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Connecting many-sorted theories

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Connecting many-sorted theories

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Abstract

Basically, the connection of two many-sorted theories is obtained by taking their disjoint union, and then connecting the two parts through connection functions that must behave like homomorphisms on the shared signature. We determine conditions under which decidability of the validity of universal formulae in the component theories transfers to their connection. In addition, we consider variants of the basic connection scheme.

Contents

1 Introduction

The combination of decision procedures for logical theories arises in many areas of logic in computer science, such as constraint solving, automated deduction, term rewriting, modal logics, and description logics. In general, one has two first-order theories T_1 and T_2 over signatures Σ_1 and Σ_2, for which validity of a certain type of formulae (e.g., universal, existential positive, etc.) is decidable. These theories are then combined into a new theory T over a combination Σ of the signatures Σ_1 and Σ_2. The question is whether decidability transfers from T_1, T_2 to their combination T.

One way of combining the theories T_1, T_2 is to build their union $T_1 \cup T_2$. Both the Nelson-Oppen combination procedure [NO79, Nel84] and combination procedures for the word problem [Pig74, SS89, Nip91, BT97] address this type of combination, but for different types of formulae to be decided. Whereas the original combination procedures were restricted to the case of theories over disjoint signatures, there are now also solutions for the non-disjoint case [DKR94, TR03, BT02, FG03, Ghi05, BGT04], but they always require some additional restrictions since it is easy to see that in the unrestricted case decidability does not transfer. Similar combination problems have also been investigated in modal logic, where one asks whether decidability of (relativized) validity transfers from two modal logics to their fusion [KW91, Spa93, Wol98, BLSW02]. The approaches in [Ghi05, BGT04] actually generalize these results from equational theories induced by modal logics to more general first-order theories satisfying certain model-theoretic restrictions: the theories T_1, T_2 must be *compatible* with their shared theory T_0, and this shared theory must be *locally finite* (i.e., its finitely generated models are finite). The theory T_i is compatible with the shared theory T_0 iff (i) $T_0 \subseteq T_i$; (ii) T_0 has a model completion T_0^*; and (iii) every model of T_i embeds into a model of $T_i \cup T_0^*$.

In [KLWZ04], a new combination scheme for modal logics, called $\mathcal{E}$-connection, was introduced, for which decidability transfer is much simpler to show than in the case of the fusion. Intuitively, the difference between fusion and $\mathcal{E}$-connection can be explained as follows. A model of the fusion is obtained from two models of the component logics by identifying their domains. In contrast, a model of the $\mathcal{E}$-connection consists of two separate models of the component logics together with certain connecting relations between their domains. There are also differences in the syntax of the combined logic. In the case of the fusion, the Boolean operators are shared, and all operators can be applied to each other without restrictions. In the case of the $\mathcal{E}$-connection, there are two copies of the Boolean operators, and operators of the different logics cannot be mixed; the only connection between the two logics are new (diamond) modal operators that are induced by the connecting relations.

If we want to adapt this approach to the more general setting of combining first-order theories, then we must consider many-sorted theories since only the sorts allow us to keep the domains separate and to restrict the way function symbols can be applied to each other. Let T_1, T_2 be two many-sorted theories that may share some sorts as well as function and relation symbols. We first build the disjoint union $T_1 \uplus T_2$ of these two theories (by using disjoint copies of the shared parts), and then connect them by introducing *connection functions* between the shared sorts. These connection functions must behave like homomorphisms for the shared function and predicate symbols, i.e., the axioms stating this are added to $T_1 \uplus T_2$. This corresponds to the fact that the new diamond operators in the $\mathcal{E}$-connection approach distribute over disjunction and do not change the false formula $\bot$. We call the combined theory obtained this way the *connection* of T_1 and T_2.

This kind of connection between theories has already been considered in automated deduction (see, e.g., [AK97, Zar02]), but only in very restricted cases where both T_1 and T_2 are fixed theories (e.g., the theory of sets and the theory of integers in [Zar02]) and the connection functions have a fixed meaning (like yielding the length of a list). In categorical logic, this type of connection can be seen as an instance of a more general co-comma construction in bicategories associated with theories and syntactic interpretations, see for instance [Zaw95]. However, in this general setting, computational properties of the combined theories have not been considered yet.

This paper is a first step towards providing general results on the transfer of decidability from component theories to their connection. We start by considering the simplest case where there is just one connection function, and show that decidability transfers whenever certain model-theoretic conditions are satisfied. These conditions are weaker than the ones required in [BGT04] for the case of the union of theories.[1] In addition, both the combination procedure and its proof of correctness are much simpler than the ones in [Ghi05, BGT04]. The approach easily extends to the case of several connection functions. We will also consider variants of the general combination scheme where the connection function must satisfy additional properties (like being surjective, an embedding, an isomorphism), or where a theory is connected with itself. The first variant is, for example, interesting since the combination result for the union of theories shown in [Ghi05] can be obtained from the variant where one has an isomorphism as connection function. The second case is interesting since it can be used to reduce the global consequence problem in the modal logic **K** to propositional satisfiability, which is a surprising result.

[1] Our conditions are in general not weaker than the ones in [Ghi05], alhough this is the case for all the theories we have considered until now.

2 Notation and definitions

In this section, we fix the notation and give some important definitions, in particular a formal definition of the connection of two theories.

2.1 Many-sorted first-order logic

We use standard *many-sorted first-order logic* (see, e.g., [Gal86]), but try to avoid the notational overhead caused by the presence of sorts as much as possible. Thus, a *signature* Ω consists of a non-empty set of sorts $\mathcal{S}$ together with a set of function symbols $\mathcal{F}$ and a set of predicate symbols $\mathcal{P}$. The function and predicate symbols are equipped with arities from $\mathcal{S}^*$ in the usual way. For example, if the arity of $f \in \mathcal{F}$ is $S_1 S_2 S_3$, then this means that the function f takes tuples consisting of an element of sort S_1 and an element of sort S_2 as input, and produces an element of sort S_3. We consider logic with equality, i.e., the set of predicate symbols contains a symbol $\approx_S$ for equality in every sort S. Usually, we will just use $\approx$ without explicitly specifying the sort. In this paper we usually assume that signatures are countable.

Terms and first-order formulae over Ω are defined in the usual way, i.e., they must respect the arities of function and predicate symbols, and the variables occurring in them are also equipped with sorts. An Ω-*atom* is a predicate symbol applied to (sort-conforming) terms, and an Ω-*literal* is an atom or a negated atom. A *ground* literal is a literal that does not contain variables. We use the notation $\phi(\underline{x})$ to express that ϕ is a formula whose free variables are among the ones in the tuple of variables $\underline{x}$. An Ω-*sentence* is a formula over Ω without free variables. An Ω-*theory* T is a set of Ω-sentences (called the axioms of T). If T, T' are Ω-theories, then we write (by a sleight abuse of notation) $T \subseteq T'$ to express that all the axioms of T are logical consequences of the axioms of T'.

From the semantic side, we have the standard notion of an Ω-*structure* $\mathcal{A}$, which consists of non-empty and pairwise disjoint domains A_S for every sort S, and interprets function symbols f and predicate symbols P by functions $f^{\mathcal{A}}$ and predicates $P^{\mathcal{A}}$ according to their arities. By A (or sometimes by $|\mathcal{A}|$) we denote the union of all domains A_S. Validity of a formula ϕ in an Ω-structure $\mathcal{A}$ ($\mathcal{A} \models \phi$), satisfiability, and logical consequence are defined in the usual way. The Ω-structure $\mathcal{A}$ is a *model* of the Ω-theory T iff all axioms of T are valid in $\mathcal{A}$. If $\phi(\underline{x})$ is a formula with free variables $\underline{x} = x_1, \ldots, x_n$ and $\underline{a} = a_1, \ldots, a_n$ is a (sort-conforming) tuple of elements of A, then we write $\mathcal{A} \models \phi(\underline{a})$ to express that $\phi(\underline{x})$ is valid in $\mathcal{A}$ under the assignment $\{x_1 \mapsto a_1, \ldots, x_n \mapsto a_n\}$. Note that $\phi(\underline{x})$ is valid in $\mathcal{A}$ iff it is valid under all assignments iff its universal closure is valid in $\mathcal{A}$.

An Ω-*homomorphism* between two Ω-structures $\mathcal{A}$ and $\mathcal{B}$ is a mapping $\mu : A \to B$ that is sort-conforming (i.e., maps elements of sort S in $\mathcal{A}$ to elements of sort S

in $\mathcal{B}$), and satisfies the condition

$$(*) \qquad \mathcal{A} \models A(a_1, \ldots, a_n) \quad \text{implies} \quad \mathcal{B} \models A(\mu(a_1), \ldots, \mu(a_n))$$

for all Ω-atoms $A(x_1, \ldots, x_n)$ and (sort-conforming) elements $a_1, \ldots, a_n$ of A. In case the converse of $(*)$ holds too, μ is called an *embedding*. Note that an embedding is something more than just an injective homomorphism since the stronger condition must hold not only for the equality predicate, but for all predicate symbols. If the embedding μ is the identity on A, then we say that $\mathcal{A}$ is a *substructure* of $\mathcal{B}$. In case $(*)$ holds for all first order formulae, then μ is said to be an *elementary* embedding. If the elementary embedding μ is the identity on A, then we say that $\mathcal{A}$ is an *elementary substructure* of $\mathcal{B}$ or that $\mathcal{B}$ is an *elementary extension* of $\mathcal{A}$. An *isomorphism* is a surjective embedding.

We say that Σ is a subsignature of Ω (written $\Sigma \subseteq \Omega$) iff Σ is a signature that can be obtained from Ω by removing some of its sorts and function and predicate symbols. If $\Sigma \subseteq \Omega$ and $\mathcal{A}$ is an Ω-structure, then the Σ-*reduct* of $\mathcal{A}$ is the Σ-structure $\mathcal{A}_{|\Sigma}$ obtained from $\mathcal{A}$ by forgetting the interpretations of sorts, function and predicate symbols from Ω that do not belong to Σ. Conversely, $\mathcal{A}$ is called an *expansion* of the Σ-structure $\mathcal{A}_{|\Sigma}$ to the larger signature Ω. If $\mu : \mathcal{A} \to \mathcal{B}$ is an Ω-homomorphism, then the Σ-*reduct* of μ is the Σ-homomorphism $\mu_{|\Sigma} : \mathcal{A}_{|\Sigma} \to \mathcal{B}_{|\Sigma}$ obtained by restricting μ to the sorts that belong to Σ, i.e., by restricting the mapping to the domain of $\mathcal{A}_{|\Sigma}$.

Given a set X of constant symbols not belonging to the signature Ω, but each equipped with a sort from Ω, we denote by Ω^X the extension of Ω by these new constants. If $\mathcal{A}$ is an Ω-structure, then we can view the elements of A as a set of new constants, where $a \in A_S$ has sort S. By interpreting each $a \in A$ by itself, $\mathcal{A}$ can also be viewed as an Ω^A-structure. The *positive diagram* $\Delta^+_\Omega(\mathcal{A})$ of $\mathcal{A}$ is the set of all ground Ω^A-atoms that are true in $\mathcal{A}$, the *diagram* $\Delta_\Omega(A)$ of $\mathcal{A}$ is the set of all ground Ω^A-literals that are true in $\mathcal{A}$, and the *elementary diagram* $\Delta^e_\Omega(\mathcal{A})$ of $\mathcal{A}$ is the set of all Ω^A-sentences that are true in $\mathcal{A}$. The subscript Ω in $\Delta^+_\Omega(\mathcal{A})$, $\Delta_\Omega(\mathcal{A})$ and $\Delta^e_\Omega(\mathcal{A})$ is sometimes omitted if there is no danger of confusion. *Robinson's diagram theorems* [CK90] say that there is a homomorphism (embedding, elementary embedding) between the Ω-structures $\mathcal{A}$ and $\mathcal{B}$ iff it is possible to expand $\mathcal{B}$ to an Ω^A-structure in such a way that it becomes a model of the positive diagram (diagram, elementary diagram) of $\mathcal{A}$.

2.2 Basic connections

In the remainder of this section, we introduce our basic scheme for connecting many-sorted theories, and illustrate it with the example of $\mathcal{E}$-connections of modal logics. Let T_1, T_2 be theories over the respective signatures Ω_1, Ω_2, and let Ω_0 be a common subsignature of Ω_1 and Ω_2. We call Ω_0 the *connecting* signature. In addition, let T_0 be an Ω_0-theory[2] that is contained in both T_1 and T_2. We defined the new theory $T_1 >_{T_0} T_2$ (called the *connection of* T_1 *and* T_2 *over* T_0) as follows.

The *signature* Ω of $T_1 >_{T_0} T_2$ contains the disjoint union $\Omega_1 \uplus \Omega_2$ of the signatures Ω_1 and Ω_2, where the shared sorts and the shared function and predicate symbols are appropriately renamed, e.g., by attaching labels 1 and 2. Thus, if S (f, P) is a sort (function symbol, predicate symbol) contained in both Ω_1 and Ω_2, then S^i (f^i, P^i) for $i = 1, 2$ are its renamed variants in the disjoint union, where the arities are accordingly renamed. In addition, Ω contains a *new function symbol* h_S of arity $S^1 S^2$ for every sort S of Ω_0.

The *axioms* of $T_1 >_{T_0} T_2$ are obtained as follows. Given an Ω_i-formula ϕ, its renamed variant ϕ^i is obtained by replacing all shared symbols by their renamed variants with label i. The axioms of $T_1 >_{T_0} T_2$ consist of

$$\{\phi^1 \mid \phi \in T_1\} \cup \{\phi^2 \mid \phi \in T_2\},$$

together with the universal closures of the formulae

$$\begin{array}{l} h_S(f^1(x_1, \ldots, x_n)) \approx f^2(h_{S_1}(x_1), \ldots, h_{S_n}(x_n)), \\ P^1(x_1, \ldots, x_n) \to P^2(h_{S_1}(x_1), \ldots, h_{S_n}(x_n)), \end{array}$$

for every function (predicate) symbol f (P) in Ω_0 of arity $S_1 \ldots S_n S$ ($S_1 \ldots S_n$).

Since the signatures Ω_1 and Ω_2 have been made disjoint, and since the additional axioms state that the family of mappings h_S behaves like an Ω_0-homomorphism, it is easy to see that the *models of* $T_1 >_{T_0} T_2$ are formed by triples of the form $(\mathcal{M}^1, \mathcal{M}^2, h^{\mathcal{M}})$, where $\mathcal{M}^1$ is a model of T_1, $\mathcal{M}^2$ is a model of T_2 and $h^{\mathcal{M}}$ is an Ω_0-homomorphism

$$h^{\mathcal{M}} : \mathcal{M}^1_{|\Omega_0} \to \mathcal{M}^2_{|\Omega_0}$$

between the respective Ω_0-reducts.

Example 2.1 The most basic variant of an $\mathcal{E}$-connection [KLWZ04] is an instance of our approach if one translates it into the algebraic setting. The abstract description systems considered in [KLWZ04], which cover all the usual modal and description logics, correspond to Boolean-based equational theories [BGT04]. The theory E is called *Boolean-based equational theory* iff its signature Σ has just one sort, equality is the only predicate symbol, the set of function symbols contains the Boolean operators $\sqcap, \sqcup, \neg, \top, \bot$, and its set of axioms consists of identities (i.e., the universal closures of atoms $s \approx t$) and contains the Boolean algebra axioms.
For example, consider the basic modal logic **K**, where we use only the modal operator $\Diamond$ (since $\Box$ can then be defined). The Boolean-based equational theory

[2]When *defining* the connection of T_1, T_2, the theory T_0 is actually irrelevant; all we need is its signature Ω_0. However, for our decidability transfer results to hold, T_0 and the T_i must satisfy certain model-theoretic properties.

$E_{\mathbf{K}}$ corresponding to $\mathbf{K}$ is obtained from the theory of Boolean algebras by adding the identities $\Diamond(x \sqcup y) \approx \Diamond(x) \sqcup \Diamond(y)$ and $\Diamond(\bot) \approx \bot$.
Let us illustrate the notion of an $\mathcal{E}$-connection also on this simple example (see Appendix A for a more general description of $\mathcal{E}$-connections and their relationship to the notion of a connection introduced in this report). To build the $\mathcal{E}$-connection of $\mathbf{K}$ with itself, one takes two disjoint copies of $\mathbf{K}$, obtained by renaming the Boolean operators and the diamonds, e.g., into $\sqcap_i, \sqcup_i, \neg_i, \top_i, \bot_i, \Diamond_i$ for $i = 1, 2$. The signature of the $\mathcal{E}$-connection contains all these renamed symbols together with a new symbol $\Diamond$. However, it is now a two-sorted signature, where symbols with index i are applied to elements of sort S_i and yield as results an element of this sort. The new symbol has arity S_1S_2.[3] The semantics of this $\mathcal{E}$-connection can be given in terms of Kripke structures. A Kripke structure for the $\mathcal{E}$-connection consists of two Kripke structures $\mathcal{K}_1, \mathcal{K}_2$ for $\mathbf{K}$ over disjoint domains W_1 and W_2, together with an additional connecting relation $E \subseteq W_2 \times W_1$. The symbols with index i are interpreted in $\mathcal{K}_i$, and the new symbol $\Diamond$ is interpreted as the diamond operator induced by E, i.e., for every $X \subseteq W_1$ we have

$$\Diamond(X) := \{x \in W_2 \mid \exists y \in W_1.\ (x, y) \in E \wedge y \in X\}.$$

This interpretation of the new operator implies that it satisfies the usual identities of a diamond operator, i.e., $\Diamond(x \sqcup_1 y) \approx \Diamond(x) \sqcup_2 \Diamond(y)$ and $\Diamond(\bot_1) \approx \bot_2$, and that these identities are sufficient to characterize its semantics. Thus, the equational theory corresponding to the $\mathcal{E}$-connection of $\mathbf{K}$ with itself consists of these two axioms, together with the axioms of $E_{\mathbf{K}_1}$ and $E_{\mathbf{K}_2}$.
Obviously, this theory is also obtained as the connection of the theory $E_{\mathbf{K}}$ with itself, if the connecting signature Ω_0 consists of the single sort of $E_{\mathbf{K}}$, the predicate symbol $\approx$, and the function symbols $\sqcup, \bot$. As theory T_0 we can take the theory of semilattices, i.e., the axioms that say that $\sqcup$ is associative, commutative, and idempotent, and that $\bot$ is a unit for $\sqcup$.

Example 2.2 The previous example can be varied by additionally including $\sqcap$ in the connecting signature, and taking as theory T_0 the theory of distributive lattices with a least element $\bot$. It is easy to see that this corresponds to the case of an $\mathcal{E}$-connection where the connecting relation E is required to be a partial function (we call such an $\mathcal{E}$-connection *deterministic*). Finally, if we additionally include both $\sqcap$ and $\top$ in the connecting signature, and take T_0 to be the theory of bounded distributive lattices (i.e., distributive lattices with a least and a greatest element), then the equational theory obtained through our connection corresponds to the case of an $\mathcal{E}$-connection where the connecting relation E is a (total) function (we call such an $\mathcal{E}$-connection *functional*).

[3] In the general $\mathcal{E}$-connection scheme, there is also be an inverse diamond operator $\Diamond^-$ with arity S_2S_1, but we currently cannot treat this case (see the conclusion for a discussion).

