An Exact 2.9416ⁿ Algorithm for the Three Domatic Number Problem^{*}

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June 24, 2005

Abstract

The three domatic number problem asks whether a given undirected graph can be partitioned into at least three dominating sets, i.e., sets whose closed neighborhood equals the vertex set of the graph. Since this problem is NP-complete, no polynomial-time algorithm is known for it. The naive deterministic algorithm for this problem runs in time 3^n , up to polynomial factors. In this paper, we design an exact deterministic algorithm for this problem running in time 2.9416^n . Thus, our algorithm can handle problem instances of larger size than the naive algorithm in the same amount of time. We also present another deterministic and a randomized algorithm for this problem that both have an even better performance for graphs with small maximum degree.

Key words: Exact algorithms, domatic number problem

1 Introduction

In this paper, we design a deterministic algorithm for the three domatic number problem, which is one of the standard NP-complete problems, see Garey and Johnson [GJ79]. This problem asks, given an undirected graph *G*, whether or not the vertex set of *G* can be partitioned into three dominating sets. A dominating set is a subset of the vertex set that "dominates" the graph in that its closed neighborhood covers the entire graph. Motivated by the tasks of distributing resources in a computer network and of locating facilities in a communication network, this problem and the related problem of finding a minimum dominating set in a given graph have been thoroughly studied, see, e.g., [CH77,Far84,Bon85,KS94,HT98,FHK00,RR04].

^{*}Work supported in part by the DFG under Grant RO 1202/9-1.

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The exact (i.e., deterministic) algorithm designed in this paper runs in exponential time. However, its running time is better than that of the naive exact algorithm for this problem. That is, we improve the trivial $\tilde{\mathcal{O}}(3^n)$ time bound to a time bound of $\tilde{\mathcal{O}}(2.9416^n)$, where the $\tilde{\mathcal{O}}$ notation neglects polynomial factors as is common for exponential-time algorithms. The point of such an improvement is that a $\tilde{\mathcal{O}}(c^n)$ algorithm, where c < 3 is a constant, can deal with larger instances than the trivial $\tilde{\mathcal{O}}(3^n)$ algorithm in the same amount of time before the exponential growth rate eventually hits and the running time becomes infeasible. For example, if $c = \sqrt{3} \approx 1.732$ then we have $\tilde{\mathcal{O}}\left(\sqrt{3}^{2n}\right) = \tilde{\mathcal{O}}(3^n)$, so one can deal with inputs twice as large as before. Doubling the size of inputs that can be handled by some algorithm can make quite a difference in practice.

Exact exponential-time algorithms with improved running times have been designed for various other important NP-complete problems. For example, Dantsin et al. [DGH⁺02] pushed the trivial $\tilde{\mathcal{O}}(2^n)$ bound for the three satisfiability problem down to $\tilde{\mathcal{O}}(1.481^n)$, which was further improved to $\tilde{\mathcal{O}}(1.473^n)$ by Brueggemann and Kern [BK04]. Schöning [Sch02], Hofmeister et al. [HSSW02] and Paturi et al. [PPSZ98] proposed even better randomized algorithms for the satisfiability problem. Combining their ideas, the currently best randomized algorithm for this problem is due to Iwama and Tamaki [IT03], who achieve a time bound of $\tilde{\mathcal{O}}(1.324^n)$.

The currently best exact time bound of $\mathcal{O}(1.211^n)$ for the independent set problem is due to Robson [Rob86]. Eppstein [Epp01a,Epp01b] achieved a $\tilde{\mathcal{O}}(2.415^n)$ time bound for graph coloring and a $\tilde{\mathcal{O}}(1.3289^n)$ for the special case of graph three colorability. Fomin, Kratsch, and Woeginger [FKW04] improved the trivial $\tilde{\mathcal{O}}(2^n)$ bound for the dominating set problem to $\tilde{\mathcal{O}}(1.93782^n)$. Comprehensive surveys on this subject have been written by Woeginger [Woe03] and Schöning [Sch05].

In designing domatic number algorithms, it might be tempting to exploit known results (such as Eppstein's $\tilde{\mathcal{O}}(1.3289^n)$ bound) for the graph three colorability problem, which resembles the three domatic number problem in that both are partitioning problems. However, as Cockayne and Hedetniemi [CH77] point out, the theory of domination is dual to the theory of coloring in the following sense. Coloring is based on the hereditary property of independence. A graph property is *hereditary* if whenever some set of vertices has the property then so does every subset of it. In contrast, domination is an *expanding* property in that every superset of a dominating set also is a dominating set of the graph. Further, graph colorability is a minimum problem, whereas the domatic number problem is a maximum problem. Independence (and thus colorability) can be seen as a local property, since it suffices to check the immediate neighborhood of a set of vertices to determine whether or not it is independent. In contrast, dominance is a *global* property, since in order to check it one has to consider the relation between the given set of vertices and the entire graph. In this sense, determining the domatic number of a graph intuitively appears to be harder than computing its chromatic number, notwithstanding that both problems are NP-complete. More to the point, the algorithms developed for graph coloring seem to be of no help in designing algorithms for dominating set or domatic number problems.

After introducing some definitions and notation in Section 2, we describe and analyze our algorithm in Section 3; the actual pseudo-code is shifted to the appendix.

In Section 4, we give another deterministic and a randomized algorithm, which have an even better running time for graphs with small maximum degree. Finally, we summarize and discuss our results in Section 5.

2 Preliminaries and Simple Observations

We start by introducing some graph-theoretical notation. We only consider simple, undirected graphs without loops in this paper. Let G = (V, E) be a graph. Unless stated otherwise, n denotes the number of vertices in G. The *neighborhood of a vertex v in* V is defined by $N(v) = \{u \in V \mid \{u, v\} \in E\}$, and the *closed neighborhood of v* is defined by $N[v] = N(v) \cup \{v\}$. For any subset $S \subseteq V$ of the vertices of G, define $N[S] = \bigcup_{v \in S} N[v]$ and N(S) = N[S] - S. The *degree of a vertex v in* Gis the number of vertices adjacent to v, i.e., $deg_G(v) = ||N(v)||$. If the graph G is clear from the context, we omit the subscript G. Define the *minimum degree in* Gby $min-deg(G) = \min_{v \in V} deg(v)$, and the maximum degree in G by $max-deg(G) = \max_{v \in V} deg(v)$. A path $P_k = u_1 u_2 \cdots u_k$ of length k is a sequence of k vertices, where each vertex is adjacent to its successor, i.e., $\{u_i, u_{i+1}\} \in E$ for $1 \le i \le k - 1$. If, in addition, $\{u_k, u_1\} \in E$, then path P_k is said to be a cycle, and we write C_k instead of P_k .

