# An Exact $2.9416^{n}$ Algorithm for the Three Domatic Number Problem* 

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#### Abstract

The three domatic number problem asks whether a given undirected graph can be partitioned into at least three dominating sets, i.e., sets whose closed neighborhood equals the vertex set of the graph. Since this problem is NP-complete, no polynomial-time algorithm is known for it. The naive deterministic algorithm for this problem runs in time $3^{n}$, up to polynomial factors. In this paper, we design an exact deterministic algorithm for this problem running in time $2.9416^{n}$. Thus, our algorithm can handle problem instances of larger size than the naive algorithm in the same amount of time. We also present another deterministic and a randomized algorithm for this problem that both have an even better performance for graphs with small maximum degree.


Key words: Exact algorithms, domatic number problem

## 1 Introduction

In this paper, we design a deterministic algorithm for the three domatic number problem, which is one of the standard NP-complete problems, see Garey and Johnson [GJ79]. This problem asks, given an undirected graph $G$, whether or not the vertex set of $G$ can be partitioned into three dominating sets. A dominating set is a subset of the vertex set that "dominates" the graph in that its closed neighborhood covers the entire graph. Motivated by the tasks of distributing resources in a computer network and of locating facilities in a communication network, this problem and the related problem of finding a minimum dominating set in a given graph have been thoroughly studied, see, e.g., [CH77 Far84 Bon85 KS94]HT98[FHK00 RR04].

[^0]The exact (i.e., deterministic) algorithm designed in this paper runs in exponential time. However, its running time is better than that of the naive exact algorithm for this problem. That is, we improve the trivial $\tilde{\mathcal{O}}\left(3^{n}\right)$ time bound to a time bound of $\tilde{\mathcal{O}}\left(2.9416^{n}\right)$, where the $\tilde{\mathcal{O}}$ notation neglects polynomial factors as is common for exponential-time algorithms. The point of such an improvement is that a $\tilde{\mathcal{O}}\left(c^{n}\right)$ algorithm, where $c<3$ is a constant, can deal with larger instances than the trivial $\tilde{\mathcal{O}}\left(3^{n}\right)$ algorithm in the same amount of time before the exponential growth rate eventually hits and the running time becomes infeasible. For example, if $c=\sqrt{3} \approx$ 1.732 then we have $\tilde{\mathcal{O}}\left(\sqrt{3}^{2 n}\right)=\tilde{\mathcal{O}}\left(3^{n}\right)$, so one can deal with inputs twice as large as before. Doubling the size of inputs that can be handled by some algorithm can make quite a difference in practice.

Exact exponential-time algorithms with improved running times have been designed for various other important NP-complete problems. For example, Dantsin et al. [DGH ${ }^{+} 02$ pushed the trivial $\tilde{\mathcal{O}}\left(2^{n}\right)$ bound for the three satisfiability problem down to $\tilde{\mathcal{O}}\left(1.481^{n}\right)$, which was further improved to $\tilde{\mathcal{O}}\left(1.473^{n}\right)$ by Brueggemann and Kern [BK04]. Schöning [Sch02], Hofmeister et al. HSSW02] and Paturi et al. PPSZ98] proposed even better randomized algorithms for the satisfiability problem. Combining their ideas, the currently best randomized algorithm for this problem is due to Iwama and Tamaki [【T03], who achieve a time bound of $\tilde{\mathcal{O}}\left(1.324^{n}\right)$.

The currently best exact time bound of $\tilde{\mathcal{O}}\left(1.211^{n}\right)$ for the independent set problem is due to Robson Rob86. Eppstein [Epp01a Epp01b achieved a $\tilde{\mathcal{O}}\left(2.415^{n}\right)$ time bound for graph coloring and a $\tilde{\mathcal{O}}\left(1.3289^{n}\right)$ for the special case of graph three colorability. Fomin, Kratsch, and Woeginger [FKW04] improved the trivial $\tilde{\mathcal{O}}\left(2^{n}\right)$ bound for the dominating set problem to $\tilde{\mathcal{O}}\left(1.93782^{n}\right)$. Comprehensive surveys on this subject have been written by Woeginger [Woe03] and Schöning [Sch05].

In designing domatic number algorithms, it might be tempting to exploit known results (such as Eppstein's $\tilde{\mathcal{O}}\left(1.3289^{n}\right)$ bound) for the graph three colorability problem, which resembles the three domatic number problem in that both are partitioning problems. However, as Cockayne and Hedetniemi [CH77] point out, the theory of domination is dual to the theory of coloring in the following sense. Coloring is based on the hereditary property of independence. A graph property is hereditary if whenever some set of vertices has the property then so does every subset of it. In contrast, domination is an expanding property in that every superset of a dominating set also is a dominating set of the graph. Further, graph colorability is a minimum problem, whereas the domatic number problem is a maximum problem. Independence (and thus colorability) can be seen as a local property, since it suffices to check the immediate neighborhood of a set of vertices to determine whether or not it is independent. In contrast, dominance is a global property, since in order to check it one has to consider the relation between the given set of vertices and the entire graph. In this sense, determining the domatic number of a graph intuitively appears to be harder than computing its chromatic number, notwithstanding that both problems are NP-complete. More to the point, the algorithms developed for graph coloring seem to be of no help in designing algorithms for dominating set or domatic number problems.

After introducing some definitions and notation in Section 2 we describe and analyze our algorithm in Section 3, the actual pseudo-code is shifted to the appendix.

In Section 4 we give another deterministic and a randomized algorithm, which have an even better running time for graphs with small maximum degree. Finally, we summarize and discuss our results in Section 5

## 2 Preliminaries and Simple Observations

We start by introducing some graph-theoretical notation. We only consider simple, undirected graphs without loops in this paper. Let $G=(V, E)$ be a graph. Unless stated otherwise, $n$ denotes the number of vertices in $G$. The neighborhood of a vertex $v$ in $V$ is defined by $N(v)=\{u \in V \mid\{u, v\} \in E\}$, and the closed neighborhood of $v$ is defined by $N[v]=N(v) \cup\{v\}$. For any subset $S \subseteq V$ of the vertices of $G$, define $N[S]=\bigcup_{v \in S} N[v]$ and $N(S)=N[S]-S$. The degree of a vertex $v$ in $G$ is the number of vertices adjacent to $v$, i.e., $\operatorname{deg}_{G}(v)=\|N(v)\|$. If the graph $G$ is clear from the context, we omit the subscript $G$. Define the minimum degree in $G$ by $\min -\operatorname{deg}(G)=\min _{v \in V} \operatorname{deg}(v)$, and the maximum degree in $G$ by $\max -\operatorname{deg}(G)=$ $\max _{v \in V} \operatorname{deg}(v)$. A path $P_{k}=u_{1} u_{2} \cdots u_{k}$ of length $k$ is a sequence of $k$ vertices, where each vertex is adjacent to its successor, i.e., $\left\{u_{i}, u_{i+1}\right\} \in E$ for $1 \leq i \leq k-1$. If, in addition, $\left\{u_{k}, u_{1}\right\} \in E$, then path $P_{k}$ is said to be a cycle, and we write $C_{k}$ instead of $P_{k}$.

