Behavioral Extensions of Institutions*

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Abstract. We show that any institution $\mathcal{I}$ satisfying some reasonable conditions can be transformed into another institution, $\mathcal{I}_{beh}$, which captures formally and abstractly the intuitions of adding support for behavioral equivalence and reasoning to an existing, particular algebraic framework. We call our transformation an "extension" because $\mathcal{I}_{beh}$ has the same sentences as $\mathcal{I}$ and because its entailment relation includes that of $\mathcal{I}$. Many properties of behavioral equivalence in concrete hidden logics follow as special cases of corresponding institutional results. As expected, the presented constructions and results can be instantiated to other logics satisfying our requirements as well, thus leading to novel behavioral logics, such as partial or infinitary ones, that have the desired properties.

1 Introduction

Many approaches to behavioral equivalence are defined as extensions of more standard algebraic frameworks, following relatively well understood methodologies. For example, hidden algebra is defined as an extension of algebraic specification: it adds appropriate machinery for experiments and then uses it to define behavioral equivalence as "indistinguishability under experiments", also known to be the largest behavioral congruence consistent with the visible data.

Here we explore this problem from an abstract model theoretical perspective. We investigate conditions under which an institution admits behavioral extensions. The intuition of a behavioral signature extending an algebraic signature is captured categorically in a general way covering all cases of operations in current use, including the ones that tend to be problematic: constants of hidden sorts and operations with multiple arguments of hidden sort. Let the original institution be $\mathcal{I} = (Sign, Sen, Mod, \models)$, let Ψ be a fixed signature in $Sign$ called the *visible signature*, and let D be a Ψ-model called the *data model*. Then we build the *behavioral extension of $\mathcal{I}$ over* (Ψ, D), say $\mathcal{I}_{beh} = (Sign_{beh}, Sen_{beh}, Mod_{beh}, \models\!\!\!\equiv)$, as follows. The objects in $Sign_{beh}$ are those in the comma category $\Psi/Sign$; the $(\varphi : \Psi \to \Sigma, \Sigma)$-sentences in $\mathcal{I}_{beh}$ are exactly the Σ-sentences in $\mathcal{I}$, while the $(\varphi : \Psi \to \Sigma, \Sigma)$-models in $\mathcal{I}_{beh}$ are the *data-consistent* Σ-models in $\mathcal{I}$; finally, satisfaction $A \models\!\!\!\equiv_{(\varphi,\Sigma)} \rho$ in $\mathcal{I}_{beh}$ is defined as $A_\varphi \models_\Sigma \rho$ in $\mathcal{I}$, for a carefully chosen model A_φ that symbolizes the "quotient" of A by its behavioral equivalence. An appropriate novel notion of *quotient system* is introduced for this purpose.

The abstract relationship between behavioral and normal satisfactions is studied via a model-theoretic notion of "visibility", and some structural properties preserved by the behavioral extension are pointed out. We show that many of

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the relevant properties of particular hidden logics can be proved at institutional level. The motivation for such a generalization is, as usual, its logic-independent status: a plethora of concrete algebraic logics formalizable as institutions satisfy our mild restrictions, so they all admit behavioral extensions.

Notice that from the way we define the concepts, we restrict ourselves to the *fixed-data* approach. An adaptation of our construction to the *loose-data* setting seems possible, and we shall sketch it in Section 7. Due to space limitations, proofs of our results are omitted, but they can all be found in [29].

Preliminaries. We assume the reader familiar with basic categorical notions: functor, colimit, etc. We use the terminology and notation from [25], with the following exceptions: we let ";" denote the morphisms' composition, which is considered in diagrammatic order; by colimit and limit we mean small colimit and small limit; by a *filtered (chain) colimit* we mean a colimit of a functor defined on a *non-empty* filtered (total respectively) ordered set. We use the following *comma category* notations: if $A \in |\mathcal{C}|$, $A/\mathcal{C}$ denotes the category whose objects are pairs (h, B), where $h : A \to B$ is a morphism in $\mathcal{C}$, and whose morphisms $u : (h, B) \to (g, C)$ are such that $u : B \to C$ is a morphism in $\mathcal{C}$ with $h; u = g$; there is a canonical forgetful functor U from $A/\mathcal{C}$ to $\mathcal{C}$, which maps each object (h, B) to B and each morphism $u : (h, B) \to (g, C)$ to $u : B \to C$; when $u : A \to A'$ is a morphism in $\mathcal{C}$, there is a canonical comma functor $u/\mathcal{C}$ between $A'/\mathcal{C}$ and $A/\mathcal{C}$, mapping each object (h, B) to $(u; h, B)$ and each morphism to itself; to each functor $F : \mathcal{C} \to \mathcal{D}$ and object A in $\mathcal{C}$, one can associate a functor between comma categories $F_A : A/\mathcal{C} \to F(A)/\mathcal{D}$, which maps each object (h, B) to $(F(h), F(B))$ and each morphism g to $F(g)$.

Since we need a special notion of quotient object, we define a parameterized notion of co-well-powered-ness: let $\mathcal{C}$ be a category and $\mathcal{E}$ be a class of morphisms in $\mathcal{C}$. $|\mathcal{C}|$ is said to be *$\mathcal{E}$-co-well-powered* if for each $A \in |\mathcal{C}|$ there is some *set* $\mathcal{D}$ of morphisms in $\mathcal{E}$ of source A, such that any morphism of source A in $\mathcal{E}$ is isomorphic in $A/\mathcal{C}$ to some morphism in $\mathcal{D}$. If $\mathcal{E}$ is taken to be the class of all epimorphisms, we get the usual notion of co-well-powered-ness. If $\mathcal{C}$ is a category, $\mathcal{C}^{op}$ denotes its dual. We let Set denote the category of sets and functions and Cat the category of categories and functors.

2 Institutions

In this section, we discuss several institutional concepts, many already known.

An *institution* [17] consists of: a category $Sign$, whose objects are called *signatures*; a functor $Sen : Sign \to Set$, giving for each signature Σ a set whose elements are called *Σ-sentences*; a functor $Mod : Sign \to Cat^{op}$ giving for each signature Σ a category whose objects are called *Σ-models* and whose arrows are called *Σ-morphisms*; a *Σ-satisfaction* relation $\models_\Sigma \subseteq |Mod(\Sigma)| \times Sen(\Sigma)$ for each $\Sigma \in |Sign|$, such that for each morphism $\varphi : \Sigma \to \Sigma'$ in $Sign$, the *satisfaction condition* "$M' \models_{\Sigma'} Sen(\varphi)(e)$ iff $Mod(\varphi)(M') \models_\Sigma e$" holds for all $M' \in |Mod(\Sigma')|$ and $e \in Sen(\Sigma)$. As usual, we may let $\_\!\restriction_\varphi$ denote the reduct functor $Mod(\varphi)$ and φ denote $Sen(\varphi)$. When $M = M'\!\restriction_\varphi$ we say that M' is a *φ-expansion of M* and M is the *φ-reduct of M'*.

The satisfaction relation is extended to sets of Σ-sentences and classes of Σ-models: if $E \subseteq Sen(\Sigma)$ and $\mathcal{M} \subseteq |Mod(\Sigma)|$, then we write $\mathcal{M} \models_\Sigma E$ whenever $M \models_\Sigma e$ for each $e \in E$ and $M \in \mathcal{M}$. We let E^* denote the class $\{M \mid M \models_\Sigma E\}$ and dually, $\mathcal{M}^*$ the set of Σ-sentences $\{e \mid \mathcal{M} \models_\Sigma e\}$. The two "$*$" operators form a Galois connection [17]; we let "$\bullet$" denote the two corresponding closure operators. The satisfaction relation is also extended to a (semantic) consequence relation, for which we use the same symbol, following classical logic tradition: if $E, E' \subseteq Sen(\Sigma)$, we write $E \models_\Sigma E'$ whenever $E^* \subseteq E'^*$. To simplify notation, we may write $\models$ instead of $\models_\Sigma$. A *presentation* [17] is a pair (Σ, E), where $E \subseteq Sen(\Sigma)$. A *theory* [17] is a presentation (Σ, E) with E with $E^\bullet = E$. A *presentation morphism* $\varphi : (\Sigma, E) \to (\Sigma', E')$ is a signature morphism $\varphi : \Sigma \to \Sigma'$ with $\varphi(E) \subseteq E'^\bullet$. A presentation morphism between theories is called a *theory morphism*. We let $Mod(\Sigma, E)$ denote the full sub-category of $Mod(\Sigma)$ having as objects all the Σ-models which satisfy E. An institution is *ω-exact* if Mod preserves colimits of functors defined on the ordered set of natural numbers.

A signature morphism $\varphi : \Sigma \to \Sigma'$ is *representable* [10] if there exists a Σ-model $T_{[\varphi]}$ (called the *representation* of φ) and an isomorphism of categories $I_\varphi : Mod(\Sigma') \to T_{[\varphi]}/Mod(\Sigma)$ such that $I_\varphi ; U = Mod(\varphi)$, where $U : T_{[\varphi]}/Mod(\Sigma) \to Mod(\Sigma)$ is the usual forgetful functor. Representable signature morphisms capture the idea of first-order variable. For instance, in the institution of first-order predicate logic with equality (FOPL$_=$; see Example 1.(1)), given a set of constant symbols X, the inclusion of $\Sigma = (S, F, P)$ into $\Sigma' = (S, F \cup X, P)$ is represented by $T_\Sigma(X)$, the term algebra over variables X and operations in F, with all the relations in P empty.

The sentences of an institution $\mathcal{I}$ can be naturally extended with first-order-like constructions [34]: if $\varphi : \Sigma \to \Sigma'$, $\rho, \delta \in Sen(\Sigma)$, $\rho' \in Sen(\Sigma')$, and $E \subseteq Sen(\Sigma)$, one can build the sentences $\bigwedge E$, $\bigvee E$, $\neg\rho$, $\delta \Rightarrow \rho$, $(\forall\varphi)\rho'$, $(\exists\varphi)\rho'$, with the following semantics, for each Σ-model M: $M \models \bigwedge E$ iff $M \models E$; $M \models \bigvee E$ iff $M \models e$ for some $e \in E$; $M \models \neg\rho$ iff $M \not\models \rho$; $M \models \delta \Rightarrow \rho$ iff $M \models \delta$ implies $M \models \rho$; $M \models (\forall\varphi)\rho'$ iff $M' \models \rho'$ for all φ-expansions M' of M; $M \models (\exists\varphi)\rho'$ iff there exists some φ-expansion M' of M such that $M' \models \rho'$. It might be the case that the newly constructed sentences are equivalent to some existing sentences in $\mathcal{I}$ - we take the convention that whenever we mention such a sentence, say $(\forall\varphi)\rho'$, we tacitly assume that it is equivalent to an existing one in $\mathcal{I}$ and we simply identify them, i.e., consider that $(\forall\varphi)\rho' \in Sen(\Sigma)$.

Given a signature Σ, a Σ-sentence ρ is called: *basic* [10] if there exits a Σ-model T_ρ such that for each Σ-model M, $M \models \rho$ iff there exists some morphism $T_\rho \to M$; *universal* if there exists a signature morphism $\varphi : \Sigma \to \Sigma'$ and a basic sentence $\rho' \in Sen(\Sigma')$ such that ρ is of the form $(\forall\varphi)\rho'$; *positive* if it is either basic or is obtained from basic sentences by a finite number of conjunctions ($\bigwedge E$), disjunctions ($\bigvee E$), universal quantification ($(\forall\varphi)\rho'$), and existential quantification ($(\exists\varphi)\rho'$). The notion of basic sentence is an institutional generalization for ground atom (equation, predicate etc.) - in our examples of institutions, the basic sentences are the primary bricks used to construct the more complicated sentences. For instance, in FOPL$_=$, the basic sentences are just finite conjunctions of ground term equalities $t_1 = t_2$ and/or of relational statements over ground terms $R(t_1, \ldots, t_n)$; in the institution of equational logic (EQL - see Example 1.(2)), the basic sentences are just ground term equalities. Universal sentences capture institutionally the universally quantified atoms. Universal sentences contain basic sentences: any basic sentence $\rho \in Sen(\Sigma)$ is equivalent to $(\forall 1_\Sigma)\rho$. The institution $\mathcal{I}$ is said to: *have basic Horn implications* iff for each signature Σ, each set of basic sentences $E \subseteq Sen(\Sigma)$, and each basic sentence $\rho \in Sen(\Sigma)$, the sentence $(\bigwedge E) \Rightarrow e$ is in $Sen(\Sigma)$; *have finitary basic Horn implications* if the above condition is satisfied for E finite.

A signature morphism $\varphi : \Sigma \to \Sigma'$ is called *liberal* [17] iff $Mod(\varphi)$ has a left adjoint. An institution is called *liberal* iff each of its signature morphisms is liberal. Let $\mathcal{I}$ be an institution, $\mathcal{U}$ be a $|Sign|$-indexed class of model morphisms closed under composition and images by reduct functors, and $\varphi : \Sigma \to \Sigma'$ be a morphism in $Sign$. We say that: φ *creates $\mathcal{U}$-morphisms* iff for any $A' \in |Mod(\Sigma')|$ and any $h : A'\restriction_\varphi \to B$ in $\mathcal{U}_\Sigma$, there exists $f : A' \to B'$ in $\mathcal{U}_{\Sigma'}$ such that $f\restriction_\varphi = h$; also, φ *weakly creates* $\mathcal{U}$-morphisms iff for any $A' \in |Mod(\Sigma')|$ and any $h : A'\restriction_\varphi \to B$ in $\mathcal{U}_\Sigma$, there exist $g : B \to C$ in $\mathcal{U}_\Sigma$ and $f : A' \to B'$ in $\mathcal{U}_{\Sigma'}$ such that $f\restriction_\varphi = h; g$. Morphism creation condition is used in [12] and [10] (under the name *lifting*) for institution-independent interpolation and ultraproducts results. We shall use weak creation at the bare definition of hidden signature morphisms.