3 Positive algebraic completions and compatibility

In order to transfer decidability results from the component theories T_1, T_2 to their connection $T_1 >_{T_0} T_2$ over T_0, the theories T_0, T_1, T_2 must satisfy certain model-theoretic conditions, which we introduce below. The most important one is that T_0 has a positive algebraic completion. Before we can define this concept, we must introduce some notions from model theory.

The formula ϕ is called *open* iff it does not contain quantifiers; it is called *universal* iff it is obtained from an open formula by adding a prefix of universal quantifiers; and it is called *geometric* iff it is built from atoms by using conjunction, disjunction, and existential quantifiers. The latter formulae are called “geometric” in categorical logic [MR77] since they are preserved under inverse image geometric morphisms.

The main property of geometric formulae is that they are preserved under homomorphisms in the following sense: if $\mu : \mathcal{A} \to \mathcal{B}$ is a homomorphism between Ω-structures and $\phi(x_1, \ldots, x_n)$ is a geometric formula over Ω, then

$$\mathcal{A} \models \phi(a_1, \ldots, a_n) \quad \text{implies} \quad \mathcal{B} \models \phi(\mu(a_1), \ldots, \mu(a_n))$$

for all (sort-conforming) $a_1, \ldots, a_n \in A$.

Open formulae are related to embeddings in various way. First, they are preserved under building sub- and superstructures, i.e., if $\mathcal{A}$ is a substructure of $\mathcal{B}$, $\phi(x_1, \ldots, x_n)$ is an open formula, and $a_1, \ldots, a_n \in A$ are sort-conforming, then $\mathcal{A} \models \phi(a_1, \ldots, a_n)$ iff $\mathcal{B} \models \phi(a_1, \ldots, a_n)$. The following lemma is well-known [CK90]:

Lemma 3.1 *Two Ω-theories T, T' entail the same set of open formulae iff every model of T can be embedded into a model of T' and vice versa.*

Proof. The direction from right to left follows from the fact that open formulae are preserved under building substructures.

For the other direction, assume that T and T' entail the same set of open formulae, and take any model $\mathcal{M}$ of T (for T' the argument is symmetric). First observe that $T' \cup \Delta(\mathcal{M})$ is consistent. Otherwise, by compactness of first-order logic, $T' \models \phi(\underline{a})$ for some ground sentence $\phi(\underline{a})$ with additional free constants $\underline{a}$ from M that is false in $\mathcal{M}$. Since $\underline{a}$ consists of free constants, it follows that $T' \models \phi(\underline{x})$, and consequently $T \models \phi(\underline{x})$ by assumption. Since $T \models \phi(\underline{x})$ iff $T \models \forall \underline{x}.\phi(\underline{x})$, this is a contradiction since $\phi(\underline{a})$ is false in $\mathcal{M}$.

Now, let $\mathcal{N}$ be a model of $T' \cup \Delta(\mathcal{M})$. Thus, $\mathcal{N}$ is a model of T', and by Robinson's diagram theorem, $\mathcal{M}$ can be embedded into $\mathcal{N}$. $\dashv$

Since a theory entails an open formula iff it entails its universal closure, the lemma also says that two theories T, T' entail the same universal sentences iff every model of T can be embedded into a model of T' and vice versa.

The theory T is a *universal theory* iff its axioms are universal sentences; it is a *geometric theory* iff it can be axiomatized by using universal closures of geometric sequents, where a geometric sequent is an implication between two geometric formulae. Note that any universal theory is geometric since open formulae are conjunctions of clauses and clauses can be rewritten as geometric sequents.

Definition 3.2 *Let T be a universal and T^* a geometric theory over Ω. We say that T^* is a* positive algebraic completion *of T iff the following properties hold:*

1. *$T \subseteq T^*$;*
2. *every model of T embeds into a model of T^*;*[4]
3. *for every geometric formula $\phi(\underline{x})$ there is an* open *geometric formula $\phi^*(\underline{x})$ such that $T^* \models \phi \leftrightarrow \phi^*$.*

It can be shown that the models of T^* are exactly the algebraically closed models of T (see Appendix B below). In particular, this means that the positive algebraic completion of T is unique, provided that it exists.

When trying to show that Property 3 of Definition 3.2 holds for given theories T, T^*, then it is sufficient to consider *simple existential formulae* $\phi(\underline{x})$, i.e., formulae that are obtained from conjunctions of atoms by adding an existential quantifier prefix. In fact, any geometric formula ϕ can be normalized to a disjunction $\phi_1 \vee \ldots \vee \phi_n$ of simple existential formulae ϕ_i by using distributivity of conjunction and existential quantification over disjunction. In addition, if $T^* \models \phi_i \leftrightarrow \phi_i^*$ for geometric open formulae ϕ_i^* $(i = 1, \ldots, n)$, then $\phi_1^* \vee \ldots \vee \phi_n^*$ is also a geometric open formula and $T^* \models (\phi_1 \vee \ldots \vee \phi_n) \leftrightarrow (\phi_1^* \vee \ldots \vee \phi_n^*)$.

The following lemma will turn out to be useful later on.

Lemma 3.3 *Assume that T, T^* satisfy Property 1 and 2 of Definition 3.2. If $\phi(\underline{x})$ is a simple existential formula and $\phi^*(\underline{x})$ is an open formula, then $T^* \models \phi \to \phi^*$ iff $T \models \phi \to \phi^*$.*

This is an immediate consequence of the fact that $\phi \to \phi^*$ is then equivalent to an open formula, and hence Lemma 3.1 applies.

The first ingredient of our combinability condition is the following notion of compatibility, which is a variant of analogous compatibility conditions introduced in [Ghi05, BGT04] for the case of the union of theories.

[4] equivalently, T and T^* entail the same universal sentences.

Definition 3.4 *Let $T_0 \subseteq T$ be theories over the respective signatures $\Omega_0 \subseteq \Omega_1$. We say that T* is T_0-algebraically compatible *iff T_0 is universal, has a positive algebraic completion T_0^*, and every model of T embeds into a model of $T \cup T_0^*$.*

The second ingredient is that T_0 must be locally finite, i.e., all finitely generated models of T_0 are finite. To be more precise, we need the following effective variant of local finiteness defined in [Ghi05, BGT04].

Definition 3.5 *Let T_0 be a universal theory over the finite signature Ω_0. Then T_0 is called* effectively locally finite *iff for every tuple of variables $\underline{x}$, one can effectively determine terms $t_1(\underline{x}), \ldots, t_k(\underline{x})$ such that, for every further term $u(\underline{x})$, we have that $T_0 \models u \approx t_i$ for some $i = 1, \ldots, k$.*

4 The main combination results

We are interested in deciding the universal fragments of our theories, i.e., validity of universal formulae (or, equivalently open formulae) in a theory T. This is the decision problem also treated by the Nelson-Oppen combination method (albeit for the union of theories). It is well know that this problem is equivalent to the problem of deciding whether a set of literals is satisfiable in some model of T. We call such a set of literals a *constraint*.

By introducing new free constants (i.e., constants not occurring in the axioms of the theory), we can assume without loss of generality that such constraints contain no variables. In addition, we can transform any ground constraint into an equisatisfiable set of *ground flat literals*, i.e., literals of the form

$$a \approx f(a_1, \ldots, a_n), \quad P(a_1, \ldots, a_n), \quad \text{or} \quad \neg P(a_1, \ldots, a_n),$$

where $a, a_1, \ldots, a_n$ are (sort-conforming) free constants, f is a function symbol, and P is a predicate symbol (possibly also equality).

In the following, we first treat the case of a basic connection, as introduced in Section 2. Then, we show that the combination result can be extended to connections with several connection functions, possibly going in both directions. Finally, we give examples of theories satisfying our combinability conditions.

4.1 Basic connections

In this subsection we show under what conditions decidability of the universal fragments of T_1, T_2 transfers to their connection $T_1 >_{T_0} T_2$.

Theorem 4.1 *Let T_0, T_1, T_2 be theories over the respective signatures $\Omega_0, \Omega_1, \Omega_2$, where Ω_0 is a common subsignature of Ω_1 and Ω_2. Assume that $T_0 \subseteq T_1$ and $T_0 \subseteq T_2$, that T_0 is universal and locally finite, and that T_2 is T_0-algebraically compatible. Then the decidability of the universal fragments of T_1 and T_2 entails the decidability of the universal fragment of $T_1 >_{T_0} T_2$.*

To prove the theorem, we consider a finite set Γ of ground flat literals over the signature Ω of $T_1 >_{T_0} T_2$ (with additional free constants), and show how it can be tested for satisfiability in $T_1 >_{T_0} T_2$. Since all literals in Γ are flat, we can divide Γ into three disjoint sets $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$, where Γ_i $(i = 1, 2)$ is a set of literals in the signature Ω_i (expanded with free constants), and Γ_0 is of the form

$$\Gamma_0 = \{h(a_1) \approx b_1, \ldots, h(a_n) \approx b_n\}$$

for free constants $a_1, b_1, \ldots, a_n, b_n$.

Proposition 4.2 *The constraint $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$ is satisfiable in $T_1 >_{T_0} T_2$ iff there exists a triple $(\mathcal{A}, \mathcal{B}, \nu)$ such that*

1. *$\mathcal{A}$ is an Ω_0-model of T_0, which is generated by $\{a_1^{\mathcal{A}}, \ldots, a_n^{\mathcal{A}}\}$;*
2. *$\mathcal{B}$ is an Ω_0-model of T_0, which is generated by $\{b_1^{\mathcal{B}}, \ldots, b_n^{\mathcal{B}}\}$;*
3. *$\nu : \mathcal{A} \to \mathcal{B}$ is an Ω_0-homomorphism such that $\nu(a_j^{\mathcal{A}}) = b_j^{\mathcal{B}}$ for $j = 1, \ldots, n$;*
4. *$\Gamma_1 \cup \Delta_{\Omega_0}(\mathcal{A})$ is satisfiable in T_1;*
5. *$\Gamma_2 \cup \Delta_{\Omega_0}(\mathcal{B})$ is satisfiable in T_2.*

Proof. The only-if direction is simple. In fact, as noted in Section 2, a model $\mathcal{M}$ of $T_1 >_{T_0} T_2$ is given by a triple $(\mathcal{M}^1, \mathcal{M}^2, h^{\mathcal{M}})$, where $\mathcal{M}^1$ is a model of T_1, $\mathcal{M}^2$ is a model of T_2 and $h^{\mathcal{M}} : \mathcal{M}^1_{|\Omega_0} \to \mathcal{M}^2_{|\Omega_0}$ is an Ω_0-homomorphism between the respective Ω_0-reducts. Assume that this model $\mathcal{M}$ satisfies Γ. We can take as $\mathcal{A}$ the substructure of $\mathcal{M}^1_{|\Omega_0}$ generated by (the interpretations of) $a_1, \ldots, a_n$, as $\mathcal{B}$ the substructure of $\mathcal{M}^2_{|\Omega_0}$ generated by (the interpretations of) $b_1, \ldots, b_n$, and as homomorphism ν the restriction of $h^{\mathcal{M}}$ to $\mathcal{A}$. It is easy to see that the triple $(\mathcal{A}, \mathcal{B}, \nu)$ obtained this way satisfies 1.–5. of the proposition.

Conversely, assume that $(\mathcal{A}, \mathcal{B}, \nu)$ is a triple satisfying 1.–5. of the proposition. Because of 4. and 5., there is an Ω_1-model $\mathcal{N}'$ of T_1 satisfying $\Gamma_1 \cup \Delta_{\Omega_0}(\mathcal{A})$ and an Ω_2-model $\mathcal{N}''$ of T_2 satisfying $\Gamma_2 \cup \Delta_{\Omega_0}(\mathcal{B})$. By Robinson's diagram theorem, $\mathcal{N}'$ has $\mathcal{A}$ as an Ω_0-substructure and $\mathcal{N}''$ has $\mathcal{B}$ as an Ω_0-substructure. We assume without loss of generality that $\mathcal{N}'$ is at most countable and that $\mathcal{N}''$ is a model of $T_2 \cup T_0^*$. The latter assumption is by T_0-algebraic compatibility of T_2, and the former assumption is by the Löwenheim-Skolem theorem since our signatures are at most countable. Let us enumerate the elements of $\mathcal{N}'$ as

$$c_1, c_2, \dots, c_n, c_{n+1}, \dots$$

where we assume that $c_i = a_i^{\mathcal{A}}$ $(i = 1, \dots, n)$, i.e., $c_1, \dots, c_n$ are generators of $\mathcal{A}$. We define an increasing sequence of sort-conforming functions $\nu_k : \{c_1, \dots c_k\} \to N''$ (for $k \geq n$) such that, for every ground $\Omega_0^{\{c_1,\dots,c_k\}}$-atom A we have

$$\mathcal{N}'_{|\Omega_0} \models A(c_1, \dots, c_k) \quad \text{implies} \quad \mathcal{N}''_{|\Omega_0} \models A(\nu_k(c_1), \dots, \nu_k(c_k)).$$

We first take ν_n to be ν. To define ν_{k+1} (for $k \geq n$), let us consider the conjunction $\psi(c_1, \dots, c_n, c_{n+1})$ of the $\Omega_0^{\{c_1,\dots,c_{n+1}\}}$-atoms that are true in $\mathcal{N}'_{|\Omega_0}$: this conjunction is finite (modulo taking representative terms, thanks to local finiteness of T_0). Let $\phi(x_1, \dots, x_n)$ be $\exists x_{n+1}.\psi(x_1, \dots, x_n, x_{n+1})$ and let $\phi^*(x_1, \dots, x_n)$ be a geometric open formula such that $T_0^* \models \phi \leftrightarrow \phi^*$.

By Lemma 3.3, $T_0 \models \phi \to \phi^*$, and thus we have $\mathcal{N}'_{|\Omega_0} \models \phi^*(c_1, \dots, c_k)$ and also $\mathcal{N}''_{|\Omega_0} \models \phi^*(\nu_k(c_1), \dots, \nu_k(c_k))$ by the induction hypothesis. Since $\mathcal{N}''_{|\Omega_0}$ is a model of T_0^*, there is a b such that $\mathcal{N}''_{|\Omega_0} \models \psi(\nu_k(c_1), \dots, \nu_k(c_k), b)$ for some b. We now obtain the desired extension ν_{k+1} of ν_k by setting $\nu_{k+1}(c_{k+1}) := b$. Taking $\nu_\infty = \bigcup_{k \geq n} \nu_k$, we finally obtain a homomorphism $\nu_\infty : \mathcal{N}'_{|\Omega_0} \to \mathcal{N}''_{|\Omega_0}$ such that the triple $(\mathcal{N}', \mathcal{N}'', \nu_\infty)$ is a model of $T_1 >_{T_0} T_2$ that satisfies $\Gamma_0 \cup \Gamma_1 \cup \Gamma_2$. $\dashv$

The above proof uses the assumption that T_0 is locally finite. By using heavier model-theoretic machinery, one can also prove the proposition without using local finiteness of T_0 (see Appendix C below). However, since the proof of Theorem 4.1 needs this assumption anyway (see below), we gave the above proof since it is simpler.

To conclude the proof of Theorem 4.1, we describe a *non-deterministic decision procedure* that effectively guesses an appropriate triple $(\mathcal{A}, \mathcal{B}, \nu)$ and then checks whether it satisfies 1.–5. of Proposition 4.2. To guess an Ω_0-model of T_0 that is generated by a finite set X, one uses effective local finiteness of T_0 to obtain an effective bound on the size of such a model and guesses an Ω_0-structure that satisfies this size bound.

Once the Ω_0-structures $\mathcal{A}, \mathcal{B}$ are given, one can build their diagrams, and use the decision procedures for T_1 and T_2 to check whether 4. and 5. of Proposition 4.2 are satisfied. If the answer is yes, then $\mathcal{A}, \mathcal{B}$ are also models of T_0: in fact, if for instance $\Gamma_1 \cup \Delta_{\Omega_0}(\mathcal{A})$ is satisfiable in the model $\mathcal{M}$ of T_1, then $\mathcal{M}$ has $\mathcal{A}$ as a substructure, and this implies $\mathcal{A} \models T_0$ because T_0 is universal and $T_0 \subseteq T_1$.

Finally, one can guess a mapping $\nu : A \to B$ that satisfies $\nu(a_j^{\mathcal{A}}) = b_j^{\mathcal{B}}$, and then use the diagrams of $\mathcal{A}, \mathcal{B}$ to check whether ν satisfies the homomorphism condition $(*)$.

4.2 Two-side connections

The proof of Proposition 4.2 basically shows that our decidability transfer result can easily be extended to the case of *several connection functions*, possibly going in both directions. For simplicity, we examine only the case of two connection functions, going in the two opposite directions.

The theory $T_1 >_{T_0}< T_2$ is defined as the union of $T_1 >_{T_0} T_2$ and $T_2 >_{T_0} T_1$. Thus, a model of $T_1 >_{T_0}< T_2$ is a 4-tuple given by a model $\mathcal{M}^1$ of T_1, a model $\mathcal{M}^2$ of T_2 and two homomorphisms

$$h^{\mathcal{M}} : \mathcal{M}^1_{|\Omega_0} \longrightarrow \mathcal{M}^2_{|\Omega_0} \qquad \text{and} \qquad g^{\mathcal{M}} : \mathcal{M}^2_{|\Omega_0} \longrightarrow \mathcal{M}^1_{|\Omega_0}$$

among the respective Ω_0-reducts.

Theorem 4.3 *Let T_0, T_1, T_2 be theories over the respective signatures $\Omega_0, \Omega_1, \Omega_2$, where Ω_0 is a common subsignature of Ω_1 and Ω_2. Assume that $T_0 \subseteq T_1$ and $T_0 \subseteq T_2$, that T_0 is universal and locally finite, and that T_1, T_2 are both T_0-algebraically compatible. Then the decidability of the universal fragments of T_1 and T_2 entails the decidability of the universal fragment of $T_1 >_{T_0}< T_2$.*

To prove the Theorem, notice that any finite set of ground flat literals (with free constants) Γ to be tested for $T_1 >_{T_0}< T_2$-consistency can be divided into four disjoint sets

$$\Gamma = \Theta_1 \cup \Theta_2 \cup \Gamma_1 \cup \Gamma_2,$$

where Γ_i $(i = 1, 2)$ are sets of literals in the signature Ω_i (expanded with free constants), and

$$\Theta_1 = \{h(a_1) \approx b_1, \ldots, h(a_n) \approx b_n\} \ \text{ and } \ \Theta_2 = \{g(b'_1) \approx a'_1, \ldots, g(b'_m) \approx a'_m\}.$$

Theorem 4.3 is an easy consequence of the following proposition.