Definition 1 Let $G=(V, E)$ be a graph. A subset $D \subseteq V$ is a dominating set of $G$ if and only if $N[D]=V$, i.e., if and only if every vertex in $G$ either belongs to $D$ or has some neighbor in $D$. The domination number of $G$, denoted $\gamma(G)$, is the minimum size of a dominating set of $G$. The domatic number of $G$, denoted $\delta(G)$, is the maximum number of disjoint dominating sets of $G$, i.e., $\delta(G)$ is the maximum $k$ such that $V=$ $V_{1} \cup V_{2} \cup \ldots \cup V_{k}$, where $V_{i} \cap V_{j}=\emptyset$ for $1 \leq i<j \leq k$, and each $V_{i}$ is a dominating set of $G$. The dominating set problem asks, given a graph $G$ and a positive integer $k$, whether or not $\gamma(G) \leq k$. The domatic number problem asks, given a graph $G$ and a positive integer $k$, whether or not $\delta(G) \geq k$.

For fixed $k \geq 3$, both the dominating set problem and the domatic number problem are known to be NP-complete, see Garey and Johnson [GJ79]. Thus, they are not solvable in deterministic polynomial time unless $P=N P$, and all we can hope for is to design an exponential-time algorithm having a better running time than the trivial exponential time bound. For exponential-time algorithms, it is common to drop polynomial factors, as indicated by the $\tilde{\mathcal{O}}$ notation: For functions $f$ and $g$, we write $f \in \tilde{\mathcal{O}}(g)$ if and only if $f \in \mathcal{O}(p \cdot g)$ for some polynomial $p$. The naive deterministic algorithm for the dominating set problem runs in time $\tilde{\mathcal{O}}\left(2^{n}\right)$. Fomin, Kratsch, and Woeginger [FKW04] improved this trivial upper bound to $\tilde{\mathcal{O}}\left(1.93782^{n}\right)$. For various restricted graph classes, they achieve even better bounds.

The naive deterministic algorithm for the domatic number problem works as follows: Given a graph $G$ and an integer $k$, it sequentially checks every potential solution (i.e., every possible partition of the vertex set of $G$ into $k$ sets $D_{1}, D_{2}, \ldots, D_{k}$ ), and accepts if and only if a correct solution is found (i.e., if and only if each $D_{i}$ is a dominating set). How many potential solutions are there? The number of ways of partitioning a set with $n$ elements into $k$ nonempty, disjoint
subsets can be calculated by the Stirling number of the second kind: $S_{2}(n, k)=$ $\frac{1}{k!} \sum_{i=0}^{k-1}(-1)^{i}\binom{k}{i}(k-i)^{n}$, which yields a running time of $\tilde{\mathcal{O}}\left(k^{n}\right)$. A better result can be achieved via the dynamic programming across the subsets technique, which was introduced by Lawler [Law76] to compute the chromatic number of a graph by exploiting the fact that every minimum chromatic partition contains at least one maximum independent set. By suitably modifying this technique, one can compute the domatic number of a graph in time $\tilde{\mathcal{O}}\left(3^{n}\right)$. This is done by generating all dominating sets of the graph with increasing cardinality, which takes time

$$
\sum_{k=0}^{n}\binom{n}{k} 2^{k}=(1+2)^{n}=3^{n}
$$

The difference to Lawler's algorithm lies in the fact that all dominating sets need to be checked, whereas only maximum independent sets are relevant to compute the chromatic number.

Proposition 2 Let $G=(V, E)$ be a graph. Then, the domatic number $\delta(G)$ can be computed in time $\tilde{\mathcal{O}}\left(3^{n}\right)$.

One tempting way of designing an improved algorithm for the domatic number problem might be to exploit the result for the dominating set problem mentioned above. However, we observe that no such useful connection between the two problems exists in general. The first part of Proposition 3 shows that an arbitrary given minimum dominating set is not necessarily part of a partition into a maximum number of dominating sets. The second part of Proposition 3 shows that, given an arbitrary partition into a maximum number of dominating sets, it is not necessarily the case that one set of the partition indeed is a minimum dominating set. Thus, for solving the domatic number problem, one cannot use in any obvious way the exact $\tilde{\mathcal{O}}\left(1.93782^{n}\right)$ algorithm for the dominating set problem by Fomin et al. [FKW04]. Proposition 3 is stated for graphs with domatic number 3 ; it can easily be generalized to graphs with domatic number $k \geq 3$. The proof of Proposition 3 can be found in the appendix.

Proposition 3 1. There exists some graph $G$ with $\delta(G)=3$ such that some minimum dominating set $D$ of $G$ is not part of any partition into three dominating sets of $G$.
2. There exists some graph $H=(V, E)$ with $\delta(H)=3$ such that for each partition $V=D_{1} \cup D_{2} \cup D_{3}$ into three dominating sets of $H$ and for each $i,\left\|D_{i}\right\|>\gamma(H)$.

For the three domatic number problem, no algorithm with a running time better than $\tilde{\mathcal{O}}\left(3^{n}\right)$ is known. We improve this trivial upper bound to $\tilde{\mathcal{O}}\left(2.9416^{n}\right)$.

We now define some technical notions suitable to measure how "useful" a vertex is to achieve domination of the graph $G=(V, E)$. Intuitively, the vertex degree is a good (local) measure, since the larger the neighborhood of a vertex is, the more vertices are potentially dominated by the set to which it belongs. The technical notions introduced in Definition 4 will be used later on to describe our algorithm.

Definition 4 Let $G=(V, E)$ be a graph with $n$ vertices, and let $\mathcal{P}=\left(D_{1}, D_{2}, D_{3}, R\right)$ be a partition of $V$ into four sets, $D_{1}, D_{2}, D_{3}$, and $R$. The subsets $D_{i}$ of $V$ will eventually yield a partition of $V$ into the three dominating sets (if they exist) to be constructed, and the subset $R \subseteq V$ collects the remaining vertices not yet assigned at the current point in the computation of the algorithm. Let $r=\|R\|$ be the number of these remaining vertices, and let $d=n-r$ be the number of vertices already assigned to some set $D_{i}$. The area of $G$ covered by $\mathcal{P}$ is defined as $\operatorname{area}_{\mathcal{P}}(G)=\sum_{i=1}^{3}\left\|N\left[D_{i}\right]\right\|$. Note that $\operatorname{area}_{\mathcal{P}}(G)=3 n$ if and only if $D_{1}, D_{2}$, and $D_{3}$ are dominating sets of $G$. For a partition $\mathcal{P}$, we also define the surplus of graph $G$ as $\operatorname{surplus}_{\mathcal{P}}(G)=\operatorname{area}_{\mathcal{P}}(G)-3 d$.

Some of the vertices in $R$ may be assigned to three, not necessarily disjoint, auxiliary sets $A_{1}, A_{2}$, and $A_{3}$ arbitrarily. Let $\mathcal{A}=\left(A_{1}, A_{2}, A_{3}\right)$. For each vertex $v \in R$ and for each $i$ with $1 \leq i \leq 3$, define the gap of vertex $v$ with respect to set $D_{i}$ by

$$
\operatorname{gap}_{\mathcal{P}, \mathcal{A}}(v, i)= \begin{cases}\|N[v]\|-\left\|\left\{u \in N[v] \mid(\exists w \in N[u])\left[w \in D_{i}\right]\right\}\right\| & \text { if } v \notin A_{i} \\ \perp & \text { otherwise }\end{cases}
$$

where $\perp$ is a special symbol that indicates that $\operatorname{gap}_{\mathcal{P}, \mathcal{A}}(v, i)$ is undefined for this $v$ and $i$. (Our algorithm will make sure to properly handle the cases of undefined gaps.)