Example 1. We briefly discuss two important institutions that will be used as working examples. Their detailed descriptions, as well as several other examples of institutions on which our results apply, are discussed in Appendix C.
(1) $\mathrm{FOPL}_=$ [17] - the institution of (many-sorted) first order predicate logic with equality. The signatures are triples (S, F, P), where S is a set of *sorts*, $F = \bigcup\{F_{w,s} | w \in S^*, s \in S\}$ is a set of (S-sorted) *operation symbols*, and $P = \bigcup\{P_w | w \in S^*\}$ is a set of (S-sorted) *relation symbols*. A signature morphism is a triple $\varphi = (\varphi^{sort}, \varphi^{op}, \varphi^{rel}) : (S, F, P) \to (S', F', P')$, where $\varphi^{sort} : S \to S'$, $\varphi^{op} : F \to F'$, and $\varphi^{rel} : P \to P'$ are mappings such that $\varphi^{op}(F_{w,s}) \subseteq F'_{\varphi^{sort}(w), \varphi^{sort}(s)}$ and $\varphi^{rel}(P_w) \subseteq P'_{\varphi^{sort}(w)}$ for each $w \in S^*$ and $s \in S$. (We may write φ instead of φ^{sort}, φ^{rel} and φ^{op}.) Given a signature $\Sigma = (S, F, P)$, a Σ-model is a triple $M = (\{M_s\}_{s \in S}, \{M_{w,s}(\sigma)\}_{(w,s) \in S^* \times S}, \{M_w(\sigma)\}_{w \in S^*})$ interpreting each sort as a set, each operation symbol as a function, and each relation symbol as a relation, with appropriate arities. (We may write M_σ and M_π instead of $M_{w,s}(\sigma)$ and $M_w(\pi)$.) The model morphisms are S-sorted functions which preserve operations and relations. The set of Σ-sentences and the satisfaction relation are the usual first-order ones. Each $Sen(\varphi)$ translates sentences symbol-wise, and $Mod(\varphi)$ is the usual forgetful functor.
(2) EQL, the institution of equational logic [17], is a restriction of $\mathrm{FOPL}_=$, with no relation symbols (its signatures are pairs (S, F)), and with only conditional equations $(\forall X) t_1 = t'_1 \wedge \ldots t_n = t'_n \Rightarrow t = t')$ as sentences.

3 Hidden Algebra Logic and Behavioral Satisfaction

Hidden algebra extends algebraic specification to handle states naturally, using behavioral equivalence. Systems need only satisfy their requirements behaviorally, in the sense of *appearing* to satisfy them under all possible experiments. Hidden algebra was introduced in [16] and developed further in [18, 19, 21, 32] among many other places. CafeOBJ [14] and BOBJ [21], are executable specification languages that support behavioral specification and reasoning. One distinctive feature of hidden algebra logics is to split sorts into *visible* for data and *hidden* for states. A model, or *hidden algebra*, is an abstract implementation of a system, consisting of its possible states, with functions for operations. The restriction of a model to the visible subsignature is called *data*. *Hidden logics* refer to close relatives of hidden algebra, including both *fixed-data* and *loose-data* variants. This paper is concerned with the fixed-data approach. Hidden algebra is constructed on top of many-sorted algebra and equational logic - we shall use the notations of EQL (see Example 1).

Given a set V of *visible sorts*, a V-sorted signature Ψ called the *data signature*, and a Ψ-algebra D called the *data algebra*, then a *fixed-data hidden* (Ψ, D)-*signature* is a $(V \cup H)$-sorted signature Σ with $\Sigma{\restriction_V} = \Psi$, where H is a set disjoint from V of *hidden sorts*. Hereafter we write "hidden signature" instead of "fixed-data hidden (Ψ, D)-signature". The operations in Σ with one hidden argument and visible result are called *attributes*, those with one hidden argument and hidden result are called *methods*, those with two hidden arguments and hidden result are called *binary methods*, and so on; those with only visible arguments and hidden result are called *hidden constants*. Let $\Sigma = (S, F)$ be a hidden signature, where $S = V \cup H$. A *hidden Σ-algebra* is a Σ-algebra A with $A{\restriction_\Psi} = D$; it can be regarded as a universe of possible states of a system. A system can be seen as a "black-box," the inside of which is not seen, one being only concerned with its behavior under "experiments". A *hidden Σ-morphism* between two hidden Σ-algebras A and B is a usual Σ-homomorphism $h : A \to B$ such that $h{\restriction_\Psi} = 1_D$.

An *experiment* is an observation of a system after it has been perturbed; the $\bullet$ below is a placeholder for the state being experimented upon. A *context for sort* s is a term in $T_\Sigma(\{\bullet : s\} \cup Z)$ having exactly one occurrence of a special variable $\bullet$ of sort s, where Z is an S-indexed componentwise infinite set of special variables. Let $\mathbb{C}[\bullet : s]$ denote the S-indexed set of all contexts for sort s, and $var(c)$ the finite set of variables in a context c except $\bullet$. A context with visible result sort is called an *experiment*; let $\mathbb{E}[\bullet : s]$ denote the V-indexed set of all experiments for sort s. The interesting experiments are those for hidden sorts $s \in H$. We sometimes say that an experiment or a context for sort s is *appropriate* for terms or equations of sort s. Contexts can be "applied" as follows. If $c \in \mathbb{C}_{s'}[\bullet : s]$ and $t \in T_{\Sigma,s}(X)$, then $c[t]$ denotes the term in $T_{\Sigma,s'}(var(c) \cup X)$ obtained from c by substituting t for $\bullet$. Further, c generates a map $A_c \colon A_s \to [A^{var(c)} \to A_{s'}]$ on each Σ-algebra A, defined by $A_c(a)(\theta) = a^*_\theta(c)$, where a^*_θ is the unique extension of the map (denoted a_θ) that takes $\bullet$ to a and each $z \in var(c)$ to $\theta(z)$.

We recall the important notion of behavioral equivalence. Given a hidden Σ-algebra A, the equivalence $a \equiv_\Sigma a'$ iff $A_\gamma(a)(\theta) = A_\gamma(a')(\theta)$ for all experiments γ and all maps $\theta \colon var(\gamma) \to A$ is called *behavioral equivalence on A*. A *hidden congruence* is a congruence which is the identity on visible sorts. The following supports several important results in hidden logics. Since final models may not exist when operations of zero or more than one hidden argument are allowed, the existence of a largest hidden congruence does not depend on them.

Theorem 1. *Given a hidden Σ-algebra A, the behavioral equivalence is the largest hidden congruence on A* (see [31] for a proof).

Given a hidden Σ-algebra A and a Σ-equation $(\forall X)\ t = t'$, say ρ, then A *behaviorally satisfies* ρ, written $A \models\!\!\!\equiv_\Sigma \rho$, iff $\theta(t) \equiv_\Sigma \theta(t')$ for all $\theta\colon X \to A$. Let $\mathbb{E}[\rho]$ be either the set $\{(\forall X, var(\gamma))\ \gamma[t] = \gamma[t'] \mid \gamma \in \mathbb{E}[\bullet : h]\}$ when the sort h of t, t' is hidden, or the set $\{\rho\}$ when the sort of t, t' is visible. $\mathbb{E}[E]$ is the set $\bigcup_{e\in E} \mathbb{E}[\rho]$. Behavioral satisfaction of an equation can be reduced to strict satisfaction of a potentially infinite set of equations:

Proposition 1. *If A is a hidden Σ-algebra then $A \models\!\!\!\equiv_\Sigma E$ iff $A \models_\Sigma \mathbb{E}[E]$.*

Behavioral satisfaction is "reflected" by hidden morphisms [19]:

Proposition 2. *If $h : A \to B$ is a hidden Σ-morphism and ρ a Σ-equation, then $B \models\!\!\!\equiv \rho$ implies $A \models\!\!\!\equiv \rho$.*

The notion of morphism of hidden signatures [16] reflects at a *syntactic level* the object-oriented principles of data encapsulation. A morphism of (Ψ, D)-hidden signatures $\chi : (V \cup H, F) \to (V \cup H', F')$ of (Ψ, D)-hidden signatures is a many sorted signature morphism such that: (C1) χ is an identity on Ψ; (C2) $\chi^{sort}(H) \subseteq H'$; (C3) for each operation $\sigma' \in F'$ having an argument sort in $\chi^{sort}(H)$, it is the case that $\sigma' \in \chi^{op}(F)$. These conditions have natural interpretations in terms of information encapsulation: visible data remains unchanged (C1); hidden states are not unhidden by imports (C2); and no new methods or attributes are added on imported states (C3). Condition (C3), although has a rather restrictive character, is quite faithful to the principle of "behavior-protecting" inheritance mechanism. The above conditions ensure that behavioral equivalence and satisfaction are preserved by the reduct functor:

Proposition 3. *If $\chi : \Sigma \to \Sigma'$ is a hidden signature morphism with $\Sigma = (V \cup H, F)$ and A' is a hidden Σ'-algebra, then: (1) for all $h \in H$ and $a, b \in A'_{\chi^{sort}(h)}$, $a \equiv_{\Sigma'} b$ iff $a \equiv_\Sigma b$; (2) $(A'\restriction_\chi)/_{\equiv_\Sigma} = (A'/_{\equiv_{\Sigma'}})\restriction_\chi$; (3) $A' \models\!\!\!\equiv \chi(\rho)$ iff $A'\restriction_\chi\ \models\!\!\!\equiv \rho$, for each Σ-equation ρ.*

4 Quotient Systems

Image factorization systems [1] are a categorical generalization of the system of injections and surjections from set theory. Unlike bare monics and epics, the morphisms of a factorization system work together to provide, up to an isomorphism, a unique factorization for each morphism. *Inclusion systems* [15] and *weak inclusion systems* [8], modifications of factorization systems by dropping the "up to an isomorphism" relaxation, turn out to be more suitable for the categorical study of algebraic specification concepts. In this paper, because of the coalgebraic nature of the involved notions, we introduce a variant of a factorization system that is dual to the weak inclusion system:

Definition 1. *A **quotient system** for a category $\mathcal{C}$ is a pair $(\mathcal{E}, \mathcal{M})$, where $\mathcal{E}$ and $\mathcal{M}$ are subcategories of $\mathcal{C}$ such that: (1) $\mathcal{E}$ is a* partial order*, in the sense that $\mathcal{E}(A, B)$ contains at most one morphism for any $A, B \in |\mathcal{C}|$, and $A = B$ whenever $\mathcal{E}(A, B) \neq \emptyset$ and $\mathcal{E}(B, A) \neq \emptyset$; (2) Morphisms in $\mathcal{C}$ can be factored uniquely as $e; m$, with $e \in \mathcal{E}$, $m \in \mathcal{M}$. The elements of $\mathcal{E}$ are called **quotients** and those of $\mathcal{M}$ **injections**. B is called a **quotient object** of A when $\mathcal{E}(A, B) \neq \emptyset$.*

Note that $(\mathcal{E}, \mathcal{M})$ is a quotient system for $\mathcal{C}$ iff $(\mathcal{M}, \mathcal{E})$ is a weak inclusion system for $\mathcal{C}^{op}$. Thus, w.r.t. category theory, quotient systems bring nothing essentially new. However, they model properly the important notion of *congruence*, which is not to be considered, like in the case of factorization systems, *up to an isomorphism*, but chosen in a *unique*, canonical way. This will have important semantical and technical consequences when we define behavioral satisfaction: first, we can model faithfully in an institutional framework the process of constructing the behavioral equivalence, originally defined in an *internal* fashion within the set-theoretical structure of the algebras (see Section 3); second, by regarding models as *universes for congruences*, we do not need to postulate the existence of final objects; finally, delicate technical issues regarding lifting and preserving properties can be elegantly treated using quotient systems.

The category of sets, as well as that of algebras, have natural quotient systems if we allow a slight and non-problematic *foundational modification:* we assume that all elements in the considered sets or carriers are sets themselves and in addition they are mutually disjoint. That anything is a set is a harmless principle of the Zermelo-Fraenkel Set Theory,[1] but note that we only take this assumption about algebras (models), and not about sentences. Moreover, any algebra can be isomorphically and uniformly transformed into one satisfying the above condition by simply replacing its elements x with singletons $\{x\}$. Now, we can take $\mathcal{M}$ as the category of all injective morphisms and $\mathcal{E}$ as that of those surjective morphisms $f : A \to B$ such that, for each element $b \in B$, the elements $a \in A$ with $f(a) = b$ form a partition of b. Therefore, $\mathcal{E}$ provides canonical ways to factor algebras by refining their carrier sets, viewed as partitions, in a dual manner to inclusions that give a canonical way to embed an algebra into another. We next list some properties of quotient systems, some of them dual to ones for weak inclusion systems [8]. Let $(\mathcal{E}, \mathcal{M})$ be a quotient system for $\mathcal{C}$.

Proposition 4. (see Fact 5 in [8]) *(1) Any $e \in \mathcal{E}$ in an epic; (2) $\mathcal{M}$ contains all the isomorphisms in $\mathcal{C}$; and (3) all isomorphisms in $\mathcal{E}$ are identities.*

Proposition 5. (see also Corollary 26 in [8]) *If $e, e' \in \mathcal{E}$ of same source admit pushout in $\mathcal{C}$, then they have a* unique *pushout whose morphisms are in $\mathcal{E}$. If $(I, \leq)$ is a filtered set and $c = (e_{i,j} : A_i \to A_j)_{i,j \in I, i \leq j}$ an I-diagram in $\mathcal{E}$ admitting a colimit in $\mathcal{C}$, then there is a unique colimit of c in $\mathcal{C}$ whose morphisms are in $\mathcal{E}$. In particular, if $\mathcal{C}$ is $\{$pushout and filtered$\}$-cocomplete, then so is $\mathcal{E}$.*

[1] This set-theoretical assumption that we take should be regarded as a meta-level setting, having nothing to do with the duality algebra-coalgebra. In particular, it does not imply that we are planning to treat the coalgebraic phenomena with algebraic methods; at least not to a greater extent than any other "mathematical" approach.

