 there is a simultaneous improving path from g'' to some g' such that g' is core stable and each man weakly prefers g' to g'', under the new preferences. Now we argue that the original man strictly prefers g' to g under the original preferences, which completes the claim. Since the man-optimal matching is unchanged by the preference change, the original man must be linked in g' since he is linked in the man optimal matching (again by Theorem B). Given the stability of g' under his new preferences, he strictly prefers his mate in g' to his mate in g under his original preferences.

Claim 5: If there are $k > 1$ core stable networks, then the resistance of the man optimal network is $k - 1$.

Proof of Claim 5: This follows from Claims 2 and 4 by constructing a restricted man-optimal tree by directing each core-stable g that is not man optimal to a g' as defined in Claim 4. Note that by the preference ordering of men over g and g', this results in a directed tree and there must be some network that connects to the man optimal network.

Claim 6: If there are $k > 1$ core stable networks, then the resistance of any core-stable network is $k - 1$.

Proof of Claim 6: We know this for the man-optimal (and correspondingly the woman optimal) network. Consider some other core stable g. We know that this is man-optimal under a change of preferences where each man's preferences are changed so that any woman that he preferred to his mate under g is now unacceptable. Then, by the logic of claim 5, we can find a restricted g tree where the resistance of each edge is 1. Note that from the proof of claim 1, the resistance of each of these edges is the same under the original preferences (since no one is ever linked to an unacceptable mate). Thus, we can find a restricted g-tree on the set networks that all men find weakly less preferred to g, where the resistance of each edge is 1. Similarly we can find a restricted g-tree on the set of networks that all men find weakly preferred to g (using Theorem A), where the resistance of each edge is 1. For other networks that are not uniformly ranked by men relative to g, keep the same directed edges as in the tree in Claim 5. The resulting restricted g-tree has resistance $k - 1$.

Claims 2, 3 and 6 establish the theorem.

Ordered Improving Paths:

In the definition of improving path there is no priority over addition and deletion of links. In some contexts one might require that addition of links only take place when no player wishes to delete a link (see Watts (1997) for additional). The following definition is a variation on the definition of improving paths is in this spirit.

An *ordered improving path* from g to g' is a finite sequence of graphs $g_1, \ldots, g_K$ with $g_1 = g$ and $g_K = g'$ such that for any $k \in \{1, \ldots, K-1\}$ either:

1. $g_{k+1} = g_k - ij$ for some ij such that $Y_i(g_k - ij) > Y_i(g_k)$, or
2. $g_{k+1} = g_k + ij$ for some ij such that $Y_i(g_k + ij) > Y_i(g_k)$ and $Y_j(g_k + ij) \geq Y_j(g_k)$, and for any $i'j' \in g_k$, $Y_{i'}(g_k) \geq Y_{i'}(g_k - i'j')$.

The set of ordered improving paths are a subset of the set of improving paths.

Note that implicit in the definition of ordered improving path is the idea that deletion is done sequentially, rather than simultaneously.

Version 2 of the Dynamic Process:
Same as version 1, except that trembles occur over the whole graph (independently) after each period.

Version 3 of the Dynamic Process:
Same as version 1, except that all agents may choose to delete any existing links at the end of each period, as in Watts.

Version 4 of the Dynamic Process:
Same as version 3, but with independent trembles again over whole graph.

Version 5 of the Dynamic Process:
Same as version 4, except trembles take place after deletion.

Note that with versions 2 and 4 a change of two links at once is on the order of ε^2, which is vanishing more quickly than ε. Given any one of the versions above, we can define a markov chain with different states corresponding to the graph obtained at the end of a given period (given some specification of what agents do when they are faced with several actions over which they are indifferent—e.g., severing either of two different links.) In versions 1, 2, and 5, the markov chain is irreducible (given non-zero trembles). In versions 3 and 4 one cannot end a period at a graph where an agent would like to delete a link. Thus, versions 3 and 4 do not satisfy irreducibility. In versions 1, 2, and 5 the markov chain is also aperiodic (given non-zero trembles). This is most easily seen by noting that it is possible for the graph to return to itself at the end of the period, and so the periodicity must be one. However, this is not true in versions 3 and 4. it may not be in version 4 unless the trembles occur after agents' choices to delete any existing links. A finite state irreducible, aperiodic markov chain has a unique corresponding stationary distribution. Thus, this is guaranteed for versions 1, 2, and 5.