Additionally, given $\mathcal{P}$ and $\mathcal{A}$, define for all vertices $v \in R$ :

$$
\begin{aligned}
\operatorname{maxgap}_{\mathcal{P}, \mathcal{A}}(v) & =\max \left\{\operatorname{gap}_{\mathcal{P}, \mathcal{A}}(v, i) \mid 1 \leq i \leq 3\right\} \\
\operatorname{mingap}_{\mathcal{P}, \mathcal{A}}(v) & =\min \left\{\operatorname{gap}_{\mathcal{P}, \mathcal{A}}(v, i) \mid 1 \leq i \leq 3\right\} \\
\operatorname{sumgap}_{\mathcal{P}, \mathcal{A}}(v) & =\sum_{i=1}^{3} \operatorname{gap}_{\mathcal{P}, \mathcal{A}}(v, i)
\end{aligned}
$$

Given $G, \mathcal{P}$, and $\mathcal{A}$, define the maximum gap of $G$ and the minimum gap of $G$ by taking the maximum and minimum gaps over all vertices in $G$ not yet assigned:

$$
\begin{aligned}
\operatorname{maxgap}_{\mathcal{P}, \mathcal{A}}(G) & =\max \left\{\operatorname{maxgap}_{\mathcal{P}, \mathcal{A}}(v) \mid v \in R\right\} \\
\operatorname{mingap}_{\mathcal{P}, \mathcal{A}}(G) & =\min \left\{\operatorname{mingap}_{\mathcal{P}, \mathcal{A}}(v) \mid v \in R\right\}
\end{aligned}
$$

Let $\mathcal{P}$ be given. A vertex $u \in V$ is called an open neighbor of $v \in V$ if $u \in N[v]$ and $u$ has not been assigned to any set $D_{1}, D_{2}$, or $D_{3}$ yet. A potential dominating set $D_{i}, 1 \leq i \leq 3$, is called an open set of $v \in V$ if its closed neighborhood does not include $v$, i.e., $v$ is not dominated by $D_{i}$. The balance of $v \in V$ is defined as the difference between the number of open vertices and the number of open sets. Formally, define

$$
\begin{aligned}
\text { openNeighbors }_{\mathcal{P}}(v) & =\{u \in N[v] \mid u \in R\} \\
\operatorname{openSets}_{\mathcal{P}}(v) & =\left\{i \in\{1,2,3\} \mid v \notin N\left[D_{i}\right]\right\} \\
\operatorname{balance}_{\mathcal{P}}(v) & =\left\|\operatorname{openNeighbors}_{\mathcal{P}}(v)\right\|-\left\|\operatorname{openSets}_{\mathcal{P}}(v)\right\| .
\end{aligned}
$$

We call a vertex $v \in V$ critical if and only if balance $_{\mathcal{P}}(v) \leq 0$ and $\left\|\operatorname{openSets}_{\mathcal{P}}(v)\right\|>$ 0.

The proof of the next proposition is straightforward. Once balance $\mathcal{P}_{\mathcal{P}}(v)=0$, no two vertices remaining in $N[v] \cap R$ can be assigned to the same dominating set $D_{i}$, $1 \leq i \leq 3$, since balance $\mathcal{P}(v)$ would then be negative.

Proposition 5 Let $\mathcal{P}=\left(D_{1}, D_{2}, D_{3}, R\right)$ be given as in Definition 4 , and $v \in V$ be a critical vertex for this partition. The only way to modify $\mathcal{P}$ so as to contain three dominating sets is to assign all vertices $u \in N[v] \cap R$ to distinct dominating sets $D_{i}$.

## 3 The Algorithm

Our strategy is to recursively assign the vertices $v \in V$ to obtain a correct potential solution consisting of a partition into three dominating sets, $D_{1}, D_{2}$, and $D_{3}$. Once a previous assignment of $v$ to some set $D_{i}$ turns out to be wrong, we remember this by adding this vertex to $A_{i}$. More precisely, the basic idea is to first pick those vertices with the highest maximum gap. While the algorithm is progressing, it dynamically updates the gaps for every vertex in each step. We now state our main result.

Theorem 6 The three domatic number problem can be solved by a deterministic algorithm running in time $\tilde{\mathcal{O}}\left(2.9416^{n}\right)$.

Proof. Let $G=(V, E)$ be the given graph. The algorithm seeks to find a partition of $V$ into three disjoint dominating sets. Note that every vertex $v \in V$ is contained in one of these sets and is dominated by the remaining two sets, i.e., it is adjacent to at least one of their elements. The algorithm is described in pseudo-code in the appendix, see Figures 2 3 3 5 5 and 6 Since $\delta(G) \leq \min -\operatorname{deg}(G)+1$, we may assume that $\min -\operatorname{deg}(G) \geq 2$.

The algorithm starts by initializing the potential dominating sets $D_{1}, D_{2}$, and $D_{3}$ and the auxiliary sets $A_{1}, A_{2}$, and $A_{3}$, setting each to the empty set. The initial partition thus is $\mathcal{P}=(\emptyset, \emptyset, \emptyset, V)$ and the initial triple of auxiliary sets is $\mathcal{A}=(\emptyset, \emptyset, \emptyset)$.

Then, the recursive function DOMinate is called for the first time. It is always invoked with graph $G$, a partition $\mathcal{P}=\left(D_{1}, D_{2}, D_{3}, R\right)$, and a triple $\mathcal{A}=\left(A_{1}, A_{2}, A_{3}\right)$ of not necessarily disjoint auxiliary sets. $\mathcal{P}$ and $\mathcal{A}$ represent a situation in which the vertices in $V-R$ have been assigned to $D_{1}, D_{2}$, and $D_{3}$, and $v \in A_{i}$ means that in some previous recursive call to function Dominate the vertex $v$ has been assigned to $D_{i}$ without successfully changing $\mathcal{P}$ to contain three dominating sets.

Function Dominate starts by calling Recalculate-Gaps, which calculates all gaps with respect to $\mathcal{P}$ and $\mathcal{A}$. Additionally, openNeighbors ${ }_{\mathcal{P}}(v)$, openSets ${ }_{\mathcal{P}}(v)$, and balance $_{\mathcal{P}}(v)$ are determined for every vertex $v \in V$. Four trivial cases can occur.

Case 1: The sets $D_{1}, D_{2}$, and $D_{3}$ are dominating sets of graph $G$. In this case, we are done and may add the remaining vertices $v \in R$ to any set $D_{i}$, say to $D_{1}$.

Case 2: For some vertex $v \in V$, we have balance $\mathcal{P}_{\mathcal{P}}(v)<0$. That is, there are less vertices in $R \cap N[v]$ than dominating sets with $v \notin N\left[D_{i}\right]$. Thus, no matter how the vertices in $R \cap N[v]$ are assigned, $\mathcal{P}$ won't contain three dominating sets. We have run into a dead-end and return to the previous level of the recursion.

Case 3: There exists a vertex $v \in R$ that is also a member of two of the auxiliary sets $A_{1}, A_{2}$, and $A_{3}$. Hence, vertex $v$ was previously assigned to two distinct sets $D_{i}$ and $D_{j}, 1 \leq i<j \leq 3$, but the recursion returned without success. We assign $v$ to the only possible set $D_{k}$ left, with $i \neq k \neq j$.