Example 2. For each signature (S, F) in EQL, $\mathcal{E}_{(S,F)}$ consists of all surjective morphisms $h : A \to B$ such that $b = \bigcup_{a \in A, h_s(a)=b} a$ for each sort $s \in S$ and $b \in B_s$, and $\mathcal{M}_{(S,F)}$ consists of all injective morphisms. In the case of FOPL$_=$, we can consider two canonical ways to provide quotient systems, following the idea of inclusion systems for FOPL$_=$ [13]. Let (S, F, P) be a signature. An (S, F, P)-morphism $f : A \to B$ is called *strong* if, for each (n-ary) relation symbol $R \in P$ and each $(a_1, \ldots, a_n)$, it holds that $(a_1, \ldots, a_n) \in A_R$ iff $(f(a_1), \ldots, f(a_n)) \in B_R$. (1) The quotients are morphisms $h : A \to B$ such that h is a (S, F)-quotient in EQL; the injections are the strong injective morphisms; (2) The quotients are morphisms $h : A \to B$ such that h is a strong (S, F)-quotient in EQL; the injections are the injective morphisms.

All the institutions that use some form of set-theoretical notion of model tend to have quotient systems on models, although the choice is not always unique.

5 The Behavioral Extension of an Institution

Next we provide an institutional generalization of fixed-data hidden logic.

Definition 2. *An* ***institution with quotients*** *is an institution equipped with quotient systems $(\mathcal{E}_\Sigma, \mathcal{M}_\Sigma)$ on each category of models $Mod(\Sigma)$, such that all reducts $Mod(\varphi)$ along signature morphisms $\varphi : \Sigma \to \Sigma'$ preserve quotients and injections. (That is, for each e in $\mathcal{E}_{\Sigma'}$ and m in $\mathcal{M}_{\Sigma'}$, it holds that $e\restriction_\varphi$ is in $\mathcal{E}_\Sigma$ and $m\restriction_\varphi$ is in $\mathcal{M}_\Sigma$.) An institution with quotients is* ***co-well-powered*** *if each $Mod(\Sigma)$ is $\mathcal{E}_\Sigma$-co-well-powered.*

Notice that the notion of $\mathcal{E}_\Sigma$-co-well-powered-ness becomes particularly simple thanks to Proposition 4.(3): one only asks that, for each $A \in |Mod(\Sigma)|$, the class of morphisms in $\mathcal{E}_\Sigma$ of source A is a set. All throughout this section, we shall work inside the following framework:

> **Framework 1:** A co-well-powered institution with quotients $\mathcal{I}$, having filtered colimits and pushouts of models, such that all reducts $Mod(\varphi)$ along signature morphisms $\varphi : \Sigma \to \Sigma'$ preserve filtered colimits and pushouts of quotient diagrams (i.e., diagrams consisting of morphisms in $\mathcal{E}$).

Our examples of institutions with quotients all satisfy the above conditions. While these institutions have not only filtered colimits and pushouts, but also arbitrary colimits on models, the arbitrary colimits are usually not preserved by reduct functors. The only property that needs explanation is the preservation of pushouts of quotients. In EQL, this follows from the fact that the supremum of two congruences of a model does not depend on the signature where the supremum is taken - see Appendix D. As for the case of the two possible families of quotient systems in FOPL$_=$, the quotient preservation property follows from the equational case, using the fact that the forgetful functor $Mod(S, F, P) \to Mod(S, F, \emptyset)$ creates colimits (and pushouts in particular).

Let Ψ be a fixed signature of $\mathcal{I} = (Sign, Mod, Sen, \models)$, that we call the *visible signature*, and D be a fixed Ψ-model, that we call the *data model*. We define an institution $\mathcal{I}_{beh}(\Psi, D)$, the *behavioral extension of* $\mathcal{I}$ *over* (Ψ, D). We let $\mathcal{I}_{beh} = (Sign_{beh}, Mod_{beh}, Sen_{beh}, \models\!\!\!\equiv)$ denote $\mathcal{I}_{beh}(\Psi, D)$ without forgetting though that our construction is parameterized by Ψ and D.

Signatures. The signatures of $\mathcal{I}_{beh}$ are pairs $(\varphi : \Psi \to \Sigma, \Sigma)$, where Σ is a signature in $\mathcal{I}$. (Instead of the entire class of objects of $\Psi/Sign$, one could also consider, without adding any technical difficulties, only a subclass, like the class of inclusions [21].) We postpone the definition of signature morphisms.

Sentences. For a signature (φ, Σ) in $\mathcal{I}_{beh}$, let $Sen_{beh}(\varphi, \Sigma)$ be precisely $Sen(\Sigma)$. However, the sentences will get in $\mathcal{I}_{beh}$ a different meaning than in $\mathcal{I}$.

Models. For a signature (φ, Σ) in $\mathcal{I}_{beh}$, let $Mod_{beh}(\varphi, \Sigma)$ be the *fiber category* [2] $D\!\upharpoonright_{\varphi}^{-1}$ of the functor $\_\!\upharpoonright_{\varphi}$: $Mod(\Sigma) \to Mod(\Psi)$ over D: its objects are those $A \in |Mod(\Sigma)|$ with $A\!\upharpoonright_{\varphi} = D$ and its morphisms are those $h : A \to B$ in $Mod(\Sigma)$ with $h\!\upharpoonright_{\varphi} = 1_D$. Interestingly, this fiber category captures precisely the intuition of hidden algebra: models protect data and morphisms are data-consistent.

We are next going to define behavioral satisfaction (in $\mathcal{I}_{beh}$) as satisfaction in $\mathcal{I}$ on smallest data-consistent quotient objects. We first need to introduce some notation and show that such objects indeed exist.

Definition 3. *For a signature* (φ, Σ) *and a* (φ, Σ)*-model* A *in* $\mathcal{I}_{beh}$*, let* $A/_D\mathcal{E}_\Sigma$ *be the category of* ***data-consistent quotients*** *of* A*: its objects are morphisms* $e : A \to B$ *in* $\mathcal{E}_\Sigma$ *with* $e\!\upharpoonright_{\varphi} = 1_D$ *and its morphisms* $h : (e : A \to B) \to (e' : A \to B')$ *are morphisms* $h : B \to B'$ *with* $h\!\upharpoonright_D = 1_D$ *and* $e; h = e'$.

It follows from the above definition that all the mentioned morphisms $h : B \to B'$ are actually in $\mathcal{E}_\Sigma$ (one can see that by decomposing h as $e_h; i_h$ and using the unique factorization property for $e; e_h; i_h = e'$). Moreover, the category $A/_D\mathcal{E}_\Sigma$ is isomorphic to the full subcategory of $\mathcal{E}_\Sigma$ having the class of objects restricted to quotient objects of A.

Proposition 6. *The category* $A/_D\mathcal{E}_\Sigma$ *has a unique final object,* $e_{A,\varphi} : A \to A_\varphi$.

The morphism $e_{A,\varphi}$ can be intuitively regarded as the "largest congruence on A that is data-consistent", or the "behavioral equivalence" on A. Note that the construction of A_φ follows a final approach, without assuming the existence of *globally final models* - rather, we get a final model, i.e., a greatest congruence, starting from any given model. This allows our formalization to capture non-coalgebraic variants of hidden algebra at no additional cost.

Satisfaction relation. We can now define satisfaction in $\mathcal{I}_{beh}$, called *behavioral satisfaction* and written $\models\!\!\!\equiv$, as follows: for a signature (φ, Σ), a (φ, Σ)-model A and a (φ, Σ)-sentence ρ, let $A \models\!\!\!\equiv_{(\varphi,\Sigma)} \rho$ in $\mathcal{I}_{beh}$ iff $A_\varphi \models_\Sigma \rho$ in $\mathcal{I}$.

The only thing left to define in $\mathcal{I}_{beh}$ is the morphism of signatures. As discussed in Section 3, this is a delicate concept to define even in the concrete framework of hidden algebra, because it needs to imply the property that its semantic counterpart, the reduct, preserves behavioral equivalences on models. Whether the morphisms in $Sign_{beh}$ can be defined categorically in some "syntactic" way capturing the conditions (C1), (C2), (C3) from Section 3 seems to be a difficult problem and perhaps not worthwhile the effort. Our approach, instead, is to define morphisms of signatures by capturing precisely the above crucial property.

Proposition 7. *Let $\varphi : \Psi \to \Sigma$, $\varphi' : \Psi \to \Sigma'$ and $\chi : \Sigma \to \Sigma'$ be three signature morphisms in $\mathcal{I}$ such that $\varphi;\chi = \varphi'$. Then the following are equivalent: (a) χ weakly creates data-consistent quotients; and (b) for each Σ'-model A' with $A'{\restriction_{\varphi}} = D$, it is the case that $(e_{A',\varphi'}){\restriction_{\chi}} = e_{(A'{\restriction_{\chi}}),\varphi}$.*

Signature morphisms. The morphisms $\chi : (\varphi, \Sigma) \to (\varphi', \Sigma')$ in $Sign_{beh}$ are now defined to be morphisms $\chi : \Sigma \to \Sigma'$ in $Sign$ such that $\varphi;\chi = \varphi'$ and the equivalent conditions in Proposition 7 hold. It is not hard to see that $Sign_{beh}$ is now a (broad) subcategory of $\Psi/Sign$. Sen_{beh} and Mod_{beh} can be defined on signature morphisms $\chi : (\varphi, \Sigma) \to (\varphi', \Sigma')$ as expected, that is, exactly as the functors Sen and Mod are defined on $\chi : \Sigma \to \Sigma'$, but using the appropriate restricted classes of models and model morphisms.

Condition (b) in Proposition 7 provides the motivation for the definition of signature morphisms: one wants the "behavioral equivalence", i.e. the largest hidden quotient, to be preserved by reduct functors - this is in fact the main reason for the conditions (C2) and (C3) in the definition of hidden signature morphisms (see Section 3). As for condition (a), one can use the following intuition for the weak creation property stated there. Let $\chi : \Sigma \to \Sigma'$ be a morphism in $\Psi/Sign$. Also, let $A \in Mod_{beh}(\varphi, \Sigma)$ and $A' \in Mod_{beh}(\varphi', \Sigma')$ such that $A = A'{\restriction_{\chi}}$. The existence of a quotient $e : A \to B$ with $e{\restriction_{\varphi}} = 1_D$ means that the hidden structure of A can be flattened in a behaviorally consistent way, i.e., not affecting the data. This situation should not depend on notation, so one should be able to alternatively perform this flattening on A'. Yet, because of the larger number of expressible entities in Σ', here consistent flattening might cause more effects - hence the "weak" nature of creation.

Theorem 2. *$\mathcal{I}_{beh}$ is an institution with quotients, where, for each $(\varphi, \Sigma) \in |Sign|$, $\mathcal{E}_{(\varphi,\Sigma)}$ and $\mathcal{M}_{(\varphi,\Sigma)}$ are the restrictions of $\mathcal{E}_{\Sigma}$ and $\mathcal{M}_{\Sigma}$ to $Mod_{beh}(\Sigma, \varphi)$, respectively. Moreover, there exists a canonical morphism of institutions (in the sense of [17]) between $\mathcal{I}_{beh}$ and $\mathcal{I}$, projecting each $\mathcal{I}_{beh}$ signature (φ, Σ) into Σ, not changing the sentences, and mapping each (φ, Σ)-model A to A_{φ}.*

The institution $\mathcal{I}_{beh}$ above generalizes the institutions of variants of fixed-data hidden algebra [16, 21, 31], constructed in a similar fashion on top of many-sorted equational logic. Theorem 2 tells us that similar behavioral extensions of many other logics are possible, in for particular those in Appendix C, including partial and infinitary ones. A first important property of behavioral satisfaction is that entailment in $\mathcal{I}$ is "sound" in $\mathcal{I}_{beh}$. The next proposition generalizes former results on "behavioral soundness of equational deduction" [32], with syntactic proofs in the concrete hidden algebraic framework.

Proposition 8. *If $(\varphi, \Sigma) \in |Sign_{beh}|$, $\rho \in Sen(\Sigma)$ and $E \subseteq Sen(\Sigma)$, then $E \models_{\Sigma} \rho$ implies $E \models\!\!\!\equiv_{(\varphi,\Sigma)} \rho$.*

The following proposition generalizes another standard result in hidden algebra, namely that behavioral satisfaction coincides with usual satisfaction on sentences over the visible syntax.

Proposition 9. *Let* $(\varphi, \Sigma) \in |Sign_{beh}|$, $\rho \in Sen_{\mathcal{I}}(\Psi)$ *and* $A \in |Mod_{beh}(\varphi, \Sigma)|$. *Then* $A \models\!\!\!\equiv_{(\varphi,\Sigma)} \varphi(\rho)$ *iff* $A \models_{\Sigma} \varphi(\rho)$ *iff* $D \models_{\Psi} \rho$.

In hidden algebra, "visibility" does not concern only sentences over the visible signature. The sentences of visible sort need not contain only data constructs; indeed, sentences of visible sort may involve several attributes and methods. There is no notion of "visible sort" in our abstract framework. However, we can still define an institutional generalization of "sentences of visible sorts", that we call "visible sentences", by model-theoretic means; the visible sentences will be those preserved back and forth by data-consistent flattening, following the intuition that these sentences should sense only modifications in the visible part of a system. We also introduce "quasi-visible sentence", for which the preservation property holds only backwards. But let us set some terminology first:

Definition 4. *Let* $(\varphi, \Sigma) \in |Sign_{beh}|$, $\rho \in Sen(\Sigma)$, *and* $\mathcal{K}$ *a subcategory of* $Mod_{beh}(\varphi, \Sigma)$. *Then* ρ *is* ***closed (behaviorally closed) under*** $\mathcal{K}$ *if, for each* $A \to B$ *in* $\mathcal{K}$, $A \models \rho$ *implies* $B \models \rho$ *(*$A \models\!\!\!\equiv \rho$ *implies* $B \models\!\!\!\equiv \rho$*, respectively).*

Definition 5. *Let* (φ, Σ) *be a signature in* $\mathcal{I}_{beh}$. *Then* $\rho \in Sen_{beh}(\varphi, \Sigma)$ *is* φ-***visible*** *if it is closed under both* $\mathcal{E}_{(\Sigma,\varphi)}$ *and* $\mathcal{E}^{op}_{(\Sigma,\varphi)}$ *and* φ-***quasi-visible*** *if it is closed under* $\mathcal{E}^{op}_{(\Sigma,\varphi)}$. *If the signature* φ *is clear, we shall say "visible" ("quasi-visible") instead of "*φ*-visible" ("*φ*-quasi-visible").*

Proposition 10. *Let* $(\varphi, \Sigma) \in |Sign_{beh}|$ *and* $\rho \in Sen_{beh}(\varphi, \Sigma)$. *Then: (1)* ρ *is visible iff, for each* $A \in |Mod_{beh}(\varphi, \Sigma)|$, *[*$A \models \rho$ *iff* $A \models\!\!\!\equiv \rho$*]; (2) if* ρ *is quasi-visible then, for each* $A \in |Mod_{beh}(\varphi, \Sigma)|$, *[*$A \models\!\!\!\equiv \rho$ *implies* $A \models \rho$*]; (3) if* ρ *is closed under* $\mathcal{M}^{op}_{(\varphi,\Sigma)}$ *and under* $\mathcal{E}_{(\varphi,\Sigma)}$, *then it is behaviorally closed under* $Mod_{beh}(\varphi, \Sigma)^{op}$.