Definition 1 Let G = (V, E) be a graph. A subset $D \subseteq V$ is a dominating set of G if and only if N[D] = V, i.e., if and only if every vertex in G either belongs to D or has some neighbor in D. The domination number of G, denoted $\gamma(G)$, is the minimum size of a dominating set of G. The domatic number of G, denoted $\delta(G)$, is the maximum number of disjoint dominating sets of G, i.e., $\delta(G)$ is the maximum k such that V = $V_1 \cup V_2 \cup \ldots \cup V_k$, where $V_i \cap V_j = \emptyset$ for $1 \le i < j \le k$, and each V_i is a dominating set of G. The dominating set problem asks, given a graph G and a positive integer k, whether or not $\gamma(G) \le k$. The domatic number problem asks, given a graph G and a positive integer k, whether or not $\delta(G) \ge k$.

For fixed $k \ge 3$, both the dominating set problem and the domatic number problem are known to be NP-complete, see Garey and Johnson [GJ79]. Thus, they are not solvable in deterministic polynomial time unless P = NP, and all we can hope for is to design an exponential-time algorithm having a better running time than the trivial exponential time bound. For exponential-time algorithms, it is common to drop polynomial factors, as indicated by the $\tilde{\mathcal{O}}$ notation: For functions f and g, we write $f \in \tilde{\mathcal{O}}(g)$ if and only if $f \in \mathcal{O}(p \cdot g)$ for some polynomial p. The naive deterministic algorithm for the dominating set problem runs in time $\tilde{\mathcal{O}}(2^n)$. Fomin, Kratsch, and Woeginger [FKW04] improved this trivial upper bound to $\tilde{\mathcal{O}}(1.93782^n)$. For various restricted graph classes, they achieve even better bounds.

The naive deterministic algorithm for the domatic number problem works as follows: Given a graph G and an integer k, it sequentially checks every potential solution (i.e., every possible partition of the vertex set of G into k sets D_1, D_2, \ldots, D_k), and accepts if and only if a correct solution is found (i.e., if and only if each D_i is a dominating set). How many potential solutions are there? The number of ways of partitioning a set with n elements into k nonempty, disjoint subsets can be calculated by the Stirling number of the second kind: $S_2(n,k) = \frac{1}{k!} \sum_{i=0}^{k-1} (-1)^i {k \choose i} (k-i)^n$, which yields a running time of $\tilde{\mathcal{O}}(k^n)$. A better result can be achieved via the dynamic programming across the subsets technique, which was introduced by Lawler [Law76] to compute the chromatic number of a graph by exploiting the fact that every minimum chromatic partition contains at least one maximum independent set. By suitably modifying this technique, one can compute the domatic number of a graph in time $\tilde{\mathcal{O}}(3^n)$. This is done by generating all dominating sets of the graph with increasing cardinality, which takes time

$$\sum_{k=0}^{n} \binom{n}{k} 2^{k} = (1+2)^{n} = 3^{n}.$$

The difference to Lawler's algorithm lies in the fact that all dominating sets need to be checked, whereas only maximum independent sets are relevant to compute the chromatic number.

Proposition 2 Let G = (V, E) be a graph. Then, the domatic number $\delta(G)$ can be computed in time $\tilde{\mathcal{O}}(3^n)$.

One tempting way of designing an improved algorithm for the domatic number problem might be to exploit the result for the dominating set problem mentioned above. However, we observe that no such useful connection between the two problems exists in general. The first part of Proposition 3 shows that an arbitrary given minimum dominating set is not necessarily part of a partition into a maximum number of dominating sets. The second part of Proposition 3 shows that, given an arbitrary partition into a maximum number of dominating sets, it is not necessarily the case that one set of the partition indeed is a minimum dominating set. Thus, for solving the domatic number problem, one cannot use in any obvious way the exact $\tilde{O}(1.93782^n)$ algorithm for the dominating set problem by Fomin et al. [FKW04]. Proposition 3 is stated for graphs with domatic number 3; it can easily be generalized to graphs with domatic number $k \geq 3$. The proof of Proposition 3 can be found in the appendix.

- **Proposition 3** 1. There exists some graph G with $\delta(G) = 3$ such that some minimum dominating set D of G is not part of any partition into three dominating sets of G.
 - 2. There exists some graph H = (V, E) with $\delta(H) = 3$ such that for each partition $V = D_1 \cup D_2 \cup D_3$ into three dominating sets of H and for each i, $||D_i|| > \gamma(H)$.

For the three domatic number problem, no algorithm with a running time better than $\tilde{\mathcal{O}}(3^n)$ is known. We improve this trivial upper bound to $\tilde{\mathcal{O}}(2.9416^n)$.

We now define some technical notions suitable to measure how "useful" a vertex is to achieve domination of the graph G = (V, E). Intuitively, the vertex degree is a good (local) measure, since the larger the neighborhood of a vertex is, the more vertices are potentially dominated by the set to which it belongs. The technical notions introduced in Definition 4 will be used later on to describe our algorithm.

Definition 4 Let G = (V, E) be a graph with n vertices, and let $\mathcal{P} = (D_1, D_2, D_3, R)$ be a partition of V into four sets, D_1 , D_2 , D_3 , and R. The subsets D_i of V will eventually yield a partition of V into the three dominating sets (if they exist) to be constructed, and the subset $R \subseteq V$ collects the remaining vertices not yet assigned at the current point in the computation of the algorithm. Let r = ||R|| be the number of these remaining vertices, and let d = n - r be the number of vertices already assigned to some set D_i . The area of G covered by \mathcal{P} is defined as $\operatorname{area}_{\mathcal{P}}(G) = \sum_{i=1}^{3} ||N[D_i]||$. Note that $\operatorname{area}_{\mathcal{P}}(G) = 3n$ if and only if D_1 , D_2 , and D_3 are dominating sets of G. For a partition \mathcal{P} , we also define the surplus of graph G as $\operatorname{surplus}_{\mathcal{P}}(G) = \operatorname{area}_{\mathcal{P}}(G) - 3d$.

Some of the vertices in R may be assigned to three, not necessarily disjoint, auxiliary sets A_1 , A_2 , and A_3 arbitrarily. Let $\mathcal{A} = (A_1, A_2, A_3)$. For each vertex $v \in R$ and for each i with $1 \le i \le 3$, define the gap of vertex v with respect to set D_i by

$$\operatorname{gap}_{\mathcal{P},\mathcal{A}}(v,i) = \begin{cases} ||N[v]|| - ||\{u \in N[v] \mid (\exists w \in N[u])[w \in D_i]\}|| & \text{if } v \notin A_i \\ \bot & \text{otherwise,} \end{cases}$$

where \perp is a special symbol that indicates that $gap_{\mathcal{P},\mathcal{A}}(v,i)$ is undefined for this vand i. (Our algorithm will make sure to properly handle the cases of undefined gaps.) Additionally, given \mathcal{P} and \mathcal{A} , define for all vertices $v \in R$:

$$\begin{aligned} \max_{\substack{\mathcal{P},\mathcal{A}}}(v) &= \max_{\substack{\mathcal{P},\mathcal{A}}}(v,i) \mid 1 \leq i \leq 3 \},\\ \min_{\substack{\mathcal{P},\mathcal{A}}}(v) &= \min_{\substack{\mathcal{P},\mathcal{A}}}(v,i) \mid 1 \leq i \leq 3 \},\\ \operatorname{sumgap}_{\mathcal{P},\mathcal{A}}(v) &= \sum_{i=1}^{3} \operatorname{gap}_{\mathcal{P},\mathcal{A}}(v,i). \end{aligned}$$

