

# Defensive forecasting for linear protocols

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## Abstract

We consider a general class of forecasting protocols, called “linear protocols”, and discuss several important special cases, including multi-class forecasting. Forecasting is formalized as a game between three players: Reality; Forecaster, whose goal is to predict Reality’s next move; and Skeptic, who tries to make money on any lack of agreement between Forecaster and Reality. Our main mathematical result is that for any continuous strategy for Skeptic in a linear protocol there exists a strategy for Forecaster that does not allow Skeptic’s capital to grow. This result is a meta-theorem that allows one to transform any constructive law of probability in a linear protocol into a forecasting strategy whose predictions are guaranteed to satisfy this law. We apply this meta-theorem to a weak law of large numbers in inner product spaces to obtain a version of the K29 prediction algorithm for linear protocols and show that this version also satisfies the attractive properties of proper calibration and resolution under a suitable choice of its kernel parameter, with no assumptions about the data-generating mechanism.

## 1 Introduction

In [14] we suggested a new methodology for designing forecasting strategies. Considering only the simplest case of binary forecasting, we showed that any constructive, in the sense explained below, law of probability can be translated into a forecasting strategy that satisfies this law. In this paper this result is extended to a general class of protocols including multi-class forecasting. In

proposing this approach to forecasting we were inspired by [3] and papers further developing [3], although our methods and formal results appear to be completely different.

Whereas the meta-theorem stated in [14] is mathematically trivial, we have to overcome some technical difficulties in the generalization considered in this paper. Our general meta-theorem is stated in §4. The general forecasting protocols covered by this theorem are introduced and discussed in §§2–3.

In [14] we demonstrated the value of the meta-theorem by applying it to the strong law of large numbers, obtaining from it a kernel forecasting strategy which we called K29. The derivation, however, was informal, involving heuristic transitions to a limit, and this made it impossible to state formally any properties of K29. In this paper we deduce K29 in a much more direct way from the weak law of large numbers and state its properties. (For binary forecasting, this was also done in [13], and the reader might prefer to read that paper first.) The weak law of large numbers is stated and proved in §5, and K29 is derived and studied in §6.

We call the approach to forecasting using our meta-theorem “defensive forecasting”: Forecaster is trying to defend himself when playing against Skeptic. The justification of this approach given in this paper and in [13] is K29’s properties of proper calibration and resolution. Another justification, in a sense the ultimate justification of any forecasts, is given in [12]: defensive forecasts lead to good decisions; this result, however, is obtained for rather simple decision problems requiring only binary forecasts.

The exposition of probability theory needed for this paper is given in [9]. The standard exposition is based on Kolmogorov’s measure-theoretic axioms of probability, whereas [9] states several key laws of probability in terms of a game between the forecaster, the reality, and a third player, the skeptic. The game-theoretic laws of probability in [9] are constructive in that we explicitly construct computable winning strategies for the forecaster in various games of forecasting.

## 2 Forecasting as a game

Following [9] and [14] we consider the following general forecasting protocol:

FORECASTING GAME 1

**Players:** Reality, Forecaster, Skeptic

**Parameters:**  $\mathbf{X}$  (*object space*),  $\mathbf{Y}$  (*label space*),  $\mathbf{F}$  (*Forecaster’s move space*),  $\mathbf{S}$  (*Skeptic’s move space*),  $\lambda : \mathbf{S} \times \mathbf{F} \times \mathbf{Y} \rightarrow \mathbb{R}$  (*Skeptic’s gain function and Forecaster’s loss function*)

**Protocol:**

$\mathcal{K}_0 := 1.$

FOR  $n = 1, 2, \dots$ :

Reality announces  $x_n \in \mathbf{X}.$

Forecaster announces  $f_n \in \mathbf{F}.$

Skeptic announces  $s_n \in \mathbf{S}$ .  
Reality announces  $y_n \in \mathbf{Y}$ .  
 $\mathcal{K}_n := \mathcal{K}_{n-1} + \lambda(s_n, f_n, y_n)$ .  
END FOR

**Restriction on Skeptic:** Skeptic must choose the  $s_n$  so that his capital is always nonnegative ( $\mathcal{K}_n \geq 0$  for all  $n$ ) no matter how the other players move.