Thus, according to Proposition 10, the visible sentences are precisely those for which behavioral satisfaction coincides with usual satisfaction. On the other hand, the quasi-visible sentences have the property that, in order to satisfy them behaviorally, one has to satisfy them strictly. Moreover, (3) in Proposition 10 is the abstract version of the hidden algebraic result (Proposition 2) saying that equational behavioral satisfaction is preserved by reflexions of arbitrary hidden morphisms. (Recall that in the usual algebraic settings, equations are closed under arbitrary quotients and reflexions of embedding.)

Proposition 11. *Visible and quasi-visible sentences are preserved by signature morphisms and closed under conjunctions, disjunctions, universal and existential quantifications. In addition, visible sentences are also closed under negation.*

An immediate consequence of the above proposition is that both visible and quasi-visible sentences provide subinstitutions of $\mathcal{I}_{beh}$. Also, in the case of positive sentences (a very wide class, containing the basic and the universal sentences), the notions of visibility and quasi-visibility coincide:

Corollary 1. *Let (φ, Σ) be a signature in $\mathcal{I}_{beh}$ and ρ be a positive Σ-sentence in $\mathcal{I}$. Then ρ is φ-visible iff it is φ-quasi-visible.*

The next proposition deals with some structural properties inherited from $\mathcal{I}$ to $\mathcal{I}_{beh}$: filtered colimits of models and signatures. The former are usually important for Birkhoff-like axiomatizability results, while the latter, which also bring filtered colimits of theories [17], can be used for approximating finite refinements towards a fixed point. The comma nature of the signatures in $\mathcal{I}_{beh}$ "invite" us to construct filtered colimits, starting from those of $\mathcal{I}$.

Proposition 12. *(1) If (φ, Σ) is a signature in $\mathcal{I}_{beh}$ such that φ creates isomorphisms in $\mathcal{I}$, then $Mod_{beh}(\varphi, \Sigma)$ has filtered colimits; (2) If $\mathcal{I}$ has countable filtered colimits of signatures and is ω-exact, then $\mathcal{I}_{beh}$ also has countable filtered colimits of signatures.*

In the case of many-sorted algebraic signatures, the signature morphisms that create model isomorphisms are precisely those that are injective on sorts. In particular, Proposition 12.(1) holds for the case, usually considered for hidden algebra, of φ being an inclusion.

6 Behavioral Satisfaction of Universal Sentences

We next focus our study on basic and universal sentences. As already mentioned, these are institutional generalizations of ground equations and arbitrary equations, respectively. Some important properties of hidden logics depend on the equational character of these special sentences.

Before we define our next framework, let us first recall that, in $\mathrm{FOPL}_=$ or EQL, if ρ is some ground Σ-equation, then T_ρ is the quotient by ρ of the ground Σ-term model; then because of the special way to construct direct sums in these logics, it follows that for any Σ-model A, the direct sum $A \amalg T_\rho$ is actually isomorphic to A "factored" by ρ, i.e., the least restrictive "flattening" of A that satisfies ρ (this property is actually institution-independent). Following this intuition, from here on we assume:

> **Framework 2:** An institution $\mathcal{I}$ satisfying Framework 1, such that for any Σ, any $A \in |\mathit{Mod}(\Sigma)|$, and any basic $\rho \in Sen(\Sigma)$, the coproduct $(\amalg_A : A \to A \amalg T_\rho, \amalg_{T_\rho} : T_\rho \to A \amalg T_\rho)$ exists and can be taken such that $\amalg_A \in \mathcal{E}_\Sigma$. Then $A \amalg T_\rho$ is unique with this property and we denote it $A/_\rho$.

The following says that behavioral satisfaction of basic sentences can be equivalently regarded as data-consistent factoring:

Proposition 13. *If (φ, Σ) is a signature, A is a (φ, Σ)-model in $\mathcal{I}_{beh}$, and ρ is a basic Σ-sentence (in $\mathcal{I}$), then $A \models\!\!\!\equiv \rho$ iff $(\amalg_A)\restriction_\varphi = 1_D$.*

In what follows, we shall place the discussion in the context of elementary diagrams. Diagrams are a main concept in classical model theory [7]. The diagram of a model M consists of a set of sentences in its parameterized language which describe its structure well enough in order to axiomatize the class of morphisms of source M. A first institutional definition of diagrams was given in

[34]. We shall make use of a more recent definition in [11], which has the advantage that asks the morphisms between models and signatures to yield smooth translations of the diagram sentences. An institution $\mathcal{I} = (\mathit{Sign}, \mathit{Sen}, \mathit{Mod}, \models)$ is said to have *elementary diagrams* [11] if: (1) for each signature Σ and each Σ-model M there exists a signature morphism $\iota_\Sigma(M) : \Sigma \to \Sigma_M$ (called the *elementary extension of Σ via M*) and a set E_M of Σ_M-sentences (called *the elementary diagram* of the model M) such that $\mathit{Mod}(\Sigma_M, E_M)$ and $M/\mathit{Mod}(\Sigma)$ are isomorphic by an isomorphism $i_{\Sigma,M}$ such that $i_{\Sigma,M}; U = \mathit{Mod}(\iota_\Sigma(M))^r$, where $U : M/\mathit{Mod}(\Sigma) \to \mathit{Mod}(\Sigma)$ is the usual forgetful functor from the comma category and $\mathit{Mod}(\iota_\Sigma(M))^r : \mathit{Mod}(\Sigma_M, E_M) \to \mathit{Mod}(\Sigma)$ is the restriction of $\mathit{Mod}(\iota_\Sigma(M)) : \mathit{Mod}(\Sigma_M) \to \mathit{Mod}(\Sigma)$; (2) ι is *functorial*, i.e., for each signature morphism $\varphi : \Sigma \to \Sigma'$, each $M \in |\mathit{Mod}(\Sigma)|$, $M' \in |\mathit{Mod}(\Sigma')|$ and $h : M \to M'\restriction_\varphi$, there exists a presentation morphism $\iota_\varphi(h) : (\Sigma_M, E_M) \to (\Sigma'_{M'}, E_{M'})$ such that $\iota_\Sigma(M); \iota_\varphi(h) = \varphi; \iota_{\Sigma'}(M')$; (3) i is *natural*, i.e., for each signature morphism $\varphi : \Sigma \to \Sigma'$, each $M \in |\mathit{Mod}(\Sigma)|$, $M' \in |\mathit{Mod}(\Sigma')|$ and $h : M \to M'\restriction_\varphi$ in $\mathit{Mod}(\Sigma)$, $i_{\Sigma',M'}; \mathit{Mod}(\varphi)_{M'}; (h/\mathit{Mod}(\varphi)) = \mathit{Mod}(\iota_\varphi(h))^{rcr}; \imath_{\Sigma,M}$, where $h/\mathit{Mod}(\varphi) : M/\mathit{Mod}(\Sigma) \to (M'\restriction_\varphi)/\mathit{Mod}(\Sigma')$ and $\mathit{Mod}(\varphi)_{M'} : (M'\restriction_\varphi)/\mathit{Mod}(\Sigma') \to M'/Mod(\Sigma')$ are the usual functors between comma categories (see the end of Section 1), and $\mathit{Mod}(\iota_\varphi(h))^{rcr} : \mathit{Mod}(\Sigma_M, E_M) \to \mathit{Mod}(\Sigma'_{M'}, E_{M'})$ is the restriction and corestriction of $\mathit{Mod}(\iota_\varphi(h)) : \mathit{Mod}(\Sigma_M) \to \mathit{Mod}(\Sigma'_{M'})$.

For each $h : A \to B$ in $\mathit{Mod}(\Sigma)$, we shall write $\iota_\Sigma(h)$ instead of $\iota_{1_\Sigma}(h)$.

An important result in hidden algebra is that behavioral satisfaction of unconditional equational sentences can be reduced to usual satisfaction *in the same model* of a set of visible sentences (see Proposition 1). We shall provide an institutional version of this result. For this, we further assume that the institution $\mathcal{I}$ is liberal and either has basic Horn implications, or {is compact and has finitary basic Horn implications}. Regarding the elementary diagrams, we assume that they are: *basic*, in the sense that, for each signature Σ and Σ-model A, each $\rho \in E_A$ is basic and $(E_A)^\bullet \cap \mathit{Basic}(\Sigma) = (A_A)^* \cap \mathit{Basic}(\Sigma)$;[2] *D-representable*, i.e., $\iota_\Sigma(D)$ is representable; *basic-sensitive*, i.e., for each signature Σ, Σ-model A and basic Σ-sentence ρ, $\iota_\Sigma(i_A)^{-1}((E_{A\amalg T_\rho})^\bullet) = (E_A \cup \iota_\Sigma(A)(\rho))^\bullet$ (thus, if a model is factored by a basic sentence, its diagram gains precisely that sentence); *quotient-sensitive*, i.e., for each Σ-quotient $e : A \to B$, if $A \neq B$, there exists a basic Σ_A-sentence α such that $A_A \not\models \alpha$ and $B_e \models \alpha$ (so the fact that B is smaller than A by a quotient is expressible in the language of A as a simple sentence).

For each $(\varphi, \Sigma) \in |\mathit{Sen}_{beh}|$ and $\rho \in \mathit{Sen}_{beh}(\varphi, \Sigma)$, define $\mathcal{QV}_\rho = \{(\forall\phi)\alpha \mid \phi$ signature morphism of source Σ, α quasi-visible sentence, $\rho \models (\forall\phi)\alpha\}$.

Proposition 14. *Let $(\varphi, \Sigma) \in |\mathit{Sen}_{beh}|$, let ρ be a universal Σ-sentence, and let $A \in |\mathit{Mod}_{beh}(\varphi, \Sigma)|$. Then $A \models_{(\varphi,\Sigma)} \rho$ iff $A \models_\Sigma \mathcal{QV}_\rho$.*

Our two working examples of institutions, as well as the others listed in Appendix C, satisfy the hypotheses from our Frameworks 1 and 2, as well as those needed for Proposition 14. Let us take $\mathrm{FOPL}_=$ for instance. The only properties

[2] $\mathit{Basic}(\Sigma)$ denotes the set of basic Σ-sentences.

which might not be clear (like the existence of basic Horn implications) or well-known (like liberality or semi-exact-ness), are some of those regarding diagrams: $(E_A)^\bullet \cap Basic(\Sigma) = (A_A)^* \cap Basic(\Sigma)$ simply because the first-order entailment system extends conservatively the ground equational entailment system; each $\iota_\Sigma(A)$ is representable: it only adds some constants to the source signature; basic-sensitivity asks that, if A is a model factored by a ground equation or atomic relation ρ becoming $A/_\rho$, all that one can infer from E_{A_ρ}, can be equivalently inferred from E_A together with ρ, which is obviously true; quotient-sensitivity is fulfilled as follows: if B is a quotient object of A (by $h : A \to B$), different from A, then there exists a sort s and $a, b \in A_s$ such that $a \neq b$ and $h_s(a) \neq h_s(b)$ - then $a = b$ is the desired sentence α from E_A.

In the case of EQL, it happens that the quasi-visible sentences α can be taken to be basic, hence visible (since "quasi-visible" plus "basic" implies "visible"), so the concrete equational result actually says more than we were able to prove at our institutional level. Yet, it is not clear that a similar neater result as the equational one holds for our other examples of institutions (like $\mathrm{FOPL}_=$). Another question would be whether Proposition 14 holds for other types of sentences besides universal ones - one could easily find examples of conditional equations and existentially quantified sentences for which the property of reducing behavioral satisfaction to normal satisfaction in the same model does not hold; thus the class of universal sentences of an institution might be close to maximality w.r.t. this property, if one wants to cover the classical relevant cases. Note that universal sentences cover the cases when second-order quantification, i.e., over relation and function symbols, are considered (see also [23] for a higher-order result related to our Proposition 14).

7 Related Work and Concluding Remarks

The paper [30] was, at our knowledge, the first to introduce the notion of behavioral, or observational equivalence as we interpret it in this paper, and [33] was the first to sketch a treatment of observational equivalence in arbitrary institutions, where it is defined as existential elementary equivalence w.r.t. some signature morphism. Then [6] considered the notions of hiding and behavior in institutions; since this paper was an important source of inspiration for us, we shall discuss it below. The framework there was inspired by the following situation from "monadic" hidden algebra: the hidden models can be seen as *behavior algebras*, some forms of Lawvere-like algebras, equipped with a distinguished terminal object, having a fixed interpretation; moreover, the category of behavior algebras has a final object constructed using the sets of all possible behaviors of the (hidden) states; hence, thanks to a smooth back and forth communication between the categories of hidden algebras and behavioral algebras, a final semantics can be given for behavioral satisfaction of a sentence by a hidden model. This situation is generalized in [6] to the institutional level, where the notion of behavior algebra is provided as an extra data: a functor from a subcategory, of *hidden signatures*, to Cat^{op}, for which the relevant properties (finality, communication to the hidden models, etc.) are postulated. Our approach shares with [6] the idea of defining behavioral satisfaction as (normal) satisfaction inside a quotient. However, our approach is not tributary to the monadic framework, which only considers hidden operations with *precisely* one hidden argument, framework which loses two important cases: that of hidden constants (in particular, that of different cases of classical automata used in formal languages), and that of operations having multiple hidden-sort arguments; also we do not use data provided "from outside" the institution (as is the case of abstract behavior algebras in [6]), but construct the behavioral extension only by *internal* means of the considered institution. A quasi-abstract treatment of behavioral equivalence can also be found in [5], where a setting similar to the institutional one is used, but localized to a fixed *satisfaction frame*; the behavioral satisfaction (in one of the proposed variants) is also defined as usual satisfaction in a quotient, but in order for the quotient to enjoy good set-theoretical properties, a concrete many-sorted "carrier" set is considered attached to each model, through a *concretization functor*. Another paper in the vicinity of our work, but more concerned with *hiding* than with *behavior*, is [22], discussing compositional operations on modules that can hide some of the information.

