# Modular Forms for GL(3) and Galois Representations 

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#### Abstract

A description and an example are given of numerical experiments which look for a relation between modular forms for certain congruence subgroups of $\mathrm{SL}(3, \mathbb{Z})$ and Galois representations.


## 1 Introduction

In this paper we review a recently discovered relation between some modular forms for congruence subgroups of $\mathrm{SL}(3, Z Z)$ and three dimensional representions of $\operatorname{Gal}(\overline{\mathbb{Q}} / \mathbb{Q})$ (see vG-T and GKTV). This relation is the equality of local $L$-factors, for primes $p \leq 173$, attached to the modular forms and to the Galois representation, see Theorem 4.5. The result gives some evidence for general conjectures of Langlands and Clozel C1].

The first three section follow closely the notes from a seminar talk of the first author at the séminaire de théorie des nombres de Paris in January 1995. In the first section we briefly recall an instance of the relation between elliptic modular forms and Galois representations. In the second section we introduce the modular forms for GL(3) and the Galois representations are discussed in section three.

In section four we give some new examples of non-cusp forms for congruence subgroups of $\mathrm{SL}(3, Z Z)$ and we describe many of these in terms of classical modular forms for congruence subgroups of $\mathrm{SL}(2, Z Z)$. The last section deals with a Hodge theoretical aspect of the algebraic varieties (motives in fact) we used to define the Galois representations.

It is a pleasure to thank Avner Ash, Kevin Buzzard, Bas Edixhoven and Jasper Scholten, especially for their interest and help concerning Sect. 5.5.

## 2 Modular Forms: the GL(2) Case

Let $S_{2}(N)$ be the space of cusp forms of weight two for the congruence subgroup $\Gamma_{0}(N) \subset \mathrm{SL}(2, Z Z)$. Let $f=\sum a_{n} e^{2 \pi i n} \in S_{2}(N)$ be a newform, thus $a_{1}=1$
and $f$ is an eigenform for the Hecke algebra: $T_{p} f=a_{p} f$ for all prime numbers $p$ which do not divide $N$. For such a prime $p$ one defines the local $L$-factor of $f$ as

$$
L_{p}(f, s):=\left(1-a_{p} p^{-s}+p^{1-2 s}\right)^{-1}
$$

note that $L_{p}(f, s)$ is determined by the eigenvalue $a_{p}$.
In case all $a_{p}$ are in $\mathbb{Z}, f$ defines an elliptic curve $E_{f}$, defined over $\mathbb{Q}\left(E_{f}\right.$ is a subvariety of the Jacobian of the modular curve $X_{0}(N)$ ). The Galois group $\operatorname{Gal}(\overline{\mathbb{Q}} / \mathbb{Q})$ acts on the $\ell^{n}$-torsion points of this curve which gives an $\ell$-adic representation:

$$
\rho_{f, \ell}: \operatorname{Gal}(\overline{\mathbb{Q}} / \mathbb{Q}) \longrightarrow \mathrm{GL}_{2}\left(\mathbb{Q}_{\ell}\right) .
$$

The local $L$-factor of this representation for primes $p$ as above does not depend on the choice of the prime $\ell \neq p$ and is defined by

$$
L_{p}\left(\rho_{f}, s\right):=\operatorname{det}\left(I-\rho_{f, \ell}\left(F_{p}\right) p^{-s}\right)^{-1}=\left(1-\operatorname{trace}\left(\rho_{f, \ell}\left(F_{p}\right)\right) p^{-s}+p^{1-2 s}\right)^{-1}
$$

with $F_{p} \in \operatorname{Gal}(\overline{\mathbb{Q}} / \mathbb{Q})$ a Frobenius element at $p$.
The Eichler-Shimura congruence relation asserts that

$$
a_{p}=\operatorname{trace}\left(\rho_{f, \ell}\left(F_{p}\right)\right) \quad \text { so } \quad L_{p}(f, s)=L_{p}\left(\rho_{f}, s\right)
$$

(again with $p$ a prime not dividing $N \ell$ ). Thus we have a method to associate to a newform $f$ a (compatible system of $\ell$-adic) Galois representation(s) $\rho_{f, \ell}$ such that the $L$-factors agree. This construction has been generalized to newforms of any weight (and arbitrary Hecke eigenvalues) by Deligne [D] using Galois representations on certain etale cohomology groups of certain $\ell$-adic sheaves on the modular curve $X_{0}(N)$.

It is a pleasure to observe that recently Wiles proved a partial inverse to the construction sketched above: he shows that for a certain class of elliptic curves defined over $\mathbb{Q}$ the corresponding Galois $L$-series are the $L$-series of newforms. As is well known, this has been used to prove Fermat's Last Theorem.

## 3 Modular Forms for GL(3)

## 3.1

One can also define modular forms, a Hecke algebra and local $L$-factors for congruence subgroups of $\operatorname{SL}(3, Z Z)$, see below. However, the upper half plane

$$
\mathbb{H}=\{z \in \mathbb{C}: \Im(z)>0\} \cong \mathrm{SL}(2, \mathbb{R}) / \mathrm{SO}(2)
$$

which has a complex structure, is now replaced by $\mathrm{SL}(3, \mathbb{R}) / \mathrm{SO}(3)$ (see AGG), a real variety of dimension 5 which, for dimension reasons(!), cannot have a complex structure.

In particular, one does not know how to associate algebraic varieties to congruence subgroups of $\mathrm{SL}(3, \not Z)$ (in contrast to the modular curves in the GL(2)case). Therefore there are no a priori given Galois representations on etale cohomology groups which could be related to modular forms for such congruence subgroups.

## 3.2

In the case of $\mathrm{SL}(2, Z Z)$, the space of holomorphic modular forms of weight two for a congruence subgroup $\Gamma$ is a subspace of the cohomology group $H^{1}(\Gamma, \mathbb{C})$. This generalizes as follows.