Proposition 4.4 *The constraint $\Gamma = \Theta_1 \cup \Theta_2 \cup \Gamma_1 \cup \Gamma_2$ is satisfiable in $T_1 >_{T_0}< T_2$ iff there exist two triples $(\mathcal{A}, \mathcal{B}, \nu)$ and $(\mathcal{A}', \mathcal{B}', \nu')$ such that*

1. *$\mathcal{A}$ is a Ω_0-model of T_0 that is generated by $\{a_1^{\mathcal{A}}, \ldots, a_n^{\mathcal{A}}\}$, $\mathcal{B}$ is a Ω_0-model of T_0 which is generated by $\{b_1^{\mathcal{B}}, \ldots, b_n^{\mathcal{B}}\}$ and $\nu : \mathcal{A} \to \mathcal{B}$ is a Ω_0-homomorphism such that $\nu(a_j^{\mathcal{A}}) = b_j^{\mathcal{B}}$ for all $j = 1, \ldots, n$;*

2. *$\mathcal{A}'$ is a Ω_0-model of T_0 that is generated by $\{{a'_1}^{\mathcal{A}'}, \ldots, {a'_m}^{\mathcal{A}'}\}$, $\mathcal{B}$ is a Ω_0-model of T_0 that is generated by $\{{b'_1}^{\mathcal{B}'}, \ldots, {b'_m}^{\mathcal{B}'}\}$ and $\mu : \mathcal{B}' \to \mathcal{A}'$ is a Ω_0-homomorphism such that $\nu'({b'_j}^{\mathcal{B}'}) = {a'_j}^{\mathcal{A}'}$ for all $j = 1, \ldots, m$;*

3. *$\Gamma_1 \cup \Delta_{\Omega_0}(\mathcal{A}) \cup \Delta_{\Omega_0}(\mathcal{A}')$ is satisfiable in T_1, and*
   $\Gamma_2 \cup \Delta_{\Omega_0}(\mathcal{B}) \cup \Delta_{\Omega_0}(\mathcal{B}')$ is satisfiable in T_2.

Proof. The only-if direction is again simple. To proof the if direction, assume that for some $\nu : \mathcal{A} \to \mathcal{B}$ and $\mu : \mathcal{B}' \to \mathcal{A}'$, the set of literals $\Gamma_1 \cup \Delta_{\Omega_0}(\mathcal{A}) \cup \Delta_{\Omega_0}(\mathcal{A}')$ is satisfiable in an Ω_1-model $\mathcal{N}'$ of T_1, and the set of literals $\Gamma_2 \cup \Delta_{\Omega_0}(\mathcal{B}) \cup \Delta_{\Omega_0}(\mathcal{B}')$ is satisfiable in an Ω_2-model $\mathcal{N}''$ of T_2. By Robinson's diagram theorem, $\mathcal{N}'$ has $\mathcal{A}$ and $\mathcal{A}'$ as Ω_0-substructures, and $\mathcal{N}''$ has $\mathcal{B}$ and $\mathcal{B}'$ as Ω_0-substructures. We assume without loss of generality that $\mathcal{N}'$ and $\mathcal{N}''$ are at most countable models of $T_1 \cup T_0^*$ and $T_1 \cup T_0^*$, respectively.

Now, an argument identical to the one used in the proof of Proposition 4.2 yields the homomorphisms

$$\nu_\infty : \mathcal{N}'_{|\Omega_0} \longrightarrow \mathcal{N}''_{|\Omega_0} \quad \text{and} \quad \nu'_\infty : \mathcal{N}''_{|\Omega_0} \longrightarrow \mathcal{N}'_{|\Omega_0},$$

which are needed in order to obtain a full model of $T_1 >_{T_0}< T_2$. $\dashv$

It should be clear how to adapt this proof to the case of more than one connection function going in each direction.

4.3 Examples

When trying to axiomatize the positive algebraic completion T_0^* of a given universal theory T_0, it is sufficient to produce for every simple existential formula $\phi(\underline{x})$ an appropriate geometric and open formula $\phi^*(\underline{x})$. Take as theory T_0^* the one axiomatized by T_0 together with the formulae $\phi \leftrightarrow \phi^*$ for every simple existential formula ϕ. In order to complete the job, it is sufficient to show that every model of T_0 embeds into a model of T_0^*. It should also be noted that one can without loss of generality restrict the attention to simple existential formulae with just one existential quantifier since more than one quantifier can then be treated by iterated elimination of single quantifiers.

In the next example we encounter a special case where the formulae $\phi \leftrightarrow \phi^*$ are already valid in T_0. In this case, we have $T_0 = T_0^*$, and thus the model-embedding condition is trivially satisfied. In addition, any theory T with $T_0 \subseteq T$ is T_0-algebraically compatible.

Example 4.5 Recall from [BGT04] the definition of a Gaussian theory. Let us call a conjunction of atoms an *e-formula.* The universal theory T_0 is *Gaussian* iff for every e-formula $\phi(\underline{x}, y)$ it is possible to compute an e-formula $\psi(\underline{x})$ and a term $s(\underline{x}, \underline{z})$ with fresh variables $\underline{z}$ such that

$$T_0 \models \phi(\underline{x}, y) \leftrightarrow (\psi(\underline{x}) \wedge \exists \underline{z}.(y \approx s(\underline{x}, \underline{z}))). \tag{1}$$

Any Gaussian theory T_0 is its own positive algebraic completion. In fact, it is easy to see that (1) implies $T_0 \models (\exists y.\phi(\underline{x}, y)) \leftrightarrow \psi(\underline{x})$, and thus the comment given above this example applies.

As a consequence, our combination result applies to all the examples of effectively locally finite Gaussian theories given in [BGT04] (e.g., Boolean algebras, vector spaces over a finite field, empty theory over a signature whose sets of predicates consists of $\approx$ and whose set of function symbols is empty): if the universal theory T_0 is effectively locally finite and Gaussian, and T_1, T_2 are arbitrary theories containing T_0 and with decidable universal fragment, then the universal fragment of $T_1 >_{T_0} T_2$ is also decidable.

Example 4.6 Let T_0 be the theory of semilattices (see Example 2.1). This theory is obviously effectively locally finite. In the following, we use the disequation $s \sqsubseteq t$ as an abbreviation for the equation $s \sqcup t \approx t$. Obviously, any equation $s \approx t$ can be expressed by the disequations $s \sqsubseteq t \wedge t \sqsubseteq s$.
The theory T_0 has a positive algebraic completion, which can be axiomatized as follows. Let $\phi(\underline{x})$ be a simple existential formula with just one existential quantifier. Using the fact that $z_1 \sqcup \ldots \sqcup z_n \sqsubseteq z$ is equivalent to $z_1 \sqsubseteq z \wedge \ldots \wedge z_n \sqsubseteq z$, it is easy to see that $\phi(\underline{x})$ is equivalent to a formula of the form

$$\exists y.((y \sqsubseteq t_1) \wedge \cdots \wedge (y \sqsubseteq t_n) \wedge (u_1 \sqsubseteq s_1 \sqcup y) \wedge \cdots \wedge (u_m \sqsubseteq s_m \sqcup y)), \qquad (2)$$

where t_i, s_j, u_k are terms not involving y. Let $\phi^*(\underline{x})$ be the formula

$$\bigwedge_{i=1}^{n} \bigwedge_{j=1}^{m} (u_j \sqsubseteq s_j \sqcup t_i), \qquad (3)$$

and let T_0^* be obtained from T_0 by adding to it the universal closures of all formulae $\phi \leftrightarrow \phi^*$.
We prove that T_0^* is contained in the theory of Boolean algebras. In fact, the system of disequations (2) is equivalent, in the theory of Boolean algebras, to

$$\exists y.((y \sqsubseteq t_1) \wedge \cdots \wedge (y \sqsubseteq t_n) \wedge (u_1 \sqcap \neg s_1 \sqsubseteq y) \wedge \cdots \wedge (u_m \sqcap \neg s_m \sqsubseteq y), \qquad (4)$$

and hence to

$$(u_1 \sqcap \neg s_1 \sqsubseteq t_1 \sqcap \ldots \sqcap t_n) \wedge \cdots \wedge (u_m \sqcap \neg s_m \sqsubseteq t_1 \sqcap \ldots \sqcap t_n). \qquad (5)$$

Finally, it is easy to see that (5) and (3) are equivalent.
It is well-known that every semilattice embeds into a Boolean algebra. This can, for example, be shown as follows. Given a semilattice $\mathcal{S} = (S, \sqcup, \bot)$, just consider the Boolean algebra $\mathcal{B} = (2^S, \cap, S, \cup, \emptyset, \overline{(\cdot)})$ given by the dual of the usual Boolean algebra formed by the powerset of S: this means that as join in $\mathcal{B}$ we take the intersection of sets, as the least element S, as the meet the union of sets, as the greatest element $\emptyset$, and as the negation operation the set complement. It is easy to see that the map associating with $s \in S$ the set $\{s' \mid s \sqsubseteq s'\}$ is a semilattice embedding from $\mathcal{S}$ into $\mathcal{B}$.
This shows that T_0^* is the positive algebraic completion of T_0. In addition, this implies that any Boolean-based theory T is T_0-algebraically compatible since T_0^* is contained in T. Consequently, Theorem 4.1 covers the case of a basic $\mathcal{E}$-connection, as introduced in Example 2.1 (see Appendix A for details).

Example 4.7 Let us now turn to Example 2.2, i.e., to connections over the theory T_0 of distributive lattices with a least element $\bot$. This theory is obviously effectively locally finite, and it has a positive algebraic completion, which can be obtained as follows. Every term is equivalent modulo T_0 both to (i) a term that is a (possibly empty) finite join of (non-empty) finite meets of variables, and to (ii) a term that is a (non-empty) finite meet of (possibly empty) finite joins of variables. A simple existential formula with just one existential quantifier $\phi(\underline{x})$ is then easily seen to be equivalent to a formula of the form

$$\exists y.(\bigwedge_i (y \sqsubseteq u_i) \wedge \bigwedge_j (t_j \sqcap y \sqsubseteq z_j) \wedge \bigwedge_k (v_k \sqsubseteq y \sqcup w_k)), \tag{6}$$

where u_i, t_j, v_k, w_k are terms not involving y. Let $\phi^*(\underline{x})$ be the formula

$$\bigwedge_{i,k} (v_k \sqsubseteq u_i \sqcup w_k) \wedge \bigwedge_{j,k} (v_k \sqcap t_j \sqsubseteq w_k \sqcup z_j), \tag{7}$$

and let T_0^* be obtained from T_0 by adding to it the universal closures of all formulae $\phi \leftrightarrow \phi^*$.

We prove that T_0^* is contained in the theory of Boolean algebras. In fact, the system of disequations (6) is equivalent, in the theory of Boolean algebras, to

$$\exists y.(\bigwedge_i (y \sqsubseteq u_i) \wedge \bigwedge_j (y \sqsubseteq \neg t_j \sqcup z_j) \wedge \bigwedge_k (v_k \sqcap \neg w_k \sqsubseteq y)), \tag{8}$$

and hence to

$$\bigwedge_{i,k} (v_k \sqcap \neg w_k \sqsubseteq u_i) \wedge \bigwedge_{j,k} (v_k \sqcap \neg w_k \sqsubseteq \neg t_j \sqcup z_j). \tag{9}$$

Finally, it is easy to see that (9) and (7) are equivalent.

Since every distributive lattice with least element embeds into a Boolean algebra,[5] this shows that T_0^* is the positive algebraic completion of T_0. In addition, this implies that any Boolean-based equational theory T is T_0-algebraically compatible since T_0^* is contained in T. Consequently, Theorem 4.1 covers the case of a basic *deterministic* $\mathcal{E}$-connection, as introduced in Example 2.2 (see Appendix A for details).

Example 4.8 The previous example can be sleightly varied, by considering the theory T_0 of bounded distributive lattices (i.e., distributive lattices with a least and a greatest element). Let us prove that its positive algebraic completion is the theory T_0^* axiomatized by T_0 together with the (universal closure of the) formula

$$\exists y.((x \sqcap y \approx 0) \wedge (x \sqcup y \approx 1)).$$

[5] It is well-known that distributive lattices with least and greatest elements embed into Boolean algebras, and it is easy to embed a distributive lattice with least element into one with least and greatest elements by just adding a greatest element.

Thus, T_0^* is simply the theory of Boolean algebras, formulated in a complement-free signature. Since every bounded distributive lattice embeds into a Boolean algebra, and since the theory of Boolean algebras coincides with its own positive algebraic completion because it is Gaussian (see Example 4.5), it is sufficient to show that every e-formula ϕ in the signature of Boolean algebras is equivalent to an e-formula in the complement-free subsignature. In fact, we can assume that ϕ is a conjunction of identities of the form

$$1 \approx \overline{x}_1 \sqcup \cdots \sqcup \overline{x}_n \sqcup y_1 \sqcup \cdots \sqcup y_m;$$

these identities are in turn trivially equivalent to the inequations

$$x_1 \sqcap \cdots \sqcap x_n \sqsubseteq y_1 \sqcup \cdots \sqcup y_m,$$

which can obviously be transformed into identities between term in the complement-free subsignature.
Again this implies that every Boolean-based equational theory is T_0-compatible and that Theorem 4.1 covers the case of a basic *functional* $\mathcal{E}$-connection, as introduced in Example 2.2 (see again Appendix A for details).

Example 4.9 Here we give an example with a non-functional signature. Let T_0 be the (obviously locally finite) theory of partial orders (posets). The positive algebraic completion T_0^* of T_0 is the theory axiomatized by T_0 together with the axioms

$$\exists x.(\bigwedge_i (x \sqsubseteq a_i) \wedge \bigwedge_j (b_j \sqsubseteq x)) \leftrightarrow \bigwedge_{i,j} (b_j \sqsubseteq a_i),$$

where i, j range over a finite index set and a_i, b_j are variables.
To embed a model $(P, \sqsubseteq)$ of T_0 into a model of T_0^*, just take the poset of downward closet subsets of $(P, \sqsubseteq)$. A downward closed subset of P is a set $X \subseteq P$ such that $x \in X$ and $y \sqsubseteq x$ imply $y \in X$. These sets are ordered by set inclusion. It is easy to see that this yields a model of T_0^*. In fact, it is enough to show that, given downward closed sets A_i, B_j satisfying $\bigwedge_{i,j}(B_j \sqsubseteq A_i)$, there is a downward closed set X such that $\bigwedge_i(X \sqsubseteq A_i) \wedge \bigwedge_j(B_j \sqsubseteq X)$. Since the union of downward closed sets is again downward closed, we can take the union of the B_j as the set X. The embedding of $(P, \sqsubseteq)$ into downward closed sets is obtained by associating with $a \in P$ the cone $a{\downarrow} := \{b \mid b \sqsubseteq a\}$. It is easy to see that $a \sqsubseteq a'$ iff $a{\downarrow} \subseteq a'{\downarrow}$.
In order to obtain a T_0-algebraically compatible theory, we consider again the theory T of semilattices, but now we assume that the symbol $\sqsubseteq$ belongs to the signature, and satisfies the axiom $x \sqsubseteq y \leftrightarrow x \wedge y \approx y$. The theory T is T_0-algebraically compatible since every model of T is a model of T_0^*: in fact

$$\exists x.(\bigwedge_i (x \sqsubseteq a_i) \wedge \bigwedge_j (b_j \sqsubseteq x))$$

is equivalent (in the theory T) to

$$\exists x.(\bigwedge_i (x \sqsubseteq a_i) \wedge (\bigsqcup_j b_j \sqsubseteq x)),$$

i.e., to

$$\bigwedge_i (\bigsqcup_j b_j \sqsubseteq a_i)$$

and thus to $\bigwedge_{i,j}(b_j \sqsubseteq a_i)$.
Other theories that extend T_0^* (and are hence T_0-algebraically compatible) are theories that extend the theory of total orders, as is easily seen.

5 A variant of the connection scheme

Here we consider a slightly different combination scheme where a theory T is connected with itself w.r.t. a subtheory T_0. Let $T_0 \subseteq T$ be theories over the respective signatures $\Omega_0 \subseteq \Omega$. We use $T_{>T_0}$ to denote the theory whose models are models $\mathcal{M}$ of T endowed with a homomorphism $h : \mathcal{M}_{|\Omega_0} \to \mathcal{M}_{|\Omega_0}$. Thus, the *signature* Ω' of $T_{>T_0}$ is obtained from the signature Ω of T by adding a new function symbol h_S of arity SS for every sort S of Ω_0. The axioms of $T_{>T_0}$ are obtained from the axioms of T by adding

$$\begin{aligned}
&h_S(f(x_1,\ldots,x_n)) \approx f(h_{S_1}(x_1),\ldots,h_{S_n}(x_n)),\\
&P(x_1,\ldots,x_n) \to P(h_{S_1}(x_1),\ldots,h_{S_n}(x_n)),
\end{aligned}$$

for every function (predicate) symbol f (P) in Ω_0 of arity $S_1 \ldots S_n S$ $(S_1 \ldots S_n)$.

Example 5.1 An interesting example of a theory obtained as such a connection is the theory $E_{\mathbf{K}}$ corresponding to the basic modal logic $\mathbf{K}$. In fact, let T be the theory of Boolean algebras, and T_0 the theory of semilattices over the signature Ω_0 as defined in Example 2.1. If we use the symbol $\Diamond$ for the connection function, then $T_{>T_0}$ is exactly the theory $E_{\mathbf{K}}$.

5.1 A non-deterministic combination procedure

In this subsection we state the main decidability transfer result. The approach is analogous to the one chosen in Section 4, and it leads to a non-deterministic combination procedure. In the next subsection we show that, under certain additional restrictions, this non-deterministic procedure can be replaced by a deterministic one.

Theorem 5.2 *Let T_0, T be theories over the respective signatures Ω_0, Ω, where Ω_0 is a subsignature of Ω. Assume that $T_0 \subseteq T$, that T_0 is universal and locally finite, and that T is T_0-algebraically compatible. Then the decidability of the universal fragment of T entails the decidability of the universal fragment of $T_{>T_0}$.*

To prove the theorem, we consider a finite set $\Gamma \cup \Gamma_0$ of ground flat literals over the signature Ω' of $T_{>T_0}$, where Γ is a set of literals in the signature Ω of T (expanded with free constants), and Γ_0 is of the form

$$\Gamma_0 = \{h(a_1) \approx b_1, \ldots, h(a_n) \approx b_n\}.$$

The theorem is an easy consequence of the following proposition, whose proof is similar to the one of Proposition 4.2.

Proposition 5.3 *The constraint $\Gamma \cup \Gamma_0$ is satisfiable in $T_{>T_0}$ iff there exists a triple $(\mathcal{A}, \mathcal{B}, \nu)$ such that*

1. *$\mathcal{A}$ is an Ω_0-model of T_0, which is generated by $\{a_1^{\mathcal{A}}, \ldots, a_n^{\mathcal{A}}\}$;*
2. *$\mathcal{B}$ is an Ω_0-model of T_0, which is generated by $\{b_1^{\mathcal{B}}, \ldots, b_n^{\mathcal{B}}\}$;*
3. *$\nu : \mathcal{A} \to \mathcal{B}$ is an Ω_0-homomorphism such that $\nu(a_j^{\mathcal{A}}) = b_j^{\mathcal{B}}$ for $j = 1, \ldots, n$;*
4. *$\Gamma \cup \Delta_{\Omega_0}(\mathcal{A}) \cup \Delta_{\Omega_0}(\mathcal{B})$ is satisfiable in T.*

Proof. The only-if direction is again simple. To proof the if direction, assume that there is a triple $(\mathcal{A}, \mathcal{B}, \nu)$ satisfying 1.–4. of the proposition. In particular, this means that $\Gamma \cup \Delta_{\Omega_0}(\mathcal{A}) \cup \Delta_{\Omega_0}(\mathcal{B})$ is satisfiable in a model $\mathcal{N}$ of T. We can assume without loss of generality that $\mathcal{N}$ is an at most countable model of $T \cup T_0^*$. By Robinson's diagram theorem, $\mathcal{A}, \mathcal{B}$ are Ω_0-substructures of $\mathcal{N}$. Using the same argument as in the proof of Proposition 4.2, we can extend the Ω_0-homomorphism $\nu : \mathcal{A} \to \mathcal{B}$ to an Ω_0-endomorphism $\nu_\infty : \mathcal{N}_{|\Omega_0} \to \mathcal{N}_{|\Omega_0}$. The pair $(\mathcal{N}, \nu_\infty)$ yields a model of $T_{>T_0}$ that satisfies $\Gamma \cup \Gamma_0$. $\dashv$

Obviously, this proposition gives rise to a non-deterministic decision procedure for the universal fragment of $T_{>T_0}$, which is analogous to the one described in the proof of Theorem 4.1

Applied to the connection of BA with itself w.r.t. the theory of semilattices considered in Example 5.1, the proof of Theorem 5.2 shows that deciding the universal theory of $E_{\mathbf{K}}$ can be reduced to deciding the universal theory of BA. It is well-known that deciding the universal theory of $E_{\mathbf{K}}$ is equivalent to deciding global consequence in $\mathbf{K}$, and that deciding the universal theory of BA is equivalent to propositional reasoning. Thus, we have shown the (rather surprising) result that the global consequence problem in $\mathbf{K}$ can be reduced to purely propositional reasoning. However, if we directly apply the non-deterministic combination algorithm suggested by Proposition 5.3, then the complexity of the obtained decision procedure is worse then the known ExpTime-complexity [Spa93] of the problem. The deterministic combination procedure described below overcomes this problem.