Case 4: For some vertex $v \in V$, we have balance $\mathcal{P}_{\mathcal{P}}(v)=0$ and $\left\|\operatorname{openSets}_{\mathcal{P}}(v)\right\|>0$. That is, $v$ is a critical vertex, since it is not dominated by all three sets $D_{1}, D_{2}$, and $D_{3}$ contained in the current $\mathcal{P}$, and there are as many open neighbors as open sets left for it. Note that this is the case for each vertex $v$ with $\operatorname{deg}(v)=2$ and $N[v] \cap R \neq \emptyset$, as $v$ and its two neighbors have to be assigned to three different dominating sets. We select one of the at most three vertices left in $N[v] \cap R$, say $u$, and call function $\operatorname{Assign}(G, \mathcal{P}, \mathcal{A}, u, i)$ for all $i$ with $u \notin A_{i}$.

Function Handle-Critical-Vertex deals with the latter three of these trivial cases. After they have been ruled out, one of the remaining vertices $v \in R$ is selected and assigned to one of the three sets $D_{i}$, under the constraint that a vertex $v \in R$ cannot be added to $D_{i}$ if it is already a member of $A_{i}$. This case occurs whenever the recursion returns because no three dominating sets could be found with this combination. The recursion continues by calling $\operatorname{Assign}(G, \mathcal{P}, \mathcal{A}, v, i)$, which adds $v$ to $D_{i}$, and then calls Dominate $(G, \mathcal{P}, \mathcal{A})$. If no three dominating sets are found by this choice, we remember this by adding $v$ to the set $A_{i}$. A final call to Dominate is made without assigning a vertex to one potential dominating set $D_{i}$. If this call fails, the recursion returns to the previous level. This completes the description of the algorithm. We now argue that it is correct and estimate its running time.

To see that the algorithm works correctly, note that it outputs three sets $D_{1}, D_{2}$, and $D_{3}$ only if they each are dominating sets of $G$. It remains to prove that these sets are definitely found in the recursion tree. All drop-backs within the recursion occur when, for the current $\mathcal{P}=\left(D_{1}, D_{2}, D_{3}, R\right)$, we have balance $\mathcal{P}(v)<0$ for some vertex $v \in V$. Thus, $\mathcal{P}$ cannot be modified so as to contain a correct partition into three dominating sets on this branch of the recursion tree. Since the algorithm checks every possible partition of $G$ into three sets, unless it is stopped by such a drop-back, a partition into three dominating sets will be found, if it exists. If the algorithm does not find three dominating sets, it eventually terminates when returning from the first recursive call of function DOMINATE. It reports the failure, and thus always yields the correct output.

To estimate the running time of the algorithm, an important observation is that the recalculation of the gaps takes no more than quadratic time in $n$, the number of vertices of the graph $G$. Thus, in terms of the $\tilde{\mathcal{O}}$-notation, the running time of the algorithm depends solely on the number of recursive calls. Let $T(m)$ be the number of steps of the algorithm, where $m$ is the number of potential dominating sets left for all vertices that have not been selected as yet. Initially, every vertex may be a member of any of the three dominating sets to be constructed (if they exist), hence $m=3 n$.

There are two scenarios where the algorithm calls function DOMINATE recursively. If Handle-Critical-Vertex detects a vertex $v \in V$ as being critical, it selects a vertex $u \in N[v] \cap R$ and calls function ASSIGN (and thus Dominate) for each $i$ with $u \notin A_{i}$. Since every critical vertex $v \in V$ remains critical as long as $N[v] \cap R \neq \emptyset$,
function Handle-Critical-VERTEX will be called until all vertices in $N[v] \cap R$ have been assigned to any of $D_{1}, D_{2}$, and $D_{3}$. Since $\|$ openSets $_{\mathcal{P}}(v) \| \leq 3$, at most three vertices in the closed neighborhood of $v$ have not been assigned when $v$ turns critical. By Proposition 5] all vertices in $N[v] \cap R$ have to be assigned to different dominating sets. If $\|$ openNeighbors $\mathcal{P}_{\mathcal{P}}(v) \|=3$, we have at most six combinations; if we have two open neighbors for a critical vertex, there are at most two combinations left; and finally, for one open neighbor $u \in N[v] \cap R$, there remains only one possible choice to assign $u$ to one of the sets $D_{1}, D_{2}$, and $D_{3}$. Thus, in the worst case, we have $T(m) \leq 6 T(m-6)$, as we will handle three vertices for which at least two choices for dominating sets are left. With $m=3 n$, it follows that $T(m) \leq 6^{m / 6}=6^{n / 2}$, i.e., $T(m)=\tilde{\mathcal{O}}\left(2.4495^{n}\right)$.

The only other branching into two different recursive calls happens in the main body of function Dominate, when selecting a vertex $v$ with the currently highest maximum gap with respect to $\mathcal{P}$ and $\mathcal{A}$. Two cases might occur. On the one hand, we might have considered a correct dominating set $D_{i}$ for $v$. If $v$ had not been looked at so far, i.e., if $v$ is not contained in any set $A_{j}, 1 \leq j \leq 3, j \neq i$, we have eliminated all three possible sets for $v$ to belong to. Thus, in this case, $T(m)=T(m-3)$. On the other hand, if the algorithm returns from the recursion and thus did not make the right choice for $v$, we have $T(m)=T(m-1)$, since $v$ is added to $A_{i}$, and function DOMINATE is called without assigning any vertex. Summing up, we have $T(m) \leq$ $T(m-1)+T(m-3)$. In the second case, we have already visited vertex $v$ in a previous stage of the algorithm and unsuccessfully tried to assign it to some set $D_{j}$, with $1 \leq j \leq 3$. There are only two dominating sets for $v$ left. Either way, if we put $v$ into the correct dominating set right away or fail the first time, we have $T(m)=$ $T(m-2)$. Summing up both cases, we have $T(m) \leq 2 T(m-2)$. Suppose that the first and the second case occur equally often, i.e., the algorithm considers every vertex twice. It then follows that

$$
T(m) \leq \frac{1}{2}(T(m-1)+T(m-3))+\frac{1}{2}(2 T(m-2))
$$

with $m=3 n$. Thus, we have $T(m)=\tilde{\mathcal{O}}\left(3^{n}\right)$, and the trivial time bound cannot be beaten. To improve this running time, we have to make sure that the recursion tree will not reach its full depth, i.e., not all vertices are considered by the algorithm or function Handle-Critical-VERTEX will be called for a sufficiently large portion of the vertices. It is clear that the algorithm has found three dominating sets once $\operatorname{area}_{\mathcal{P}}(G)=3 n$ (recall the notions from Definition 4). By selecting the maximum gap possible for a partition $\mathcal{P}$, we try to reach this goal as fast as possible. For every vertex $v \in R$ that we assign to one of the potential dominating sets $D_{i}, 1 \leq i \leq 3$, we increase $\operatorname{area}_{\mathcal{P}}(G)$ by $\operatorname{gap}_{\mathcal{P}, \mathcal{A}}(v, i)$, and additionally we $\operatorname{add}\left(\operatorname{gap}_{\mathcal{P}, \mathcal{A}}(v, i)-3\right)$ to surplus $_{\mathcal{P}}(G)$.

Since the vertices of degree two are critical, they and their neighbors can be handled in time $\tilde{\mathcal{O}}\left(2.4495^{n}\right)$, as argued above. So assume that $\min -\operatorname{deg}(G) \geq 3$. Then, we have $\operatorname{maxgap}_{\mathcal{P}, \mathcal{A}}(G)>3$ at the start of the algorithm. If this condition remains to hold for at least $3 n / 4$ steps, we have reached $\operatorname{area}_{\mathcal{P}}(G)=3 n$, and the algorithm terminates successfully. To make use of more than $3 n / 4$ vertices, $\operatorname{maxgap}_{\mathcal{P}, \mathcal{A}}(G)$ has to drop below four at one point of the computation. We exploit the fact that up to this point,
the surplus has grown sufficiently large with respect to $n$. Decreasing it will force $\operatorname{maxgap}_{\mathcal{P}, \mathcal{A}}(G)$ to drop below three, and this condition can hold only for a certain portion of the remaining vertices until the algorithm terminates. To see this, we now analyze the remaining steps of the algorithm after the given graph $G$ has reached a certain maximum gap with respect to the current $\mathcal{P}$ and $\mathcal{A}$.