Given G, \mathcal{P} , and \mathcal{A} , define the maximum gap of G and the minimum gap of G by taking the maximum and minimum gaps over all vertices in G not yet assigned:

$$\max_{\mathcal{P},\mathcal{A}}(G) = \max\{\max_{\mathcal{P},\mathcal{A}}(v) \mid v \in R\},\\ \min_{\mathcal{P},\mathcal{A}}(G) = \min\{\max_{\mathcal{P},\mathcal{A}}(v) \mid v \in R\}.$$

Let \mathcal{P} be given. A vertex $u \in V$ is called an open neighbor of $v \in V$ if $u \in N[v]$ and u has not been assigned to any set D_1 , D_2 , or D_3 yet. A potential dominating set D_i , $1 \leq i \leq 3$, is called an open set of $v \in V$ if its closed neighborhood does not include v, i.e., v is not dominated by D_i . The balance of $v \in V$ is defined as the difference between the number of open vertices and the number of open sets. Formally, define

We call a vertex $v \in V$ critical if and only if $\text{balance}_{\mathcal{P}}(v) \leq 0$ and $||\text{openSets}_{\mathcal{P}}(v)|| > 0$.

The proof of the next proposition is straightforward. Once $\text{balance}_{\mathcal{P}}(v) = 0$, no two vertices remaining in $N[v] \cap R$ can be assigned to the same dominating set D_i , $1 \leq i \leq 3$, since $\text{balance}_{\mathcal{P}}(v)$ would then be negative.

Proposition 5 Let $\mathcal{P} = (D_1, D_2, D_3, R)$ be given as in Definition 4, and $v \in V$ be a critical vertex for this partition. The only way to modify \mathcal{P} so as to contain three dominating sets is to assign all vertices $u \in N[v] \cap R$ to distinct dominating sets D_i .

3 The Algorithm

Our strategy is to recursively assign the vertices $v \in V$ to obtain a correct potential solution consisting of a partition into three dominating sets, D_1 , D_2 , and D_3 . Once a previous assignment of v to some set D_i turns out to be wrong, we remember this by adding this vertex to A_i . More precisely, the basic idea is to first pick those vertices with the highest maximum gap. While the algorithm is progressing, it dynamically updates the gaps for every vertex in each step. We now state our main result.

Theorem 6 The three domatic number problem can be solved by a deterministic algorithm running in time $\tilde{O}(2.9416^n)$.

Proof. Let G = (V, E) be the given graph. The algorithm seeks to find a partition of V into three disjoint dominating sets. Note that every vertex $v \in V$ is contained in one of these sets and is dominated by the remaining two sets, i.e., it is adjacent to at least one of their elements. The algorithm is described in pseudo-code in the appendix, see Figures 2, 3, 3, 4, 5, and 6. Since $\delta(G) \leq min-deg(G) + 1$, we may assume that $min-deg(G) \geq 2$.

The algorithm starts by initializing the potential dominating sets D_1, D_2 , and D_3 and the auxiliary sets A_1, A_2 , and A_3 , setting each to the empty set. The initial partition thus is $\mathcal{P} = (\emptyset, \emptyset, \emptyset, V)$ and the initial triple of auxiliary sets is $\mathcal{A} = (\emptyset, \emptyset, \emptyset)$.

Then, the recursive function DOMINATE is called for the first time. It is always invoked with graph G, a partition $\mathcal{P} = (D_1, D_2, D_3, R)$, and a triple $\mathcal{A} = (A_1, A_2, A_3)$ of not necessarily disjoint auxiliary sets. \mathcal{P} and \mathcal{A} represent a situation in which the vertices in V - R have been assigned to D_1 , D_2 , and D_3 , and $v \in A_i$ means that in some previous recursive call to function DOMINATE the vertex v has been assigned to D_i without successfully changing \mathcal{P} to contain three dominating sets.

Function DOMINATE starts by calling RECALCULATE-GAPS, which calculates all gaps with respect to \mathcal{P} and \mathcal{A} . Additionally, openNeighbors_{\mathcal{P}}(v), openSets_{\mathcal{P}}(v), and balance_{\mathcal{P}}(v) are determined for every vertex $v \in V$. Four trivial cases can occur.

- **Case 1:** The sets D_1 , D_2 , and D_3 are dominating sets of graph G. In this case, we are done and may add the remaining vertices $v \in R$ to any set D_i , say to D_1 .
- **Case 2:** For some vertex $v \in V$, we have $\text{balance}_{\mathcal{P}}(v) < 0$. That is, there are less vertices in $R \cap N[v]$ than dominating sets with $v \notin N[D_i]$. Thus, no matter how the vertices in $R \cap N[v]$ are assigned, \mathcal{P} won't contain three dominating sets. We have run into a dead-end and return to the previous level of the recursion.

- **Case 3:** There exists a vertex $v \in R$ that is also a member of two of the auxiliary sets A_1, A_2 , and A_3 . Hence, vertex v was previously assigned to two distinct sets D_i and $D_j, 1 \le i < j \le 3$, but the recursion returned without success. We assign v to the only possible set D_k left, with $i \ne k \ne j$.
- **Case 4:** For some vertex $v \in V$, we have $\operatorname{balance}_{\mathcal{P}}(v) = 0$ and $||\operatorname{openSets}_{\mathcal{P}}(v)|| > 0$. That is, v is a critical vertex, since it is not dominated by all three sets D_1 , D_2 , and D_3 contained in the current \mathcal{P} , and there are as many open neighbors as open sets left for it. Note that this is the case for each vertex v with deg(v) = 2 and $N[v] \cap R \neq \emptyset$, as v and its two neighbors have to be assigned to three different dominating sets. We select one of the at most three vertices left in $N[v] \cap R$, say u, and call function $\operatorname{ASSIGN}(G, \mathcal{P}, \mathcal{A}, u, i)$ for all i with $u \notin A_i$.

Function HANDLE-CRITICAL-VERTEX deals with the latter three of these trivial cases. After they have been ruled out, one of the remaining vertices $v \in R$ is selected and assigned to one of the three sets D_i , under the constraint that a vertex $v \in R$ cannot be added to D_i if it is already a member of A_i . This case occurs whenever the recursion returns because no three dominating sets could be found with this combination. The recursion continues by calling $ASSIGN(G, \mathcal{P}, \mathcal{A}, v, i)$, which adds v to D_i , and then calls $DOMINATE(G, \mathcal{P}, \mathcal{A})$. If no three dominating sets are found by this choice, we remember this by adding v to the set A_i . A final call to DOMINATE is made without assigning a vertex to one potential dominating set D_i . If this call fails, the recursion returns to the previous level. This completes the description of the algorithm. We now argue that it is correct and estimate its running time.