## 3.3

From now on we use the following definition:

$$
\Gamma_{0}(N)=\left\{\left(a_{i j}\right) \in \mathrm{SL}(3, Z Z) \mid a_{21} \equiv 0 \bmod N \text { and } a_{31} \equiv 0 \bmod N\right\}
$$

The modular forms for $\Gamma_{0}(N)$ we consider are elements of $H^{3}\left(\Gamma_{0}(N), \mathbb{C}\right)$. To compute this vector space, we introduce a finite set:
$\mathbb{P}^{2}(\mathbb{Z} / N)=\left\{(\bar{x}, \bar{y}, \bar{z}) \in(\mathbb{Z} / N)^{3} \mid \bar{x} \not \mathbb{Z} / N+\bar{y} \not Z / N+\bar{z} \not \mathbb{Z} / N=\mathbb{Z} / N\right\} /(\mathbb{Z} / N)^{\times}$.
When the elements of this set are viewed as column vectors, there is a natural left action of $\mathrm{SL}(3, Z Z)$ on $\mathbb{P}^{2}(Z Z / N)$. This action is transitive, and the stabilizer of $(\overline{1}: \overline{0}: \overline{0})$ equals $\Gamma_{0}(N)$. Therefore

$$
\mathrm{SL}(3, \not Z) / \Gamma_{0}(N) \cong \mathbb{P}^{2}(\not Z / N)
$$

This relation between $\Gamma_{0}(N)$ and $\mathbb{P}^{2}(\mathbb{Z} / N)$ leads to a very concrete description of the vector space $H^{3}\left(\Gamma_{0}(N), \mathbb{C}\right)$. In fact, its dual $H_{3}\left(\Gamma_{0}(N), \mathbb{C}\right)$ can be computed as follows:

### 3.4 Theorem.

(AGG, Thm 3.2, Prop 3.12)
There is a canonical isomorphism between $H_{3}\left(\Gamma_{0}(N), \mathbb{C}\right)$ and the vector space of mappings $f: \mathbb{P}^{2}(\mathbb{Z} / N) \rightarrow \mathbb{C}$ that satisfy

1. $f(\bar{x}: \bar{y}: \bar{z})=-f(-\bar{y}: \bar{x}: \bar{z})$,
2. $f(\bar{x}: \bar{y}: \bar{z})=f(\bar{z}: \bar{x}: \bar{y})$,
3. $f(\bar{x}: \bar{y}: \bar{z})+f(-\bar{y}: \bar{x}-\bar{y}: \bar{z})+f(\bar{y}-\bar{x}:-\bar{x}: \bar{y})=0$.

## 3.5

For any $\alpha \in \mathrm{GL}(3, \mathbb{Q})$ one has a ( $\mathbb{C}$-linear) Hecke operator:

$$
T_{\alpha}: H^{3}\left(\Gamma_{0}(N), \mathbb{C}\right) \longrightarrow H^{3}\left(\Gamma_{0}(N), \mathbb{C}\right)
$$

The adjoint operator $T_{\alpha}^{*}$ on the dual space $H_{3}\left(\Gamma_{0}(N), \mathbb{C}\right)$ can be explicitly computed using modular symbols.

The Hecke algebra $\mathcal{T}$ is defined to be the subalgebra of $\operatorname{End}\left(H^{3}\left(\Gamma_{0}(N), \mathbb{C}\right)\right)$ generated by the $T_{\alpha}$ 's with $\operatorname{det}(\alpha)$ relatively prime with $N$. The Hecke algebra
is a commutative algebra and we are interested in eigenforms $F \in H^{3}\left(\Gamma_{0}(N), \mathbb{C}\right)$ for the Hecke algebra:

$$
T F=\lambda(T) F, \quad \text { with } \quad \lambda: \mathcal{T} \rightarrow \mathbb{C} \quad(\text { for all } T \in \mathcal{T})
$$

Of particular interest are the Hecke operators $E_{p}=T_{\alpha_{p}}$, which are for a prime $p$ not dividing $N$ defined using $\alpha_{p}=\left(\begin{array}{ccc}p & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{array}\right) \in \operatorname{GL}(3, \mathbb{Q})$.

Let $a_{p}:=\lambda\left(E_{p}\right)$, for a (given) character $\lambda$ of $\mathcal{T}$ and a prime $p$ not dividing $N$, then the local $L$-factor of a Hecke eigenform $F \in H^{3}\left(\Gamma_{0}(N), \mathbb{C}\right.$ ) (with the additional condition that $F$ is cuspidal) corresponding to $\lambda$ (so $E_{p} F=a_{p} F$ ) is

$$
L_{p}(F, s)=\left(1-a_{p} p^{-s}+\bar{a}_{p} p^{1-2 s}-p^{3-3 s}\right)^{-1}
$$

where $\bar{a}_{p}$ is the complex conjugate of $a_{p}$. The field $K_{F}:=\mathbb{Q}\left(\ldots, a_{p}, \ldots\right)$ generated by the Hecke eigenvalues of an eigenform $F$ is known to be either totally real or is a CM field (a degree 2, non-real extension of a totally real field).

## 3.6

In GKTV, a list of the $a_{p}$ 's with $p \leq 173$ is given for several eigenforms with $N \leq 245$. Here we list some $a_{p}$ 's of three particularly interesting eigenforms (these eigenforms are uniquely determined by their level $N$ and the $a_{p}$ 's listed). In case $p$ divides $N$ we write $* *$ for $a_{p}$. In the three cases listed here $K_{F}=\mathbb{Q}(i)$ with $i^{2}=-1$. The complex conjugates of the $a_{p}$ 's for a given $F$ are the Hecke eigenvalues for another modular form $G$ of the same level.

| $p=$ | 2 | 3 | 5 | 7 | 11 | 13 | 17 | 101 | 173 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $N$ | eigenvalue $a_{p}$ |  |  |  |  |  |  |  |  |
| 128 | $* *$ | $1+2 i$ | $-1-4 i$ | $1+4 i$ | $-7-10 i$ | $-1+4 i$ | 7 | $-105-100 i$ | $-49-188 i$ |
| 160 | $* *$ | $1+2 i$ | $* *$ | $1-2 i$ | $-3-12 i$ | $-5-8 i$ | -5 | $-33+64 i$ | $99+104 i$ |
| 205 | -1 | $1+2 i$ | $* *$ | $1+2 i$ | $-7-10 i$ | $3-8 i$ | -5 | $115-40 i$ | $-153-288 i$ |

## 4 Galois Representations

## 4.1

We are interested in relating Hecke eigenforms and Galois representations. In particular, given a Hecke eigenform $F$ we would like to find (a compatible system of) $\lambda$-adic Galois representations

$$
\rho_{F, \lambda}: \operatorname{Gal}(\overline{\mathbb{Q}} / \mathbb{Q}) \longrightarrow \operatorname{GL}\left(W_{\lambda}\right)
$$

having the same local $L$-factors as $F$. Here $\lambda$ is a prime in a finite extension $K_{\lambda}$ of $\mathbb{Q}_{\ell}$ and $W_{\lambda}$ is a (finite dimensional) $K_{\lambda}$ vector space. The local $L$-factors of
$\rho_{F, \lambda}$ (again independent of $\lambda$ ) being defined as before (for unramified primes, conjecturally those not dividing $N \ell$ ):

$$
L_{p}\left(\rho_{F}, s\right):=\operatorname{det}\left(I-\rho_{F, \lambda}\left(F_{p}\right) p^{-s}\right)^{-1}
$$

In particular, we want $\operatorname{dim} W_{\lambda}=3$.