5.2 A deterministic combination procedure

As pointed out in [Opp80], Nelson-Oppen style combination procedures can be made deterministic in the presence of a certain convexity condition. Let T be a theory over the signature Ω, and let Ω_0 be a subsignature of Ω. Following [Tin03], we say that T is Ω_0-*convex* iff every finite set of ground Ω^X-literals (using additional free constants from X) T-entailing a disjunction of $n > 1$ Ω_0^X-atoms, already T-entails one of the disjuncts. Note that universal Horn Ω-theories are always Ω-convex. In particular, this means that equational theories (like BA) are convex w.r.t. any subsignature.

Let $T_0 \subseteq T$ be theories over the respective signatures Ω_0, Ω, where Ω_0 is a subsignature of Ω. If T is Ω_0-convex, then Theorem 5.2 can be shown with the help of a deterministic combination procedure. (The same is actually also true for Theorem 4.1 and Theorem 4.3, but this will not explicitly be shown here.)

Let $\Gamma \cup \Gamma_0$ be a finite set of ground flat literals (with free constants) in the signature of $T_{>T_0}$; suppose also that Γ does not contain the symbol h and that $\Gamma_0 = \{h(a_1) \approx b_1, \ldots, h(a_n) \approx b_n\}$. We say that Γ is Γ_0-*saturated* iff for every Ω_0-atom $A(x_1, \ldots, x_n)$, $T \cup \Gamma \models A(a_1, \ldots, a_n)$ implies $A(b_1, \ldots, b_n) \in \Gamma$.

Theorem 5.4 *Let T_0, T be theories over the respective signatures Ω_0, Ω, where Ω_0 is a subsignature of Ω. Assume that $T_0 \subseteq T$, that T_0 is universal and locally finite, and that T is Ω_0-convex and T_0-algebraically compatible. Then the following deterministic procedure decides whether $\Gamma \cup \Gamma_0$ is satisfiable in $T_{>T_0}$ (where Γ, Γ_0 are as above):*

1. *Γ_0-saturate Γ;*
2. *check whether the Γ_0-saturated set $\widehat{\Gamma}$ obtained this way is satisfiable in T.*

Proof. The saturation process (and thus the procedure) terminates because T_0 is locally finite (it should be clear that saturation is done modulo T_0). In addition, if $\Gamma \cup \Gamma_0$ is satisfied in a model $\mathcal{M}$ of $T_{>T_0}$, then the reduct of $\mathcal{M}$ to the signature Ω obviously satisfies $\widehat{\Gamma}$.

Conversely, if the Γ_0-saturated set $\widehat{\Gamma}$ is satisfiable in T, then we use $\widehat{\Gamma}$ to construct a triple $(\mathcal{A}, \mathcal{B}, \nu)$ satisfying 1.–4 of Proposition 5.3. Since $\widehat{\Gamma}$ is satisfiable in T, and T is Ω_0-convex, the following two finite[6] sets of literals are both satisfiable in T_0 (where $\underline{a}$ abbreviate $a_1, \ldots, a_n$ and let $\underline{b}$ abbreviate $b_1, \ldots, b_n$):

$$\begin{aligned} \Gamma_{\underline{a}} &:= \{A(\underline{a}) \mid T \cup \widehat{\Gamma} \models A(\underline{a})\} \;\cup\; \{\neg A(\underline{a}) \mid T \cup \widehat{\Gamma} \not\models A(\underline{a})\}, \\ \Gamma_{\underline{b}} &:= \{A(\underline{b}) \mid T \cup \widehat{\Gamma} \models A(\underline{b})\} \;\cup\; \{\neg A(\underline{b}) \mid T \cup \widehat{\Gamma} \not\models A(\underline{b})\}, \end{aligned}$$

[6] It goes without saying that "finiteness" here means "finiteness modulo T_0;" see the definition of local finiteness.

where $A(\underline{x})$ ranges over Ω_0-atoms (modulo T_0). In fact, assume (without loss of generality) that $\Gamma_{\underline{a}}$ is not satisfiable in T_0. This means that

$$T_0 \cup \{A(\underline{a}) \mid T \cup \widehat{\Gamma} \models A(\underline{a})\} \models \bigvee_{T\cup\widehat{\Gamma}\not\models A(\underline{a})} A(\underline{a}),$$

Since $T_0 \subseteq T$ and T is Ω_0-convex, this implies that $T \cup \{A(\underline{a}) \mid T \cup \widehat{\Gamma} \models A(\underline{a})\} \models A'(\underline{a})$ for some Ω_0-atom $A'(\underline{x})$ such that $T \cup \widehat{\Gamma} \not\models A'(\underline{a})$. However, $T \cup \{A(\underline{a}) \mid T \cup \widehat{\Gamma} \models A(\underline{a})\} \models A'(\underline{a})$ obviously implies $T \cup \widehat{\Gamma} \models A'(\underline{a})$, which yields the desired contradiction.

Pick a pair of models of T_0 satisfying $\Gamma_{\underline{a}}$ and $\Gamma_{\underline{b}}$, and let $\mathcal{A}$, $\mathcal{B}$ be their Ω_0-substructures generated by (the interpretations of) $\underline{a}$ and $\underline{b}$, respectively. Since T_0 is universal, $\mathcal{A}$ and $\mathcal{B}$ are models of T_0. Moreover, by construction, for every Ω_0-atom $A(\underline{x})$ we have that $T\cup\widehat{\Gamma} \models A(\underline{a})$ iff $\mathcal{A} \models A(\underline{a})$ and, similarly, $T\cup\widehat{\Gamma} \models A(\underline{b})$ iff $\mathcal{B} \models A(\underline{b})$. As a consequence, the Γ_0-saturatedness of $\widehat{\Gamma}$ and Robinson's diagram theorem guarantee that the map associating b_i with a_i can be extended to a homomorphism $\nu : \mathcal{A} \to \mathcal{B}$.

It remains to show that $\widehat{\Gamma} \cup \Delta_{\Omega_0}(\mathcal{A}) \cup \Delta_{\Omega_0}(\mathcal{B})$ is satisfiable in T (since $\Gamma \subseteq \widehat{\Gamma}$, this implies that $\Gamma \cup \Delta_{\Omega_0}(\mathcal{A}) \cup \Delta_{\Omega_0}(\mathcal{B})$ is satisfiable in T). Taking into consideration the Ω_0-convexity of T and the fact that $\widehat{\Gamma}$ is satisfiable in T, satisfiability of $\widehat{\Gamma}\cup\Delta_{\Omega_0}(\mathcal{A})\cup\Delta_{\Omega_0}(\mathcal{B})$ in T means that for no atom $A(\underline{a})$ false in $\mathcal{A}$ ($A(\underline{b})$ false in $\mathcal{B}$) we have that $T\cup\widehat{\Gamma}\cup\Delta^+_{\Omega_0}(\mathcal{A})\cup\Delta^+_{\Omega_0}(\mathcal{B}) \models A(\underline{a})$ ($T\cup\widehat{\Gamma}\cup\Delta^+_{\Omega_0}(\mathcal{A})\cup\Delta^+_{\Omega_0}(\mathcal{B}) \models A(\underline{b})$).[7] However, as remarked above, $T\cup\widehat{\Gamma} \models A(\underline{a})$ holds iff $\mathcal{A} \models A(\underline{a})$ holds (and similarly for $\mathcal{B}$). This means that $T\cup\widehat{\Gamma}\cup\Delta^+_{\Omega_0}(\mathcal{A})\cup\Delta^+_{\Omega_0}(\mathcal{B})$ is the same theory as $T\cup\widehat{\Gamma}$. But then the claim that "for no atom $A(\underline{a})$ false in $\mathcal{A}$ (or $A(\underline{b})$ false in $\mathcal{B}$) we have that $T \cup \widehat{\Gamma} \models A(\underline{a})$ ($T \cup \widehat{\Gamma} \models A(\underline{b})$)" becomes trivial, once again because $T \cup \widehat{\Gamma} \models A(\underline{a})$ is equivalent to $\mathcal{A} \models A(\underline{a})$ ($T \cup \widehat{\Gamma} \models A(\underline{b})$ is equivalent to $\mathcal{B} \models A(\underline{b})$). $\dashv$

Example 5.1 (continued) Let us come back to the connection of $T := BA$ with itself w.r.t. the theory T_0 of semilattices, which yields as combined theory the equational theory $E_{\mathbf{K}}$ corresponding to the basic modal logic **K**. In this case, checking during the saturation process whether $T \cup \Gamma \models A(\underline{a})$ amounts to checking whether a propositional formula ϕ_Γ (whose size is linear in the size of Γ) implies a propositional formula of the form $\psi_1 \Leftrightarrow \psi_2$, where ψ_1, ψ_2 are disjunctions of the propositional variables from $\underline{a}$. Since propositional reasoning can be done in time exponential in the number of propositional variables, and there are only exponentially many different formulae of the form $\psi_1 \Leftrightarrow \psi_2$, the saturation process needs at most exponential time. The size of the Γ_0-saturated set $\widehat{\Gamma}$ may be exponential in the size of Γ, but it still contains only the free constants $\underline{a}$. Consequently, testing satisfiability of $\widehat{\Gamma}$ in T is again a propositional

[7] Recall that $\Delta^+_{\Omega_0}(\mathcal{A})$ denotes the positive diagram of $\mathcal{A}$, i.e., it consists of those atoms true in $\mathcal{A}$. Also note that $\neg A(\underline{a}) \in \Delta_{\Omega_0}(\mathcal{A}) \setminus \Delta^+_{\Omega_0}(\mathcal{A})$ iff the atom $A(\underline{a})$ is false in $\mathcal{A}$.

reasoning problem that can be done in time exponential in the number of free constants $\underline{a}$.

Consequently, we have shown that Theorem 5.4 yields an ExpTime decision procedure for the global consequence relation in $\mathbf{K}$, which thus matches the known worst-case complexity of the problem.

6 Conditions on the connection functions

Until now, we have considered connection functions that are arbitrary homomorphisms. In this section we impose the additional conditions that the connection functions be surjective, embeddings, or isomorphisms: in this way, we obtain new combined theories, which we denote by $T_1 >^{em}_{T_0} T_2, T_1 >^{s}_{T_0} T_2, T_1 >^{iso}_{T_0} T_2$, respectively. This defines the combined theories in a model-theoretic way. One can also give an axiomatic description of $T_1 >^{em}_{T_0} T_2, T_1 >^{s}_{T_0} T_2$, and $T_1 >^{iso}_{T_0} T_2$. For example, the axioms of $T_1 >^{s}_{T_0} T_2$ are obtained from the ones of $T_1 >_{T_0} T_2$ by adding axioms expressing that h is surjective, i.e., for every sort S in Ω_0 we add the axiom

$$\forall y. \exists x. h_S(x) = y,$$

where x is a variable of sort S^1 and y a variable of sort S^2.

For these combined theories one can show combination results that are analogous to Theorem 4.1: one just needs different compatibility conditions. To treat embeddings and isomorphisms, we use the compatibility condition introduced in [Ghi05, BGT04] for the case of unions of theories. Following [Ghi05, BGT04], we call this condition T_0-compatibility in the following.

In order to define this notion of compatiblity, we need to introduce the notion of a model completion. The definition given below differs from the one given in [Ghi05, BGT04]. However, the two notions can be shown to be equivalent (see Proposition 9.6 in Appendix B below). The reason for giving an alternative formulation is that it makes the connection between a model completion and a positive algebraic completion more transparent.

Definition 6.1 *Let T be a universal Ω-theory and let T^* be an Ω-theory. We say that T^* is a* model completion *of T iff the following conditions are satisfied:*

(i) $T \subseteq T^*$;

(ii) *every model of T embeds into a model of T^*;*

(iii) *for every formula $\phi(\underline{x})$ there is an open formula $\phi^*(\underline{x})$ such that*

$$T^* \models \phi \leftrightarrow \phi^*.$$

It can be shown that models of T^* are just the existentially closed models of T (see [CK90] or Appendix B below).

Definition 6.2 *Let $T_0 \subseteq T$ be theories over the respective signatures $\Omega_0 \subseteq \Omega$. We say that T is T_0-*compatible *iff T_0 is universal, has a model completion T_0^*, and every model of T embeds into a model of $T \cup T_0^*$.*

6.1 Embeddings as connection functions

Let us first investigate the case of connection functions that are embeddings.

Theorem 6.3 *Let T_0, T_1, T_2 be theories over the respective signatures $\Omega_0, \Omega_1, \Omega_2$, where Ω_0 is a common subsignature of Ω_1 and Ω_2. Assume that $T_0 \subseteq T_1$ and $T_0 \subseteq T_2$, and that T_0 is universal and locally finite. If T_2 is T_0-compatible, then the decidability the universal fragments of T_1 and T_2 entails the decidability of the universal fragment of $T_1 >^{em}_{T_0} T_2$.*

As usual, in order to prove the Theorem, we consider a finite set Γ of ground flat literals over the signature Ω of $T_1 >^{em}_{T_0} T_2$ (with additional free constants), and show how it can be tested for satisfiability in $T_1 >^{em}_{T_0} T_2$. Since all literals in Γ are flat, we can divide Γ into three disjoint sets $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$, where Γ_i $(i = 1, 2)$ is a set of literals in the signature Ω_i (expanded with free constants), and Γ_0 is of the form

$$\Gamma_0 = \{h(a_1) \approx b_1, \ldots, h(a_n) \approx b_n\}$$

for free constants $a_1, b_1, \ldots, a_n, b_n$. Theorem 6.3 easily follows from the next proposition:

Proposition 6.4 *The constraint $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$ is satisfiable in $T_1 >^{em}_{T_0} T_2$ iff there exists a triple $(\mathcal{A}, \mathcal{B}, \nu)$ such that*

1. *$\mathcal{A}$ is an Ω_0-model of T_0, which is generated by $\{a_1^{\mathcal{A}}, \ldots, a_n^{\mathcal{A}}\}$;*
2. *$\mathcal{B}$ is an Ω_0-model of T_0, which is generated by $\{b_1^{\mathcal{B}}, \ldots, b_n^{\mathcal{B}}\}$;*
3. *$\nu : \mathcal{A} \to \mathcal{B}$ is an Ω_0-embedding such that $\nu(a_j^{\mathcal{A}}) = b_j^{\mathcal{B}}$ for $j = 1, \ldots, n$;*
4. *$\Gamma_1 \cup \Delta_{\Omega_0}(\mathcal{A})$ is satisfiable in T_1;*
5. *$\Gamma_2 \cup \Delta_{\Omega_0}(\mathcal{B})$ is satisfiable in T_2.*

Proof. Again, the only-if direction is simple. Conversely, assume that $(\mathcal{A}, \mathcal{B}, \nu)$ is a triple satisfying 1.–5. of the proposition. Because of 4. and 5, there is an Ω_1-model $\mathcal{N}'$ of T_1 satisfying $\Gamma_1 \cup \Delta_{\Omega_0}(\mathcal{A})$ and an Ω_2-model $\mathcal{N}''$ of T_2 satisfying $\Gamma_2 \cup \Delta_{\Omega_0}(\mathcal{B})$. By Robinson's diagram theorem, $\mathcal{N}'$ has $\mathcal{A}$ as an Ω_0-substructure and $\mathcal{N}''$ has $\mathcal{B}$ as an Ω_0-substructure. As in the proof of Proposition 4.2, we assume without loss of generality that $\mathcal{N}'$ is at most countable and that $\mathcal{N}''$ is a model of $T_2 \cup T_0^*$. Let us enumerate the elements of $\mathcal{N}'$ as

$$c_1, c_2, \ldots, c_n, c_{n+1}, \ldots$$

where we assume that $c_i = a_i^{\mathcal{A}}$ $(i = 1, \ldots, n)$, i.e., $c_1, \ldots, c_n$ are generators of $\mathcal{A}$. We define an increasing sequence of sort-conforming functions $\nu_k : \{c_1, \ldots c_k\} \to N''$ (for $k \geq n$) such that, for every ground $\Omega_0^{\{c_1,\ldots,c_k\}}$-*literal* A we have

$$\mathcal{N}'_{|\Omega_0} \models A(c_1, \ldots, c_k) \quad \text{implies} \quad \mathcal{N}''_{|\Omega_0} \models A(\nu_k(c_1), \ldots, \nu_k(c_k))$$

Since this condition is asked for literals and not just for atoms, it follows that the mappings ν_k are injective.

We first take ν_n to be ν. To define ν_{k+1} (for $k \geq n$), let us consider the conjunction $\psi(c_1, \ldots, c_n, c_{n+1})$ of the $\Omega_0^{\{c_1,\ldots,c_{n+1}\}}$-literals that are true in $\mathcal{N}'_{|\Omega_0}$: this conjunction is finite (modulo taking representative terms, thanks to local finiteness of T_0). Let $\phi(x_1, \ldots, x_n)$ be $\exists x_{n+1}.\psi(x_1, \ldots, x_n, x_{n+1})$ and let $\phi^*(x_1, \ldots, x_n)$ be an open formula such that $T_0^* \models \phi \leftrightarrow \phi^*$.

By (i) and (ii) of Definition 6.1, Lemma 3.1, and the fact that $\phi \to \phi^*$ is equivalent to an open formula, we have $T_0 \models \phi \to \phi^*$. This implies $\mathcal{N}'_{|\Omega_0} \models \phi^*(c_1, \ldots, c_k)$, and thus $\mathcal{N}''_{|\Omega_0} \models \phi^*(\nu_k(c_1), \ldots, \nu_k(c_k))$ by the induction hypothesis. Since $\mathcal{N}''_{|\Omega_0}$ is a model of T_0^* and $T_0^* \models \phi^* \to \phi$, there is an element b of $\mathcal{N}''_{|\Omega_0}$ such that $\mathcal{N}''_{|\Omega_0} \models \psi(\nu_k(c_1), \ldots, \nu_k(c_k), b)$. We now obtain the desired extension ν_{k+1} of ν_k by setting $\nu_{k+1}(c_{k+1}) := b$. Taking $\nu_\infty = \bigcup_{k \geq n} \nu_k$, we finally obtain an embedding $\nu_\infty : \mathcal{N}'_{|\Omega_0} \to \mathcal{N}''_{|\Omega_0}$ such that the triple $(\mathcal{N}', \mathcal{N}'', \nu_\infty)$ is a model of $T_1 >^{em}_{T_0} T_2$ that satisfies $\Gamma_0 \cup \Gamma_1 \cup \Gamma_2$. $\dashv$

6.2 Surjective connections

To treat $T_1 >^{s}_{T_0} T_2$, we must dualize the notions "algebraic completion" and "algebraic compatibility". These notions are based on co-geometric formulae, which the dual of geometric formulae in the sense that existential quantification is replaced by universal quantification. A *co-geometric formula* is a formula built from atoms by using conjunction, disjunction and universal quantification. Similarly, a *co-geometric* theory is a theory axiomatized by (universal closure of) implications of co-geometric formulae.