To see that the algorithm works correctly, note that it outputs three sets D_1 , D_2 , and D_3 only if they each are dominating sets of G. It remains to prove that these sets are definitely found in the recursion tree. All drop-backs within the recursion occur when, for the current $\mathcal{P} = (D_1, D_2, D_3, R)$, we have $\text{balance}_{\mathcal{P}}(v) < 0$ for some vertex $v \in V$. Thus, \mathcal{P} cannot be modified so as to contain a correct partition into three dominating sets on this branch of the recursion tree. Since the algorithm checks every possible partition of G into three sets, unless it is stopped by such a drop-back, a partition into three dominating sets will be found, if it exists. If the algorithm does not find three dominating sets, it eventually terminates when returning from the first recursive call of function DOMINATE. It reports the failure, and thus always yields the correct output.

To estimate the running time of the algorithm, an important observation is that the recalculation of the gaps takes no more than quadratic time in n, the number of vertices of the graph G. Thus, in terms of the \tilde{O} -notation, the running time of the algorithm depends solely on the number of recursive calls. Let T(m) be the number of steps of the algorithm, where m is the number of potential dominating sets left for all vertices that have not been selected as yet. Initially, every vertex may be a member of any of the three dominating sets to be constructed (if they exist), hence m = 3n.

There are two scenarios where the algorithm calls function DOMINATE recursively. If HANDLE-CRITICAL-VERTEX detects a vertex $v \in V$ as being critical, it selects a vertex $u \in N[v] \cap R$ and calls function ASSIGN (and thus DOMINATE) for each *i* with $u \notin A_i$. Since every critical vertex $v \in V$ remains critical as long as $N[v] \cap R \neq \emptyset$, function HANDLE-CRITICAL-VERTEX will be called until all vertices in $N[v] \cap R$ have been assigned to any of D_1 , D_2 , and D_3 . Since $||\text{openSets}_{\mathcal{P}}(v)|| \leq 3$, at most three vertices in the closed neighborhood of v have not been assigned when v turns critical. By Proposition 5, all vertices in $N[v] \cap R$ have to be assigned to different dominating sets. If $||\text{openNeighbors}_{\mathcal{P}}(v)|| = 3$, we have at most six combinations; if we have two open neighbors for a critical vertex, there are at most two combinations left; and finally, for one open neighbor $u \in N[v] \cap R$, there remains only one possible choice to assign u to one of the sets D_1 , D_2 , and D_3 . Thus, in the worst case, we have $T(m) \leq 6T(m-6)$, as we will handle three vertices for which at least two choices for dominating sets are left. With m = 3n, it follows that $T(m) \leq 6^{m/6} = 6^{n/2}$, i.e., $T(m) = \tilde{\mathcal{O}}(2.4495^n)$.

The only other branching into two different recursive calls happens in the main body of function DOMINATE, when selecting a vertex v with the currently highest maximum gap with respect to \mathcal{P} and \mathcal{A} . Two cases might occur. On the one hand, we might have considered a correct dominating set D_i for v. If v had not been looked at so far, i.e., if v is not contained in any set A_j , $1 \le j \le 3$, $j \ne i$, we have eliminated all three possible sets for v to belong to. Thus, in this case, T(m) = T(m-3). On the other hand, if the algorithm returns from the recursion and thus did not make the right choice for v, we have T(m) = T(m-1), since v is added to A_i , and function DOMINATE is called without assigning any vertex. Summing up, we have $T(m) \leq 1$ T(m-1) + T(m-3). In the second case, we have already visited vertex v in a previous stage of the algorithm and unsuccessfully tried to assign it to some set D_i , with $1 \leq j \leq 3$. There are only two dominating sets for v left. Either way, if we put v into the correct dominating set right away or fail the first time, we have T(m) =T(m-2). Summing up both cases, we have $T(m) \leq 2T(m-2)$. Suppose that the first and the second case occur equally often, i.e., the algorithm considers every vertex twice. It then follows that

$$T(m) \le \frac{1}{2}(T(m-1) + T(m-3)) + \frac{1}{2}(2T(m-2))$$

with m = 3n. Thus, we have $T(m) = \tilde{\mathcal{O}}(3^n)$, and the trivial time bound cannot be beaten. To improve this running time, we have to make sure that the recursion tree will not reach its full depth, i.e., not all vertices are considered by the algorithm or function HANDLE-CRITICAL-VERTEX will be called for a sufficiently large portion of the vertices. It is clear that the algorithm has found three dominating sets once $\operatorname{area}_{\mathcal{P}}(G) = 3n$ (recall the notions from Definition 4). By selecting the maximum gap possible for a partition \mathcal{P} , we try to reach this goal as fast as possible. For every vertex $v \in R$ that we assign to one of the potential dominating sets D_i , $1 \le i \le 3$, we increase $\operatorname{area}_{\mathcal{P}}(G)$ by $\operatorname{gap}_{\mathcal{P},\mathcal{A}}(v,i)$, and additionally we add $(\operatorname{gap}_{\mathcal{P},\mathcal{A}}(v,i)-3)$ to $\operatorname{surplus}_{\mathcal{P}}(G)$.

Since the vertices of degree two are critical, they and their neighbors can be handled in time $\tilde{\mathcal{O}}(2.4495^n)$, as argued above. So assume that $min\text{-}deg(G) \geq 3$. Then, we have $\max \operatorname{gap}_{\mathcal{P},\mathcal{A}}(G) > 3$ at the start of the algorithm. If this condition remains to hold for at least 3n/4 steps, we have reached $\operatorname{area}_{\mathcal{P}}(G) = 3n$, and the algorithm terminates successfully. To make use of more than 3n/4 vertices, $\max \operatorname{gap}_{\mathcal{P},\mathcal{A}}(G)$ has to drop below four at one point of the computation. We exploit the fact that up to this point, the surplus has grown sufficiently large with respect to n. Decreasing it will force $\max_{\mathcal{P},\mathcal{A}}(G)$ to drop below three, and this condition can hold only for a certain portion of the remaining vertices until the algorithm terminates. To see this, we now analyze the remaining steps of the algorithm after the given graph G has reached a certain maximum gap with respect to the current \mathcal{P} and \mathcal{A} .

If $\max_{\mathcal{A}}(G) = 0$, the recursion stops immediately. Either we have already found three disjoint dominating sets (in which case we put the remaining vertices $v \in R$ into set D_1 and halt), or one vertex has not been dominated by one set D_i in \mathcal{P} yet. Since no positive gaps exist for the vertices $v \in R$, \mathcal{P} cannot be modified to a valid partition into three dominating sets. Function HANDLE-CRITICAL-VERTEX returns true immediately after detecting balance $\mathcal{P}(v) < 0$ for some vertex $v \in V$, and function DOMINATE drops back one recursion level. The question is how many vertices are left in R when we reach $\max_{\mathcal{A}} \mathcal{P}_{\mathcal{A}}(G) = 0$.