If $\operatorname{maxgap}_{\mathcal{P}, \mathcal{A}}(G)=0$, the recursion stops immediately. Either we have already found three disjoint dominating sets (in which case we put the remaining vertices $v \in R$ into set $D_{1}$ and halt), or one vertex has not been dominated by one set $D_{i}$ in $\mathcal{P}$ yet. Since no positive gaps exist for the vertices $v \in R, \mathcal{P}$ cannot be modified to a valid partition into three dominating sets. Function Handle-Critical-VERTEX returns true immediately after detecting balance $\mathcal{P}_{\mathcal{P}}(v)<0$ for some vertex $v \in V$, and function DOMINATE drops back one recursion level. The question is how many vertices are left in $R$ when we reach $\operatorname{maxgap}_{\mathcal{P}, \mathcal{A}}(G)=0$.

Lemma 7 Let $G=(V, E)$ be a graph and $\mathcal{P}=\left(D_{1}, D_{2}, D_{3}, R\right)$ be a partition of $V$ as in Definition 4 Let $r=\|R\|$ and $\operatorname{maxgap}_{\mathcal{P}, \mathcal{A}}(G)=3$. Then, for at least $r / 64$ vertices in $R$, the algorithm will not recursively call function DOMINATE.

Proof of Lemma 7 Let $\operatorname{maxgap}_{\mathcal{P}, \mathcal{A}}(G)=k$ with $k>0$. Since $\operatorname{gap}_{\mathcal{P}, \mathcal{A}}(v, i) \leq k$ for each $v \in R$ and for each $i, 1 \leq i \leq 3$, we have $\sum_{v \in R} \operatorname{sumgap}_{\mathcal{P}, \mathcal{A}}(v) \leq 3 k r$. Every vertex $v$ that is selected for a set $D_{i}$ with $_{\operatorname{gap}}^{\mathcal{P}, \mathcal{A}}(~ v, i)=k$ decreases at least $k$ gaps of the vertices in $R-\{v\}$ by one. Otherwise, Handle-Critical-Vertex would have found a critical vertex $u \in N[v]$ with $N[u] \cap R=\{v\}$. Then, either $\left\|\operatorname{openSets}_{\mathcal{P}}(u)\right\|>1$ (which implies balance $\mathcal{P}_{\mathcal{P}}(u)<0$ and we abort), or $\|$ openSets $_{\mathcal{P}}(u) \|=1$, in which case $v$ is added to the appropriate set $D_{i}$ without further branching of function DOMINATE. Thus, if no critical vertex is detected, selecting a vertex $v \in R$ for some set $D_{i}$ decreases at least $k$ gaps, and since $v$ does not belong to $R$ anymore, additionally all gaps previously defined for $v$ are now undefined. So the lowest possible rate at which the gaps are decreased is related to the maximum gap of $G$.

Now suppose that $\operatorname{maxgap}_{\mathcal{P}, \mathcal{A}}(G)=3$ and $\operatorname{sumgap}_{\mathcal{P}, \mathcal{A}}(v)=9$ for all vertices $v \in R$. We always select a vertex $v$ with the highest summation gap of all vertices $u \in R$ with maxgap $\mathcal{P}_{\mathcal{A}}(u)=3$. As long as there exists a vertex $v \in R$ with $\operatorname{gap}_{\mathcal{P}, \mathcal{A}}(v, i)=3$ for all $i$, it will be selected by the algorithm. After calling function RECALCULATE-GAPS, the number of gaps equal to three will be decreased at least by six. If exactly three other gaps of vertices in $R-\{v\}$ decrease by one in every step, it takes at least $r / 4$ vertices until sumgap $\mathcal{P}_{\mathcal{P}, \mathcal{A}}(v)<9$ for all $v \in R$. Another $1 / 4$ of the $3 r / 4$ vertices remaining have to be selected until $\operatorname{sumgap}_{\mathcal{P}, \mathcal{A}}(v)<8$. Adding $1 / 4$ of the $9 r / 16$ vertices left in $R$, we have reached $\operatorname{maxgap}_{\mathcal{P}, \mathcal{A}}(G)=2$ with sumgap $\mathcal{P}_{\mathcal{A}}(v)=6$ for all vertices $v \in R$. This implies that every defined gap is equal to two. Summing up, we have selected

$$
\frac{1}{4} \cdot r+\frac{1}{4} \cdot \frac{3}{4} r+\frac{1}{4} \cdot \frac{9}{16} r=\frac{37}{64} r
$$

vertices until $\operatorname{maxgap}_{\mathcal{P}, \mathcal{A}}(G)=2$, under the constraint that a minimum number of gaps is reduced in each step, while simultaneously trying to reduce the maximum summation
gap in the fastest possible way. This way we reach level $\operatorname{maxgap}_{\mathcal{P}, \mathcal{A}}(G)=0$ with as few vertices left in $R$ as possible, which describes the worst case that might happen.

Analogously, we can show that $\operatorname{maxgap}_{\mathcal{P}, \mathcal{A}}(G)$ drops from 2 to 1 after selecting another $19 r / 64$ vertices. And once we have $\operatorname{maxgap}_{\mathcal{P}, \mathcal{A}}(G)=1$, it takes $7 r / 64$ vertices to get to $\operatorname{maxgap}_{\mathcal{P}, \mathcal{A}}(G)=0$. Now, there are $r / 64$ vertices remaining in $R$, which do not have to be processed recursively.

Lemma 7
Continuing the proof of Theorem 6 note that we assumed $\min -\operatorname{deg}(G) \geq 3$, so when the gaps are initialized for graph $G$, we have $\operatorname{mingap}_{\mathcal{P}, \mathcal{A}}(v) \geq 4$ for each vertex $v \in V$. Thus, more than three vertices are dominated by the selected set $D_{i}$ for vertex $v$. As long as $\operatorname{maxgap}_{\mathcal{P}, \mathcal{A}}(G)>3$ is true, $\operatorname{surplus}_{\mathcal{P}}(G)$ is increasing. The only way to lower the surplus is by adding vertices $v$ to a set $D_{i}$ with $\operatorname{gap}_{\mathcal{P}, \mathcal{A}}(v, i)<3$. The surplus decreases by one when $\operatorname{gap}_{\mathcal{P}, \mathcal{A}}(v, i)=2$, and it decreases by two when $\operatorname{gap}_{\mathcal{P}, \mathcal{A}}(v, i)=1$.