## 4.2

The case that $K_{F}$ is totally real is analyzed by Clozel C2. We just recall that if in this case such a Galois representation $\rho_{F, \lambda}$ exists then $\rho_{F, \lambda}$ is selfdual in the following sense.

Consider the Tate-twisted dual Galois representation:
$\rho_{F, \lambda}^{*}:={ }^{t} \rho_{F, \lambda}^{-1}(-2): \operatorname{Gal}(\overline{\mathbb{Q}} / \mathbb{Q}) \longrightarrow \mathrm{GL}\left(W_{\lambda}\right), \quad$ so $\quad \rho_{F, \lambda}^{*}\left(F_{p}\right):=p^{2 t} \rho_{F, \lambda}^{-1}\left(F_{p}\right)$.
Let $\alpha_{i}, i=1,2,3$ be the eigenvalues of $\rho_{F, \ell}\left(F_{p}\right)$, then the eigenvalues of $\rho_{F, \ell}^{*}\left(F_{p}\right)$ are $\beta_{i}:=p^{2} / \alpha_{i}$. Since $\sum \alpha_{i}=a_{p}, \sum \alpha_{i} \alpha_{j}=p a_{p}$ (since now $\bar{a}_{p}=a_{p}$ ) and $\prod \alpha_{i}=p^{3}$, the sets of eigenvalues $\left\{\alpha_{i}\right\}$ and $\left\{\beta_{i}\right\}$ coincide.

Thus $L_{p}\left(\rho_{F}, s\right)=L_{p}\left(\rho_{F}^{*}, s\right)$ for all $p$ not dividing $N$ and so the (semisimplifications of the) Galois representations are the same. It implies also that a subgroup of finite index of the image of $\operatorname{Gal}(\overline{\mathbb{Q}} / \mathbb{Q})$ is contained in a group $G \subset \mathrm{GL}\left(W_{\lambda}\right)$ with $G \cong \mathrm{PGL}\left(2, K_{\lambda}\right)$. Examples of this are the $\mathrm{Sym}^{2}$ of Galois representations in $\mathrm{GL}\left(2, \mathbb{Q}_{\ell}\right)$.

## 4.3

We will be especially interested in the non-selfdual case. Since we found several examples of Hecke eigenforms $F$ with $K_{F}=\mathbb{Q}(i)$ we will consider that case here. To find corresponding Galois representations we use the fact that for any algebraic variety $X$ defined over $\mathbb{Q}$, one has a Galois representation on the etale cohomology:

$$
\operatorname{Gal}(\overline{\mathbb{Q}} / \mathbb{Q}) \longrightarrow \operatorname{GL}\left(H_{e t}^{n}\left(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_{\ell}\right)\right) .
$$

The point is to find a suitable $X$ and (a subspace of) a suitable $H_{e t}^{n}$. In case $X$ is smooth, projective, and has good reduction mod $p$, theorems of Grothendieck and Deligne imply that the eigenvalue polynomial of $F_{p}$ acting on $H_{e t}^{n}\left(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_{\ell}\right)$ has coefficients in $\mathbb{Z}$, is independent of $\ell$ and the eigenvalues of $F_{p}$ have absolute value $p^{n / 2}$.

The desired equality $L_{p}(F, s)=L_{p}\left(\rho_{F}, s\right)$ for the eigenforms $F$ from (3.6) (and one expects the same more generally for certain cusp forms, 'Ramanujan conjecture'), implies that the absolute value of the eigenvalues of $\rho_{F}\left(F_{p}\right)$ must be $p$. Therefore we will consider $H_{e t}^{2}$ and take $\operatorname{dim} X>1$ since $\operatorname{dim} H_{e t}^{2}=1$ for curves.

A well-known theorem implies that $H_{e t}^{2}\left(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_{\ell}\right) \hookrightarrow H_{e t}^{2}\left(S_{\overline{\mathbb{Q}}}, \mathbb{Q}_{\ell}\right)$ where $S$ is a suitable surface contained in $X$. Thus we restrict ourselves to considering $H_{e t}^{2}\left(S_{\overline{\mathbb{Q}}}, \mathbb{Q}_{\ell}\right)$ for a surface $S$.

The Galois representation on this $\mathbb{Q}_{\ell}$-vector space is reducible in general, a decomposition is:

$$
H_{e t}^{2}\left(S_{\overline{\mathbb{Q}}}, \mathbb{Q}_{\ell}\right)=T_{\ell} \oplus \mathrm{NS}\left(S_{\overline{\mathfrak{Q}}}\right) \otimes_{Z Z} \mathbb{Q}_{\ell}
$$

where $\mathrm{NS}\left(S_{\overline{\mathbb{Q}}}\right)$ is the Nèron-Severi group of the surface $S$ over $\overline{\mathbb{Q}}$ (the Galois group permutes the classes of divisors modulo a Tate twist) and $T_{\ell}$ is the orthogonal complement of $\mathrm{NS}\left(S_{\overline{\mathbb{Q}}}\right)$ with respect to the intersection form. The intersection form is the cup product $H_{e t}^{2} \times H_{e t}^{2} \rightarrow H_{e t}^{4} \cong \mathbb{Q}_{\ell}$. The eigenvalues of Frobenius on $\operatorname{NS}\left(S_{\overline{\mathbb{Q}}}\right) \otimes \mathbb{Q}$ are roots of unity multiplied by $p$, so $\rho_{F, \lambda}$, if it exists, should be a representation on a subspace of $T_{\ell} \otimes_{\mathbb{Q}_{\ell}} K_{\lambda}$.