Lemma 7 Let G = (V, E) be a graph and $\mathcal{P} = (D_1, D_2, D_3, R)$ be a partition of V as in Definition 4. Let r = ||R|| and $\max_{\mathcal{P},\mathcal{A}}(G) = 3$. Then, for at least r/64 vertices in R, the algorithm will not recursively call function DOMINATE.

Proof of Lemma 7. Let $\max \operatorname{gap}_{\mathcal{P},\mathcal{A}}(G) = k$ with k > 0. Since $\operatorname{gap}_{\mathcal{P},\mathcal{A}}(v,i) \leq k$ for each $v \in R$ and for each $i, 1 \leq i \leq 3$, we have $\sum_{v \in R} \operatorname{sumgap}_{\mathcal{P},\mathcal{A}}(v) \leq 3kr$. Every vertex v that is selected for a set D_i with $\operatorname{gap}_{\mathcal{P},\mathcal{A}}(v,i) = k$ decreases at least kgaps of the vertices in $R - \{v\}$ by one. Otherwise, HANDLE-CRITICAL-VERTEX would have found a critical vertex $u \in N[v]$ with $N[u] \cap R = \{v\}$. Then, either $||\text{openSets}_{\mathcal{P}}(u)|| > 1$ (which implies $\operatorname{balance}_{\mathcal{P}}(u) < 0$ and we abort), or $||\text{openSets}_{\mathcal{P}}(u)|| = 1$, in which case v is added to the appropriate set D_i without further branching of function DOMINATE. Thus, if no critical vertex is detected, selecting a vertex $v \in R$ for some set D_i decreases at least k gaps, and since v does not belong to R anymore, additionally all gaps previously defined for v are now undefined. So the lowest possible rate at which the gaps are decreased is related to the maximum gap of G.

Now suppose that $\max_{\mathcal{P},\mathcal{A}}(G) = 3$ and $\operatorname{sumgap}_{\mathcal{P},\mathcal{A}}(v) = 9$ for all vertices $v \in R$. We always select a vertex v with the highest summation gap of all vertices $u \in R$ with $\max_{\mathcal{P},\mathcal{A}}(u) = 3$. As long as there exists a vertex $v \in R$ with $\operatorname{gap}_{\mathcal{P},\mathcal{A}}(v,i) = 3$ for all i, it will be selected by the algorithm. After calling function RECALCULATE-GAPS, the number of gaps equal to three will be decreased at least by six. If exactly three other gaps of vertices in $R - \{v\}$ decrease by one in every step, it takes at least r/4 vertices until $\operatorname{sumgap}_{\mathcal{P},\mathcal{A}}(v) < 9$ for all $v \in R$. Another 1/4 of the 3r/4 vertices remaining have to be selected until $\operatorname{sumgap}_{\mathcal{P},\mathcal{A}}(v) < 8$. Adding 1/4 of the 9r/16 vertices left in R, we have reached $\operatorname{maxgap}_{\mathcal{P},\mathcal{A}}(G) = 2$ with $\operatorname{sumgap}_{\mathcal{P},\mathcal{A}}(v) = 6$ for all vertices $v \in R$. This implies that every defined gap is equal to two. Summing up, we have selected

$$\frac{1}{4} \cdot r + \frac{1}{4} \cdot \frac{3}{4}r + \frac{1}{4} \cdot \frac{9}{16}r = \frac{37}{64}r$$

vertices until maxgap_{\mathcal{P},\mathcal{A}}(G) = 2, under the constraint that a minimum number of gaps is reduced in each step, while simultaneously trying to reduce the maximum summation

gap in the fastest possible way. This way we reach level $\max_{\mathcal{P},\mathcal{A}}(G) = 0$ with as few vertices left in R as possible, which describes the worst case that might happen.

Analogously, we can show that $\max_{\mathcal{P},\mathcal{A}}(G)$ drops from 2 to 1 after selecting another 19r/64 vertices. And once we have $\max_{\mathcal{P},\mathcal{A}}(G) = 1$, it takes 7r/64vertices to get to $\max_{\mathcal{P},\mathcal{A}}(G) = 0$. Now, there are r/64 vertices remaining in R, which do not have to be processed recursively.

Continuing the proof of Theorem 6, note that we assumed $\min\text{-deg}(G) \geq 3$, so when the gaps are initialized for graph G, we have $\operatorname{mingap}_{\mathcal{P},\mathcal{A}}(v) \geq 4$ for each vertex $v \in V$. Thus, more than three vertices are dominated by the selected set D_i for vertex v. As long as $\operatorname{maxgap}_{\mathcal{P},\mathcal{A}}(G) > 3$ is true, $\operatorname{surplus}_{\mathcal{P}}(G)$ is increasing. The only way to lower the surplus is by adding vertices v to a set D_i with $\operatorname{gap}_{\mathcal{P},\mathcal{A}}(v,i) < 3$. The surplus decreases by one when $\operatorname{gap}_{\mathcal{P},\mathcal{A}}(v,i) = 2$, and it decreases by two when $\operatorname{gap}_{\mathcal{P},\mathcal{A}}(v,i) = 1$.

Let $S = \operatorname{surplus}_{\mathcal{P}}(G)$ be the surplus collected for a partition \mathcal{P} until we reach a point where $\operatorname{maxgap}_{\mathcal{P},\mathcal{A}}(G) = 3$. To make use of the most recursive calls and to even out the surplus completely, there have to be at least r = ||R|| vertices remaining with

$$0 \cdot \frac{37r}{64} + 1 \cdot \frac{19r}{64} + 2 \cdot \frac{7r}{64} = S,$$

so $r \ge 64S/33$. A fraction of 1/64 of these vertices will be handled by the algorithm without branching into more than one recursive call, which is at least S/33. The question is how big the surplus S might grow and how many vertices are left in Rbefore maxgap_{\mathcal{P},\mathcal{A}}(G) = 3 is reached. The lowest surplus with as few vertices in R as possible occurs if min-deg(G) = max-deg(G) = 3. Surplus S is increased by one in each step until we arrive at maxgap_{\mathcal{P},\mathcal{A}}(G) = 3. When selecting a vertex v of degree 3 for a set D_i , the gap of its neighbors $u \in N(v)$ and the gaps of the neighbors of every u might be decreased. Summing up, at most $1 + 3 + 3 \cdot 2 = 10$ vertices can have decreased their gaps for some i. After selecting at least n/10 vertices for each i, we have mingap_{\mathcal{P},\mathcal{A}}(G) = 3 (in the worst case). From this point on, we cannot be sure if the next vertex selected for some D_i satisfies gap_{\mathcal{P},\mathcal{A}}(v,i) > 3. But so far we have already collected a surplus of S = 3n/10, and applying this we obtain $64n/110 \le r \le 7n/10$. Thus, for at least n/110 vertices we never branch into two different recursive calls. Setting m = 3(109n/110), we obtain a running time of $\tilde{\mathcal{O}}(2.9416^n)$.