Theorem 12 *Consider version 5 of the dynamic process. The set of evolutionarily stable networks is the set* $\{g|\ or\ (g) \leq or\ r(g')$ *for all* $g'\}$ *and thus includes only closed cycles and pairwise stable graphs of least resistance, where* $or(g)$ *is defined relative to ordered improving paths (with simultaneous deletion—a corresponding definition applies for sequential deletion).*

The following corollaries (to Theorem 2) give sufficient conditions for a network (or cycle) to be evolutionarily stable.

Reciprical resistance

The resistance between g and g' is *reciprocal* if $r(g, g') = r(g', g)$.

Proposition 13 *If resistance is reciprocal between all networks in the set of pairwise stable or closed cycle networks, then all such networks are evolutionarily stable.*

Proof: Consider some g in this set and t, a resistance minimizing restricted g-tree. For any other g', a restricted g'-tree with the same resistance as t can be constructed by keeping the same links as in t, but reversing them where necessary (i.e., on the directed path between g and g' only) to obtain a restricted g'-tree. This implies that the resistance of g' must be at least as low as that of g. Since g and g' were arbitrary, g and g' must have the same resistance, which by Theorem 2 (and Lemma 7) establishes the result. ∎

Corollary 14 *Consider a case where there exists a directed graph with vertices the elements of PSC, with one directed edge leaving each vertex, and such that for any* x *in PSC, a minimal resistance* x*-tree may be found by deleting the directed edge leaving* x*. Note that this directed graph must be a directed circle. Then* x *is evolutionarily stable if and only if* $r(x, x')$ *is maximal where the directed edge leaving* x *goes to* x'*.*

Corollary 15 *Let PSC consist of three elements; PSC=* $\{x, x', x''\}$*. Then* $r(x, x') \geq r(x', x)$ *and* $r(x, x'') \geq r(x'', x)$ *imply that* x *is evolutionarily stable (and uniquely so if both inequalities are strict). While* $r(x, x') < r(x', x)$ *and* $r(x, x'') < r(x'', x)$ *imply that* x *is not evolutionarily stable.*

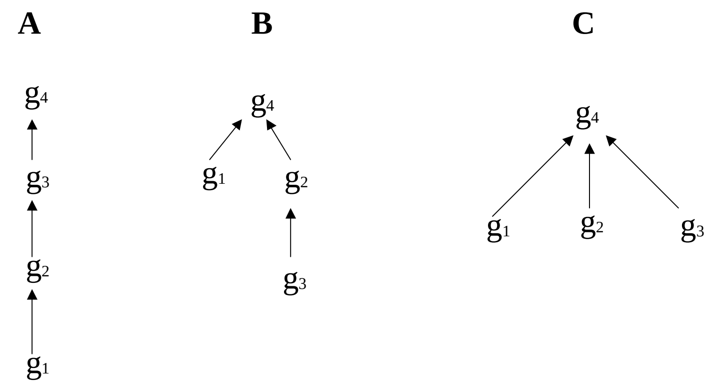

Figure 1

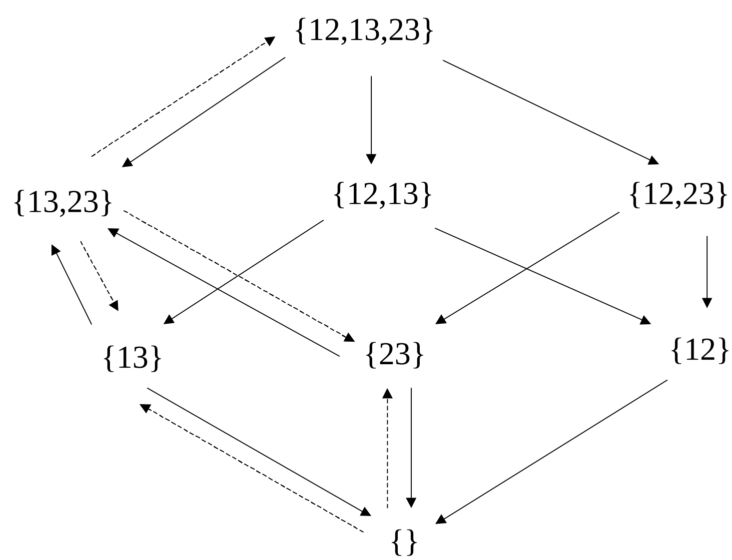

Figure 2

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