In case $T_{\ell}$ has dimension 3 , the Galois representation on it will be selfdual (due to the intersection form). To find a 3 dimensional Galois representations with trace $F_{p} \in \mathbb{Z}[i]$ as desired we assume that the surface has an automorphism, defined over $\mathbb{Q}$ :

$$
\phi: S \longrightarrow S, \quad \text { with } \quad \phi^{4}=i d_{S} .
$$

Thus $\phi^{*}: H_{e t}^{2} \rightarrow H_{e t}^{2}$ will commute with the Galois representation.
Assume moreover that $\operatorname{dim} T_{\ell}=6$ and $\phi^{*}: T_{\ell} \rightarrow T_{\ell}$ has two 3-dimensional eigenspaces $W_{\lambda}, W_{\lambda}^{\prime}$ (with eigenvalue $\pm i$ ):

$$
T_{\lambda}:=T_{\ell} \otimes_{\mathbb{Q}_{\ell}} K_{\lambda}=W_{\lambda} \oplus W_{\lambda}^{\prime}
$$

with $K_{\lambda}$ an extension of $\mathbb{Q}_{\ell}$ containing $i$. Then we have a 3-dimensional Galois representation $\sigma^{\prime}$ on $W_{\lambda}$. The determinant of $\sigma^{\prime}\left(F_{p}\right)$ is in general not equal to $p^{3}$ but is $\chi(p) p^{3}$ for a Dirichlet character $\chi$. Twisting $\sigma^{\prime}$ by this character we get a Galois representation

$$
\sigma_{S, \lambda}: \operatorname{Gal}(\overline{\mathbb{Q}} / \mathbb{Q}) \longrightarrow \mathrm{GL}\left(W_{\lambda}\right)
$$

whose $L$-factors $L_{p}\left(\sigma_{S}, s\right)$ are similar to the $L_{p}(F, s)$ for the eigenforms in the example above.

Note that the intersection form $(\cdot, \cdot)$ restricted to $W_{\lambda}$ is trivial (because it is invariant under pull-back by $\phi^{*}$ and extends $K_{\lambda}$-linearly: $\left(w_{1}, w_{2}\right)=\left(\phi^{*} w_{1}, \phi^{*} w_{2}\right)$ $=\left(i w_{1}, i w_{2}\right)=i^{2}\left(w_{1}, w_{2}\right)=-\left(w_{1}, w_{2}\right)$ with $\left.w_{1}, w_{2} \in W_{\lambda}\right)$. Thus there is no obvious reason for $\sigma_{S}$ to be selfdual.

## 4.4

Now one has to search for such surfaces. The main problem is that in general $\operatorname{dim} H_{e t}^{2}$ will be large but rankNS will be small. Thus it is not so easy to get $\operatorname{dim} T_{\ell}=6$, see however vG-T and vG-T2 for various examples.

The most interesting example is given by the one parameter family of surfaces $S_{a}$ which are the smooth, minimal, projective model of the singular, affine surface defined in $x, y, t$-space by

$$
t^{2}=x y\left(x^{2}-1\right)\left(y^{2}-1\right)\left(x^{2}-y^{2}+a x y\right), \quad \text { and } \quad(x, y, t) \longmapsto(y,-x, t)
$$

defines the automorphism $\phi$. In vG-T, 3.7-3.9, we explain how to determine eigenvalue polynomials of $\sigma_{S, \lambda}\left(F_{p}\right)$, and thus the $L$-factors, basically using the Lefschetz trace formula and counting points on $S$ over finite fields. The main result is:

### 4.5 Theorem.

(vG-T], 3.11; GKTV], 3.9) The local $L$-factors of the modular forms for $N=$ $128,160,205$ in $\$ 3.6$ are the same as the local $L$-factors of the Galois representations $\sigma_{S_{a}, \lambda}$, with $a=2,1, \frac{1}{16}$ respectively, for all odd primes $p \leq 173$ not dividing $N$.

## 4.6

In vG-T2 we gave another construction of surfaces $S$ which define 3 dimensional Galois representations. These surfaces are degree 4 cyclic base changes of elliptic surfaces $\mathcal{E} \rightarrow \mathbb{P}^{1}$. By taking the orthogonal complement to a large algebraic part in $H_{e t}^{2}$ together with all cohomology coming from the intermediate degree 2 base change, one obtains a representation space, similar to $T_{\ell}$, for $\operatorname{Gal}(\overline{\mathbb{Q}} / \mathbb{Q})$. Taking an eigenspace $W_{\lambda}$ of the action of the automorphism of order 4 defining the cyclic base change finally gives 3 dimensional Galois representations.

Our main (technical) result is a formula for the traces of Frobenius elements on this space in terms of the number of points on $\mathcal{E}$ and $S$ over a finite field (VG-T2 ,Theorem 3.4). This formula allows us to compute the characteristic polynomial of Frobenius in many cases.

We use this result to prove that certain examples yield selfdual representations, while others do not. For some of the selfdual cases we can actually exhibit 2-dimensional Galois representations (defined by elliptic curves) whose symmetric square seems to coincide with the 3-dimensional Galois representation.

We did not find new examples of non-selfdual Galois representations with the same local $L$-factors as modular forms, probably because the conductor of these Galois representations is too large. We would like to point out that there does not seem to be an explicit way to determine the conductor of the Galois representation $\sigma_{S}$ in terms of the geometry of $S$ (a surface over $\left.\operatorname{Spec}(\mathbb{Q})\right)$.

## 5 Non-Cusp Forms and Galois Representions

## 5.1

In this section we give an example of the decomposition in Hecke eigenspaces of a cohomology group $H^{3}\left(\Gamma_{0}(N), \mathbb{C}\right)$. We will take $N=245$. This example is also mentioned in GKTV, $\S 3.5$ where it is shown that a certain 8 dimensional Hecke invariant subspace of $H^{3}\left(\Gamma_{0}(245), \mathbb{C}\right)$ contains no cusp forms. Here we extend this by interpreting most of the 83 dimensional space $H^{3}\left(\Gamma_{0}(245), \mathbb{C}\right)$ in terms of so-called Eisenstein liftings of classical elliptic cusp forms and of Eisenstein series.

As before, if $F \in H^{3}\left(\Gamma_{0}(245), \mathbb{C}\right)$ is an eigenform for all Hecke operators, we denote by $K_{F}$ the field generated by all eigenvalues of the Hecke operators on $F$. As a first step towards the decomposition we have the following Proposition.