4 Graphs with Bounded Maximum Degree

As seen in the last section, the running time of the algorithm crucially depends on the degrees of the vertices of G. If we restrict ourselves to graphs G with bounded maximum degree (say $\Delta = max \cdot deg(G)$), we can optimize our strategy in finding three disjoint dominating sets. In this section, we present a simple deterministic algorithm, which has a better running time than the algorithm from Theorem 6, provided that Δ is low. By using randomization, we can further improve the running time for graphs G with low maximum degree. Before stating the two results, note that graphs with maximum degree two can trivially be partitioned into three dominating sets, if such a partition exists. Every component of such a graph is either an isolated vertex, a path, or a cycle, and each such property can be recognized in polynomial time.

Proposition 8 Let G = (V, E) be a given graph with max-deg(G) = 2. There exists a partition of the vertices of G into three dominating sets if and only if every component of G is a cycle of length k such that 3 divides k.

We use the terms from Definition 4 in Section 3 to describe a snapshot within the algorithm. For any partition $\mathcal{P} = (D_1, D_2, D_3, R)$, some vertices of V have already been assigned to the potential dominating sets D_1, D_2 , and D_3 , while all the remaining vertices are in R. The auxiliary sets $\mathcal{A} = (A_1, A_2, A_3)$ will not be needed in this section. Only connected graphs are considered, as it is possible to treat every connected component separately, producing the desired output within the same time bounds.

Table 1 lists the running times of both the deterministic and the random algorithm, where the maximum degree of the input graph is bounded by Δ , $3 \le \Delta \le 8$. Note that the exact deterministic algorithm from Theorem 6 in Section 3 beats the deterministic algorithm from Theorem 9 whenever $\Delta \ge 7$.

Δ	3	4	5	6	7	8
deterministic	2.2894^{n}	2.6591^{n}	2.8252^{n}	2.9058^{n}	2.9473^{n}	2.9697^{n}
randomized	2^n	2.3570^{n}	2.5820^{n}	2.7262^{n}	2.8197^{n}	2.8808^{n}

Table 1: Results for max-deg(G) = k, where $3 \le k \le 8$

Theorem 9 Let G = (V, E) be a graph with max-deg $(G) = \Delta$, where $\Delta \geq 3$. There exists a deterministic algorithm solving the three domatic number problem in time $\tilde{O}(d^{\frac{n}{\Delta}})$, where

$$d = \sum_{a=0}^{\Delta-2} \left[\begin{pmatrix} \Delta \\ a \end{pmatrix} \sum_{b=1}^{\Delta-a-1} \begin{pmatrix} \Delta-a \\ b \end{pmatrix} \right].$$
(4.1)

Proof. The algorithm works as follows. We start with an arbitrary vertex $v \in V$ and assign it to the first set D_1 . In each step, we first check whether we found a partition $\mathcal{P} = (D_1, D_2, D_3, R)$ into dominating sets D_1, D_2 , and D_3 . If not, one vertex $v \in V$ is selected that is not dominated by all three sets D_1, D_2 , and D_3 , and additionally has a vertex $u \in N[v]$ in its closed neighborhood that has already been added to some set $D_i, 1 \leq i \leq 3$. It follows that $1 \leq ||\text{openSets}_{\mathcal{P}}(v)|| \leq 2$.

If balance_{\mathcal{P}}(v) < 0, we return within the recursion. Otherwise, we try all combinations to partition the vertices in $N[v] \cap R$, so that after this step vertex v is dominated by all three potential dominating sets. If no such combination leads to a valid partition, we again return within the recursion.

Suppose now that $\text{balance}_{\mathcal{P}}(v) \ge 0$, $||\text{openSets}_{\mathcal{P}}(v)|| = 2$, and $N[v] \cap D_1 \neq \emptyset$. To obtain three disjoint dominating sets, at least one vertex in N[v] has to be assigned to D_2 , and at least one vertex in N[v] has to be added to D_3 . This limits our choices, especially if the degree of v is bounded by some constant Δ . To measure the running time of the algorithm, we consider the worst case with the most possible combinations that might yield a partition into three dominating sets. This occurs when only one vertex $u \in N[v]$ has already been added to one set, i.e., $||N[v] \cap (D_1 \cup D_2 \cup D_3)|| = 1$. If $N[v] \cap D_1 \neq \emptyset$, then any number between 0 and $\Delta - 2$ of vertices in $N[v] \cap R$ may be assigned to the same set D_1 . Let this number be a. It follows that from one to $\Delta - a - 1$ vertices remaining in $N[v] \cap R$ are allowed to be in the next potential dominating set D_2 . This is how Equation 4.1 for d is derived. After assigning the last vertices in $N[v] \cap R$ to the dominating set D_3 , exactly Δ vertices have been removed from R. Thus, we have a worst case running time of $\tilde{\mathcal{O}}(d^{\frac{\pi}{\Delta}})$. Table 1 lists the running time for graphs with maximum degree from three to nine.

In the next theorem, randomization is used to speed up this procedure. Instead of assigning all vertices in the closed neighborhood of some vertex $v \in V$ in one step, only one or two vertices in $N[v] \cap R$ are added to the potential dominating sets D_1 , D_2 , and D_3 . The goal is to dominate one vertex by all three sets in one step. We will select the one or two vertices that are missing for this goal at random.

Theorem 10 Let G = (V, E) be a graph with max-deg $(G) = \Delta$, where $\Delta \ge 3$, and let d be defined as in Equation (4.1) in Theorem 9. For each constant c > 0, there exists a randomized algorithm solving the three domatic number problem with error probability at most e^{-c} in time $\tilde{O}(r^{\frac{n}{2}})$, where

$$r = \frac{d}{3^{\Delta - 2}}.\tag{4.2}$$

Proof. Let graph G = (V, E) be given with max-deg(G) = k. As in the deterministic algorithm, we start by adding a random vertex to the set D_1 . In every following step, a vertex $v \in V$ is selected with $0 < ||\text{openSets}_{\mathcal{P}}(v)|| < 3$, so it is $N[v] \cap (D_1 \cup D_2 \cup D_3) \neq \emptyset$. If $||\text{openSets}_{\mathcal{P}}(v)|| = 1$, we have $N[v] \cap D_i = \emptyset$ for one i with $1 \le i \le 3$. We randomly choose a vertex $u \in N[v] \cap R$ and assign it to set D_i , in order that v is dominated by all three sets afterwards. If $||\text{openSets}_{\mathcal{P}}(v)|| = 2$, we randomly select two vertices $u_1, u_2 \in R$ in the closed neighborhood of v. Another random choice is made when deciding how to distribute these two vertices among the two potential dominating sets that have not dominated v up to now.