Let $S=\operatorname{surplus}_{\mathcal{P}}(G)$ be the surplus collected for a partition $\mathcal{P}$ until we reach a point where $\operatorname{maxgap}_{\mathcal{P}, \mathcal{A}}(G)=3$. To make use of the most recursive calls and to even out the surplus completely, there have to be at least $r=\|R\|$ vertices remaining with

$$
0 \cdot \frac{37 r}{64}+1 \cdot \frac{19 r}{64}+2 \cdot \frac{7 r}{64}=S
$$

so $r \geq 64 S / 33$. A fraction of $1 / 64$ of these vertices will be handled by the algorithm without branching into more than one recursive call, which is at least $S / 33$. The question is how big the surplus $S$ might grow and how many vertices are left in $R$ before $\operatorname{maxgap}_{\mathcal{P}, \mathcal{A}}(G)=3$ is reached. The lowest surplus with as few vertices in $R$ as possible occurs if $\min -\operatorname{deg}(G)=\max -\operatorname{deg}(G)=3$. Surplus $S$ is increased by one in each step until we arrive at $\operatorname{maxgap}_{\mathcal{P}, \mathcal{A}}(G)=3$. When selecting a vertex $v$ of degree 3 for a set $D_{i}$, the gap of its neighbors $u \in N(v)$ and the gaps of the neighbors of every $u$ might be decreased. Summing up, at most $1+3+3 \cdot 2=10$ vertices can have decreased their gaps for some $i$. After selecting at least $n / 10$ vertices for each $i$, we have $\operatorname{mingap}_{\mathcal{P}, \mathcal{A}}(G)=3$ (in the worst case). From this point on, we cannot be sure if the next vertex selected for some $D_{i}$ satisfies $\operatorname{gap}_{\mathcal{P}, \mathcal{A}}(v, i)>3$. But so far we have already collected a surplus of $S=3 n / 10$, and applying this we obtain $64 n / 110 \leq r \leq 7 n / 10$. Thus, for at least $n / 110$ vertices we never branch into two different recursive calls. Setting $m=3(109 n / 110)$, we obtain a running time of $\tilde{\mathcal{O}}\left(2.9416^{n}\right)$.

## 4 Graphs with Bounded Maximum Degree

As seen in the last section, the running time of the algorithm crucially depends on the degrees of the vertices of $G$. If we restrict ourselves to graphs $G$ with bounded maximum degree (say $\Delta=\max -\operatorname{deg}(G)$ ), we can optimize our strategy in finding three disjoint dominating sets. In this section, we present a simple deterministic algorithm, which has a better running time than the algorithm from Theorem6 provided that $\Delta$ is low. By using randomization, we can further improve the running time for graphs $G$ with low maximum degree.

Before stating the two results, note that graphs with maximum degree two can trivially be partitioned into three dominating sets, if such a partition exists. Every component of such a graph is either an isolated vertex, a path, or a cycle, and each such property can be recognized in polynomial time.

Proposition 8 Let $G=(V, E)$ be a given graph with max-deg $(G)=2$. There exists a partition of the vertices of $G$ into three dominating sets if and only if every component of $G$ is a cycle of length $k$ such that 3 divides $k$.

We use the terms from Definition 4 in Section 3 to describe a snapshot within the algorithm. For any partition $\mathcal{P}=\left(D_{1}, D_{2}, D_{3}, R\right)$, some vertices of $V$ have already been assigned to the potential dominating sets $D_{1}, D_{2}$, and $D_{3}$, while all the remaining vertices are in $R$. The auxiliary sets $\mathcal{A}=\left(A_{1}, A_{2}, A_{3}\right)$ will not be needed in this section. Only connected graphs are considered, as it is possible to treat every connected component separately, producing the desired output within the same time bounds.

Table 1 lists the running times of both the deterministic and the random algorithm, where the maximum degree of the input graph is bounded by $\Delta, 3 \leq \Delta \leq 8$. Note that the exact deterministic algorithm from Theorem 6 in Section 3 beats the deterministic algorithm from Theorem 9 whenever $\Delta \geq 7$.

| $\Delta$ | 3 | 4 | 5 | 6 | 7 | 8 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| deterministic | $2.2894^{n}$ | $2.6591^{n}$ | $2.8252^{n}$ | $2.9058^{n}$ | $2.9473^{n}$ | $2.9697^{n}$ |
| randomized | $2^{n}$ | $2.3570^{n}$ | $2.5820^{n}$ | $2.7262^{n}$ | $2.8197^{n}$ | $2.8808^{n}$ |

Table 1: Results for $\max -\operatorname{deg}(G)=k$, where $3 \leq k \leq 8$

Theorem 9 Let $G=(V, E)$ be a graph with max-deg $(G)=\Delta$, where $\Delta \geq 3$. There exists a deterministic algorithm solving the three domatic number problem in time $\tilde{\mathcal{O}}\left(d^{\frac{n}{\Delta}}\right)$, where

$$
\begin{equation*}
d=\sum_{a=0}^{\Delta-2}\left[\binom{\Delta}{a} \sum_{b=1}^{\Delta-a-1}\binom{\Delta-a}{b}\right] . \tag{4.1}
\end{equation*}
$$

Proof. The algorithm works as follows. We start with an arbitrary vertex $v \in V$ and assign it to the first set $D_{1}$. In each step, we first check whether we found a partition $\mathcal{P}=\left(D_{1}, D_{2}, D_{3}, R\right)$ into dominating sets $D_{1}, D_{2}$, and $D_{3}$. If not, one vertex $v \in V$ is selected that is not dominated by all three sets $D_{1}, D_{2}$, and $D_{3}$, and additionally has a vertex $u \in N[v]$ in its closed neighborhood that has already been added to some set $D_{i}, 1 \leq i \leq 3$. It follows that $1 \leq \|$ openSets $_{\mathcal{P}}(v) \| \leq 2$.

If balance $\mathcal{P}_{\mathcal{P}}(v)<0$, we return within the recursion. Otherwise, we try all combinations to partition the vertices in $N[v] \cap R$, so that after this step vertex $v$ is dominated by all three potential dominating sets. If no such combination leads to a valid partition, we again return within the recursion.

Suppose now that balance $_{\mathcal{P}}(v) \geq 0,\left\|\operatorname{openSets}_{\mathcal{P}}(v)\right\|=2$, and $N[v] \cap D_{1} \neq \emptyset$. To obtain three disjoint dominating sets, at least one vertex in $N[v]$ has to be assigned to $D_{2}$, and at least one vertex in $N[v]$ has to be added to $D_{3}$. This limits our choices, especially if the degree of $v$ is bounded by some constant $\Delta$.

To measure the running time of the algorithm, we consider the worst case with the most possible combinations that might yield a partition into three dominating sets. This occurs when only one vertex $u \in N[v]$ has already been added to one set, i.e., $\left\|N[v] \cap\left(D_{1} \cup D_{2} \cup D_{3}\right)\right\|=1$. If $N[v] \cap D_{1} \neq \emptyset$, then any number between 0 and $\Delta-2$ of vertices in $N[v] \cap R$ may be assigned to the same set $D_{1}$. Let this number be $a$. It follows that from one to $\Delta-a-1$ vertices remaining in $N[v] \cap R$ are allowed to be in the next potential dominating set $D_{2}$. This is how Equation 4.1 for $d$ is derived. After assigning the last vertices in $N[v] \cap R$ to the dominating set $D_{3}$, exactly $\Delta$ vertices have been removed from $R$. Thus, we have a worst case running time of $\tilde{\mathcal{O}}\left(d^{\frac{n}{\Delta}}\right)$. Table 1 lists the running time for graphs with maximum degree from three to nine.

In the next theorem, randomization is used to speed up this procedure. Instead of assigning all vertices in the closed neighborhood of some vertex $v \in V$ in one step, only one or two vertices in $N[v] \cap R$ are added to the potential dominating sets $D_{1}$, $D_{2}$, and $D_{3}$. The goal is to dominate one vertex by all three sets in one step. We will select the one or two vertices that are missing for this goal at random.