Proposition 1. The cohomology group $H^{3}\left(\Gamma_{0}(245), \mathbb{C}\right)$ decomposes as

$$
H^{3}\left(\Gamma_{0}(245), \mathbb{C}\right)=V_{1} \oplus V_{2} \oplus V_{3} \oplus V_{4} \oplus V_{5}
$$

(as a module over the Hecke algebra), with
$-\operatorname{dim} V_{1}=25$ and $V_{1}$ is generated by eigenforms $F$ with $K_{F}=\mathbb{Q}$;
$-\operatorname{dim} V_{2}=16$ and $V_{2}$ is generated by eigenforms $F$ with $K_{F}=\mathbb{Q}(\sqrt{2})$;
$-\operatorname{dim} V_{3}=16$ and $V_{3}$ is generated by eigenforms $F$ with $K_{F}=\mathbb{Q}(\sqrt{17})$;
$-\operatorname{dim} V_{4}=8$ and $V_{4}$ is generated by eigenforms $F$ with $K_{F}=\mathbb{Q}(\sqrt{2}, \sqrt{-3})$;
$-\operatorname{dim} V_{5}=18$ and $V_{5}$ is generated by eigenforms $F$ with $K_{F}=\mathbb{Q}(\sqrt{-3})$.
None of the spaces $V_{1}, \ldots, V_{5}$ contains a non-zero cuspform; in fact, these spaces are generated by Eisenstein liftings or (in the case of $V_{4}$ and $V_{5}$ ) twists of such by cubic Dirichlet characters.

## 5.2

With notations as given in GKTV §3.5, one has $V_{4}=V_{a} \oplus V_{b}$, hence this case of the above proposition is already described in loc. sit.

We briefly recall the two types of Eisenstein liftings of classical modular forms here. Let $f$ be a normalized elliptic cuspform of level $N$ and weight 2 , which is an eigenform for the Hecke operators $T_{n}$ with $(n, N)=1$. Also, we allow $f$ to be the normalized Eisenstein series of weight 2: $f=-B_{2} / 4+\sum_{n=1}^{\infty} \sigma_{1}(n) q^{n}$; so $a_{p}=$ $p+1$, compare e.g. for notations. The Fourier coefficients in the $q$-expension $f=q+a_{2} q^{2}+a_{3} q^{3}+\cdots$ define a Dirichlet series $L(f, s)=\sum_{n} a_{n} n^{-s}$. This series has an Euler product expansion with Euler factors $\left(1-a_{p} p^{-s}+p^{1-2 s}\right)^{-1}$ for primes $p$ which do not divide $N$ (in case $f$ is the Eisenstein series, these factors are $\left.\left(1-p^{-s}\right)^{-1}\left(1-p^{1-s}\right)^{-1}\right)$.

Given $f$, one constructs two eigenclasses $F_{1}, F_{2} \in H^{3}\left(\Gamma_{0}(N), \mathbb{C}\right)$. The $F_{1}$ has eigenvalue $p a_{p}+1$ for the $p$ th Hecke operator $E_{p}$, and $F_{2}$ eigenvalue $a_{p}+p^{2}$.

On the Galois side of the Langlands correspondence, it is relatively easy to describe these liftings. Namely, if $f$ corresponds to a 2 dimensional $\lambda$-adic representation space $V$ for $\operatorname{Gal}(\overline{\mathbb{Q}} / \mathbb{Q})$, then $F_{1}$ corresponds to $V(-1) \oplus \mathbb{Q}_{\lambda}(0)$ and $F_{2}$ to $V \oplus \mathbb{Q}_{\lambda}(-2)$ where $\mathbb{Q}_{\lambda}(n)$ is the 1 dimensional $\lambda$-adic representation space on which the Galois group acts by the $-n$-th power of the cyclotomic character (thus $F_{p}$ acts as $p^{-n}$ ). In case $f$ is the Eisenstein series, we have $V=\mathbb{Q}_{\lambda}(0) \oplus \mathbb{Q}_{\lambda}(-1)$ and the two lifted representations coincide (both are $\left.\mathbb{Q}_{\lambda}(0) \oplus \mathbb{Q}_{\lambda}(-1) \oplus \mathbb{Q}_{\lambda}(-2)\right)$.

## 5.3

There exists a unique normalized cuspform of weight 2 and level 35 which has Q-rational Fourier coefficients. This form yields 2 eigenclasses in $H^{3}\left(\Gamma_{0}(35), \mathbb{C}\right)$;
from the theory of oldforms Ree, each of these appears three times at level $35 \cdot 7=245$.

Similarly, the modular form corresponding to the CM elliptic curve of conductor 49 gives rise to six oldforms which are Eisenstein liftings.

Starting from the Eisenstein series, one finds 7 forms at level 245 all with eigenvalues $1+p+p^{2}$.

Finally, from tables of Cremona (as well as from unpublished tables of Cohen, Skoruppa and Zagier) it follows that there exist 3 (elliptic) newforms of level 245 which are Hecke eigenforms with rational eigenvalues. Each of them gives us two Eisenstein liftings.

Adding up, we now have $6+6+7+6=25$ eigenclasses of level 245 with rational eigenvalues. Our calculations made for the tables in GKTV revealed that, e.g., the Hecke operator $E_{2}$ has precisely 25 rational eigenvalues (counted with multiplicity). Hence the conclusion is, that the space $V_{1}$ given in Proposition 11 indeed has $\operatorname{dim} V_{1}=25$, and it is generated by Eisenstein liftings as claimed.

## 5.4

The cases $V_{2}, V_{3}$ are completely analogous. For $V_{2}$, we note that there exist newforms of weight 2 and level 245 with $q$-expansion $q+\sqrt{2} q^{2}+(1+\sqrt{2}) q^{3}+\ldots$ and $q+(1+\sqrt{2}) q^{2}+(1-\sqrt{2}) q^{3}+\ldots$ respectively. These together with their Galois conjugate forms and their twists by the quadratic Dirichlet character modulo 7 give us 8 newforms of level 245 . Each of them yields two Eisenstein liftings, and this precisely describes the space $V_{2}$ of dimension 16 .

Similarly, there are exactly two (conjugate) newforms of level 35 with Fourier coefficients generating $\mathbb{Q}(\sqrt{17})$. They provide $2 \cdot 2=4$ Eisenstein liftings of level 35 , and hence $3 \cdot 4=12$ oldforms of level 245 . Twisting the newforms by the quadratic character modulo 7 yields newforms of level 245 , and from these we find another 4 Eisenstein liftings. In this way, $V_{3}$ is generated.

## 5.5

Having described $V_{1}, \ldots, V_{4}$ (the latter space was already treated in GKTV), and observing from Table 3.3 that $\operatorname{dim} H^{3}\left(\Gamma_{0}(245), \mathbb{C}\right)=83$, we conclude we still have to describe a Hecke-invariant space of dimension $83-(25+16+16+8)=18$. To this end, we mention that at level $49=245 / 5$, our programs found a 6 dimensional Hecke invariant subspace on which the operator $E_{2}$ acts with 6 (pairwise conjugate, pairwise different) eigenvalues in $\mathbb{Q}(\sqrt{-3})$. Hence this space yields eigenforms with $K_{F}=\mathbb{Q}(\sqrt{-3})$. Moreover, it lifts to a Hecke invariant subspace of dimension $3 \cdot 6=18$ at level 245 , which therefore exactly equals the summand $V_{5}$ of $H^{3}$ we did not describe yet.