We believe that our results can be adapted to also cover loose-data behavioral approach, such as *observational logic* [3, 4]. The main point towards such an adaptation is that the loose-data setting is still based on a notion of behavioral equivalence, called *observational equality* in [3, 4], hence it can still be formalized by our final construction in a fiber category. The main difference is that loose-data behavioral logics allows arrows between algebras that do not have the same data reduct. However, roughly speaking, if we express the concepts in [4] using our notations, we find that the arrows between two (φ, Σ)-models A and B are the usual morphisms between their quotients A_φ and B_φ, quotients which can be constructed independently, taking the data model D to be first $A\restriction_\varphi$ and then $B\restriction_\varphi$. One can show that this construction yields yet another institution, which takes only the data signature Ψ as a parameter this time. The latter institution could be seen as a form of Grothendieck construction (in the style of [9]) obtained by flattening the "indexed" institution $\{\mathcal{I}_{beh}(\Psi, D)\}_{D\in|Mod(\Psi)|}$.

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A Lemmas and proofs of the stated results

Terminology reminded. If (Σ, φ) is a signature of $\mathcal{I}_{beh}$ and h is a morphism in $Mod(\Sigma)$, we call h *data-consistent* if h is a morphism in $Mod_{beh}(\Sigma, \varphi)$.

The following easy lemmas [13] shall be used in proofs:

Lemma 1. *Let $\phi : \Sigma' \to \Sigma$ be a signature morphism and $\rho, \delta \in Sen(\Sigma)$. Furthermore, assume $(\forall\phi)\rho$ and $(\forall\phi)\rho$ are in $Sen(\Sigma')$. Then $\rho \models \delta$ implies $(\forall\phi)\rho \models (\forall\phi)\rho$.*

Lemma 2. *(Lemma of Constants) Let $\phi : \Sigma \to \Sigma'$ be a signature morphism, $\alpha \in Sen(\Sigma)$, $\beta \in Sen(\Sigma')$. Then $\alpha \models (\forall\phi)\beta$ iff $\phi(\alpha) \models \beta$.*

Lemma 3. *Let $\phi : \Sigma \to \Sigma'$ be a signature morphism in $\mathcal{I}$ and $\rho \in Sen(\Sigma)$ such that ϕ is liberal and ρ basic. Then $\phi(\rho)$ is also basic.*

Hint: The Σ'-model $T_\rho^{\#}$, the free extension of T_ρ, represents $\phi(\rho)$.

Below we list all the new propositions from the paper, together with their proofs.

Proposition 5. (see also Corollary 26 in [8]) *If $e, e' \in \mathcal{E}$ of same source admit pushout in $\mathcal{C}$, then they have a* unique *pushout whose morphisms are in $\mathcal{E}$. If $(I, \leq)$ is a filtered set and $c = (e_{i,j} : A_i \to A_j)_{i,j \in I, i \leq j}$ an I-diagram in $\mathcal{E}$ admitting a colimit in $\mathcal{C}$, then there is a unique colimit of c in $\mathcal{C}$ whose morphisms are in $\mathcal{E}$. In particular, if $\mathcal{C}$ is $\{$pushout and filtered$\}$-cocomplete, then so is $\mathcal{E}$.*

Proof. For the pushout property, we refer to Corollary 26 from [8].

Let now $(\mu_i : A_i \to A)_{i \in I}$ be a colimit of c. Each μ_i can be uniquely factored as $e_i; j_i$, with $e_i : A_i \to B_i$ in $\mathcal{E}$ and $j_i : B_i \to A$ in $\mathcal{M}$. For each $i \leq k$ in I, since $e_i; j_i = \mu_i = e_{i,j}; e_k; j_k$, by the uniqueness of decomposition, we have $B_i = B_k$, $j_i = j_k$, and $e_i = e_{i,k}; e_k$. By filtration, the fact that $B_i = B_k$ holds even without assuming $i \leq k$. denote $B = B_i$ and $j = j_i$, for some $i \in I$. We wish to prove that the cocone $d = (e_i : A_i \to B)_{i \in I}$ is also a colimit of $(e_{i,j})_{i,j \in I}$. By the colimit property of c, there exists a unique $u : A \to B$ such that $\mu_i; u = e_i$ for each $i \in I$. Applying again the fact that c is a colimit, together with $\mu_i; u; j = \mu_i$, we obtain $u; j = 1_A$. On the other hand, if we take an arbitrary $i \in I$, we have that $e_i; j; u = \mu_i; u = e_i$ and, since, by Proposition A, e_i is epi, we get $j; u = 1_B$. Thus, u is the inverse of j, hence an isomorphism, which makes d a colimit.

The uniqueness of such a colimit follows easily.

Lemma 4. *Let Σ an $\mathcal{I}$-signature, $(J, \leq)$ a filtered set and $c = (e_{i,j} : A_i \to A_j)_{i,j \in J, i \leq j}$ be a (filtered) J-cocone in $\mathcal{C}$, such that each $e_{i,j}$ is a data-consistent quotient. Then there exists a unique colimit of c such that all the structural morphisms are data-consistent quotients.*

Proof. The existence and uniqueness of a colimit $(\mu_i : A_i \to A)_{i \in J}$ with quotients as structural morphisms follows from Proposition 5. Moreover, because the signature morphisms preserve quotients and filtered colimits, $(\mu_i\restriction_\chi : D \to A\restriction_\chi)_{i \in J}$ is a colimit of $(1_D : D \to D)_{i,j \in J, i \leq j}$; and this colimit has quotients structural morphisms - we apply again Proposition 5 (the uniqueness part), to conclude that $\mu_i\restriction_\chi = 1_D$, thus the μ_i's are also data-consistent.

Proposition 6. *The category $A/_D\mathcal{E}_\Sigma$ has a unique final object, $e_{A,\varphi} : A \to A_\varphi$.*

Proof. Since the quotient system of $Mod(\Sigma)$ is $\mathcal{E}_\Sigma$-co-well-powered and any two isomorphic quotients are equal by Proposition A.(3), all the quotients with source A form a set. Consider $\mathcal{F}$ the family (indexed by a set) of all morphisms of $A/_D\mathcal{E}_\Sigma$. We claim that $\mathcal{F}$ is a filtered diagram in $Mod(\Sigma)$. In order to prove this, all we need is to show that, for each morphisms $g_1 : (e : A \to B) \to (e_1 : A \to B_1)$ and $g_2 : (e : A \to B) \to (e_2 : A \to B_2)$ in $A/_D\mathcal{E}_\Sigma$, there exist two morphisms $h_1 : (e_1 : A \to B_1) \to (e' : A \to C)$ and $h_2 : (e_2 : A \to B_2) \to (e' : A \to C)$ such that $g_1; h_1 = g_2; h_2$. Notice that the latter condition is superfluous, since a $A/_D\mathcal{E}_\Sigma$ is a partial order category. Since e_1 and e_2 are quotients, by Proposition 5, we can take (h_1, h_2) to be their pushout such that h_1, h_2 are also quotients. Moreover, $(h_1\restriction_\varphi, h_2\restriction_\varphi)$ is a pushout of $(e_1\restriction_\varphi, e_2\restriction_\varphi)$, i.e., of $(1_D, 1_D)$. But $(1_D, 1_D)$ is also a pushout of $(1_D, 1_D)$. Hence, again by Proposition 5 (the uniqueness part), $h_1\restriction_\varphi = h_2\restriction_\varphi = 1_D$.

It now suffices to take the colimit in $Mod(\Sigma)$ of the diagram $\mathcal{F}$ as in Lemma 4, to get a final object in $A/_D\mathcal{E}_\Sigma$ as the structural morphism with target A of the colimit. Its uniqueness follows immediately from the fact that the morphisms of $A/_D\mathcal{E}_\Sigma$ are quotients.

Proposition 7. *Let $\varphi : \Psi \to \Sigma$, $\varphi' : \Psi \to \Sigma'$ and $\chi : \Sigma \to \Sigma'$ be three signature morphisms in $\mathcal{I}$ such that $\varphi; \chi = \varphi'$. Then the following are equivalent: (a) χ weakly creates data-consistent quotients; and (b) for each Σ'-model A' with $A'\restriction_\varphi = D$, it is the case that $(e_{A',\varphi'})\restriction_\chi = e_{(A'\restriction_\chi),\varphi}$.*

Proof. Denote $A = A'\restriction_\varphi$.
(a) implies (b): Let $\mathcal{F}$ and $\mathcal{F}'$ be the families of morphisms considered in the proof of Proposition 6 when defining $e_{A,\phi}$ and $e_{A',\phi'}$, respectively. Since $Mod(\chi)$ preserves quotients, $(e_{A',\varphi'})\restriction_\chi$ is a data-consistent quotient of A, and thus, by the finality of A_ϕ, there exists a data-consistent quotient $e'' : (A'_{\varphi'})\restriction_\chi \to A_\varphi$. But $Mod(\chi)$ weakly creates data-consistent quotients, and creates e'' in particular, so there exist two data-consistent quotients $h : A'_{\varphi'} \to B$ and $g : A_\phi \to B\restriction_\chi$ such that $e''; g = h\restriction_\chi$. By the finality of A_φ and $A'_{\varphi'}$, g and h are identities, hence e'' is also an identity, implying $(e'_{A',\varphi'})\restriction_\chi = e_{A,\varphi}$.
(b) implies (a): Let $g : A \to B$ be a data-consistent quotient in $Mod(\Sigma)$. By construction, $A_\varphi = B_\varphi$. Thus, $(A'_{\varphi'})\restriction_\chi = B_\varphi$, which means that g is weakly created as $g; e_{B,\varphi} = (e_{A',\varphi'})\restriction_\chi$.

Theorem 2. *$\mathcal{I}_{beh}$ is an institution with quotients, where, for each $(\varphi, \Sigma) \in |Sign|$, $\mathcal{E}_{(\varphi,\Sigma)}$ and $\mathcal{M}_{(\varphi,\Sigma)}$ are the restrictions of $\mathcal{E}_\Sigma$ and $\mathcal{M}_\Sigma$ to $Mod_{beh}(\Sigma, \varphi)$,* *respectively. Moreover, there exists a canonical morphism of institutions between $\mathcal{I}_{beh}$ and $\mathcal{I}$, flattening each $\mathcal{I}_{beh}$ signature (φ, Σ) into Σ, not changing the sentences, and mapping each (φ, Σ)-model A to A_φ.*

Proof. The fact that $Sign_{beh}$ is indeed a category and Sen_{beh}, Mod_{beh} are indeed functors are routine check. In order to show that the satisfaction condition holds, consider $\chi : (\varphi, \Sigma) \to (\varphi', \Sigma')$ a morphism in $Sign_{beh}$, $\rho \in Sen_{beh}(\varphi, \Sigma)$, and $A' \in |Mod_{beh}(\varphi', \Sigma')|$. Then $[A' \models\!\!\!\equiv Sen_{beh}(\rho)$ iff $Mod_{beh}(A') \models\!\!\!\equiv \rho]$ simply because $(A'_{\varphi'})\restriction_\chi = (A'\restriction_\chi)_\varphi$. Furthermore, it is immediate that $(\varphi, \Sigma) \in |Sign|$, $\mathcal{E}_{(\varphi,\Sigma)}$ and $\mathcal{M}_{(\varphi,\Sigma)}$ inherit from $\mathcal{E}_\Sigma$ and $\mathcal{M}_\Sigma$ the quotient systems properties. Finally, the described mapping between $\mathcal{I}$ and $\mathcal{I}_{beh}$ is seen to be a morphism by the bare definition of $\models\!\!\!\equiv$.

Proposition 8. *If $(\varphi, \Sigma) \in |Sign_{beh}|$, $\rho \in Sen(\Sigma)$ and $E \subseteq Sen(\Sigma)$, then $E \models_\Sigma \rho$ implies $E \models\!\!\!\equiv_{(\varphi,\Sigma)} \rho$.*

Proof. Assume $E \models \rho$. Let A such that $A \models\!\!\!\equiv E$. We get, consecutively, $A_\varphi \models E$, $A_\varphi \models \rho$, $A \models\!\!\!\equiv \rho$.

Proposition 9. *Let $(\varphi, \Sigma) \in |Sign_{beh}|$, $\rho \in Sen_{\mathcal{I}}(\Psi)$ and $A \in |Mod_{beh}(\varphi, \Sigma)|$. Then $A \models\!\!\!\equiv_{(\varphi,\Sigma)} \varphi(\rho)$ iff $A \models_\Sigma \varphi(\rho)$.*

Proof. We know that $A\restriction_\varphi = (A_\varphi)\restriction_\varphi = D$. So, by the satisfaction condition, $A \models\!\!\!\equiv \varphi(\rho)$ iff $A_\varphi \models \varphi(\rho)$ iff $D \models \rho$ iff $A \models \varphi(\rho)$.