Theorem 10 Let $G=(V, E)$ be a graph with max-deg $(G)=\Delta$, where $\Delta \geq 3$, and let d be defined as in Equation (4.1) in Theorem 9 For each constant $c>0$, there exists a randomized algorithm solving the three domatic number problem with error probability at most $e^{-c}$ in time $\tilde{\mathcal{O}}\left(r^{\frac{n}{2}}\right)$, where

$$
\begin{equation*}
r=\frac{d}{3^{\Delta-2}} \tag{4.2}
\end{equation*}
$$

Proof. Let graph $G=(V, E)$ be given with $\max -\operatorname{deg}(G)=k$. As in the deterministic algorithm, we start by adding a random vertex to the set $D_{1}$. In every following step, a vertex $v \in V$ is selected with $0<\left\|\operatorname{openSets}_{\mathcal{P}}(v)\right\|<3$, so it is $N[v] \cap\left(D_{1} \cup D_{2} \cup D_{3}\right) \neq \emptyset$. If $\|$ openSets $_{\mathcal{P}}(v) \|=1$, we have $N[v] \cap D_{i}=\emptyset$ for one $i$ with $1 \leq i \leq 3$. We randomly choose a vertex $u \in N[v] \cap R$ and assign it to set $D_{i}$, in order that $v$ is dominated by all three sets afterwards. If $\left\|\operatorname{openSets}_{\mathcal{P}}(v)\right\|=2$, we randomly select two vertices $u_{1}, u_{2} \in R$ in the closed neighborhood of $v$. Another random choice is made when deciding how to distribute these two vertices among the two potential dominating sets that have not dominated $v$ up to now.

Suppose $G$ indeed has a partition into three dominating sets. We have to measure the error rate when making our random choices to estimate the success probability of the algorithm. In every step, a vertex $v \in V$ is selected with at least one vertex $u \in N[v]$ in its closed neighborhood that has already been added to one of the sets $D_{1}$, $D_{2}$, or $D_{3}$. The highest error occurs when exactly one vertex in $N[v]$ is not included in $R$, so we restrict our analysis to this case. To obtain a valid partition into three dominating sets, there are at most $d$ choices left to partition the vertices remaining in $N[v] \cap R$. Here, $d$ is the number from Equation 4.1 Once we selected and assigned two vertices from $N[v] \cap R$ at random, there are $3^{k-2}$ possibilities left to partition the vertices in the closed neighborhood of $v$ that are still left in $R$. Our success rate when selecting the two vertices is therefore $3^{k-2} / d$.

To achieve an error probability of below $e^{-c}$, the algorithm needs to be executed more than once. The repetition number of the algorithm equals the reciprocal of the
success rate, which explains Equation 4.2 Since two vertices are processed in every step, the overall running time is $\tilde{\mathcal{O}}\left(r^{\frac{n}{2}}\right)$.

## 5 Conclusion

We have shown that the three domatic number problem can be solved by a deterministic algorithm in time $\tilde{\mathcal{O}}\left(2.9416^{n}\right)$. Furthermore, we presented two algorithms solving the three domatic number problem for graphs with bounded maximum degree, improving the above time bound for graphs with small maximum degree. Although our running times seem to be not too big of an improvement of the trivial $\tilde{\mathcal{O}}\left(3^{n}\right)$ bound, they are to our knowledge the first such algorithms breaking this barrier. For $k>3$, the decision problem of whether $\delta(G) \geq k$ can be solved in time $\tilde{\mathcal{O}}\left(3^{n}\right)$ by Lawler's dynamic programming algorithm for the chromatic number, appropriately modified for the domatic number problem. Therefore, it would not be reasonable to use our gap approach of Section 3 to decide if $\delta(G) \geq k$ for a graph $G$ and $k>3$.

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## A Proof of Proposition 3

Proof. Figure 1 shows the graphs $G$ and $H$ whose existence is claimed. In this figure, the numbers $i \mid j$ within a vertex have the following meaning: $i$ indicates which dominating set $D_{i}$ this vertex belongs to in a fixed partition into three dominating sets, and $j$ indicates a specific choice of a minimum dominating set $S$ of the graph by setting $j=1$ if and only if this vertex belongs to $S$.


Figure 1: Graphs $G$ and $H$ for Proposition 3

For the first assertion, look at the graph $G$ shown on the left-hand side of Figure 1 Note that $\gamma(G)=2$. In particular, $D=\left\{u_{3}, u_{5}\right\}$ is a minimum dominating set of $G$. Note further that $\delta(G)=3$. In particular, a partition into three dominating sets of $G$ is given by $D_{1}=\left\{u_{1}, u_{4}, u_{7}\right\}, D_{2}=\left\{u_{2}, u_{5}\right\}$, and $D_{3}=\left\{u_{3}, u_{6}\right\}$. However, $D$ cannot be part of any partition into three dominating sets, since the only neighbors of $u_{4}$, namely $u_{3}$ and $u_{5}$, belong to $D$.

Note that the minimum dominating set $D_{2}=\left\{u_{2}, u_{5}\right\}$ of $G$ defined above indeed is part of a partition into three dominating sets. The second part of the proposition, however, shows that this is not always the case. Consider the graph $H=(V, E)$ shown on the right-hand side of Figure 1 We have $\gamma(H)=2$ by choosing the minimum dominating set $D=\left\{v_{1}, v_{2}\right\}$, which is unique in this case. Again, $\delta(H)=3$. The only way, up to isomorphism, to partition the vertex set of $H$ into three dominating sets is given by $D_{1}=\left\{v_{1}, v_{7}, v_{8}\right\}, D_{2}=\left\{v_{2}, v_{6}, v_{9}\right\}$, and $D_{3}=\left\{v_{3}, v_{4}, v_{5}\right\}$. Thus, $\min \left\{\left|\mid D_{1}\|,\| D_{2}\|,\| D_{3} \|\right\}>\gamma(H)\right.$ for each partition into three dominating sets.

## B Pseudo-Code of the Algorithm from Theorem6

Figures 2, 3, 4, 5ad 6describe the algorithm from Theorem6in pseudo-code.

```
Algorithm for the Three Domatic Number Problem
    Input: Graph G = (V,E) with vertex set V ={v},\mp@subsup{v}{2}{},\ldots,\mp@subsup{v}{n}{}}\mathrm{ and edge set E
    Output: Partition of V into three dominating sets D}\mp@subsup{D}{1}{},\mp@subsup{D}{2}{},\mp@subsup{D}{3}{}\subseteqV\mathrm{ or "failure"
    Set each of D}\mp@subsup{D}{1}{},\mp@subsup{D}{2}{},\mp@subsup{D}{3}{},\mp@subsup{A}{1}{},\mp@subsup{A}{2}{}\mathrm{ , and }\mp@subsup{A}{3}{}\mathrm{ to the empty set;
    Set R=V;
    Set \mathcal{P}=(\mp@subsup{D}{1}{},\mp@subsup{D}{2}{},\mp@subsup{D}{3}{},R);
    Set \mathcal{A = ( }\mp@subsup{A}{1}{},\mp@subsup{A}{2}{},\mp@subsup{A}{3}{})\mathrm{ ;}
    Dominate (G,\mathcal{P},\mathcal{A}); // Start recursion
    output "failure" and halt;
```