As an example, the eigenvalues of the operator $E_{3}$ on $V_{5}$ are $a_{3}, \overline{a_{3}}, a_{3} \omega$, $\overline{a_{3}} \omega, a_{3} \bar{\omega}$ and $\overline{a_{3}} \omega$ where $\omega^{2}+\omega+1=0$ and $a_{3}=-5-3 \sqrt{-3}$. This situation is explained as follows. The Euler factor that corresponds to a Hecke eigenclass is obtained using the polynomial $X^{3}-a_{p} X^{2}+p b_{p} X-p^{3}$, where $a_{p}$ is the eigenvalue
of the operator $E_{p}$. The number $b_{p}$ similarly corresponds to the operator $D_{p}=$ $T_{\beta_{p}}$, defined using $\beta_{p}:=\left(\begin{array}{ccc}p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & 1\end{array}\right) \in \mathrm{GL}(3, \mathbb{Q})$. If the eigenclass is cuspidal, then $b_{p}$ is the complex conjugate of $a_{p}$. This in fact follows from the fact that the associated automorphic representation is unitary in that case. In our situation however, a computation shows that $b_{3}=a_{3} \neq \overline{a_{3}}$. Hence the representation cannot be unitary and therefore the eigenclasses here are not cuspidal.

Based on calculations for primes $\leq 131$, the Hecke eigenvalues seem to be as follows. For $p \neq 5,7$ we have $b_{p}=a_{p}=\chi(p)\left(\psi(p)+p+\psi^{2}(p) p^{2}\right)$ with $\chi, \psi$ Dirichlet characters modulo 7 of order dividing 3 . This corresponds to the sum of 1-dimensional Galois representations

$$
\left(\chi \psi \otimes \mathbb{Q}_{\lambda}(0)\right) \oplus\left(\chi \otimes \mathbb{Q}_{\lambda}(-1)\right) \oplus\left(\chi \psi^{2} \otimes \mathbb{Q}_{\lambda}(-2)\right) .
$$

## 6 Variations of Hodge Structures of Weight Two

## 6.1

In all our constructions for Galois representations we consider a subspace $T_{\ell} \subset$ $H_{e t}^{2}\left(S_{\overline{\mathbb{Q}}}, \mathbb{Q}_{\ell}\right)$. This subspace is defined using algebraic cycles, thus there exists also a Betti realization $T_{\mathbb{Z}} \subset H^{2}(S(\mathbb{C}), \mathbb{Z})$ (of the motive $T$ ) which is a polarized Hodge structure of weight two. We recall the relevant definitions and the main results of Griffiths and Carlson on the moduli of the $T_{Z Z}$ 's.

The main point is the essential difference with the weight one case (which is essentially the theory of abelian varieties). In the weight one case, one has a universal family of abelian varieties over suitable quotients of the Siegel space. In the weight two (and higher) case, the analogy of the Siegel space is a certain (subset of a) period domain, but in general (and in particular this is the case with the $T_{Z Z}$ under consideration), the (polarized) Hodge structures obtained from algebraic varieties do not fill up the period space. In fact we will see that the Hodge structures like $T_{Z Z}$ are parametrized by a 4 -dimensional space, but those that come from geometry have at most a 2 -dimensional deformation space (and imposing an automorphism of order 4 as we do implies a 1-dimensional deformation space).

It is not clear whether these period spaces (or the subvarieties parametrizing 'geometrical' Hodge structures) have good arithmetical properties like Shimura varieties.

## 6.2

Recall that a $\mathbb{Z}$-Hodge structure $V$ of weight $n$ is a free $\mathbb{Z}$-module of finite rank together with decomposition:

$$
V_{\mathbb{C}}:=V \otimes_{\mathbb{Z}} \mathbb{C}=\oplus_{p+q=n} V^{p, q}, \quad \text { with } \quad \overline{V^{p, q}}=V^{q, p},
$$

where the $V^{p, q}$ are complex vector spaces and the bar indicates complex conjugation (given by $\overline{v \otimes z}=v \otimes \bar{z}$ ).

A rational Hodge structure $V_{\mathbb{Q}}$ is a finite dimensional $\mathbb{Q}$-vector space with a similar decomposition of $V_{\mathbb{C}}:=V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$. Thus a $\mathbb{Z}$-Hodge structure $V$ determines a rational Hodge structure on $V_{\mathbb{Q}}:=V \otimes_{Z Z} \mathbb{Q}$.

A (rational) Hodge structure $V_{\mathbb{Q}}$ determines an $\mathbb{R}$-linear map, the Weil operator:

$$
J: V_{\mathbb{R}}:=V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R} \longrightarrow V_{\mathbb{R}} \quad \text { with } \quad J_{\mathbb{C}} v_{p, q}=i^{p-q} v_{p, q}
$$

for all $v_{p, q} \in V^{p, q}$ and $J_{\mathbb{C}}$ is the $\mathbb{C}$-linear extension of $J$. One has $J^{2}=(-1)^{n}$ since $i^{2 p-2 q}=(-1)^{p-q}=(-1)^{p+q}$. Thus $J$ determines a complex structure on $V_{\mathbb{R}}$ in case $V$ has odd weight.

A polarization on a rational Hodge structure $V_{\mathbb{Q}}$ of weight $n$ is a bilinear map

$$
\Psi: V_{\mathbb{Q}} \times V_{\mathbb{Q}} \longrightarrow \mathbb{Q}, \quad \Psi_{\mathbb{C}}\left(v_{p, q}, v_{r, s}\right)=0 \quad \text { unless } \quad p+r=q+s=n
$$

(intrinsically: $\Psi: V_{\mathbb{Q}} \otimes V_{\mathbb{Q}} \rightarrow \mathbb{Q}(-n)$ is a morphism of Hodge structures) which satisfies the Riemann relations, that is, for all $v, w \in V_{\mathbb{R}}$ :

$$
\Psi(v, J w)=\Psi(w, J v), \quad \Psi(v, J v)>0 \quad(\text { if } v \neq 0)
$$

thus $\Psi$ defines an inner product $\Psi(-, J-)$ on $V_{\mathbb{R}}$.
One easily verifies, using the first property, that $\Psi(J v, J w)=\Psi(v, w)$, since also $\Psi(J v, J w)=\Psi\left(w, J^{2} v\right)=(-1)^{n} \Psi(w, v)$, a polarization is symmetric if $n$ is even and antisymmetric if $n$ is odd.