Proposition 10. *Let $(\varphi, \Sigma) \in |Sign_{beh}|$ and $\rho \in Sen_{beh}(\varphi, \Sigma)$. Then: (1) ρ is visible iff, for each $A \in |Mod_{beh}(\varphi, \Sigma)|$, $[A \models \rho$ iff $A \models\!\!\!\equiv \rho]$; (2) if ρ is quasi-visible then, for each $A \in |Mod_{beh}(\varphi, \Sigma)|$, $[A \models\!\!\!\equiv \rho$ implies $A \models \rho]$; (3) if ρ is closed under $\mathcal{M}^{op}_{(\varphi,\Sigma)}$ and under $\mathcal{E}_{(\varphi,\Sigma)}$, then it is behaviorally closed under $Mod_{beh}(\varphi, \Sigma)^{op}$.*

Proof. (1): The "only if" part is obvious. "if": Let $h : A \to B$ be a data-consistent quotient. Then $A_\varphi = B_\varphi$, hence $A \models \rho$ iff $A \models\!\!\!\equiv \rho$ iff $A_\varphi \models \rho$ iff $B \models\!\!\!\equiv \rho$ iff $B \models \rho$.
(2): This is obvious.
(3): Let $h : A \to B$ be a morphism in $Mod_{beh}(\varphi, \Sigma)$, and assume $B \models\!\!\!\equiv \rho$, i.e., $B_\varphi \models \rho$. We need to show $A \models\!\!\!\equiv \rho$. For this, factor h as $e; j$, where $e : A \to C$ is a quotient and $j : C \to B$ is an injection. Since h is data-consistent, it follows that e and j also data-consistent - this follows from the unique factorization of 1_D and the fact that reducts preserve quotients and injections. As a matter of fact, all morphisms that we shall discuss will be data consistent. Factor $j; e_{B,\varphi}$ as $e''; j''$, where $e'' : C \to F$ and $j'' : F \to B_\varphi$. By the finality of A_φ and the fact that $e; e''$ is a data-consistent quotient, there exists a data-consistent quotient $e''' : F \to A_\varphi$. We now apply the closure hypotheses for j'' and e''' to obtain $A_\varphi \models \rho$, i.e., $A \models\!\!\!\equiv \rho$.

Lemma 5. *Let (φ, Σ) be a signature in $\mathcal{I}_{beh}$ and ρ be a positive Σ-sentence in $\mathcal{I}$. Then, for each $A \in |Mod_{beh}(\varphi, \Sigma)|$, $A \models_\Sigma \rho$ implies $A \models\!\!\!\equiv_{(\Sigma,\varphi)} \rho$.*

Proof. The proof will go by induction on the structure of ρ, according to the definition of positive sentences.

In the case of basic sentences, the desired property is a simple consequence of the fact that satisfaction of basic sentences is preserved along any model morphism, in particular along $e_{A,\varphi} : A \to A_\varphi$. Also, the inductive step of arbitrary conjunctions and disjunctions is trivial.

Let now ρ be of the form $(\forall\phi)\rho'$, where $\phi : \Sigma \to \Sigma'$ is a signature morphism, and let $A \in |Mod_{beh}(\varphi, \Sigma)|$ such that $A \models \rho$. In order to show that $A \models\!\!\equiv \rho$, let $A' \in |Mod_{beh}(\varphi;\phi, \Sigma')|$ such that $Mod_{beh}(\phi)(A') = A$. Then $A'\restriction_\phi = A$, so $A' \models \rho'$, and, by the inductive hypothesis, $A' \models\!\!\equiv \rho'$. Thus $A \models\!\!\equiv (\forall\phi)\rho'$, that is $A \models\!\!\equiv \rho$.

Finally, let ρ be of the form $(\exists\phi)\rho'$, where $\phi : \Sigma \to \Sigma'$ is a signature morphism, and let $A \in |Mod_{beh}(\varphi, \Sigma)|$ such that $A \models \rho$. Then there exists $A' \in |Mod(\Sigma)|$ such that $Mod(\phi)(A') = A$ and $A' \models \rho'$. Because $A'\restriction_{(\varphi;\phi)} = A\restriction_\phi = D$, $A' \in |Mod_{beh}(\varphi;\phi, \Sigma')|$, and thus $Mod_{beh}(\phi)(A') = A$; furthermore, by the induction hypothesis, $A' \models\!\!\equiv \rho'$; so $A \models (\exists\phi)\rho'$, that is $A \models \rho$.

Proposition 11. *Visible and quasi-visible sentences are preserved by signature morphisms and closed under conjunctions, disjunctions, universal and existential quantifications. In addition, visible sentences are also closed under negation.*

Proof. This is immediate by the definition of behavioral satisfaction.

Corollary 2. *Let (φ, Σ) be a signature in $\mathcal{I}_{beh}$ and ρ be a positive Σ-sentence in $\mathcal{I}$. Then ρ is φ-visible iff it is φ-quasi-visible.*

Proof. This follows from Lemma 5.

Proposition 12. *(1) If (φ, Σ) is a signature in $\mathcal{I}_{beh}$ such that φ creates isomorphisms in $\mathcal{I}$, then $Mod_{beh}(\varphi, \Sigma)$ has filtered colimits; (2) If $\mathcal{I}$ has countable filtered colimits of signatures and is ω-exact, then $\mathcal{I}_{beh}$ also has countable filtered colimits of signatures.*

Proof. (1): Let $(h_{i,j} : A_i \to A_j)_{i,j\in I, i\leq j}$ be a countable filtered cocone in $Mod_{beh}(phi, \Sigma)$. Let $(h_i : A_i \to A)_{i\in I}$ be a colimit of this cocone in $Mod(\Sigma)$. We have that $(h_{i,j})\restriction_\varphi = 1_D$ for each $i \leq j$ and, since $Mod(\varphi)$ preserves filtered colimits, $((h_i)\restriction_\varphi : D \to A\restriction_\varphi)_{i\in I}$ is a colimit in $Mod(\chi)$ of the filtered cocone $(1_D : D \to D)_{i,j\in I, i\leq j}$; but $(1_D : D \to A\restriction_\varphi D)_{i\in I}$ is also a choice for colimit, hence there exists an isomorphism $u : A\restriction_\varphi \to D$ such that, for each i, $(h_i)\restriction_\varphi; u = 1_D$; moreover, by the creation property, there exists a Σ-model B and an isomorphism $v : A \to B$ such that $v\restriction_\varphi = u$. Thus, $((h_i; v) : A_i \to B)_{i\in I}$ is the desired colimit of the cocone in $Mod_{beh}(\varphi, \Sigma)$.

(2): Let $(I, \leq)$ be a countable filtered set, $\mathcal{F} = (\chi_{i,j} : (\varphi_i, \Sigma_i) \to (\varphi_j, \Sigma_j))_{i,j\in I, i\leq j}$ a countable filtered family of signatures in $Sign_{beh}$. We can canonically construct the colimit of $\mathcal{F}$ in $\Psi/Sign$, by reflecting the colimit $(\mu_i : \Sigma_i \to \Sigma)_{i\in I}$ of $(\chi_{i,j} : \Sigma_i \to \Sigma_j)_{i,j\in I, i\leq j}$ from $Sign$. Namely, we set $\varphi = \varphi_i; \mu_i : \Psi \to \Sigma$ for some i and notice that, because of filtration, the choice of i does not count; now we take the cocone $(\mu_i : (\varphi_i, \Sigma_i) \to (\varphi, \Sigma))_{i\in I}$ - this is the colimit of $\mathcal{F}$ in $\Psi/Sign$. In order for this to also be the colimit of $\mathcal{F}$ in $Sign_{beh}$, it would suffice that each μ_i be a morphism in $Sign_{beh}$ between (φ_i, Σ_i) and (φ, Σ). Only the weak creation condition needs to be verified.

Let k be a fixed element of I, A a model of Σ and $h : A\restriction_{\mu_k} \to B$ be a data-consistent quotient from $Mod(\Sigma_k)$. Take the filtered family of $\mathcal{I}$-signatures, $(\chi_{i,j} : \Sigma_i \to \Sigma_j)_{i,j\in I, k\leq i\leq j}$, and its colimit $(\mu_i : \Sigma_i \to \Sigma)_{i\in I, i\geq k}$. We know that any $\chi_{i,j}$ weakly creates data-consistent quotients and need to show that φ weakly creates h. We insert here a simple lemma:

Lemma: Let $(J, \leq)$ be a countable filtered set. Then there exists a subset H of J such that $(H, \leq)$ is isomorphic to the set of natural numbers ordered in the usual way and H is final in J, in the sense that, for each $j \in J$, there exists $h \in H$ such that $j \leq h$.

By applying this lemma for the set $J = \{i \in I \ / \ i \geq k\}$, we obtain a final well ordered subset of J, which, for convenience, we index by natural numbers: $(j_p)_{p\in N}$; we can consider that $j_0 = k$. Notice that $(\mu_{j_p} : \Sigma_{j_p} \to \Sigma)_{p\in N}$ is the colimit of $(\chi_{j_p,j_q} : \Sigma_{j_p} \to \Sigma_{j_q})_{p,q\in N, p\leq q}$. To simplify the denotation, we write p instead of j_p every time - thus we have a chain of signatures $(\chi_{p,q} : \Sigma_p \to \Sigma_q)_{p,q\in N, p\leq q}$ in $Sign$ and its colimit $(\mu_p : \Sigma_p \to \Sigma)_{p\in N}$ such that each each $\chi_{p,q}$ weakly creates data-consistent quotients; we want to show that μ_0 weakly creates the morphism $h : A\restriction_{\mu_0} \to B$. Now, we shall define the following:
- $(g_{p,q,q'} : C_{p,q} \to C_{p,q'})_{p,q,q'\in N, p\leq q<q'}$,
- $(h_p : A\restriction_{\mu_p} \to C_{p,p})_{p\in N}$
where h_p and $g_{p,q}$ are Σ_p-morphisms, such that:
(1) $(g_{p,q,q'})_{p,q,q'\in N, p\leq q<q'}$ is a chain cocone, for each $p \in N$;
(2) $h_p\restriction_{\chi_{p,p'}} = h_{p'}; g_{p',p',p}$, for each $p, p' \in N$, $p' < p$;
(3) $g_{p,q,q'}\restriction_{\chi_{p,p'}} = g_{p',q,q'}$, for each $p, p', q, q' \in N$, $p' < p \leq q < q'$;
(4) all the involved morphisms are data-consistent quotients.

We shall actually proceed triangularly and define $(h_{q'})_{q'\in N, q'\leq k}$ and $(g_{p,q,q'})_{p,q,q'\in N, p\leq q<q'\leq k}$, by recursion on $k \in N$:
- $h_0 = h$;
- Let $k \in N$ and assume $h_0, \ldots, h_k$ and $(g_{p,q,q'})_{p,q,q'\in N, p\leq q<q'\leq k}$ already defined. We define h_{k+1} by the weak creation of h_k along $\chi_{k,k+1}$ and using the model $A\restriction_{\mu_{k+1}}$, thus obtaining also a morphism $g_{k,k,k+1}$ such that $h_{k+1}\restriction_{\chi_{k,k+1}} = h_{k,k}; g_{k,k,k+1}$. For each $i < k$, we take $g_{i,k,k+1} = g_{k,k,k+1}\restriction_{\chi_{i,k}}$.

It is immediate that the above definition fulfills the properties (1)-(3). Now let, for each $p \in N$, $(f_{p,q} : C_{p,q} \to C_p)_{q\in N, q\geq p}$ be the colimit of $(g_{p,q,q'})_{q,q'\in N, q'>q\geq p}$ constructed as in Lemma 4, i.e. in the unique way such that the structural morphisms $f_{p,q}$ are data-consistent quotients. Because signature morphisms preserve chain colimits and quotients and using the uniqueness of colimits with data-consistent quotients as structural morphisms, it follows that $C_{p'} = C_p\restriction_{\chi_{p',p}}$ and $f_{p',q} = f_{p,q}\restriction_{\chi_{p',p}}$ for each $p, p', q \in N$, $p' \leq p \leq q$. We now apply the ω-exactness for $(h_p; f_{p,p}) : A\restriction_{\mu_p} \to C_p$ to obtain a morphism $f : A \to C$ in $Mod(\Sigma)$ such that

$f\restriction_{\mu_p} = h_p; f_{p,p}$ for each $p \in N$. In particular, f is an expansion of $h_0; f_{0,0}$, the last being a data-consistent quotient. But f might not be a quotient; however, we can factor it as $e; j$, with e quotient and j injection, and, by the preservation of quotient systems along signature morphisms, it turns out that the Σ_0-morphism $h_0; f_{0,0}$, which is already a quotient, can be further factored as $e\restriction_{\mu_0}; j\restriction_{\mu_0}$; by uniqueness of factorization, $j\restriction_{\mu_0}$ is an identity and $e\restriction_{\mu_0} = h_0; f_{0,0}$; thus e is the desired morphism to which $h = h_0$ is weakly created by μ_0.

Proposition 13. *If (φ, Σ) is a signature, A is a (φ, Σ)-model in $\mathcal{I}_{beh}$, and ρ is a basic Σ-sentence (in $\mathcal{I}$), then $A \models\!\!\!\equiv \rho$ iff $(\amalg_A)\restriction_\varphi = 1_D$.*

Proof. "only if": Assume that $A \models\!\!\!\equiv \rho$. Then $A_\varphi \models \rho$, thus there exists a morphism $h : T_\rho \to A_\varphi$. Let $g : A/_\rho \to A_\varphi$ be the morphism from the universal property of direct sums applied to h and $e_{A,\varphi} : A \to A_\varphi$. Since $\amalg_A; g = e_{A,\varphi}$, we have that $(\amalg_A)\restriction_\varphi; g\restriction_\varphi = 1_D$. Because signature reducts preserve quotients, $(\amalg_A)\restriction_\varphi$ is a quotient, hence, by Proposition .(1), an epi; and since it is also right-invertible, it is an isomorphism, and furthermore, according to Proposition .(4), it is equal to 1_D.
"if": Because $\amalg_A$ is a data-consistent quotient and by the finality of A_φ, there exists a morphism $g : A/_\rho \to A_\varphi$. Then, because of the morphism $\amalg_{T_\rho}; g$ (where $i_{T_\rho} : T_\rho \to A/_\rho$ is the structural morphism), $A_\varphi \models \rho$, i.e. $A \models\!\!\!\equiv \rho$.

Lemma 6. Let $\varphi : \Psi \to \Sigma$ be a representable signature morphism in $\mathcal{I}$. Then every sentence from $Sen_{beh}(\varphi, \Sigma)$ is visible.