Figure 2: Algorithm for the Three Domatic Number Problem

```
Function Dominate (G,\mathcal{P},\mathcal{A}){
                                    // P}\mathrm{ is a partition of graph }G
                                    // \mathcal{A is a triple of auxiliary sets}
    Recalculate-Gaps(G,\mathcal{P},\mathcal{A});
    if (each \mp@subsup{D}{i}{}}\mathrm{ is a dominating set) {
        D = D D \cupR;
        output D, D, D2, D
    }
    if ( not Handle-Critical-Vertex (G,\mathcal{P},\mathcal{A})) {
        select vertex v\inR with
            maxgap
            \mp@subsup{\operatorname{sumgap}}{\mathcal{P},\mathcal{A}}{}(v)=\operatorname{max{\mp@subsup{\operatorname{sumgap}}{\mathcal{P},\mathcal{A}}{}(u)|u\inR\wedge maxgap}
                maxgap
        find i with gap
        Assign (G,\mathcal{P},\mathcal{A},v,i);
                            // First recursive call
        A}=\mp@subsup{A}{i}{}\cup{v}; // If recursion fails, put v in Ai and try again
        Dominate(G,\mathcal{P},\mathcal{A}); // Second recursive call
    }
    return;
}
```

Figure 3: Recursive function to dominate graph $G$

```
Function Assign (G,\mathcal{P},\mathcal{A},v,i){
    Di}=\mp@subsup{D}{i}{}\cup{v}
    R=R-{v};
    Dominate(G,\mathcal{P},\mathcal{A});
}
```

Figure 4: Function to assign vertex $v$ to set $D_{i}$

```
Function Recalculate-Gaps (G,\mathcal{P},\mathcal{A}){ //\mathcal{P}\mathrm{ is a partition of graph }G\mathrm{ ,}
// \mathcal{A is a triple of auxiliary sets}
```

```
    for all (vertices \(v \in V\) ) \{
```

    for all (vertices \(v \in V\) ) \{
    if (vertex \(v \in R\) ) \(\{\)
    if (vertex \(v \in R\) ) \(\{\)
        for all \((i=1,2,3)\{\)
        for all \((i=1,2,3)\{\)
            if \(\left(v \notin A_{i}\right)\left\{\operatorname{gap}_{\mathcal{P}, \mathcal{A}}(v, i)=\|N[v]\|-\|\{u \in N[v] \mid(\exists w \in N[u])[w \in\right.\)
            if \(\left(v \notin A_{i}\right)\left\{\operatorname{gap}_{\mathcal{P}, \mathcal{A}}(v, i)=\|N[v]\|-\|\{u \in N[v] \mid(\exists w \in N[u])[w \in\right.\)
                \(\left.\left.\left.D_{i}\right]\right\} \| ;\right\}\)
                \(\left.\left.\left.D_{i}\right]\right\} \| ;\right\}\)
            else \(\operatorname{gap}_{\mathcal{P}, \mathcal{A}}(v, i)=\perp ; \quad / / \perp\) indicates that \(\operatorname{gap}_{\mathcal{P}, \mathcal{A}}(v, i)\) is undefined
            else \(\operatorname{gap}_{\mathcal{P}, \mathcal{A}}(v, i)=\perp ; \quad / / \perp\) indicates that \(\operatorname{gap}_{\mathcal{P}, \mathcal{A}}(v, i)\) is undefined
        \}
        \}
        \(\operatorname{maxgap}_{\mathcal{P}, \mathcal{A}}(v)=\max _{i \in\{1,2,3\}}\left\{\operatorname{gap}_{\mathcal{P}, \mathcal{A}}(v, i)\right\} ;\)
        \(\operatorname{maxgap}_{\mathcal{P}, \mathcal{A}}(v)=\max _{i \in\{1,2,3\}}\left\{\operatorname{gap}_{\mathcal{P}, \mathcal{A}}(v, i)\right\} ;\)
        \(\operatorname{mingap}_{\mathcal{P}, \mathcal{A}}(v)=\min _{i \in\{1,2,3\}}\left\{\operatorname{gap}_{\mathcal{P}, \mathcal{A}}(v, i)\right\} ;\)
        \(\operatorname{mingap}_{\mathcal{P}, \mathcal{A}}(v)=\min _{i \in\{1,2,3\}}\left\{\operatorname{gap}_{\mathcal{P}, \mathcal{A}}(v, i)\right\} ;\)
        \(\operatorname{sumgap}_{\mathcal{P}, \mathcal{A}}(v)=\sum_{i \in\{1,2,3\}} \operatorname{gap}_{\mathcal{P}, \mathcal{A}}(v, i) ;\)
        \(\operatorname{sumgap}_{\mathcal{P}, \mathcal{A}}(v)=\sum_{i \in\{1,2,3\}} \operatorname{gap}_{\mathcal{P}, \mathcal{A}}(v, i) ;\)
    \}
    \}
    openNeighbors \(_{\mathcal{P}}(v)=\{u \in N[v] \mid u \in R\}\);
    openNeighbors \(_{\mathcal{P}}(v)=\{u \in N[v] \mid u \in R\}\);
    openSets \(_{\mathcal{P}}(v)=\left\{i \in\{1,2,3\} \mid v \notin N\left[D_{i}\right]\right\} ;\)
    openSets \(_{\mathcal{P}}(v)=\left\{i \in\{1,2,3\} \mid v \notin N\left[D_{i}\right]\right\} ;\)
    balance \(_{\mathcal{P}}(v)=\|\) openNeighbors \(_{\mathcal{P}}(v)\|-\|\) openSets \(_{\mathcal{P}}(v) \| ;\)
    balance \(_{\mathcal{P}}(v)=\|\) openNeighbors \(_{\mathcal{P}}(v)\|-\|\) openSets \(_{\mathcal{P}}(v) \| ;\)
    \}
\}
$\operatorname{maxgap}_{\mathcal{P}, \mathcal{A}}(G)=\max _{v \in R}\left\{\operatorname{maxgap}_{\mathcal{P}, \mathcal{A}}(v)\right\} ;$
$\operatorname{maxgap}_{\mathcal{P}, \mathcal{A}}(G)=\max _{v \in R}\left\{\operatorname{maxgap}_{\mathcal{P}, \mathcal{A}}(v)\right\} ;$
$\operatorname{mingap}_{\mathcal{P}, \mathcal{A}}(G)=\min _{v \in R}\left\{\operatorname{mingap}_{\mathcal{P}, \mathcal{A}}(v)\right\} ;$
$\operatorname{mingap}_{\mathcal{P}, \mathcal{A}}(G)=\min _{v \in R}\left\{\operatorname{mingap}_{\mathcal{P}, \mathcal{A}}(v)\right\} ;$
}

```

Figure 5: Function to recalculate gaps after partition has changed
```

Function boolean Handle-Critical-VERtex (G,\mathcal{P},\mathcal{A}){
for all (vertices v\inV) {
if (\mp@subsup{b}{}{\prime\primeance}
return true;
} else if (|{i\in{1,2,3}|v\in\mp@subsup{A}{i}{}}|==2){\quad// one choice for v remaining
select i with v\not\in A
Assign (G, P},\mathcal{A},v,i)
return true;
} else if (balance}\mp@subsup{\mathcal{P}}{\mathcal{P}}{}(v)==0\mathrm{ and |oonenSets}\mp@subsup{\mathcal{P}}{\mathcal{P}}{(v)|>0){ / | v is critical
select }u\inN[v]\capR\mathrm{ ;
for all (i with }u\not\in\mp@subsup{A}{i}{}\mathrm{ and v not dominated by }\mp@subsup{D}{i}{}\mathrm{ )
Assign}(G,\mathcal{P},\mathcal{A},u,i)
return true;
}
}
return false; // no critical vertices were found
}

```

Figure 6: Function to handle the critical vertices```


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