## 6.3

For a smooth complex projective variety $X$ the cohomology groups $H^{n}(X, \mathbb{Q})$ are polarized rational Hodge structures of weight $n$. One writes $H^{p, q}(X):=$ $H^{n}(X, \mathbb{C})^{p, q}$. In case $X$ is a surface, the cup product on $H^{2}(X, \mathbb{Q})$ (note that $\left.H^{4}(X, \mathbb{Q})=\mathbb{Q}\right)$ gives ( -1 times) a polarization on the primitive cohomology $H_{\text {prim }}^{2}$. In particular it induces a polarization $\Psi$ on the sub-Hodge structure $T_{\mathbb{Q}}=\mathrm{NS}^{\perp}$ of $H^{2}(S(\mathbb{C}), \mathbb{Q})$ which we consider.

## 6.4

Let $T_{Z Z}$ be a Hodge structure of weight 2 and rank 6 with

$$
T_{\mathbb{C}}=T^{2,0} \oplus T^{1,1} \oplus T^{0,2}, \quad \operatorname{dim} T^{p, q}=2
$$

for all $p, q$. Then one easily verifies that:

$$
T_{\mathbb{R}}=W_{1} \oplus W_{2} \quad \text { with } \quad\left\{\begin{array}{l}
W_{1}:=T_{\mathbb{R}} \cap T^{1,1} \\
W_{2}:=T_{\mathbb{R}} \cap\left(T^{2,0} \oplus T^{0,2}\right)
\end{array}\right.
$$

For $v \in W_{1} \subset T^{1,1}$ we have $J v=v$ and thus $\Psi(v, v)=\Psi(v, J v)>0$, so $\Psi$ is positive definite on $W_{1}$. Hence we can choose an $\mathbb{R}$ basis $f_{1}$, $f_{2}$ of $W_{1}$ which is orthonormal w.r.t. $\Psi$ and which is a $\mathbb{C}$-basis of $T^{1,1}=W_{1} \otimes_{\mathbb{R}} \mathbb{C}$.

For $v \in W_{2}$ we have $v=v_{2,0}+v_{0,2}$ thus $J v=-v$ and so $\Psi$ is negative definite on $W_{2}$. Let $v_{1}:=e_{1}+\bar{e}_{1}, \quad v_{2}:=e_{2}+\bar{e}_{2}$ be an orthonormal basis for $(-1 / 2) \Psi$ on $W_{2}$ with $e_{1}, e_{2} \in V^{2,0}$. Then $e_{1}, e_{2}$ is a $\mathbb{C}$-basis of $T^{2,0}$ (and thus $\bar{e}_{1}, \bar{e}_{2}$ is a $\mathbb{C}$ basis of $\left.T^{0,2}\right)$. Note $-2=\Psi\left(e_{1}+\bar{e}_{1}, e_{1}+\bar{e}_{1}\right)=\Psi\left(e_{1}, \bar{e}_{1}\right)+\Psi\left(\bar{e}_{1}, e_{1}\right)=2 \Psi\left(e_{1}, \bar{e}_{1}\right)$ (since $\Psi$ is symmetric). In this way one finds $\Psi\left(e_{k}, \bar{e}_{l}\right)=-\delta_{k l}$ (Kronecker's delta) thus $\Psi_{\mathbb{C}}$ is given by the matrix $Q$ on the basis $e_{1}, e_{2}, f_{1}, f_{2}, \bar{e}_{1}, \bar{e}_{2}$ of $T_{\mathbb{C}}$ :

$$
Q=\left(\begin{array}{ccc}
0 & 0 & -I \\
0 & I & 0 \\
-I & 0 & 0
\end{array}\right)
$$

## 6.5

We consider first order deformations of the polarized Hodge structure $T_{Z Z}$ as in §6.4. Thus we fix the $Z Z$-module and the bilinear map $\Psi$ and consider deformations of the Hodge structure induced by deformations of an algebraic variety $X$ with $T_{Z Z} \subset H^{2}(X, Z Z)$, that is, of the direct sum decomposition $T_{\mathbb{C}}=\oplus T^{p, q}$.

The first order deformations of a smooth complex projective algebraic variety $X$ are parametrized by $H^{1}\left(X, \Theta_{X}\right)$ with $\Theta_{X}$ the tangent bundle of $X$ (KodairaSpencer theory). The isomorphisms $H^{p, q}(X)=H^{q}\left(X, \Omega^{p}\right)$ and the contraction $\operatorname{map} \Theta_{X} \otimes_{\mathcal{O}_{X}} \Omega_{X}^{p} \rightarrow \Omega_{X}^{p-1}$ give a cup product map:

$$
H^{1}\left(X, \Theta_{X}\right) \otimes H^{p, q}(X) \longrightarrow H^{p-1, q+1}(X)
$$

Thus, for any $n$, we obtain a map, called the infinitesimal period map:

$$
\delta: H^{1}\left(X, \Theta_{X}\right) \longrightarrow \oplus_{p+q=n} \operatorname{Hom}\left(H^{p, q}(X), H^{p-1, q+1}(X)\right)
$$

Griffiths proved that for $\theta \in H^{1}\left(X, \Theta_{X}\right)$, the deformation of the Hodge structure induced by the deformation of $X$ in the direction of $\theta$ is essentially given by $\delta(\theta)$.

The subspace $\Im(\delta)$ of $\oplus_{p+q=n} \operatorname{Hom}\left(H^{p, q}(X), H^{p-1, q+1}(X)\right)$ satisfies (at least) two conditions. The first comes from the polarization (see $\S(6.6)$, the second is an integrability condition found by Griffiths which is non-trivial only if the weight of the Hodge structure is greater than one (see $\S$ 6.8).

We will now spell out the restriction of these conditions to the sub Hodge structure $\oplus_{p+q=n} \operatorname{Hom}\left(T^{p, q}, T^{p-1, q+1}\right)$.