Definition 6.5 *Let T be a universal Ω-theory, and let T^* be an Ω-theory. We say that T^* is a* positive co-algebraic completion *of T iff the following conditions are satisfied:*

(i) $T \subseteq T^*$*;*

(ii) *every model of T embeds into a model of T^*;*

(iii) *for every co-geometric formula $\phi(\underline{x})$ there is an open co-geometric formula $\phi^*(\underline{x})$ such that*

$$T^* \models \phi \leftrightarrow \phi^*.$$

The new notion of compatibility defined below differs from the one introduced in Section 3 in that positive algebraic completions are replaced by positive co-algebraic completions.

Definition 6.6 *Let $T_0 \subseteq T$ be theories over the respective signatures $\Omega_0 \subseteq \Omega_1$. We say that T is T_0-co-algebraically compatible iff T_0 is universal, has a positive co-algebraic completion T_0^*, and every model of T embeds into a model of $T \cup T_0^*$.*

If the prerequisites of Theorem 4.1 hold and T_1 is additionally T_0-co-algebraically compatible, then decidability of the universal fragment transfers from T_1, T_2 to $T_1 >^s_{T_0} T_2$.

Theorem 6.7 *Let T_0, T_1, T_2 be theories over the respective signatures $\Omega_0, \Omega_1, \Omega_2$, where Ω_0 is a common subsignature of Ω_1 and Ω_2. Assume that $T_0 \subseteq T_1$ and $T_0 \subseteq T_2$, that T_0 is universal and locally finite, that T_1 is T_0-co-algebraically compatible, and that T_2 is T_0-algebraically compatible. Then the decidability of the universal fragments of T_1 and T_2 entails the decidability of the universal fragment of $T_1 >^s_{T_0} T_2$.*

To prove the theorem, let $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$ be a finite set of ground flat literals over the signature Ω of $T_1 >^s_{T_0} T_2$ (with additional free constants), where Γ_i $(i = 1, 2)$ is a set of literals in the signature Ω_i (expanded with free constants), and Γ_0 is of the form

$$\{h(a_1) \approx b_1, \ldots, h(a_n) \approx b_n\},$$

for free constants $a_1, b_1, \ldots, a_n, b_n$. The following proposition, whose formulation is identical to the formulation of Proposition 4.2, immediately entails Theorem 6.7.

Proposition 6.8 *The constraint $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$ is satisfiable in $T_1 >^s_{T_0} T_2$ iff there exists a triple $(\mathcal{A}, \mathcal{B}, \nu)$ such that*

1. *$\mathcal{A}$ is an Ω_0-model of T_0, which is generated by $\{a_1^{\mathcal{A}}, \ldots, a_n^{\mathcal{A}}\}$;*

2. *$\mathcal{B}$ is an Ω_0-model of T_0, which is generated by $\{b_1^{\mathcal{B}}, \ldots, b_n^{\mathcal{B}}\}$;*

3. *$\nu : \mathcal{A} \to \mathcal{B}$ is an Ω_0-homomorphism such that $\nu(a_j^{\mathcal{A}}) = b_j^{\mathcal{B}}$ for $j = 1, \ldots, n$;*

4. *$\Gamma_1 \cup \Delta_{\Omega_0}(\mathcal{A})$ is satisfiable in T_1;*

5. *$\Gamma_2 \cup \Delta_{\Omega_0}(\mathcal{B})$ is satisfiable in T_2.*

Proof. The only-if direction is again simple. The proof of the if direction requires now a back-and-forth argument. Suppose we are given $\mathcal{A}$, $\mathcal{B}$, ν as in 1.–5. of the proposition, and let $\mathcal{N}'$ be an Ω_1-model of T_1 satisfying $\Gamma_1 \cup \Delta_{\Omega_0}(\mathcal{A})$, and $\mathcal{N}''$ be an Ω_2-model of T_2 satisfying $\Gamma_2 \cup \Delta_{\Omega_0}(\mathcal{B})$. We can assume without loss of generality that $\mathcal{N}', \mathcal{N}''$ are both at most countable, that $\mathcal{N}'$ is a model of the positive co-algebraic completion of T_0, and that $\mathcal{N}''$ is a model of the positive algebraic completion of T_0. By Robinson's diagram theorem, $\mathcal{N}'$ has $\mathcal{A}$ as an Ω_0-substructure, and $\mathcal{N}''$ has $\mathcal{B}$ as an Ω_0-substructure. Let us enumerate the elements of N' as

$$c_1, c_3, \ldots, c_{2k+1}, \ldots$$

and the elements of N'' as

$$d_2, d_4, \ldots, d_{2k}, \ldots$$

(here we prefer, for uniformity, both lists to be infinite, so we may tolerate repetitions in each list). We define an increasing sequence of sort-conforming surjective mappings $\nu_k : S_k \longrightarrow T_k$, such that:

- S_k is a finite subset of N' including all the elements from A as well as c_{2j+1}, for $2j + 1 \leq k$;
- T_k is a finite subset of N'' including all the elements from B as well as d_{2j}, for $2j \leq k$;
- for all Ω_0-atoms $C(\underline{x})$ we have

$$\mathcal{N}'_{|\Omega_0} \models C(\underline{a}) \quad \text{implies} \quad \mathcal{N}''_{|\Omega_0} \models C(\nu_k(\underline{a})) \tag{10}$$

  for every tuple $\underline{a}$ from S_k.

Once this is settled, $\mathcal{N}'$ and $\mathcal{N}''$ together with the surjective homomorphism $\nu_\infty = \bigcup_{k \geq n} \nu_k$ give, as usual, the desired model of $T_1 >^s_{T_0} T_2$ satisfying Γ.

We first take ν_0 to be ν. To define ν_k ($k > 0$), we distinguish the case in which k is even from the case in which k is odd. In the latter case, we proceed as in the proof of Proposition 4.2. As to the former case, let $b = d_{2k}$ and let $\underline{a}$ be a tuple collecting all the elements from S_{k-1}. We want to find a suitable $a \in N'$ in order to extend ν_{k-1} by defining $\nu_k(a) := b$. For this purpose, it is sufficient to show that $\mathcal{N}' \not\models \forall y.\phi(\underline{a}, y)$, where $\phi(\underline{x}, y)$ is the disjunction of all atoms $C(\underline{x}, y)$ such that $\mathcal{N}'' \not\models C(\nu_{k-1}(\underline{a}), b)$. In fact, if $\mathcal{N}' \not\models \forall y.\phi(\underline{a}, y)$, then there is a (sort-conforming) $a \in N'$ such that $\mathcal{N}' \models \neg\phi(\underline{a}, a)$, and we can set $\nu_k(a) := b$. Assume that C is an atom such that $\mathcal{N}'_{|\Omega_0} \models C(\underline{a}, a)$, but $\mathcal{N}''_{|\Omega_0} \not\models C(\nu_k(\underline{a}, a)) = C(\nu_{k-1}(\underline{a}), b)$. However, this means that $C(\underline{x}, y)$ occurs as a disjunct in $\phi(\underline{x}, y)$, and thus $\mathcal{N}' \models \neg\phi(\underline{a}, a)$ implies that $\mathcal{N}' \models \neg C(\underline{a}, a)$, which is a contradiction to our assumption that $\mathcal{N}'_{|\Omega_0} \models C(\underline{a}, a)$.

To show that $\mathcal{N}' \not\models \forall y.\phi(\underline{a}, y)$, we consider the positive co-algebraic completion T_0^* of T_0. In this theory, $\forall y.\phi(\underline{x}, y) \leftrightarrow \phi^*(\underline{x})$ is provable for some (co-)geometric open formula[8] $\phi^*(\underline{x})$. As usual, the implication $\phi^*(\underline{x}) \rightarrow \forall y.\phi(\underline{x}, y)$ must already hold in T_0 because T_0 and its co-algebraic completion T_0^* entail the same open formulae, and $\phi^*(\underline{x}) \rightarrow \forall y.\phi(\underline{x}, y)$ is equivalent to the open formula $\phi^*(\underline{x}) \rightarrow \phi(\underline{x}, y)$.

Since $\mathcal{N}'$ is a model of T_0^*, and $T_0^* \models \forall y.\phi(\underline{x}, y) \rightarrow \phi^*(\underline{x})$, it is enough to prove that $\mathcal{N}' \not\models \phi^*(\underline{a})$. However, $\mathcal{N}'' \not\models \forall y.\phi(\nu_{k-1}(\underline{a}), y)$, by the definition of ϕ. Since $\mathcal{N}''$ is a model of T_0, and $T_0 \models \phi^*(\underline{x}) \rightarrow \forall y.\phi(\underline{x}, y)$, this implies $\mathcal{N}'' \not\models \phi^*(\nu_{k-1}(\underline{a}))$. Finally, the induction hypothesis on the validity of (10) yields $\mathcal{N}' \not\models \phi^*(\underline{a})$. $\dashv$

The following example shows that there are natural examples of theories T_0 admitting both a positive algebraic and a positive co-algebraic completion.

Example 6.9 Consider the theory of join semilattices with a greatest element. These are join semilattices as introduced in Example 4.6, but endowed with a further element $\top$ such that $x \sqcup \top = \top$ holds for all x. The positive algebraic completion of this theory is axiomatized as in Example 4.6 above. In order to axiomatize the co-algebraic completion of this theory, we need a theory that allows us to eliminate the universal quantifier from formulae $\forall y.\phi(\underline{x}, y)$ of the form

$$\forall y. \left((y \sqsubseteq t_1) \vee \cdots \vee (y \sqsubseteq t_n) \vee (u_1 \sqsubseteq s_1 \sqcup y) \vee \cdots \vee (u_m \sqsubseteq s_m \sqcup y)\right), \tag{11}$$

where t_i, s_j, u_k are terms not involving y. Let $\phi^*(\underline{x})$ be the formula

$$\bigvee_{i=1}^{n} (t_i \approx \top) \vee \bigvee_{j=1}^{m} (u_j \sqsubseteq s_j), \tag{12}$$

and let T_0^* be obtained from T_0 by adding to it the universal closures of the sentences $\phi \leftrightarrow \phi^*$. The theory T_0^* is included in the theory BA^* of atomless Boolean algebras (recall that a Boolean algebra is said to be atomless iff it does not have non-zero minimal elements): the axioms of T_0^* are in fact provable in BA^*, as it is evident from the quantifier elimination procedure for BA^* (see, e.g., [GZ02]). Since every join semilattice with a greatest element embeds into an

[8] In the open case, geometric and co-geometric formulae trivially coincide.

atomless Boolean algebra,[9] this shows both that T_0^* is the positive co-algebraic completion of T_0, and that the theory of Boolean algebras is co-algebraically compatible with the theory of join semilattices with a greatest element.

Since the formulation of Proposition 6.8 coincides with the one of Proposition 4.2, we know that the universal fragments of $T_1 >^s_{T_0} T_2$ and $T_1 >_{T_0} T_2$ coincide if the conditions of Theorem 6.7 are satisfied.

Corollary 6.10 *Let T_0, T_1, T_2 be theories over the respective signatures $\Omega_0, \Omega_1, \Omega_2$, where Ω_0 is a common subsignature of Ω_1 and Ω_2. Assume that $T_0 \subseteq T_1$ and $T_0 \subseteq T_2$, that T_0 is universal and locally finite, that T_1 is T_0-co-algebraically compatible, and that T_2 is T_0-algebraically compatible. Then the universal fragment of $T_1 >_{T_0} T_2$ coincides with the universal fragment of $T_1 >^s_{T_0} T_2$.*

6.3 Isomorphisms as connection functions

Finally, let us consider the problem of deciding the universal fragment of $T_1 >^{iso}_{T_0} T_2$.

Theorem 6.11 *Let T_0, T_1, T_2 be theories over the respective signatures $\Omega_0, \Omega_1, \Omega_2$, where Ω_0 is a common subsignature of Ω_1 and Ω_2. Assume that $T_0 \subseteq T_1$ and $T_0 \subseteq T_2$, that T_0 is universal and locally finite, and that T_1, T_2 are both T_0-compatible. Then the decidability of the universal fragments of T_1 and T_2 entails the decidability of the universal fragment of $T_1 >^{iso}_{T_0} T_2$.*

To prove the theorem, let $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$ be a finite set of ground flat literals over the signature Ω of $T_1 >^{iso}_{T_0} T_2$ (with additional free constants), where Γ_i $(i = 1, 2)$ is a set of literals in the signature Ω_i (expanded with free constants), and Γ_0 is of the form

$$\{h(a_1) \approx b_1, \ldots, h(a_n) \approx b_n\},$$

for free constants $a_1, b_1, \ldots, a_n, b_n$. The following proposition, whose formulation is identical to the formulation of Proposition 6.4, immediately entails Theorem 6.11.

Proposition 6.12 *The constraint $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$ is satisfiable in $T_1 >^{iso}_{T_0} T_2$ iff there exists a triple $(\mathcal{A}, \mathcal{B}, \nu)$ such that*

[9] One can embed a join semilattice with greatest element into a bounded distributive lattice by taking the dual of the lattice of non-empty upward closed subsets; that bounded distributive lattices embed into Boolean algebras, and that Boolean algebras embed into atomless Boolean algebras are standard lattice-theoretic facts.

1. $\mathcal{A}$ *is an* Ω_0*-model of* T_0*, which is generated by* $\{a_1^{\mathcal{A}}, \dots, a_n^{\mathcal{A}}\}$*;*
2. $\mathcal{B}$ *is an* Ω_0*-model of* T_0*, which is generated by* $\{b_1^{\mathcal{B}}, \dots, b_n^{\mathcal{B}}\}$*;*
3. $\nu : \mathcal{A} \to \mathcal{B}$ *is an* Ω_0*-embedding such that* $\nu(a_j^{\mathcal{A}}) = b_j^{\mathcal{B}}$ *for* $j = 1, \dots, n$*;*
4. $\Gamma_1 \cup \Delta_{\Omega_0}(\mathcal{A})$ *is satisfiable in* T_1*;*
5. $\Gamma_2 \cup \Delta_{\Omega_0}(\mathcal{B})$ *is satisfiable in* T_2*.*

Proof. To prove the if direction, we must extend ν to an isomorphism between the Ω_0-reducts of $\mathcal{N}', \mathcal{N}''$, where $\mathcal{N}', \mathcal{N}''$ are at most countable models of the diagrams of $\mathcal{A}, \mathcal{B}$ and of $T_1 \cup T_0^*$, $T_2 \cup T_0^*$, respectively. The back-and-forth argument used in the proof of Proposition 6.8 can be easily adapted to the present case: it sufficient to ask in condition (10) for truth of ground $\Omega_0^{S_k}$*-literals* rather than just atoms to be preserved.

In the case of k being odd, one can proceed as in the proof of Proposition 6.4. In the case of k being even, one must adapt the construction given in Proposition 6.8 appropriately to the stronger condition. We leave this simple adaptation to the reader. ⊣

Since the formulation of Proposition 6.12 coincides with the one of Proposition 6.4, we know that the universal fragments of $T_1 >_{T_0}^{em} T_2$ and $T_1 >_{T_0}^{iso} T_2$ coincide if the conditions of Theorem 6.11 are satisfied.

Corollary 6.13 *Let* T_0, T_1, T_2 *be theories over the respective signatures* $\Omega_0, \Omega_1, \Omega_2$*, where* Ω_0 *is a common subsignature of* Ω_1 *and* Ω_2*. Assume that* $T_0 \subseteq T_1$ *and* $T_0 \subseteq T_2$*, that* T_0 *is universal and locally finite, and that* T_1, T_2 *are* T_0*-compatible. Then the universal fragment of* $T_1 >_{T_0}^{em} T_2$ *coincides with the universal fragment of* $T_1 >_{T_0}^{iso} T_2$*.*

It is easy to see that the problem of deciding the universal fragment of $T_1 >_{T_0}^{iso} T_2$ is interreducable in polynomial time with the problem of deciding the universal fragment of $T_1 \cup T_2$. Consequently, the proof of Theorem 6.11 yields an alternative proof of the combination result in [Ghi05].

The main reason for this is that there is a close connection between models of $T_1 \cup T_2$ and $T_1 >_{T_0}^{iso} T_2$. In fact, if $\mathcal{M}$ is a model of $T_1 \cup T_2$, then it can be turned into a model $(\mathcal{M}^1, \mathcal{M}^2, \nu)$ of $T_1 >_{T_0}^{iso} T_2$ by taking as $\mathcal{M}^1$ the reduct of $\mathcal{M}$ to Ω_1, as $\mathcal{M}^2$ the reduct of $\mathcal{M}$ to Ω_2, and as isomorphism ν the identity mapping on the domain of the reduct of $\mathcal{M}$ to Ω_0. Conversely, if $(\mathcal{M}^1, \mathcal{M}^2, \nu)$ is a model of $T_1 >_{T_0}^{iso} T_2$, then one can turn it into a model of $T_1 \cup T_2$ by adapting the well-known fusion construction [TR03] to the many-sorted case.

Now, given a conjunction Γ of (sort-conforming) literals to be tested for satisfiability in $T_1 >^{iso}_{T_0} T_2$, we can simply remove the connection function h and the superscripts introduced through the renaming done in the construction of $T_1 >^{iso}_{T_0} T_2$, and test the resulting conjunction Γ' of literals for satisfiability in $T_1 \cup T_2$. If $\mathcal{M}$ is a model of $T_1 \cup T_2$ satisfying Γ', then it is easy to see that the corresponding model $(\mathcal{M}^1, \mathcal{M}^2, \nu)$ of $T_1 >^{iso}_{T_0} T_2$ satisfies Γ. Conversely, if $(\mathcal{M}^1, \mathcal{M}^2, \nu)$ is a model of $T_1 >^{iso}_{T_0} T_2$ satisfying Γ, then it is easy to see that the model $\mathcal{M}$ of $T_1 \cup T_2$ obtained from this model by applying the fusion construction satisfies Γ'.

Conversely, given a conjunction Γ of flat ground literals to be tested for satisfiability in $T_1 \cup T_2$, we can partition Γ into $\Gamma = \Gamma_1 \cup \Gamma_2$ where Γ_1 is over the signature Ω_1 and Γ_2 is over the signature Ω_2. For every free constant c occurring in Γ, we introduce two free constants c^1 and c^2. We replace c in Γ_1 by c^1 and c in Γ_2 by c^2, and also do the appropriate renamings of the shared function and predicate symbols. In addition, we add the identity $c^2 \approx h(c^1)$ for each free constant c occurring in Γ. Let Γ' be the conjunction of literals over the signature of $T_1 >^{iso}_{T_0} T_2$ obtained this way. Again, it is easy to see that Γ is satisfiable in $T_1 \cup T_2$ iff Γ' is satisfiable in $T_1 >^{iso}_{T_0} T_2$.

7 Conclusion

We have introduced a new scheme for combining many-sorted theories, and have shown under which conditions decidability of the universal fragment of the component theories transfers to their combination. Though this kind of combination has been considered before in restricted cases [KLWZ04, AK97, Zar02], it has not been investigated in the general algebraic setting considered here.