Suppose G indeed has a partition into three dominating sets. We have to measure the error rate when making our random choices to estimate the success probability of the algorithm. In every step, a vertex $v \in V$ is selected with at least one vertex $u \in N[v]$ in its closed neighborhood that has already been added to one of the sets D_1 , D_2 , or D_3 . The highest error occurs when exactly one vertex in N[v] is not included in R, so we restrict our analysis to this case. To obtain a valid partition into three dominating sets, there are at most d choices left to partition the vertices remaining in $N[v] \cap R$. Here, d is the number from Equation 4.1. Once we selected and assigned two vertices from $N[v] \cap R$ at random, there are 3^{k-2} possibilities left to partition the vertices in the closed neighborhood of v that are still left in R. Our success rate when selecting the two vertices is therefore $3^{k-2}/d$.

To achieve an error probability of below e^{-c} , the algorithm needs to be executed more than once. The repetition number of the algorithm equals the reciprocal of the success rate, which explains Equation 4.2. Since two vertices are processed in every step, the overall running time is $\tilde{O}(r^{\frac{n}{2}})$.

5 Conclusion

We have shown that the three domatic number problem can be solved by a deterministic algorithm in time $\tilde{\mathcal{O}}(2.9416^n)$. Furthermore, we presented two algorithms solving the three domatic number problem for graphs with bounded maximum degree, improving the above time bound for graphs with small maximum degree. Although our running times seem to be not too big of an improvement of the trivial $\tilde{\mathcal{O}}(3^n)$ bound, they are to our knowledge the first such algorithms breaking this barrier. For k > 3, the decision problem of whether $\delta(G) \geq k$ can be solved in time $\tilde{\mathcal{O}}(3^n)$ by Lawler's dynamic programming algorithm for the chromatic number, appropriately modified for the domatic number problem. Therefore, it would not be reasonable to use our gap approach of Section 3 to decide if $\delta(G) \geq k$ for a graph G and k > 3.

Acknowledgement. We thank Dieter Kratsch for pointing us to Lawler's algorithm.

References

- [BK04] T. Brueggemann and W. Kern. An improved local search algorithm for 3-SAT. Technical Report Memorandum No. 1709, University of Twenty, Department of Applied Mathematics, Enschede, The Netherlands, 2004.
- [Bon85] M. Bonuccelli. Dominating sets and dominating number of circular arc graphs. *Discrete Applied Mathematics*, 12:203–213, 1985.
- [CH77] E. Cockayne and S. Hedetniemi. Towards a theory of domination in graphs. *Networks*, 7:247–261, 1977.
- [DGH⁺02] E. Dantsin, A. Goerdt, E. Hirsch, R. Kannan, J. Kleinberg, C. Papadimitriou, P. Raghavan, and U. Schöning. A deterministic $(2 - 2/(k + 1))^n$ algorithm for k-SAT based on local search. *Theoretical Computer Science*, 289(1):69–83, October 2002.
- [Epp01a] D. Eppstein. Improved algorithms for 3-coloring, 3-edge-coloring, and constraint satisfaction. In *Proceedings of the 12th ACM-SIAM Symposium* on Discrete Algorithms, pages 329–337. Society for Industrial and Applied Mathematics, 2001.
- [Epp01b] D. Eppstein. Small maximal independent sets and faster exact graph coloring. In Proceedings of the 7th Workshop on Algorithms and Data Structures, pages 462–470. Springer-Verlag Lecture Notes in Computer Science #2125, 2001.
- [Far84] M. Farber. Domination, independent domination, and duality in strongly chordal graphs. *Discrete Applied Mathematics*, 7:115–130, 1984.

- [FHK00] U. Feige, M. Halldórsson, and G. Kortsarz. Approximating the domatic number. In *Proceedings of the 32nd ACM Symposium on Theory of Computing*, pages 134–143. ACM Press, May 2000.
- [FKW04] F. Fomin, D. Kratsch, and G. Woeginger. Exact (exponential) algorithms for the dominating set problem. In *Proceedings of the 30th International Workshop on Graph-Theoretic Concepts in Computer Science (WG 2004)*, pages 245–256. Springer-Verlag *Lecture Notes in Computer Science* #3353, 2004.
- [GJ79] M. Garey and D. Johnson. Computers and Intractability: A Guide to the Theory of NP-Completeness. W. H. Freeman and Company, New York, 1979.
- [HSSW02] T. Hofmeister, U. Schöning, R. Schuler, and O. Watanabe. A probabilistic 3-SAT algorithm further improved. In *Proceedings of the 19th Annual Symposium on Theoretical Aspects of Computer Science*, pages 192–202. Springer-Verlag *Lecture Notes in Computer Science #2285*, 2002.
- [HT98] P. Heggernes and J. Telle. Partitioning graphs into generalized dominating sets. *Nordic Journal of Computing*, 5(2):128–142, 1998.
- [IT03] K. Iwama and S. Tamaki. Improved upper bounds for 3-SAT. Technical Report TR03-053, Electronic Colloquium on Computational Complexity, July 2003. 3 pages.
- [KS94] H. Kaplan and R. Shamir. The domatic number problem on some perfect graph families. *Information Processing Letters*, 49(1):51–56, January 1994.
- [Law76] Eugene L. Lawler. A note on the complexity of the chromatic number problem. *Information Processing Letters*, 5(3):66–67, 1976.
- [PPSZ98] R. Paturi, P. Pudlák, M. Saks, and F. Zane. An improved exponentialtime algorithm for k-SAT. In Proceedings of the 39th IEEE Symposium on Foundations of Computer Science, pages 628–637. IEEE Computer Society Press, November 1998.
- [Rob86] J. Robson. Algorithms for maximum independent sets. Journal of Algorithms, 7:425–440, December 1986.
- [RR04] T. Riege and J. Rothe. Complexity of the exact domatic number problem and of the exact conveyor flow shop problem. *Theory of Computing Systems*, December 2004. On-line publication DOI 10.1007/s00224-004-1209-8. Paper publication to appear.
- [Sch02] U. Schöning. A probabilistic algorithm for *k*-SAT based on limited local search and restart. *Algorithmica*, 32(4):615–623, 2002.
- [Sch05] U. Schöning. Algorithmics in exponential time. In Proceedings of the 22nd Annual Symposium on Theoretical Aspects of Computer Science, pages 36–43. Springer-Verlag Lecture Notes in Computer Science #3404, 2005.

[Woe03] G. Woeginger. Exact algorithms for NP-hard problems. In M. Jünger, G. Reinelt, and G. Rinaldi, editors, *Combinatorical Optimization:* "Eureka, you shrink!", pages 185–207. Springer-Verlag Lecture Notes in Computer Science #2570, 2003.

A **Proof of Proposition 3**

Proof. Figure 1 shows the graphs G and H whose existence is claimed. In this figure, the numbers i|j within a vertex have the following meaning: i indicates which dominating set D_i this vertex belongs to in a fixed partition into three dominating sets, and j indicates a specific choice of a minimum dominating set S of the graph by setting j = 1 if and only if this vertex belongs to S.