## 6.6

The condition that $\psi \in \oplus_{p+q=n} \operatorname{Hom}\left(T^{p, q}, T^{p-1, q+1}\right) \subset \operatorname{End}\left(T_{\mathbb{C}}\right)$ preserves the polarization on $T$, is that $\Psi((I+t \psi) v,(I+t \psi) w)=\Psi(v, w)$ when $t^{2}=0$ :

$$
\Psi_{\mathbb{C}}(\psi(v), w)+\Psi_{\mathbb{C}}(v, \psi(w))=0 \quad \forall x, y \in T_{\mathbb{C}}
$$

This condition implies that if $\psi$ preserves $\Psi$, then it is determined by $\psi_{2}$ where

$$
\psi=\left(\psi_{2}, \psi_{1}\right) \in \operatorname{Hom}\left(T^{2,0}, T^{1,1}\right) \oplus \operatorname{Hom}\left(T^{1,1}, T^{0,2}\right)
$$

In fact, for all $v \in T^{2,0}$ and $w \in T^{1,1}$ we now have: $\Psi_{\mathbb{C}}\left(v, \psi_{1}(w)\right)=-\Psi_{\mathbb{C}}\left(\psi_{2}(v), w\right)$. Since $\Psi_{\mathbb{C}}$ identifies $\left(T^{0,2}\right)^{\text {dual }}$ with $T^{2,0}$, this equality thus defines $\phi_{1}(w)$ in terms of $\phi_{2}$.

## 6.7

With respect to the basis of $T_{\mathbb{C}}$ considered in 6.4, $\psi \in \operatorname{Hom}\left(T^{2,0}, T^{1,1}\right) \oplus$ $\operatorname{Hom}\left(T^{1,1}, T^{0,2}\right) \subset \operatorname{End}\left(T_{\mathbb{C}}\right)$ is given by a matrix $N$ and the condition on $\psi$ becomes ${ }^{t} N Q+Q N=0$ so:

$$
N=\left(\begin{array}{ccc}
0 & 0 & 0 \\
A & 0 & 0 \\
0 & B & 0
\end{array}\right) \quad \text { and } \quad B={ }^{t} A
$$

where the matrix $A$ (defining $\phi_{2}: T^{2,0} \rightarrow T^{1,1}$ ) can be chosen arbitrarily. This gives an isomorphism between the space $M_{2}(\mathbb{C})$ of $2 \times 2$ complex matrices and polarization preserving deformations $\psi$ :
$M_{2}(\mathbb{C}) \stackrel{\cong}{\Longrightarrow}\left(\operatorname{Hom}\left(T^{2,0}, T^{1,1}\right) \oplus \operatorname{Hom}\left(T^{1,1}, T^{0,2}\right)\right)_{\Psi}, \quad A \longmapsto N(A):=\left(\begin{array}{ccc}0 & 0 & 0 \\ A & 0 & 0 \\ 0 & { }^{t} A & 0\end{array}\right)$.
Thus we have a four dimensional deformation space. In case of Hodge structures of weight one, preserving the polarization is the only infinitesimal condition. Here, in the weight two case, there is however another condition.

## 6.8

An important restriction, discovered by Griffiths, on the image of $\delta$ is:

$$
[\operatorname{Im} \delta, \operatorname{Im} \delta]=0 \quad \text { i.e. } \quad \delta(\alpha) \circ \delta(\beta)=\delta(\beta) \circ \delta(\alpha)
$$

for all $\alpha, \beta \in H^{1}\left(X, \Theta_{X}\right)$, so $\Im(\delta)$ is an abelian subspace of $\operatorname{End}\left(T_{\mathbb{C}}\right)$. For Hodge structures of weight $n \geq 2$ this imposes non-trivial conditions on the (dimension of) the image of $\delta$. We consider again our example (cf. Ca ).

## 6.9

We already determined the polarization preserving deformations in $\oint 6.7$. Using the same notation we find that Griffiths' condition is:

$$
N(A) N(B)=N(B) N(A) \quad \text { thus } \quad{ }^{t} A B={ }^{t} B A
$$

This condition can be rephrased as saying that ${ }^{t} A B$ must be symmetric.
Thus the image of $\delta$ is at most two dimensional and if it is two dimensional with basis $N(A), N(B)$ then $A$ and $B$ span a maximal isotropic subspace of the symplectic form:

$$
E: M_{2}(\mathbb{C}) \times M_{2}(\mathbb{C}) \longrightarrow \mathbb{C}, \quad E(A, B):=a_{11} b_{12}-a_{12} b_{11}+a_{21} b_{22}-a_{22} b_{21}=0 .
$$

We recall that we also have an automorphism $\phi^{*}: T \rightarrow T$, preserving this automorphism gives another non-trivial condition on the deformations. Thus the one parameter in our surfaces $S_{a}$ (and in the other examples from vG-T2) is the maximal possible.

## References

AGG. A. Ash, D. Grayson and P. Green, Computations of Cuspidal Cohomology of Congruence Subgroups of SL(3, ZZ), J. Number Theory 19 (1984) 412436.

AR. A. Ash and L. Rudolph, The modular symbol and continued fractions in higher dimensions, Invent. Math. 55 (1979) 241-250.
Ca. J. A. Carlson, Bounds on the dimension of variations of Hodge structure, Trans. A.M.S. 294 (1986) 45-64, Erratum: Trans. A.M.S. 299 (1987) 429.
C1. L. Clozel, Motifs et formes automorphes: applications du principe de fonctorialité. In: L. Clozel and J.S. Milne (eds.), Automorphic Forms, Shimura Varieties and L-functions, Proceedings of the Ann Arbor Conference, 77159. New York London: Academic Press (1990).

C2. L. Clozel, Représentations galoisiennes associées aux représentations automorphes autoduales de GL(n). Publ. Math. IHES 73 (1991) 79-145.
D. P. Deligne, Formes modulaires et répresentations $\ell$-adiques. In: Sém. Bourbaki, Springer LNM 179 (1971) 136-186.
vG-T. B. van Geemen and J. Top, A non-selfdual automorphic representation of $\mathrm{GL}_{3}$ and a Galois representation, Invent. Math. 117 (1994) 391-401.
vG-T2. B. van Geemen and J. Top, Selfdual and non-selfdual 3-dimensional Galois representations, Compos. Math. 97 (1995) 51-70.
GKTV. B. van Geemen, W. van der Kallen, J. Top and A. Verberkmoes, Hecke eigenforms in the Cohomology of Congruence Subgroups of SL(3, $\mathbb{Z})$, Experimental Math. 6 (1997) 163-174.
Ko. N. Koblitz, Introduction to Elliptic Curves and Modular Forms. SpringerVerlag, New York etc.: 1984.
Ree. M. Reeder, Old forms on GL( $n$ ), Am. J. of Math. 113 (1991) 911-930.