In contrast to the results in [KLWZ04], our results are not restricted to Boolean-based equational theories [BGT04]. However, our results do not imply the algebraic counterpart of the more general combination results in [KLWZ04]: there, a connecting relation E (see Example 2.1) introduces *two* connection function: the diamond operators induced by E and its inverse E^{-1}. These two connection functions are not unrelated, but they are not inverses of each other (as functions). An important topic for future work is to try to extend our framework such that it can also handle this type of a connection.

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8 Appendix A: $\mathcal{E}$-connections

The purpose of this appendix is to give a more detailed comparison between the notion of an $\mathcal{E}$-connections, as introduced in [KLWZ04], and our notion of a connection of many-sorted theories.

First of all, [KLWZ04] consider connections that are more general than ours, in the sense that more complex modalities (n-ary modalities, inverse modalities, Boolean combinations of modalities, counting modalities, etc.) can be used as connection functions. Using such sophisticated modalities as connection function is, currently, beyond the scope of our methods, but they will be the subject of future research.

Here, we will content ourselves with examining the special case of plain unary modalities as connection functions, which is the most basic case of an $\mathcal{E}$-connection considered in [KLWZ04]. However, even with this restriction, there are still significant differences between our approach and the approach in [KLWZ04]. The main difference is that, seen from the modal logic point of view, our approach for defining the connection is *syntactic* (or *algebraic*), in the sense that we consider an equational axiomatization of the logic. In contrast, in [KLWZ04] the emphasis is on the model-theoretic side, meaning that $\mathcal{E}$-connections are defined at the *semantic* level as enrichments of suitable Kripke-like structures. Because of this difference, it is not a priori clear that our results specialize to decidability transfer results for $\mathcal{E}$-connections defined in the framework of [KLWZ04] (even within the limitation to plain unary modalities as connection functions). In this appendix, we show that this is indeed the case (but this proof turns out to be not entirely trivial). To simplify matters further, we will not consider abstract description systems (as used in [KLWZ04]) in their full generality, but restrict our considerations to normal modal logics and to standard uni-modal Kripke frames (most of these further restrictions are, however, without loss of generality; they are assumed just for the sake of simplicity).

Propositional modal formulae are built using the Boolean connectives and a diamond operator $\Diamond$. A *Kripke frame* is a pair $\mathcal{F} = (W, R)$, where W is a non-empty set, the set of possible worlds, and R is a binary relation on W, the transition relation. A *Kripke model* is a triple $\mathcal{M} = (W, R, V)$, where (W, R) is a Kripke frame and V is a map, called valuation, associating with each propositional letter a subset of W. The forcing relation $w \models^{\mathcal{M}} \alpha$, which expresses that the modal formula α is true in the Kripke model $\mathcal{M}$ at world w, is defined in the standard way (see, e.g., [BdRV01]).

For a given class of Kripke frames $\mathcal{C}$, the *modal constraint* problem for $\mathcal{C}$ is the problem of deciding whether a finite set of modal formulae is satisfiable w.r.t. a set of global constraints.[10]

[10]This is the kind of problem considered in [KLWZ04], where satisfiability of an A-Box con-

Definition 8.1 *A* modal constraint *is a pair of finite sets of modal formulae, written as $\alpha_1, \dots, \alpha_n; \beta_1, \dots, \beta_m$ $(n, m \geq 0)$; we say that such a modal constraint is* satisfiable in a Kripke model $\mathcal{M} = (W, R, V)$ *iff there are worlds $w_1, \dots, w_m \in W$ such that*

1. $w_1 \models^{\mathcal{M}} \beta_1, \dots, w_m \models^{\mathcal{M}} \beta_m$;
2. *for all $v \in W$ and for all $i = 1, \dots, n$, we have $v \models^{\mathcal{M}} \alpha_i$.*

The modal constraint $\alpha_1, \dots, \alpha_n; \beta_1, \dots, \beta_m$ is satisfiable in a class of Kripke frames $\mathcal{C}$ *iff it is satisfiable in some $\mathcal{M} = (W, R, V)$, for $(W, R) \in \mathcal{C}$.*

Thus, the satisfiability of a modal constraint $\alpha_1, \dots, \alpha_n; \beta_1, \dots, \beta_m$ means that there is a model in which the β_j are satisfied in some worlds w_j, and in which $\alpha_1, \dots, \alpha_n$ hold globally, i.e., in every world.

In order to algebraize the above decision problem, let us introduce the signature B_M: this is the single-sorted signature obtained by expanding the signature of Boolean algebras by a new unary operator that we still call $\Diamond$. Notice that there is an obvious bijective correspondence in this way between modal formulae and terms of the signature B_M (thus, from now on, we identify modal formulae and terms of the signature B_M). Also, a Kripke frame $\mathcal{F} = (W, R)$ can be converted into a B_M-structure called $\mathcal{B}_{\mathcal{F}}$ as follows: we take as underlying Boolean algebra the powerset Boolean algebra $\mathcal{P}(W)$ and interpret $\Diamond$ as the function associating with $X \subseteq W$ the subset of W given by

$$\Diamond(X) := \{w_2 \in W \mid \exists w_1 \in W.\ (w_2, w_1) \in R \wedge w_1 \in X\}.$$

Valuations V of $\mathcal{F}$ correspond in an obvious way to assignments of variables to elements of $\mathcal{P}(W)$. It is easy to see that, for any modal formula θ, we have $w \models^{(W,R,V)} \theta$ iff w belongs to the set obtained by evaluating the term θ in $\mathcal{B}_{\mathcal{F}}$ under the assignment V.

With every class of Kripke frames $\mathcal{C}$ we associate the B_M-theory $\mathcal{T}_{\mathcal{C}}$ whose axioms are the formulae

$$(\alpha_1 \approx \top) \wedge \cdots \wedge (\alpha_n \approx \top) \ \rightarrow \ (\beta_1 \approx \bot) \vee \cdots \vee (\beta_m \approx \bot), \tag{13}$$

where $\alpha_1, \dots, \alpha_n; \beta_1, \dots, \beta_m$ are the modal constraints that are not satisfiable in $\mathcal{C}$. If $\mathcal{F}$ is a Kripke frame in $\mathcal{C}$, then the corresponding B_M-structure $\mathcal{B}_{\mathcal{F}}$ is a model of $\mathcal{T}_{\mathcal{C}}$.

taining many individual constants, with respect to a given T-Box, is taken into consideration. Notice that, by contrast, relativized satisfiability as traditionally intended in modal logic concerns local satisfiability of just one formula with respect to a global constraint formed by a finite set of formulae.

Proposition 8.2 *The problem of deciding satisfiability of modal constraints in $\mathcal{C}$ is equivalent to the problem of deciding the universal fragment of the theory $\mathcal{T}_{\mathcal{C}}$.*

Proof. First, notice that a modal constraint

$$\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_m \tag{14}$$

is unsatisfiable in $\mathcal{C}$ iff the formula (13) is a logical consequence of $\mathcal{T}_{\mathcal{C}}$. In fact, if (14) is unsatisfiable in $\mathcal{C}$, then (13) is an axiom of $\mathcal{T}_{\mathcal{C}}$. Conversely, if (14) is satisfiable in a frame $\mathcal{F} = (W, R) \in \mathcal{C}$, then (13) cannot be a logical consequence of $\mathcal{T}_{\mathcal{C}}$, because it it is easy to see that it is then false in the B_M-structure $\mathcal{B}_{\mathcal{F}}$.

Given that, it is sufficient to observe that identities in $\mathcal{T}_{\mathcal{C}}$ are all equivalent[11] to identities of the kind $\alpha \approx \top$ as well as to identities of the kind $\beta \approx \bot$. Thus an arbitrary open formula in the signature B_M is in fact a conjunction of formulae of the kind (13). Together with what we have shown about the connection between such formulae and modal constraints, this implies the claim of the proposition. $\dashv$

Let us now show that this correspondence

$$\mathcal{C} \longmapsto \mathcal{T}_{\mathcal{C}}$$

is compatible with building connections, where on the left-hand side the connections are the $\mathcal{E}$-connections as introduced in [KLWZ04], and on the right-hand side the connections are the connections of many-sorted theories as introduced in the present paper. To show this, we need to recall the definition of an $\mathcal{E}$-connection (in the present simplified case of classes of Kripke frames).

For $\mathcal{E}$-connections, we use two-sorted propositional modal formulae. The formulae of sort 1 are just the standard propositional modal formulae (where, however, the modal operator $\Diamond$ is renamed to $\Diamond_1$); the formulae of sort 2 are built from propositional variables[12] of sort 2 and formulae of the form $\Diamond_E \phi$ where ϕ is a formula of sort 1, by applying the Boolean connectives and the modal operator $\Diamond_2$.

From the semantic side, suppose we are given two classes $\mathcal{C}_1, \mathcal{C}_2$ of Kripke frames. The class of connection frames $\mathcal{E}(\mathcal{C}_1, \mathcal{C}_2)$ is formed by all triples $\mathcal{F} = (\mathcal{F}_1, E^{\mathcal{F}}, \mathcal{F}_2)$ such that $\mathcal{F}_1 = (W_1, R_1) \in \mathcal{C}_1$, $\mathcal{F}_2 = (W_2, R_2) \in \mathcal{C}_2$ and $E^{\mathcal{F}} \subseteq W_2 \times W_1$ is an arbitrary binary relation.

An $\mathcal{E}(\mathcal{C}_1, \mathcal{C}_2)$-connection Kripke model is a 4-tuple $\mathcal{M} = (\mathcal{F}_1, E^{\mathcal{F}}, \mathcal{F}_2, V)$, where $\mathcal{F} = (\mathcal{F}_1, E^{\mathcal{F}}, \mathcal{F}_2) \in \mathcal{E}(\mathcal{C}_1, \mathcal{C}_2)$ is a connection frame and V is a map associating with propositional letters of sort i subsets of W_i ($i = 1, 2$). The forcing relation $w \models^{\mathcal{M}} \alpha$, which says that the modal formula α is true in $\mathcal{M}$ at world w, is

[11] Use Boolean bi-implication and complement to show this.

[12] Propositional variables of sort 1 are kept disjoint from propositional variables of sort 2.

defined in the standard way (see [KLWZ04]), where the only non-obvious case is the following: for $w_2 \in W_2$ and for a formula α of sort 1, we have:

$$w_2 \models^{\mathcal{M}} \Diamond_E \alpha \quad \text{iff} \quad (\exists w_1 \in W_1.\ (w_2, w_1) \in E^{\mathcal{F}} \text{ and } w_2 \models^{\mathcal{M}} \alpha).$$

Now, $\mathcal{E}(\mathcal{C}_1, \mathcal{C}_2)$-satisfiability of a modal constraint $\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_m$ is defined as above (but notice that the α_i and the β_j may be formulae of sort 1 or 2, indifferently).

When connecting the theories corresponding to two frame classes, we build the two-sorted signature B_M^2: this consists of two renamed copies of the signature B_M and, in addition, of the new unary function symbol $\Diamond_E$ of arity $S_1 S_2$ (where S_1, S_2 are the single sorts of the renamed copies of B_M). Again, terms in the signature B_M^2 can be identified with the two-sorted modal formulae introduced above; moreover any connection frame $\mathcal{F} = (\mathcal{F}_1, E^{\mathcal{F}}, \mathcal{F}_2)$ can be turned into a B_M^2-structure (which we still call $\mathcal{B}_{\mathcal{F}}$) by interpreting the two sorts by power-set Boolean algebras, as described above, and by defining $\Diamond_E$ as the function associating with $X \subseteq W_1$ the subset of W_2 given by

$$\Diamond_E(X) := \{w_2 \in W_2 \mid \exists w_1 \in W.\ (w_2, w_1) \in E^{\mathcal{F}} \wedge w_1 \in X\}.$$

We can then build the theory $\mathcal{T}_{\mathcal{E}(\mathcal{C}_1, \mathcal{C}_2)}$, whose axioms are the formulae

$$(\alpha_1 \approx \top) \wedge \cdots \wedge (\alpha_n \approx \top) \ \rightarrow \ (\beta_1 \approx \bot) \vee \cdots \vee (\beta_m \approx \bot), \qquad (15)$$

where $\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_m$ are the modal constraints that are not satisfiable in $\mathcal{E}(\mathcal{C}_1, \mathcal{C}_2)$. As in the proof of Proposition 8.2, it can be shown that the problem of deciding satisfiability of modal constraints in $\mathcal{E}(\mathcal{C}_1, \mathcal{C}_2)$ is equivalent to the problem of deciding the universal fragment of the theory $\mathcal{T}_{\mathcal{E}(\mathcal{C}_1, \mathcal{C}_2)}$.

The following proposition states a precise relationship between $\mathcal{E}$-connections and our connections of many-sorted theories.

Proposition 8.3 *Let $\mathcal{C}_1, \mathcal{C}_2$ be classes of Kripke frames; $T_{\mathcal{E}(\mathcal{C}_1, \mathcal{C}_2)}$ coincides with $T_{\mathcal{C}_1} >_{T_0} T_{\mathcal{C}_2}$, where T_0 is the theory of semilattices.*[13]

Proof. Both theories $T_{\mathcal{E}(\mathcal{C}_1, \mathcal{C}_2)}$ and $T_{\mathcal{C}_1} >_{T_0} T_{\mathcal{C}_2}$ are universal and relative to the same signature BD^2, so it is sufficient to show that a finite set of literals is satisfiable in a model of one of them iff it is satisfiable in a model of the other. First, note that a finite set of literals is satisfied in a model of $T_{\mathcal{E}(\mathcal{C}_1, \mathcal{C}_2)}$ iff it is satisfied in a model of the form $\mathcal{B}_{\mathcal{F}}$, where $\mathcal{F} = (\mathcal{F}_1, E^{\mathcal{F}}, \mathcal{F}_2)$ is such that $\mathcal{F}_1 \in \mathcal{C}_1$ and $\mathcal{F}_2 \in \mathcal{C}_2$. This can be shown by basically repeating the arguments used in the proof of Proposition 8.2: every universal B_M^2-formula is equivalent to conjunction of formulae of the kind (13), and (13) is a logical consequence of the

[13]See Example 2.1.

theory $T_{\mathcal{E}(\mathcal{C}_1,\mathcal{C}_2)}$ iff the modal constraint (14) is unsatisfiable in frames of the kind $\mathcal{F} = (\mathcal{F}_1, E^{\mathcal{F}}, \mathcal{F}_2)$ (for $\mathcal{F}_1 \in \mathcal{C}_1$ and $\mathcal{F}_2 \in \mathcal{C}_2$), i.e., iff (13) is true in models of the kind $\mathcal{B}_{\mathcal{F}}$, where $\mathcal{F} = (\mathcal{F}_1, E^{\mathcal{F}}, \mathcal{F}_2)$ is such that $\mathcal{F}_1 \in \mathcal{C}_1$ and $\mathcal{F}_2 \in \mathcal{C}_2$.

Clearly, models of the form $\mathcal{B}_{\mathcal{F}}$ for a connection frame $\mathcal{F} = (\mathcal{F}_1, E^{\mathcal{F}}, \mathcal{F}_2)$ are models of $T_{\mathcal{C}_1} >_{T_0} T_{\mathcal{C}_2}$. However, the converse is far from being true: in fact, models of $T_{\mathcal{C}_1} >_{T_0} T_{\mathcal{C}_2}$ may interpret the two sorts S_1 and S_2 by Boolean algebras that are *not* powerset Boolean algebras. Moreover, in models of $T_{\mathcal{C}_1} >_{T_0} T_{\mathcal{C}_2}$, the connecting diamond $\Diamond_E$ is taken to be any semilattice homomorphism and, as such, it need not preserve infinitary joins (as is the case, on the contrary, for the interpretation of $\Diamond_E$ in all models of the kind $\mathcal{B}_{\mathcal{F}}$).

Thus, the key point of the proof is to show that any finite set of B_M^2-literals Γ satisfiable in a model of $T_{\mathcal{C}_1} >_{T_0} T_{\mathcal{C}_2}$, is also satisfiable in a model of the form $\mathcal{B}_{\mathcal{F}}$, where $\mathcal{F} = (\mathcal{F}_1, E^{\mathcal{F}}, \mathcal{F}_2)$ is a connection frame such that $\mathcal{F}_1 \in \mathcal{C}_1$ and $\mathcal{F}_2 \in \mathcal{C}_2$.

We can, as usual, replace variables with constants and assume Γ to be flat, so that we can divide Γ into three disjoint sets $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$, where Γ_i $(i = 1, 2)$ is a set of literals in the i-th copy of the signature B_M (expanded with free constants), and Γ_0 is of the form

$$\Gamma_0 = \{\Diamond_E(a_1) \approx b_1, \dots, \Diamond_E(a_n) \approx b_n\}$$

for free constants $a_1, b_1, \dots, a_n, b_n$.

This observation is not sufficient yet: we need to modify $\Gamma_0 \cup \Gamma_1 \cup \Gamma_2$ further. Let Θ be the set of terms of the form

$$\pm a_1 \sqcap \dots \sqcap \pm a_n,$$

where $+a_i$ is a_i and $-a_i$ is $\overline{a_i}$. Notice that the equations

$$a_i \approx \bigsqcup\{\theta \mid \theta \in \Theta,\ \theta \sqsubseteq a_i\}$$

are logical consequence of the Boolean algebra axioms, and hence are always valid in our models (here $\theta \sqsubseteq a_i$ means that a_i (and not $\overline{a_i}$) appears as conjunct in θ).

Let $\tilde{\Gamma}_1$ be any set of B_M^1-literals obtained from Γ_1 by adding either $\theta \approx \bot$ or $\theta \not\approx \bot$ for every $\theta \in \Theta$. For any $\theta \in \Theta$, introduce a new constant c_θ and replace Γ_0 with

$$\tilde{\Gamma}_0 := \{\Diamond_E(\theta) \approx c_\theta \mid \theta \in \Theta\}.$$

Finally, let

$$\tilde{\Gamma}_2(\tilde{\Gamma}_1) := \Gamma_2 \cup \{c_\theta \approx \bot \mid \theta \approx \bot \in \tilde{\Gamma}_1\} \cup \{(\bigsqcup_{\theta \sqsubseteq a_i} c_\theta) \approx b_i \mid i = 1, \dots, n\}.$$

It is easily seen that $\Gamma_0 \cup \Gamma_1 \cup \Gamma_2$ is satisfiable in a model of $T_{\mathcal{C}_1} >_{T_0} T_{\mathcal{C}_2}$ iff there is a $\tilde{\Gamma}_1$ such that $\tilde{\Gamma}_0 \cup \tilde{\Gamma}_1 \cup \tilde{\Gamma}_2(\tilde{\Gamma}_1)$ is satisfiable in a model of $T_{\mathcal{C}_1} >_{T_0} T_{\mathcal{C}_2}$. The same observation applies to satisfiability in models of $T_{\mathcal{E}(\mathcal{C}_1,\mathcal{C}_2)}$. So, let us fix a set $\tilde{\Gamma}_0 \cup \tilde{\Gamma}_1 \cup \tilde{\Gamma}_2(\tilde{\Gamma}_1)$, and assume that it is satisfiable in a model of $T_{\mathcal{C}_1} >_{T_0} T_{\mathcal{C}_2}$. We must show that it is satisfiable in a model of $T_{\mathcal{E}(\mathcal{C}_1,\mathcal{C}_2)}$.