Figure 1: Graphs G and H for Proposition 3

For the first assertion, look at the graph G shown on the left-hand side of Figure 1. Note that $\gamma(G) = 2$. In particular, $D = \{u_3, u_5\}$ is a minimum dominating set of G. Note further that $\delta(G) = 3$. In particular, a partition into three dominating sets of G is given by $D_1 = \{u_1, u_4, u_7\}$, $D_2 = \{u_2, u_5\}$, and $D_3 = \{u_3, u_6\}$. However, D cannot be part of any partition into three dominating sets, since the only neighbors of u_4 , namely u_3 and u_5 , belong to D.

Note that the minimum dominating set $D_2 = \{u_2, u_5\}$ of G defined above indeed is part of a partition into three dominating sets. The second part of the proposition, however, shows that this is not always the case. Consider the graph H = (V, E) shown on the right-hand side of Figure 1. We have $\gamma(H) = 2$ by choosing the minimum dominating set $D = \{v_1, v_2\}$, which is unique in this case. Again, $\delta(H) = 3$. The only way, up to isomorphism, to partition the vertex set of H into three dominating sets is given by $D_1 = \{v_1, v_7, v_8\}$, $D_2 = \{v_2, v_6, v_9\}$, and $D_3 = \{v_3, v_4, v_5\}$. Thus, $\min\{||D_1||, ||D_2||, ||D_3||\} > \gamma(H)$ for each partition into three dominating sets.

B Pseudo-Code of the Algorithm from Theorem 6

Figures 2, 3, 4, 5, and 6 describe the algorithm from Theorem 6 in pseudo-code.

Algorithm for the Three Domatic Number Problem

Input: Graph G = (V, E) with vertex set $V = \{v_1, v_2, ..., v_n\}$ and edge set E **Output:** Partition of V into three dominating sets $D_1, D_2, D_3 \subseteq V$ or "failure" Set each of D_1, D_2, D_3, A_1, A_2 , and A_3 to the empty set; Set R = V; Set $\mathcal{P} = (D_1, D_2, D_3, R)$; Set $\mathcal{A} = (A_1, A_2, A_3)$; DOMINATE $(G, \mathcal{P}, \mathcal{A})$; // Start recursion **output** "failure" and **halt**;

Figure 2: Algorithm for the Three Domatic Number Problem

```
Function DOMINATE(G, \mathcal{P}, \mathcal{A}) {
                                                                                  // \mathcal{P} is a partition of graph G,
                                                                                 // A is a triple of auxiliary sets
   RECALCULATE-GAPS(G, \mathcal{P}, \mathcal{A});
   if (each D_i is a dominating set) {
       D_1 = D_1 \cup R;
       output D_1, D_2, D_3;
   }
   if (not HANDLE-CRITICAL-VERTEX(G, \mathcal{P}, \mathcal{A})) {
       select vertex v \in R with
           \max_{\mathcal{P},\mathcal{A}}(v) = \max_{\mathcal{P},\mathcal{A}}(G) and
          \operatorname{sumgap}_{\mathcal{P},\mathcal{A}}(v) = \max \{\operatorname{sumgap}_{\mathcal{P},\mathcal{A}}(u) \mid u \in R \land \operatorname{maxgap}_{\mathcal{P},\mathcal{A}}(u) =
              maxgap_{\mathcal{P}.\mathcal{A}}(G)\};
       find i with gap_{\mathcal{P},\mathcal{A}}(v,i) = maxgap_{\mathcal{P},\mathcal{A}}(v);
       ASSIGN(G, \mathcal{P}, \mathcal{A}, v, i);
                                                                                               // First recursive call
       A_i = A_i \cup \{v\};
                                                           // If recursion fails, put v in A_i and try again
       DOMINATE(G, \mathcal{P}, \mathcal{A});
                                                                                           // Second recursive call
   }
   return;
}
```

Figure 3: Recursive function to dominate graph G

Function ASSIGN $(G, \mathcal{P}, \mathcal{A}, v, i)$ {

```
D_i = D_i \cup \{v\};

R = R - \{v\};

Dominate(G, P, A);

}
```

Figure 4: Function to assign vertex v to set D_i

Function RECALCULATE-GAPS $(G, \mathcal{P}, \mathcal{A})$ { $//\mathcal{P}$ is a partition of graph G, //A is a triple of auxiliary sets for all (vertices $v \in V$) { if (vertex $v \in R$) { for all (i = 1, 2, 3) { $\text{if } (v \notin A_i) \{ \text{gap}_{\mathcal{P},\mathcal{A}}(v,i) = ||N[v]|| - ||\{u \in N[v] \mid (\exists w \in N[u]) | (w \in N[u]) \} \}$ D_i]}||; } else gap_{\mathcal{P},\mathcal{A}} $(v,i) = \bot$; // \bot indicates that gap_{\mathcal{P},\mathcal{A}}(v,i) is undefined }
$$\begin{split} & \max_{i \in \{1,2,3\}} \{ \operatorname{gap}_{\mathcal{P},\mathcal{A}}(v,i) \}; \\ & \operatorname{mingap}_{\mathcal{P},\mathcal{A}}(v) = \operatorname{min}_{i \in \{1,2,3\}} \{ \operatorname{gap}_{\mathcal{P},\mathcal{A}}(v,i) \}; \\ & \operatorname{sumgap}_{\mathcal{P},\mathcal{A}}(v) = \sum_{i \in \{1,2,3\}} \operatorname{gap}_{\mathcal{P},\mathcal{A}}(v,i); \end{split}$$
} openNeighbors_{\mathcal{P}} $(v) = \{u \in N[v] \mid u \in R\};$ openSets_{\mathcal{P}} $(v) = \{i \in \{1, 2, 3\} | v \notin N[D_i]\};$ $balance_{\mathcal{P}}(v) = ||openNeighbors_{\mathcal{P}}(v)|| - ||openSets_{\mathcal{P}}(v)||;$ } $\operatorname{maxgap}_{\mathcal{P},\mathcal{A}}(G) = \operatorname{max}_{v \in R} \{\operatorname{maxgap}_{\mathcal{P},\mathcal{A}}(v)\};$ $\operatorname{mingap}_{\mathcal{P},\mathcal{A}}(G) = \operatorname{min}_{v \in R} \{\operatorname{mingap}_{\mathcal{P},\mathcal{A}}(v)\};$ }

Figure 5: Function to recalculate gaps after partition has changed

Function boolean HANDLE-CRITICAL-VERTEX $(G, \mathcal{P}, \mathcal{A})$ {

```
for all (vertices v \in V) {
    if (\text{balance}_{\mathcal{P}}(v) < 0) {
                                             // impossible to three dominate v with \mathcal{P}
       return true;
    } else if (||\{i \in \{1, 2, 3\} | v \in A_i\}|| == 2) \{ // one choice for v remaining
       select i with v \notin A_i;
       ASSIGN(G, \mathcal{P}, \mathcal{A}, v, i);
       return true;
    select u \in N[v] \cap R;
       for all (i with u \notin A_i and v not dominated by D_i)
          ASSIGN(G, \mathcal{P}, \mathcal{A}, u, i);
       return true;
     }
  }
  return false;
                                                     // no critical vertices were found
}
```

Figure 6: Function to handle the critical vertices