Proof. This is immediate by the fact that the functor $Mod(\varphi)$, being isomorphic to the forgetful functor $T_\varphi/Mod(\Sigma) \to Mod(\Sigma)$, is faithful, thus any data-consistent Σ is an identity.

Here are some notational conventions about diagrams that we hope will make the reader's life easier. Let $\varphi : \Sigma \to \Sigma'$ be a signature morphism, $A' \in |Mod(\Sigma')|$, and $h : A \to B$ in $Mod(\Sigma)$. We write $\iota_\Sigma(h)$ instead of $\iota_{1_\Sigma}(h)$ and $\iota_\varphi(A'\restriction_\varphi)$ instead of $\iota_\varphi(1_{(A\restriction_\varphi)})$. Let A be a fixed object in $Mod(\Sigma)$ and let $B, C \in |Mod(\Sigma)|$ and $f : A \to B$, $g : A \to C$, $u : B \to C$ morphisms in $Mod(\Sigma)$ such that $f; u = g$. Then (f, B) and (g, C) are objects in $A/Mod(\Sigma)$ and u is also a morphism in $A/Mod(\Sigma)$ between (f, B) and (g, C). We establish the following notations: $B_f = i^{-1}_{\Sigma,A}(f, B)$ (and, similarly, $C_g = i^{-1}_{\Sigma,A}(g, C)$), $u_{f,g} = i^{-1}_{\Sigma,A}((f, B) \xrightarrow{u} (g, C))$. Thus, for instance, $f : A \to B$ be a Σ-model morphism. Then $f_{1_A,f}$ is the image through $i^{-1}_{\Sigma,A}$ of the morphism $f : (1_A, A) \to (f, B)$ in $A/Mod(\Sigma)$, and has source $A_{(1_A)}$ and target B_f. We shall usually write A_A instead of $A_{(1_A)}$ and $f_{A,f}$ instead of $f_{1_A,f}$.

Proposition 14. *Let $(\varphi, \Sigma) \in |Sen_{beh}|$, let ρ be a universal Σ-sentence, and let $A \in |Mod_{beh}(\varphi, \Sigma)|$. Then $A \models\!\!\!\equiv_{(\varphi,\Sigma)} \rho$ iff $A \models_\Sigma \mathcal{QV}_\rho$.*

Proof. "only if": This is the easy part. Suppose $A \models\!\!\!\equiv \rho$. Then, since $\rho \models \mathcal{QV}_\rho$, by Proposition 8, $\rho \models\!\!\!\equiv \mathcal{QV}_\rho$, hence $A \models\!\!\!\equiv \mathcal{QV}_\rho$. But each sentence from $\mathcal{QV}_\rho$, being of the form $(\forall\phi)\alpha$ with α quasi-visible, is quasi-visible by Proposition 11. Thus $A \models \mathcal{QV}_\rho$ by Proposition 10.

"if": Assume that $A \models \mathcal{QV}_\rho$.

I. Suppose first that ρ is basic. Assume, by absurd, that $A \not\models \rho$. Let
- B denote the model $A/_\rho$,
- g denote the structural morphism $\amalg_A : A \to B$,
- $g' : D \to F$ denote the φ-reduct of $g : A \to B$.

According to Proposition 13, g' is not an identity, so, since g' is a quotient, there exists a basic sentence $\alpha \in \mathit{Sen}(\Sigma_D)$ such that $D_D \not\models \alpha$ and $F_{g'} \models \alpha$. Then, since $(A_A)\restriction_{\iota_\varphi(D)} = D_D$ and $(B_B)\restriction_{\iota_\varphi(g')} = F_{g'}$, it follows that $A_A \not\models \iota_\varphi(D)(\alpha)$ and $B_B \models \iota_\varphi(g')(\alpha)$. Furthermore, using the fact that, by liberality and Lemma 3, $\iota_\varphi(D)(\alpha)$ and $\iota_\varphi(g')(\alpha)$ are also basic, we get $E_A \not\models \iota_\varphi(D)(\alpha)$ and $E_B \models \iota_\varphi(g')(\alpha)$, that is (since $\iota_\varphi(g') = \iota_\varphi(D);\iota_\Sigma(g)$), $(\iota_\varphi(D);\iota_\Sigma(g'))(\alpha) \in (E_B)^\bullet$, that is $\iota_\varphi(D)(\alpha) \in (\iota_\Sigma(g'))^{-1}((E_B)^\bullet)$, that is, by basic-sensitivity, $E_A \cup \{\iota_\Sigma(A)(\rho)\} \models \iota_\varphi(D)(\alpha)$.

- By compactness, there exists $E \subseteq E_A \cup \{\iota_\Sigma(A)(\rho)\}$ such that $E \models \iota_\varphi(D)(\alpha)$; then, because $E_A \not\models \iota_\varphi(D)(\alpha)$, necessarily $\iota_\Sigma(A)(\rho) \in E$; let β denote the sentence $\bigwedge(E \setminus \{\iota_\Sigma(A)(\rho)\}) \Rightarrow \iota_\varphi(D)(\alpha)$.

- Alternatively, by the existence of basic Horn implications, $(\bigwedge E_A) \Rightarrow \iota_\varphi(A')(\alpha)$ is in $\mathit{Sen}(\Sigma_A)$; let β denote this sentence.

In either case, $E_A \not\models \beta$, that is $A_A \not\models \beta$, and $\iota_\Sigma(A)(\rho) \models \beta$, that is, by Lemma 2, $\rho \models (\forall\iota_\Sigma(A))\beta$. Moreover, one can easily see that β is quasi-visible. (Notice that α is visible by Lemma 6, since $\iota_\Sigma(D)$ is representable) Because A_A is a $\iota_\Sigma(A)$-expansion of A, $A \not\models (\forall\iota_\Sigma(A))\beta$. Thus the quasi-visible sentence β and the representable signature morphism $\iota_\Sigma(A)$ contradict the hypothesis.

II. Assume now that ρ is universal, i.e., of the form $(\forall\chi)\rho_0$, where $\chi : \Sigma \to \Sigma_0$ is a signature morphism and $\rho_0 \in \mathit{Sen}(\Sigma_0)$. Let A_0 be a χ-expansion of A. We need to show $A_0 \models \rho_0$. For this, we apply case I: let $\varphi_0 : \Sigma_0 \to \Sigma_1$ a signature morphism, and $\rho_1 \in \mathit{Sen}(\Sigma_1)$ a quasi-visible sentence such that $\rho_0 \models (\forall\varphi_0)\rho_1$; it would suffice that $A_0 \models (\forall\varphi_0)\rho_1$. By Lemma 1, $(\forall\chi)\rho_0 \models (\forall\chi)(\forall\varphi_0)\rho_1$; thus, according to the hypothesis, $A \models (\forall\chi;\varphi_0)\rho_1$, that is $A \models (\forall\chi)(\forall\varphi_0)\rho_1$; thus $A_0 \models (\forall\varphi_0)\rho_1$.

B Semantic versus syntactic hidden signature morphisms

We shall prove that in the case of hidden algebra, our institutional (semantic) definition of signature morphisms is only slightly more permissive than the usual syntactic definition [16].

Let V and H be two disjoint sets and (Ψ, D) a data context as in Section 3. For a hidden (Ψ, D)-signature $(V \cup H, F)$, call a sort $s \in V \cup H$ *accessible* if it exists at least one experiment for the sort s. Notice that any visible sort is trivially accessible.

Let $(V \cup H, F))$, $(V \cup H', F')$ be two fixed data hidden signatures having the data signature (V, Fv) and the data algebra D. (Notice that $(\iota : (V, Fv) \to (V \cup H, F), (V \cup H, F))$ and $(\iota' : (V, Fv) \to (V \cup H', F'), (V \cup H', F'))$, with ι, ι' being the inclusions of signatures, are, according to our above definition, signatures in the behavioral extension of EQL, $EQL_{beh}((V, Fv), D)$.) For an EQL signature morphism $\chi : (V \cup H, F) \to (V \cup H', F')$, consider the following conditions:
(C1) $\varphi; \chi = \varphi'$;
(C2) $\chi^{sort}(H) \subseteq H'$;
(these two conditions are the same as the ones in Section 3)
(C3') For each operation $\sigma' \in F'$ which has an argument sort in $\chi^{sort}(S)$ and has the result sort accessible, $\sigma' \in \chi^{op}(F)$.
((C3') is weaker than (C3) from Section 3)

We shall call χ a *relaxed hidden morphism* if it satisfies conditions (C1), (C2), (C3'). Obviously, every hidden morphism from Section 3 is also a relaxed hidden morphism. The latter concept is more permissive in the sense of allowing new methods on old states (hidden sorts), as long as the result is totally hidden to any experiment.

Proposition 15. *With the above notations, and assuming that D has at least two elements on each carrier,*[3] *the following are equivalent:*
(1) χ is a signature morphism in $EQL_{beh}((V, Fv), D)$ between $(\iota, (V \cup H, F))$ and $(\iota', (V \cup H', F'))$;
(2) $\iota; \chi = \iota'$ and, for each Σ'-model A', $h \in H$ and $a, b \in A'_{\chi^{sort}(h)}$, a and b are behaviorally equivalent in A' iff they are so in $A' \restriction_\chi$;
(3) χ is a relaxed hidden morphism between $(V \cup H, F)$ and $(V \cup H', F')$.

Proof. For "(1) iff (2)", we simply apply Proposition 3 together with the obvious fact that, for each model M, the quotients of source M are in bijection to the congruences on M.
(3) implies (2): Let $A' \in |Mod(\Sigma)|$, $h_0 \in H$, and denote $\Sigma = (V \cup H, F)$, $\Sigma' = (V \cup H', F')$, $A = A' \restriction_\chi$. Because of condition (C2), the signature morphism χ can be naturally extended to map any (V, H, F)-experiment to a (V, H', F') experiment: $\chi^{exp}(t) = t'$, where t' is obtained from t by replacing each $\sigma \in F$ with $\chi^{op}(\sigma)$, each $x : s$ with $x : \chi^{sort}(s)$, and $\bullet : h$ with $\bullet : \chi^{sort}(h)$. In particular, any (V, H, F)-experiment of sort h_0 is mapped to a (V, H', F')-experiment of sort $\chi^{sort}(h_0)$. Moreover, this mapping is surjective, because any (V, H', F')-experiment t on $\chi(h_0)$ should use only operations σ' with accessible argument sort, which are, according to (C3'), of the form $\chi^{op}(\sigma)$; these latter operations σ can be used to build inductively a (V, H, F)-experiment t such that $\chi^{exp}(t) = t'$. Now, since the (V, H, F)-experiments perfectly parallel (V, H, F)-experiments, two elements from $A'_{\chi^{sort}(h_0)}$ are behaviorally equivalent (i.e. equal under all experiments) in A' iff they are so in $A' \restriction_\chi$.

[3] This assumption is about having non-trivial experiments, i.e. with at least two possible results: *yes* and *no*.

(2) implies (3): Note that, because ι' is an inclusion of signatures and D is non-empty on each carrier, D has at least one ι'-expansion. We assume, by absurd, that (3) does not hold. We have two cases:
I. There exists $h \in H$ such that $\chi^{sort}(h) = v \in V$. Let B' be a ι'-expansion of D and chose, for each $s \in H' \cup V$, an element $b_s \in B'_s$ (notice that, because D is non-empty on each carrier, B' can be chosen like this too). Define the algebra A' to have the same carriers as B' and the same (V, Fv)-operations, but with all the other operations to map everything to the designated elements b_s. Now let $c, d \in A'_v = D_v$ be two different elements. Then c and d are behaviorally equivalent in $A'\lceil_\chi$ as elements of sort h (because any (V, H, P)-experiment on h will certainly equalize c and d), but c and d are not behaviorally equivalent in A' as elements of (the visible) sort v.
II. (We can assume that the previous case does not hold.) There exists an operation $\sigma'_0 \in F'$ which has an argument sort in $\chi^{op}(S)$, the result sort accessible, but it is not the case that $\sigma'_0 \in \chi^{op}(F)$. To simplify the denotation, assume $\sigma'_0 \in F'_{w'_0\chi^{sort}(h_0),s'_0}$. Let B' be a ι'-expansion of D such that it has at least two elements a_v and b_v on each visible carrier B'_v and at least three elements $a_{h'}, b_{h'}, c_{h'}$ on each hidden carrier $B'_{h'}$ (this is possible, because D has at least two elements on each carrier, and because, by the fact that Case I does not hold, there is no obligation for a hidden carrier to be equal to a visible carrier). Let A' be an algebra with the same carriers and (V, Fv)-operations as B', but with the other operations as follows:
- for each $\sigma' \in F'_{w',h'}$, with $h' \in H'$ and $\sigma' \neq \sigma'_0$, $A'_{\sigma'} : A'_{w'} \to A'_{h'}$ is defined by: for $(d_1, \ldots, d_k) \in A'_{w'}$, if $w' = s'_1 \ldots s'_k$ and there exists an i such that $s'_i = h'_0$ and $d_i = c_{h'_0}$, then $A'_{\sigma'}(d_1, \ldots, d_k) = c_{h'}$; otherwise, $A'_{\sigma'}(d_1, \ldots, d_k) = a_{h'}$;
- for each $\sigma' \in F'_{w',v}$, with $v \in V$ and $\sigma' \neq \sigma'_0$, $A'_{\sigma'} : A'_{w'} \to A'_v$ is defined by: for $(d_1, \ldots, d_k) \in A'_{w'}$, if $w' = s'_1 \ldots s'_k$ and there exist an i such that $s'_i = s'_0$ and $d_i = c_{s'_0}$, then $A'\sigma'(d_1, \ldots, d_k) = b_v$; otherwise, $A'\sigma'(d_1, \ldots, d_k) = a_v$;
- if $s'_0 \in H'$, $A'_{\sigma'_0}$ is defined by: $A'_{\sigma'_0}(\ldots, a_{\chi^{sort}(h_0)}) = a_{s'_0}$ and $A'_{\sigma'_0}(\ldots) = c_{s'_0}$ otherwise;
- if $s'_0 \in V$, $A'_{\sigma'_0}$ is defined by: $A'_{\sigma'_0}(\ldots, a_{\chi^{sort}(h_0)}) = a_{s'_0}$ and $A'_{\sigma'_0}(\ldots) = b_{s'_0}$ otherwise.
Since σ'_0 is not in the image of χ^{op} and σ'_0 would have been the only operation that distinguished between them, $a_{\chi^{sort}(h_0)}$ and $b_{\chi^{sort}(h_0)}$ are behaviorally equivalent in A'_χ as elements of the sort $\chi^{sort}(h_0)$. Furthermore:
- if $s'_0 \in H'$, $a_{s'_0}$ and $c_{s'_0}$ are not behaviorally equivalent in A' on the sort s'_0 - this is because s'_0 is accessible, and thus there exists an experiment on the sort s'_0, say of visible sort v; we let the elements on all the undefined arguments of the experiment be elements of the form a_s; then the result of the experiment on $a_{s'_0}$ will be a_v, while that on $c_{s'_0}$ will be $c_{s'_0}$;
- if $s'_0 \in V$, $a_{s'_0}$ and $b_{s'_0}$ are not behaviorally equivalent in A' on the sort s'_0 simply because they are different.
Either way, because of $A_{\sigma'_0}$, which takes $a_{\chi^{sort}(h_0)}$ and $b_{\chi^{sort}(h_0)}$ to some non-equivalent elements, it follows that $a_{\chi^{sort}(h_0)}$ and $b_{\chi^{sort}(h_0)}$ are not behaviorally equivalent in A' on the sort $\chi^{sort}(h_0)$.