Now, if $\tilde{\Gamma}_0 \cup \tilde{\Gamma}_1 \cup \tilde{\Gamma}_2(\tilde{\Gamma}_1)$ is satisfiable in a model of $T_{\mathcal{C}_1} >_{T_0} T_{\mathcal{C}_2}$, then $\tilde{\Gamma}_1$ is satisfiable in a model of $T_{\mathcal{C}_1}$ and $\tilde{\Gamma}_2(\tilde{\Gamma}_1)$ is satisfiable in a model of $T_{\mathcal{C}_2}$. By the definition of $T_{\mathcal{C}_i}$, it follows that $\tilde{\Gamma}_i$ must be satisfiable in a model of the form $\mathcal{B}_{\mathcal{F}_i}$, where $\mathcal{F}_i = (W_i, R_i) \in \mathcal{C}_i$ $(i = 1, 2)$. So we simply need to define the interpretation $E^{\mathcal{F}}$ of the connecting relation E in such a way that also $\tilde{\Gamma}_0$ is satisfied in $\mathcal{F} = (\mathcal{F}_1, E^{\mathcal{F}}, \mathcal{F}_2)$. This is done as follows: pick $s_1 \in W_1$ and $s_2 \in W_2$; we say that $(s_2, s_1) \in E^{\mathcal{F}}$ iff $s_2 \in c_\theta^{\mathcal{B}_{\mathcal{F}_2}}$,[14] where θ is the unique element[15] of Θ such that $s_1 \in \theta^{\mathcal{B}_{\mathcal{F}_1}}$. This implies that, for every $\theta \in \Theta$, we have $\Diamond_E^{\mathcal{B}_{\mathcal{F}}}(\theta^{\mathcal{B}_{\mathcal{F}_1}}) \subseteq c_\theta^{\mathcal{B}_{\mathcal{F}_2}}$. For the converse inclusion, suppose that $s_2 \in c_\theta^{\mathcal{B}_{\mathcal{F}_2}}$. Then $\mathcal{B}_{\mathcal{F}_2} \not\models c_\theta \approx \bot$. By the definition of $\tilde{\Gamma}_2(\tilde{\Gamma}_1)$ and by the fact that either $\theta \approx \bot \in \tilde{\Gamma}_1$ or $\theta \not\approx \bot \in \tilde{\Gamma}_1$, we have that $\mathcal{B}_{\mathcal{F}_1} \not\models \theta \approx \bot$. This means that there is some $s_1 \in \theta^{\mathcal{B}_{\mathcal{F}_1}}$; for such s_1 we have that $(s_2, s_1) \in E^{\mathcal{F}}$, i.e. that $s_2 \in \Diamond_E^{\mathcal{B}_{\mathcal{F}}}(\theta^{\mathcal{B}_{\mathcal{F}_1}})$. ⊣

The above proposition, together with our main combination result (Theorem 4.1), and the fact that Boolean-based theories are algebraically compatible with respect to the theory of semilattices (Example 4.6), immediately entails the following result:

Corollary 8.4 *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be classes of modal frames. If the modal constraint problems for $\mathcal{C}_1$ and $\mathcal{C}_2$ are both decidable, then so is the modal constraint problem for $\mathcal{E}(\mathcal{C}_1, \mathcal{C}_2)$.*

This decidability transfer result can be proved directly by an argument similar to the one we used to prove Proposition 8.3. Notice, however, that Theorem 4.1 gives in fact more, as it applies to *any* Boolean-based theory, i.e., also to theories that are not of the kind $T_{\mathcal{C}}$ for a class $\mathcal{C}$ of Kripke frames.

Let us now turn to $\mathcal{E}$-connections that correspond to connections of theories where more than the theory of semilattices is shared. The frame classes $\mathcal{E}_d(\mathcal{C}_1, \mathcal{C}_2)$ and $\mathcal{E}_f(\mathcal{C}_1, \mathcal{C}_2)$ are defined similarly to $\mathcal{E}(\mathcal{C}_1, \mathcal{C}_2)$: the only difference is that now the connecting relation E is respectively taken to be a partial function and a function. For such deterministic or functional connections, we can show results that are analogous to Proposition 8.3.

Proposition 8.5 *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be classes of modal frames.*

[14] We use $t^{\mathcal{B}_{\mathcal{F}_2}}$ to denote the interpretation of the ground term t in the structure $\mathcal{B}_{\mathcal{F}_2}$ (and similarly for $\mathcal{F}_1$).

[15] By the definition of Θ, different elements of Θ are interpreted by disjoint sets in $\mathcal{F}_1$, and the union of the interpretations of all elements of Θ in $\mathcal{F}_1$ is W_1.

1. $T_{\mathcal{E}_d(\mathcal{C}_1,\mathcal{C}_2)}$ coincides with $T_{\mathcal{C}_1} >_{T_0} T_{\mathcal{C}_2}$, where T_0 is the theory of distributive lattices with a least element.

2. $T_{\mathcal{E}_f(\mathcal{C}_1,\mathcal{C}_2)}$ coincides with $T_{\mathcal{C}_1} >_{T_0} T_{\mathcal{C}_2}$, where T_0 is the theory of bounded distributive lattices.

Proof. Only slight modifications to the proof of Proposition 8.3 are needed. When building $\tilde{\Gamma}_2(\tilde{\Gamma}_1)$, we add also the atoms $c_{\theta_1} \sqcap c_{\theta_2} \approx \bot$, for $\theta_1 \neq \theta_2$. In the case of a functional connection, we additionally add $\top \approx \bigsqcup_{\theta \in \Theta} c_\theta$.

To define $E^{\mathcal{F}}$, we now proceed as follows: first, the definition domain of the partial function $E^{\mathcal{F}}$ is $(\bigsqcup_{\theta \in \Theta} c_\theta)^{\mathcal{B}_{\mathcal{F}_2}}$. Now notice that any s_2 in this definition domain belongs to exactly one $c_\theta^{\mathcal{B}_{\mathcal{F}_2}}$; moreover, if $s_2 \in c_\theta^{\mathcal{B}_{\mathcal{F}_2}}$, then $\mathcal{B}_{\mathcal{F}_2} \models c_\theta \not\approx \bot$ and thus $\mathcal{B}_{\mathcal{F}_1} \models \theta \not\approx \bot$. Select just one $s_1 \in \theta^{\mathcal{B}_{\mathcal{F}_1}}$ and let $E^{\mathcal{F}}(s_2) := s_1$. This definition of $E^{\mathcal{F}}$ guarantees that $\mathcal{B}_{\mathcal{F}} \models \Diamond_E \theta \approx c_\theta$ again holds for all $\theta \in \Theta$. In addition, in the case of a functional connection, the presence of $\top \approx \bigsqcup_{\theta \in \Theta} c_\theta$ in $\tilde{\Gamma}_2(\tilde{\Gamma}_1)$ enforces that the definition domain of the partial function $E^{\mathcal{F}}$ is the whole domain. $\dashv$

The algebraic compatibility of any Boolean-based theory with respect to the theory of distributive lattices with a least element and with respect to the theory of bounded distributive lattices (see Examples 4.7 and 4.8), now yields the following decidability transfer results:

Corollary 8.6 *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be classes of modal frames. If the modal constraint problems for $\mathcal{C}_1$ and $\mathcal{C}_2$ are both decidable, then so are the modal constraint problems for $\mathcal{E}_d(\mathcal{C}_1, \mathcal{C}_2)$ and $\mathcal{E}_f(\mathcal{C}_1, \mathcal{C}_2)$.*

9 Appendix B: Theory Completions

In this Appendix we develop some model theory concerning our notions of completions of a theory T. Such model theory gives further insight into some important ingredients of the paper, although it is not needed in order to understand and justify our combination procedures. We shall recall classical well-known results for model completions and show how they can be adapted to the case of positive algebraic completions.[16]

Let us call a model $\mathcal{M}$ of a theory T:

- *algebraically closed* iff every sentence of the kind $\exists \underline{x}(A_1(\underline{a}, \underline{x}) \wedge \cdots \wedge A_n(\underline{a}, \underline{x}))$ which is satisfied in some $\mathcal{N} \supseteq \mathcal{M}$ such that $\mathcal{N} \models T$, is satisfied in $\mathcal{M}$ itself (here $\underline{a}$ are parameters from $\mathcal{M}$ and the $A_i(\underline{y}, \underline{x})$ are atoms);

[16] Similar adaptations can be done also for the coalgebraic completions case, but we do not insist on them, for simplicity.

- *existentially closed* iff every sentence of the kind $\exists \underline{x}(A_1(\underline{a}, \underline{x}) \wedge \cdots \wedge A_n(\underline{a}, \underline{x}))$ which is satisfied in some $\mathcal{N} \supseteq \mathcal{M}$ such that $\mathcal{N} \models T$, is satisfied in $\mathcal{M}$ itself (here $\underline{a}$ are parameters from $\mathcal{M}$ and the $A_i(\underline{y}, \underline{x})$ are literals).

The following Lemma is taken from [CK90]:

Lemma 9.1 *If T is universal, then every model $\mathcal{M}$ of T embeds into a model of T which is existentially (hence also algebraically) closed.*

Proof. Take a well-order $\{\phi_i\}_{i<\alpha}$ of the existential sentences with parameters from $\mathcal{M}$. Define a first chain $\{\mathcal{M}_i\}_i$ of models of T, by letting $\mathcal{M}_i$ to be an extension of $\bigcup_{j<i} \mathcal{M}_j$ in which ϕ_i is true (if this extension does not exists, $\mathcal{M}_i$ is just $\bigcup_{j<i} \mathcal{M}_j$). Now let $\mathcal{M}_1$ be $\bigcup_{j<\alpha} \mathcal{M}_j$; repeating the construction,[17] we produce a countable chain $\mathcal{M} \subseteq \mathcal{M}_1 \subseteq \mathcal{M}_2 \subseteq \cdots$. The union of this chain is the desired existentially closed extension of $\mathcal{M}$ (notice that this argument works because T is preserved under union of chains, being universal). $\dashv$

Proposition 9.2 *Suppose that T has a positive algebraic (model) completion T^*; then the models of T^* are precisely those models of T which are algebraically (resp. existentially) closed.*

Proof. We show the proof just for the case of the positive algebraic completion T^* (the other case being analogous and well-known [CK90]). Recall that, according to Definition 3.2 and Lemma 3.3, for every geometric formula $\phi(\underline{x})$ there is a geometric open formula $\phi^*(\underline{x})$ such that $T \models \phi \to \phi^*$ and $T^* \models \phi^* \to \phi$.

Suppose that $\mathcal{M} \models T^*$, that $\mathcal{N} \supseteq \mathcal{M}$ is an extension of $\mathcal{M}$ which is also a model of T. Let $\phi(\underline{a})$ be a geometric sentence with parameters $\underline{a}$ from $\mathcal{M}$ which is true in $\mathcal{N}$. Then we have $\mathcal{N} \models \phi^*(\underline{a})$ and also $\mathcal{M} \models \phi^*(\underline{a})$ (because ϕ^* is open); as $\mathcal{M}$ is a model of T^*, this implies that $\mathcal{M} \models \phi(\underline{a})$.

Conversely, suppose that $\mathcal{M}$ is algebraically closed as a model of T and let $\phi(\underline{a})$ be a geometric sentence with parameters in $\mathcal{M}$ such that $\mathcal{M} \models \phi^*(\underline{a})$. By definition 3.2(ii), $\mathcal{M}$ can be embedded into a model $\mathcal{N}$ of T^*. Since ϕ^* is open and since $T^* \models \phi^* \to \phi$, in $\mathcal{N}$ we have $\mathcal{N} \models \phi(\underline{a})$ and also $\mathcal{M} \models \phi(\underline{a})$, because $\mathcal{M}$ is algebraically closed. Thus $\mathcal{M} \models \phi \leftrightarrow \phi^*$ holds for all geometric ϕ (the implication $\phi \to \phi^*$ being already a logical consequence of T). It is now easy to show that $\mathcal{M} \models T^*$: let $\phi_1 \to \phi_2$ be a geometric sequent in the axiomatization of T^*. We have that $\mathcal{M} \models \phi_1 \to \phi_2$ iff $\mathcal{M} \models \phi_1^* \to \phi_2^*$; however, from $T^* \models \phi_1 \to \phi_2$, we get $T^* \models \phi_1^* \to \phi_2^*$, hence also $T \models \phi_1^* \to \phi_2^*$, because T and T^* agree on open formulae (see Definition 6.1(i)-(ii) and Lemma 3.1). Since $\mathcal{M} \models T$, $\mathcal{M} \models \phi_1^* \to \phi_2^*$ follows; consequently we have $\mathcal{M} \models \phi_1 \to \phi_2$ (i.e. $\mathcal{M} \models T^*$). $\dashv$

[17] The construction needs to be repeated, in order to take care of existential formulae with parameters from $|\mathcal{M}_1| \setminus |\mathcal{M}|$.

Notice that Proposition 9.2 implies that T^*, when it exists, is unique. Clearly not all universal theories T have a positive algebraic or a model completion: there is no general guarantee, for instance, that the class of algebraically or existentially closed models of T is *elementary* (i.e. that it is the class of the models of some first order theory at all).

9.1 Model Completions

A classical result [CK90] says that a universal theory T has a model completion iff T has the amalgamation property and the class of the existentially closed models of T is an elementary class. We shall recall here the proof of this result and in next subsections we show how a similar statement can be proved for the case of positive algebraic completions.

We say that a theory T has the *amalgamation property* (AP for short) iff for every triple $\mathcal{M}, \mathcal{N}_1, \mathcal{N}_2$ of models of T, for every pair of embeddings $\mu_1 : \mathcal{M} \longrightarrow \mathcal{N}_1$ and $\mu_2 : \mathcal{M} \longrightarrow \mathcal{N}_2$, there are a further model $\mathcal{N}$ of T, and embeddings $\nu_1 : \mathcal{N}_1 \longrightarrow \mathcal{N}$ and $\nu_2 : \mathcal{N}_2 \longrightarrow \mathcal{N}$ such that the square

$$\begin{tikzcd} \mathcal{M} \arrow[r, tail, "\mu_1"] \arrow[d, tail, "\mu_2"'] & \mathcal{N}_1 \arrow[d, tail, "\nu_1"] \\ \mathcal{N}_2 \arrow[r, tail, "\nu_2"'] & \mathcal{N} \end{tikzcd}$$

commutes.

Proposition 9.3 *If the universal Ω-theory T has a model completion T^*, then T has AP.*

Proof. Given embeddings $\mu_1 : \mathcal{M} \longrightarrow \mathcal{N}_1$ and $\mu_2 : \mathcal{M} \longrightarrow \mathcal{N}_2$, we can freely suppose that $\mathcal{N}_1, \mathcal{N}_2$ are models of T^* and that μ_1, μ_2 are inclusions. By diagrams theorems, it is sufficient to show the consistency of $T \cup \Delta(\mathcal{N}_1) \cup \Delta(\mathcal{N}_2)$. Suppose this is not consistent; by compactness there are $\theta_1(\underline{m}, \underline{n}_1), \theta_2(\underline{m}, \underline{n}_2)$, such that $T \cup \{\theta_1(\underline{m}, \underline{n}_1), \theta_2(\underline{m}, \underline{n}_2)\}$ is inconsistent. Here: a) $\underline{m}$ are parameters from $\mathcal{M}$; b) $\underline{n}_1, \underline{n}_2$ are parameters from $\mathcal{N}_1, \mathcal{N}_2$ (not belonging to the image of μ_1, μ_2, respectively); c) $\theta_1(\underline{m}, \underline{n}_1)$ is a conjunction of ground literals true in $\mathcal{N}_1$; d) $\theta_2(\underline{m}, \underline{n}_2)$ is a conjunction of ground literals true in $\mathcal{N}_2$. Let $\phi(\underline{m})$ be $\exists \underline{y} \theta_1(\underline{m}, \underline{y})$ and recall from Definition 6.1 that there is an open formula ϕ^* such that $T^* \models \phi^* \leftrightarrow \phi$. We consequently have $\mathcal{N}_1 \models \phi^*(\underline{m})$; since $\phi^*(\underline{m})$ is open, we get that it is true in $\mathcal{M}$ and in $\mathcal{N}_2$ too. The latter is a model of T^*, hence $\mathcal{N}_2 \models \phi(\underline{m})$, contradiction because $T \cup \{\phi(\underline{m}), \theta_2(\underline{m}, \underline{n}_2)\}$ is inconsistent. $\dashv$

Lemma 9.4 *Suppose that the universal Ω-theory T has AP and that $T^* \supseteq T$ is an extension of T (in the same signature of T) whose models are all existentially closed for T. Then T^* admits quantifier elimination.*

Proof. Let $\phi(\underline{x})$ be an existential formula: it is sufficient to show that $\phi(\underline{x})$ is equivalent modulo T^* to a quantifier free formula $\phi^*(\underline{x})$. For new constants $\underline{a}$ consider the set of sentences

$$\Theta \ := \ T^* \cup \{\phi(\underline{a})\} \cup \{\neg\psi(\underline{a}) \mid \psi \text{ is quantifier free and } T^* \models \psi(\underline{a}) \to \phi(\underline{a})\}.$$

If Θ is inconsistent, then we have $T^* \models \phi(\underline{a}) \to \psi_1(\underline{a}) \vee \cdots \vee \psi_n(\underline{a})$ for quantifier-free ψ_i implying ϕ, so that we can take the disjunction of such ψ_i as ϕ^*.

Consequently it suffices to show that Θ cannot be consistent. Suppose it is and let $\mathcal{M}$ be a model of it. Let $\mathcal{A}$ be the substructure of $\mathcal{M}$ generated by the $\underline{a}$; we distinguish two cases, depending on whether we have $T^* \cup \Delta(\mathcal{A}) \models \phi(\underline{a})$ or not.

If we do not have $T^* \cup \Delta(\mathcal{A}) \models \phi(\underline{a})$, then we can build a model $\mathcal{N}$ of T^* containing $\mathcal{A}$ as a substructure and falsifying $\phi(\underline{a})$. By AP, there is a common extension $\mathcal{N}'$ of $\mathcal{M}$ and $\mathcal{N}$ (over $\mathcal{A}$); since $\mathcal{M} \models \phi(\underline{a})$ and $\phi(\underline{a})$ is existential, $\mathcal{N}' \models \phi(\underline{a})$, which cannot be because $\mathcal{N}$ is existentially closed (it is a model of T^*) and $\mathcal{N} \not\models \phi(\underline{a})$.

If we have $T^* \cup \Delta(\mathcal{A}) \models \phi(\underline{a})$, for some quantifier-free sentence $\psi(\underline{a})$ true in $\mathcal{A}$ we have that $T^* \models \psi(\underline{a}) \to \phi(\underline{a})$. According to the definition of Θ, $\neg\psi(\underline{a})$ is true in $\mathcal{M}$ and also in $\mathcal{A}$ (because it is quantifier-free), contradiction. $\dashv$

Theorem 9.5 *Let T be a universal theory; then T has a model completion iff it has AP and the class of existentially closed models of T is elementary.*

Proof. One side is covered by Propositions 9.2 and 9.3 and the other side by Lemmas 9.4 and 9.1. $\dashv$

We finally recall that the definition of a model completion given in Definition 6.1 above agrees with the standard definition used e.g. in most textbooks and a also in [Ghi05, BGT04]:[18]

Proposition 9.6 *Let T be a universal Ω-theory and let T^* be a further Ω-theory extending T. We have that T^* is a model completion of T iff the following two conditions are satisfied: (i) every model of T embeds into a model of T^*; (ii) for every Ω-structure $\mathcal{A}$ which is a model of T, we have that $T^* \cup \Delta(\mathcal{A})$ is a complete $\Omega^{|\mathcal{A}|}$-theory.*

[18] For a slightly different proof of Proposition 9.6 (which is nevertheless well-known), see [Ghi03], Appendix B. The alternative definition suggested by Proposition 9.6 is actually preferable, because it conveniently applies also to theories which might not be universal. We adopted Definition 6.1, just to make it parallel to Definition 3.2.