C Examples of institutions

We provide here some examples of institutions which fit our frameworks.

FOPL$_=$ [17] - the institution of (many-sorted) first order predicate logic with equality. The signatures are triples (S, F, P), where S is a set of *sorts*, $F = \bigcup\{F_{w,s}|w \in S^*, s \in S\}$ is the set of (S-sorted) *operation symbols*, and $P = \bigcup\{P_w|w \in S^*\}$ is the set of (S-sorted) *relation symbols*. A signature morphism $\varphi = (\varphi^{sort}, \varphi^{op}, \varphi^{rel}) : (S, F, P) \to (S', F', P')$ consists of a function between the sets of sorts $\varphi^{sort} : S \to S'$, a function between the sets of operation symbols $\varphi^{op} : F \to F'$, and a function between the sets of relation symbols $\varphi^{rel} : P \to P'$ such that $\varphi^{op}(F_{w,s}) \subseteq F'_{\varphi^{sort}(w),\varphi^{sort}(s)}$ and $\varphi^{rel}(P_w) \subseteq P'_{\varphi^{sort}(w)}$ for each word of sorts $w \in S^*$ and sort $s \in S$. (When there is no danger of confusion, we let φ denote each of φ^{sort}, φ^{rel} and φ^{op}.) Given a signature $\Sigma = (S, F, P)$, a Σ-model M is a triple $M = (\{M_s\}_{s\in S}, \{M_{w,s}(\sigma)\}_{(w,s)\in S^*\times S}, \{M_w(\sigma)\}_{w\in S^*})$ interpreting each sort s as a set M_s, each operation symbol $\sigma \in F_{w,s}$ as a function $M_{w,s}(\sigma) : M_w \to M_s$ (where M_w stands for $M_{s_1} \times ... \times M_{s_n}$ when $w = s_1...s_n$), and each relation symbol $\pi \in P_w$ as a relation $M_\pi \subseteq M_w$. (If there is no danger of confusion we may let M_σ and M_π denote $M_{w,s}(\sigma)$ and $M_w(\pi)$.)

The morphisms are the usual Σ-model homomorphisms, i.e. S-sorted functions which preserve operations and relations. The set of Σ-sentences is the least set of sentences obtained from atoms (equation atoms $t_1 = t_2$, where $t_1, t_2 \in (T_F(X))_s$,[4] or relational atoms $\pi(t_1, ..., t_n)$, where $\pi \in P_{s_1...s_n}$ and $t_i \in (T_F(X))_{s_i}$ for each $i \in \{1, ..., n\}$), by applying for a finite number of times:

- negation, conjunction, disjunction;
- universal or existential quantification over finite sets of variables.

Satisfaction is the usual first-order satisfaction. The other items of FOPL$_=$ (like the definitions of *Sen* and *Mod* on morphisms) are the natural ones: *Sen* translates sentences symbol-wise, while, for some signature morphism φ, *Mod*(φ) is the forgetful functor (forgetting, but also duplicating information in the non-injective parts of φ).

EQL - the institution of equational logic [17], a restriction of FOPL$_=$, with no relation symbols (such that the signatures are pairs (S, F), with S the set of sorts and F the set of S-ranked operation symbols), and with only conditional equations ($(\forall X)t_1 = t'_1 \wedge e_2 \wedge \ldots t_n = t'_n \Rightarrow t = t'$) as sentences.

PFOPL [13] - the institution of partial first-order predicate logic, an expansion of FOPL$_=$ which allows at signatures, besides relation and (total) operation symbols, also partial operation symbols. Signature morphisms are allowed to map partial to total operation symbols, but not *vice versa*. The model-morphism act w.r.t. partial operations as follows: if an operation from the source model is defined on some elements, then it is also defined in the target model on the

[4] $T_F(X)$ is the term algebra over F with variables from X; T_F denotes $T_F(\emptyset)$.

images of these elements, and the usual commutation of the mapping between models with the operation holds.

There exist three kinds of atoms: undefinedness $t \uparrow$, strong equality $t \stackrel{s}{=} t'$ and the existence equality $t \stackrel{e}{=} t'$. The undefinedness $t \uparrow$ of a term t holds in a model M when the interpretation M_t is undefined. The strong equality $t \stackrel{s}{=} t'$ holds when both terms are undefined or both terms are undefined and are equal. The existence equality $t \stackrel{e}{=} t'$ holds when both terms are defined and are equal.

The sentences are obtained from atoms just like in the case of $\mathrm{FOPL}_=$.

PA [13] - the institution of partial algebra, a restriction of $\mathrm{FOPL}_=$ with the signatures not having relation symbols.

IFOPL - the institution of infinitary first order predicate logic, an infinitary expansion of $\mathrm{FOPL}_=$, by allowing infinite conjunctions (but still only finite quantifications). This logic, in its unsorted form, is known under the name $L_{\infty,\omega}$ [24].

HORN - the institution of Horn clauses, a restriction of $\mathrm{FOPL}_=$, which has only sentences of the form $(\forall X)e_1 \wedge e_2 \wedge \ldots e_n \Rightarrow e$, where e and all the e_i's are either relational atoms $R(t_1, \ldots, t_n)$, or identities $t_1 = t_2$.

IHORN - an infinitary expansion of HORN, by accepting possibly infinite sets of premises.

IEQL - the corresponding equational restriction of IHORN (just like EQL is the equational restriction of HORN).

RWL [27, 13] - the institution of rewriting logic. It has the same signatures as EQL, but models have in addition a preorder on each sort, compatible with the operations, while model morphisms have to be also increasing with respect to the preordes. The sentences have the form $(\forall X)t_1 p_1 t'_1 \& \ldots t_n p_n t'_n)tpt'$, where t, t_i are terms from $T_\Sigma(X)$ and $p_i \in \{=, \rightarrow\}$, where $\rightarrow$ is semantically interpreted as the preorder relation.

OSL [20] - the institution of order-sorted (equational) logic, an extension of EQL which admits orders on the sets of sorts, such that $s \leq s'$ is interpreted as set theoretic inclusion between the corresponding model, while the operations, when overloaded, have to satisfy natural restriction-corestriction conditions.

ML [28] - the institution of membership equational logic - an extension of EQL which calls the sorts "kinds", and allows on each kind a set of sorts, that are to be interpreted, on models, as subsets of corresponding kind support. Besides equations, this logic also admits membership assertions: $(\forall X)t : s$, where $t \in (T_\Sigma(X))_k$ and s is a sort of kind k, meaning that "t is of sort s".

ExpBas - the institution of expanded basic sentences, which has the same signatures and models as $\mathrm{FOPL}_=$; its sentences are constructed from the basic sentences (each model gives a basic sentence), using negations, arbitrary conjunctions and disjunctions, and quantifications (existential and universal) over arbitrary signature morphisms.

TFOPL - the institution of total first-order predicate logic, an extension of $\mathrm{FOPL}_=$ which also accepts infinitary relation and operation symbols, having as arities families of sorts indexed by ordinals; these are to be interpreted as infinitary relations and operations. More precisely, a signature is a quadruple (S, α, F, R), where α is an ordinal, F is a family of sets $F_{w,s}$ indexed by all functions $w : \beta \to S$ with $\beta < \alpha$ and all sorts $s \in S$, and R is a family of sets R_w, where w is same as for F. Besides the $\mathrm{FOPL}_=$-like constructions, arbitrary conjunctions and quantifications over arbitrary large sets of variables are accepted. This institution corresponds to the categorical logic considered in [26].

The two families of quotient systems for $\mathrm{FOPL}_=$ obviously work for IFOPL, HORN, IHORN, ExpBas, while the one for EQL works for $IEQL$.
- For a PFOPL signature (S, F, F_p, P) (where F_p is the set of partial operation symbols), one can take the quotients to be the surjective morphisms that are also strong (S, F, P)-morphisms in $\mathrm{FOPL}_=$, and the injections to be the injective morphisms. The quotients and injections for a signature (S, F, F_p) in PA, those morphisms that are also injections and surjections for (S, F) in EQL.
- For an OSL signature $(S, \leq, F)$, the quotients and injections can be taken to be those morphisms that are quotients and injections for (S, F) in EQL.
- For ML, we have two choices of quotient systems, according to the fact that ML has an embedding into HORN which is a bijection on the model part [28].
- For TFOPL, an immediate generalization of the notion of strong morphism from $\mathrm{FOPL}_=$ provides two versions of quotient systems.

All the discussed institutions have elementary diagrams.[5]

Let us recall the canonical elementary diagrams of $\mathrm{FOPL}_=$, presented in their institutional form in [11].[6] Let $\Sigma = (S, F, P)$ be a $\mathrm{FOPL}_=$ signature and $M \in |Mod(\Sigma)|$. Then $\Sigma_M = (S, F_M, P)$, where F_M extends F by adding, for each $s \in S$, all elements from M_s as constants of sort s.

- Define $M_M \in |Mod(\Sigma_M)|$ as the expansion of M which interprets each constant $m \in M$ by itself.

[5] Diagrams for $\mathrm{FOPL}_=$, RWL, PA were presented in [11].

[6] Actually, the classical term for the elementary diagram in $\mathrm{FOPL}_=$ would be *positive diagram* [7] - we prefer the term *elementary diagram* from [11], because it is really connected to *all* the $\mathrm{FOPL}_=$ model morphisms, and not to only the "elementary ones".

– $E_M = \{t = t' \mid t, t' \in (T_{F_M})_s,\ M_M \models_{\Sigma_M} t = t'\} \cup$
$\{\pi(t_1, ..., t_n) \mid\ t_i \in (T_{F_M})_{s_i},\ i = \overline{1,n},\ \pi \in P_{s_1...s_n},\ M_M \models_{\Sigma_M} \pi(t_1, ..., t_n)\}$, where T_{F_M} denotes the algebra of ground F_M-terms;
– $\iota_\Sigma(M) : \Sigma \to \Sigma_M$ is the signature inclusion;
– The functor $i_{\Sigma,M} : Mod(\Sigma_M, E_M) \to M/Mod(\Sigma)$ is defined:
- on objects, by $i_{\Sigma,M}(N') = (M \xrightarrow{h} N, N)$, where $N = N' \restriction_{\iota_\Sigma(M)}$ and, for each $m \in M_s$, $h_s(m) = N'_m$.
- on morphisms, by $i_{\Sigma,M}(f) = f$.

The elementary diagrams of EQL are inherited from FOPL$_=$ - this is because the concept of elementary diagram uses, on signatures with no relation symbols, only the model-theoretic part and equational sentences.

Now consider the following general rule easily seen to hold:
- when an institution $\mathcal{I}$ extends another one, $\mathcal{I}'$, by taking the same signatures and functor *Mod*, but increasing the expression power with adding new sentences, elementary diagrams are inherited;
- when an institution $\mathcal{I}$ restricts another one $\mathcal{I}'$ by taking the same signatures and functor *Mod*, but removing some sentences, though not any sentence from any diagram, and is a conservative restriction of $\mathcal{I}'$ (i.e. $M \models_{\mathcal{I}} e \Leftrightarrow M \models_{\mathcal{I}'} e$ when e and M are from $\mathcal{I}$), then elementary diagrams are again inherited.

Thus the diagrams of FOPL$_=$ are also good for IFOPL, ExpBas, PFOPL, PA, HORN, IHORN, IEQL.

The elementary diagrams for the other mentioned institutions (TFOPL, RWL, ML, OSL) are constructed with a similar pattern as those of FOPL$_=$ - as remarked in [11], the sentences E_M are always the basic sentences satisfied by the model M extended in Σ_M with constants from M pointing to themselves.

Virtually all institutions constructed from particular kinds of logics have elementary diagrams - see Section C. Yet, the construction that we give in this paper for the behavioral extension of an institution does not use diagrams; the latter are used only for proving smoother results regarding the relationship between normal and behavioral satisfaction.

The fact that all our examples fit in the frameworks of this paper can be checked following similar ideas as in the cases of FOPL$_=$ and EQL.

D Preservation of quotient pushouts in EQL

Proposition 16. *Let $\chi : (S', F') \to (S, F)$ be a signature morphism, A a (S, F)-algebra, and P, Q two congruences on A, and let R be the smallest congruence that includes both P and Q. Then $(A/R)\restriction_\chi = A\restriction_\chi /R'$, where R' be the smallest congruence that includes both $P\restriction_\chi$ and $Q\restriction_\chi$.*

Proof. One can easily see that R (R') is the transitive closure $(P \cup Q)^+$ $((P\restriction_\chi \cup Q_\chi)^+)$ of the relation $P \cup Q$ ($P\restriction_\chi \cup Q\restriction_\chi$ respectively). Moreover, it is immediate that $((P \cup Q)^+)\restriction_\chi = (P\restriction_\chi \cup Q\restriction_\chi)^+$, which implies the desired conclusion.