Proof. The left-to-right side is trivial (just observe that ground formulae are preserved by both sub- and super-structures). For the other side, suppose that $T^* \cup \Delta(\mathcal{A})$ is a complete $\Sigma^{|\mathcal{A}|}$-theory for every $\mathcal{A}$ which is a model of T^*. We want to apply Lemma 9.4, so we need to show that all models of T^* are existentially closed and that T enjoys AP.

The former is shown as follows: let $\mathcal{M}$ be a model of T^* and let $\mathcal{N} \supseteq \mathcal{M}$ be a model of T in which a certain existential formula (with parameters from $\mathcal{M}$) $\phi(\underline{m})$ is true. Since models of T embeds into models of T^*, we can suppose that $\mathcal{N} \models T^*$. But then, $\mathcal{N}$ and $\mathcal{M}$ itself are both extensions of $\mathcal{M}$ to a model of T^*, whence they are both models of the complete theory $T^* \cup \Delta(\mathcal{M})$, which means that $\phi(\underline{m})$ is true in $\mathcal{M}$ (since it is true in $\mathcal{N}$).

We finally show that AP holds for T. Given embeddings $\mu_1 : \mathcal{M} \longrightarrow \mathcal{N}_1$ and $\mu_2 : \mathcal{M} \longrightarrow \mathcal{N}_2$ (to be amalgamated), we can freely suppose that $\mathcal{N}_1, \mathcal{N}_2$ are models of T^* and that μ_1, μ_2 are inclusions. Both $\mathcal{N}_1$ and $\mathcal{N}_2$ are then models of the *complete* theory $T^* \cup \Delta(\mathcal{M})$, hence the union of their elementary diagrams (in the signature of T expanded with the constants $|\mathcal{M}|$) is consistent: any model of such union gives a model of T amalgamating $\mathcal{M}_1$ and $\mathcal{M}_2$ over $\mathcal{M}$. ⊣

9.2 Positive Algebraic Completions

We wish to get a result analogous to Theorem 9.5 for the case of positive algebraic completions. To this aim, we need to identify the semantic properties playing the role of amalgamation in our context.

We say that a theory T has the *injection-transfer property* (IT for short) iff for every triple $\mathcal{M}, \mathcal{N}_1, \mathcal{N}_2$ of models of T, for every homomorphism $\mu : \mathcal{M} \longrightarrow \mathcal{N}_2$ and for every embedding $\iota : \mathcal{M} \longrightarrow \mathcal{N}_1$, there are a further model $\mathcal{N}$ of T, an embedding $\iota' : \mathcal{N}_2 \longrightarrow \mathcal{N}$ and a homomorphism $\mu' : \mathcal{N}_1 \longrightarrow \mathcal{N}$ such that the square

$$\begin{tikzcd} \mathcal{M} \arrow[r, tail, "\iota"] \arrow[d, "\mu"'] & \mathcal{N}_1 \arrow[d, "\mu'"] \\ \mathcal{N}_2 \arrow[r, tail, "\iota'"'] & \mathcal{N} \end{tikzcd}$$

commutes.

Proposition 9.7 *If the universal Ω-theory T has a positive algebraic completion T^*, then T has IT.*

Proof. Let $\mu : \mathcal{M} \longrightarrow \mathcal{N}_2$ be a homomorphism and let $\iota : \mathcal{M} \longrightarrow \mathcal{N}_1$ be an embedding ($\mathcal{M}, \mathcal{N}_1, \mathcal{N}_2$ are supposed to be models of T); by Definition 3.2(ii), we can freely suppose that $\mathcal{N}_2$ is a model of T^*. By diagrams theorems, it is sufficient to show the consistency of $T \cup \Delta^+(\mathcal{N}_1) \cup \Delta(\mathcal{N}_2)$. Suppose this is not consistent; by compactness there are $\theta_1(\underline{m}, \underline{n}_1), \theta_2(\underline{m}, \underline{n}_2)$, such that $T \cup \{\theta_1(\underline{m}, \underline{n}_1), \theta_2(\underline{m}, \underline{n}_2)\}$ is inconsistent. Here: a) $\underline{m}$ are parameters from $\mathcal{M}$; b) $\underline{n}_1, \underline{n}_2$ are parameters from $\mathcal{N}_1, \mathcal{N}_2$ (not belonging to the image of ι, μ, respectively); c) $\theta_1(\underline{m}, \underline{n}_1)$ is a conjunction of ground atoms true in $\mathcal{N}_1$; d) $\theta(\underline{m}, \underline{n}_2)$ is a conjunction of ground literals true in $\mathcal{N}_2$. Let $\phi(\underline{m})$ be $\exists \underline{y} \theta_1(\underline{m}, \underline{y})$; we have $\mathcal{N}_1 \models \phi^*(\underline{m})$, as $\phi(\underline{m}) \to \phi^*(\underline{m})$ is a logical consequence of $\overline{T}$ (see Lemma 3.1). Since $\phi^*(\underline{m})$ is geometric and open, we get that it is true in $\mathcal{M}$ and in $\mathcal{N}_2$ too. The latter is a model of T^*, hence $\mathcal{N}_2 \models \phi(\underline{m})$, contradiction because $T \cup \{\phi(\underline{m}), \theta_2(\underline{m}, \underline{n}_2)\}$ is inconsistent. $\dashv$

Propositions 9.2 and 9.7 can be inverted, in the following sense:

Theorem 9.8 *Let T be a universal theory; then T has a positive algebraic completion iff it has IT and the class of algebraically closed models of T is elementary.*

Proof. One side is covered by Propositions 9.2 and 9.7. Suppose now that T has *IT* and that there is a first-order theory T' (in principle, not necessarily a geometric one) such that the models of T' are exactly the algebraically closed models of T. Let $\phi(\underline{x})$ be a geometric formula and let $\underline{a}$ be free constants. Define Γ as the set of geometric, open and ground formulae in $\Omega^{\underline{a}}$ (here Ω is obviously the signature of T) which are logical consequences of $T' \cup \{\phi(\underline{a})\}$.

We first *claim* that $\Gamma \cup T' \models \phi(\underline{a})$. Let in fact $\mathcal{M}$ be a model of $T' \cup \Gamma$. Let $\Delta^-(\underline{a})$ be the set of negative ground $\Omega^{\underline{a}}$-literals which are true in $\mathcal{M}$. By the definition of Γ, the set $T' \cup \Delta^-(\underline{a}) \cup \{\phi(\underline{a})\}$ is consistent and hence has a model $\mathcal{N}$. Let $\mathcal{A}$ be the substructure of $\mathcal{N}$ generated by the $\underline{a}$ (notice that $\mathcal{A}$ is a model of T because T is universal): if we apply diagrams theorems and *IT*, we get a commutative square

$$\begin{tikzcd} \mathcal{A} \arrow[r, tail] \arrow[d, "\mu"'] & \mathcal{N} \arrow[d, "\mu'"] \\ \mathcal{M} \arrow[r, tail] & \mathcal{N}' \end{tikzcd}$$

From $\mathcal{N} \models \phi(\underline{a})$, we get $\mathcal{N}' \models \phi(\underline{a})$ (because ϕ is geometric) and finally $\mathcal{M} \models \phi(\underline{a})$ because $\mathcal{M}$ is algebraically closed. This ends the proof of the claim.

From the claim and compactness, we realize that for every geometric ϕ, there is a geometric open ϕ^* such that

$$T' \models \phi \to \phi^* \quad \text{and} \quad T' \models \phi^* \to \phi.$$

Let T^* be the extension of T axiomatized by the universal closure of the geometric sequents $\phi \to \phi^*$ and $\phi^* \to \phi$ (we have $T \subseteq T^* \subseteq T'$). As every model of T embeds into a model of T' by Lemma 9.1, condition (ii) of Definition 3.2 is satisfied; since condition (iii) comes directly from the construction, T^* is a positive algebraic completion of T. $\dashv$

10 Appendix C: Alternative Proofs

Here we give alternative proofs of some relevant Propositions from Sections 4 and 5, relying on some slightly deeper model theoretic machinery.[19] The main feature of these alternative proofs is that they do not use use either local finiteness of T_0 or countability of the involved signatures.

We first need the following extended *IT* property which is an interesting consequence of T_0-algebraic compatibility:

Proposition 10.1 *Let $T_0 \subseteq T$ be theories in signatures $\Omega_0 \subseteq \Omega$ such that T is T_0-algebraically compatible. Let $\mathcal{A}, \mathcal{C}$ be Ω_0-structures which are models of T_0 and let $\mathcal{M}$ be a Ω-structures which is a models of T; for every Ω_0-homomorphism $\mu : \mathcal{A} \longrightarrow \mathcal{M}_{|\Omega_0}$ and for every Ω_0-embedding $\iota : \mathcal{A} \longrightarrow \mathcal{C}$, there are a further Ω-model $\mathcal{N}$ of T, an Ω-embedding $\iota' : \mathcal{M} \longrightarrow \mathcal{N}$ and a Ω_0-homomorphism $\mu' : \mathcal{C} \longrightarrow \mathcal{N}_{|\Omega_0}$ such that the square*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\mu"] \arrow[d, "\iota"'] & \mathcal{M}_{|\Omega_0} \arrow[d, "\iota'_{|\Omega_0}"] \\
\mathcal{C} \arrow[r, "\mu'"'] & \mathcal{N}_{|\Omega_0}
\end{tikzcd}$$

commutes. Moreover, if $\mathcal{M} \models T \cup T_0^*$, then the embedding ι' can be taken to be elementary.

Proof. Similarly to the proof of Proposition 9.7, we need to show that $T \cup \Delta^+_{\Omega_0}(\mathcal{C}) \cup \Delta_\Omega(\mathcal{M})$ is consistent. Again, if this is not the case, we have that there are $\theta_1(\underline{a}, \underline{c}), \theta_2(\underline{a}, \underline{m})$, such that $T \cup \{\theta_1(\underline{a}, \underline{c}), \theta_2(\underline{a}, \underline{m})\}$ is inconsistent. Here: a) $\underline{a}$ are parameters from $\mathcal{A}$; b) $\underline{c}, \underline{m}$ are parameters from $\mathcal{C}, \mathcal{M}$ (not belonging to the image of ι, μ, respectively); c) $\theta_1(\underline{a}, \underline{c})$ is a conjunction of ground $\Omega_0^{\underline{a},\underline{c}}$-atoms

[19] Similar alternative proofs can be given also for the relevant Propositions from Section 6, but we do not insist on them. Moreover, the experienced model-theorist will realize that further alternative proofs can be obtained by using the cumbersome formalism of saturated/special models.

true in $\mathcal{C}$; d) $\theta(\underline{a}, \underline{m})$ is a conjunction of ground $\Omega^{\underline{a}, \underline{m}}$-literals true in $\mathcal{M}$. Let $\phi(\underline{a})$ be $\exists \underline{y} \theta_1(\underline{a}, \underline{y})$; we have $\mathcal{C} \models \phi^*(\underline{a})$, as $\phi(\underline{a}) \to \phi^*(\underline{a})$ is a logical consequence of T_0. Since $\phi^*(\underline{a})$ is geometric and open, we get that it is true in $\mathcal{A}$ and in $\mathcal{M}$ too. The latter can be embedded into a model $\mathcal{M}_0$ of $T \cup T_0^*$, hence $\mathcal{M}_0 \models \phi(\underline{a})$, contradiction because $T \cup \{\phi(\underline{a}), \theta_2(\underline{a}, \underline{m})\}$ was supposed to be inconsistent (notice that $\mathcal{M}_0 \models \theta_2(\underline{a}, \underline{m})$ follows from $\mathcal{M} \models \theta_2(\underline{a}, \underline{m})$ because θ_2 is open).

In case $\mathcal{M}$ is a model of $T \cup T_0^*$, we can replace $\Delta_\Omega(\mathcal{M})$ by the elementary diagram $\Delta_\Omega^e(\mathcal{M})$ of $\mathcal{M}$ and get an elementary ι', because there is no need of considering the extension $\mathcal{M}_0$. $\dashv$

Let us now give an **alternative proof of Proposition 4.2**. Such an alternative proof is indeed quite simple, from the information we have now: from the data 1-5 of Proposition 4.2, we can get a Ω_0-homomorphism $\nu : \mathcal{A} \longrightarrow \mathcal{B}$ among a Ω_0-substructure $\mathcal{A}$ of a model $\mathcal{N}'$ of T_1 and a Ω_0-substructure $\mathcal{B}$ of a model $\mathcal{N}''$ of T_2. Proposition 4.2 is proved if we build an extension of ν to a Ω_0-homomorphism $\mathcal{N}'_{|\Omega_0} \longrightarrow \mathcal{N}_{|\Omega_0}$, where $\mathcal{N}_{|\Omega_0}$ is a suitable Ω_2-superstructure of $\mathcal{N}''$. But such extension is immediately provided by an application of Proposition 10.1: take as ι the inclusion of $\mathcal{A}$ into $\mathcal{N}'$ and as μ the composition of ν with the inclusion of $\mathcal{B}$ into $\mathcal{N}''$. $\dashv$

Similar arguments (but iterations are needed!) give alternative proofs of the remaining relevant Propositions from Sections 4 and 5.

An **alternative proof of Proposition 4.4** is as follows. We are given models $\mathcal{N}^0, \mathcal{M}^0$ of T_1, T_2 respectively; $\mathcal{N}^0$ has Ω_0-substructures $\mathcal{A}, \mathcal{A}'$, whereas $\mathcal{M}^0$ has Ω_0-substructures $\mathcal{B}, \mathcal{B}'$. We are also given Ω_0-homomorphisms $\nu : \mathcal{A} \longrightarrow \mathcal{B}$ and $\mu : \mathcal{B}' \longrightarrow \mathcal{A}'$. We can freely suppose that $\mathcal{N}^0, \mathcal{M}^0$ are models of T_0^* too, by the algebraic compatibility assumptions.

The Proposition is proved, if we succeed in producing elementary extensions $\mathcal{N}^\infty, \mathcal{M}^\infty$ of $\mathcal{N}, \mathcal{M}$ endowed with Ω_0-homomorphisms

$$\nu^\infty : \mathcal{N}^\infty_{|\Omega_0} \longrightarrow \mathcal{M}^\infty_{|\Omega_0}, \qquad \mu^\infty : \mathcal{M}^\infty_{|\Omega_0} \longrightarrow \mathcal{N}^\infty_{|\Omega_0}$$

extending ν and μ, respectively. To this aim, we define elementary chains of models

$$\mathcal{N}^0 \subseteq \mathcal{N}^1 \subseteq \cdots$$
$$\mathcal{M}^0 \subseteq \mathcal{M}^1 \subseteq \cdots$$

as well as homomorphisms

$$\nu^k : \mathcal{N}^k_{|\Omega_0} \longrightarrow \mathcal{M}^{k+1}_{|\Omega_0}, \qquad \mu^j : \mathcal{M}^j_{|\Omega_0} \longrightarrow \mathcal{N}^j_{|\Omega_0}$$

($k \geq 0$, $j \geq 1$) such that $\nu \subseteq \nu^k \subseteq \nu^{k+1}$ and $\mu \subseteq \mu^j \subseteq \mu^{j+1}$ (once this is settled,[20] it is sufficient to take unions in order to get the desired $\mathcal{N}^\infty, \mathcal{M}^\infty, \nu^\infty, \mu^\infty$). All

[20] Recall the elementary chain theorem [CK90], according to which the union of an elementary chain of models is elementarily equivalent to each member of the chain.

these data can be easily built by using Proposition 10.1. For instance, to get $\mathcal{M}_1$ and ν_0 it is sufficient to fill the square

$$\begin{tikzcd}
\mathcal{A} \arrow[r] \arrow[d] & \mathcal{M}^0_{|\Omega_0} \arrow[d] \\
\mathcal{N}^0_{|\Omega_0} \arrow[r, "\nu^0_{|\Omega_0}"'] & \mathcal{M}^1_{|\Omega_0}
\end{tikzcd}$$

where the top horizontal morphism is the composite of ν with the inclusion $\mathcal{B} \subseteq \mathcal{M}^0_{|\Omega_0}$ (notice that we can get an elementary embedding $\mathcal{M}^0 \hookrightarrow \mathcal{M}'$, since $\mathcal{M}^0 \models T_0^* \cup T_2$). To get $\mathcal{N}_1$ and μ_1 it is sufficient to fill the square

$$\begin{tikzcd}
\mathcal{B}' \arrow[r] \arrow[d] & \mathcal{N}^0_{|\Omega_0} \arrow[d] \\
\mathcal{M}^1_{|\Omega_0} \arrow[r, "\mu^1_{|\Omega_0}"'] & \mathcal{N}^1_{|\Omega_0}
\end{tikzcd}$$

where the top horizontal morphism is the composite of μ with the inclusion $\mathcal{A}' \subseteq \mathcal{N}^0_{|\Omega_0}$ and the left vertical morphism is the composite inclusion $\mathcal{B}' \subseteq \mathcal{M}_0 \subseteq \mathcal{M}_1$. For the inductive cases, the same argument can be applied. $\dashv$

An **alternative proof of Proposition 5.3** is as follows. Here we are given a model $\mathcal{M}$ of T endowed with a pair of Ω_0-substructures $\mathcal{A}, \mathcal{B}$; we are also given a Ω_0-homomorphism $\nu : \mathcal{A} \longrightarrow \mathcal{B}$. Again we can suppose that $\mathcal{M} \models T \cup T_0^*$.

The Proposition is proved, if we succeed in producing an elementary extension $\mathcal{M}^\infty$ of $\mathcal{M}$ endowed with an Ω_0-homomorphism

$$\nu^\infty : \mathcal{M}^\infty_{|\Omega_0} \longrightarrow \mathcal{M}^\infty_{|\Omega_0},$$

extending ν. To this aim, we define an elementary chain of models

$$\mathcal{M}^0 \subseteq \mathcal{M}^1 \subseteq \cdots$$

as well as homomorphisms

$$\nu^k : \mathcal{M}^k_{|\Omega_0} \longrightarrow \mathcal{M}^{k+1}_{|\Omega_0},$$

($k \geq 0$) such that $\nu \subseteq \nu^k \subseteq \nu^{k+1}$ (once this is settled, it is sufficient to take unions in order to get the desired $\mathcal{M}^\infty$ and ν^∞). To get $\mathcal{M}_1$ and ν_0 it is sufficient to fill the square

$$\begin{tikzcd}
\mathcal{A} \arrow[r] \arrow[d] & \mathcal{M}^0_{|\Omega_0} \arrow[d] \\
\mathcal{M}^0_{|\Omega_0} \arrow[r, "\nu^0_{|\Omega_0}"'] & \mathcal{M}^1_{|\Omega_0}
\end{tikzcd}$$

where the top horizontal morphism is the composite of ν with the inclusion $\mathcal{B} \subseteq \mathcal{M}^0_{|\Omega_0}$. To get inductively $\mathcal{M}_{k+1}$ and ν_k, one proceeds similarly. $\dashv$