This is a perfect-information protocol: the players move in the order indicated, and each player sees the other player's moves as they are made. It specifies both an initial value for Skeptic's capital ( $\mathcal{K}_0 = 1$ ) and a lower bound on its subsequent values ( $\mathcal{K}_n \geq 0$ ). We will say that  $x_n$  are the *objects*,  $y_n$  are the *labels*,  $(x_n, y_n)$  are the *examples*, and  $f_n$  are the *forecasts*.

Book [9] contains several results (game-theoretic versions of limit theorems of probability theory) of the following form: Skeptic has a strategy that guarantees that either a property of agreement between the forecasts  $f_n$  and examples  $(x_n, y_n)$  is satisfied or Skeptic becomes very rich (without risking bankruptcy, according to the protocol). All specific strategies considered in [9] have computable versions. According to Brouwer's principle (see, e.g., §1 of [11] for a recent review of the relevant literature) they must be automatically continuous; in any case, their continuity can be checked directly. In [14] we showed that, under a special choice of the players' move spaces and Skeptic's gain function  $\lambda$ , for any continuous strategy for Skeptic Forecaster has a strategy that guarantees that Skeptic's capital never increases when he plays that strategy. Therefore, Forecaster has a strategy that ensures various properties of agreement between the forecasts and the examples.

The purpose of this paper is to extend the result of [14] to a wide class of Skeptic's gain functions  $\lambda$ . But first we consider several important special cases of Forecasting Game 1.

## Binary forecasting

The simplest non-trivial case, considered in [14], is where  $\mathbf{Y} = \{0, 1\}$ ,  $\mathbf{F} = [0, 1]$ ,  $\mathbf{S} = \mathbb{R}$ , and

$$\lambda(s_n, f_n, y_n) = s_n(y_n - f_n). \quad (1)$$

Intuitively, Forecaster gives probability forecasts for  $y_n$ :  $f_n$  is his subjective probability that  $y_n = 1$ . The operational interpretation of  $f_n$  is that it is the price that Forecaster charges for a ticket that will pay  $y_n$  at the end of the  $n$ th round of the game;  $s_n$  is the number (positive, zero, or negative) of such tickets that Skeptic chooses to buy.

## Bounded regression

This is the most straightforward extension of binary forecasting, considered in [9], §3.2. The move spaces are

$$\mathbf{Y} = \mathbf{F} = [A, B], \quad \mathbf{S} = \mathbb{R},$$

where  $A$  and  $B$  are two constants, and the gain function is, as before, (1). This protocol allows one to prove a strong law of large numbers ([9], Proposition 3.3) and a simple one-sided law of the iterated logarithm ([9], Corollary 5.1).

## Multi-class forecasting

Another extension of binary forecasting is the protocol where  $\mathbf{Y}$  is a finite set,  $\mathbf{F}$  is the set of all probability distributions on  $\mathbf{Y}$ ,  $\mathbf{S}$  is the set of all real-valued functions on  $\mathbf{Y}$ , and

$$\lambda(s_n, f_n, y_n) = s_n(y_n) - \int s_n df_n.$$

The intuition behind Skeptic's move  $s_n$  is that Skeptic buys the ticket which pays  $s_n(y_n)$  after  $y_n$  is announced; he is charged  $\int s_n df_n$  for this ticket.

The binary forecasting protocol is “isomorphic” to the special case of this protocol where  $\mathbf{Y} = \{0, 1\}$ : Forecaster's move  $f_n$  in the binary forecasting protocol is represented by the probability distribution  $f'_n$  on  $\{0, 1\}$  assigning weight  $f_n$  to  $\{1\}$  and Skeptic's move  $s_n$  in the binary forecasting protocol is represented by any function  $s'_n$  on  $\{0, 1\}$  such that  $s'_n(1) - s'_n(0) = s_n$ . The isomorphism between these two protocols follows from

$$\begin{aligned} s'_n(y_n) - \int s'_n df'_n &= s'_n(y_n) - s'_n(1)f_n - s'_n(0)(1 - f_n) \\ &= s'_n(y_n) - s'_n(0) - s_n f_n = s_n(y_n - f_n) \end{aligned}$$

(remember that  $y_n \in \{0, 1\}$ ).

## Bounded mean-variance forecasting

In this protocol,

$$\mathbf{Y} = [A, B], \quad \mathbf{F} = \mathbf{S} = \mathbb{R}^2$$

and

$$\lambda(s_n, f_n, y_n) = \lambda((M_n, V_n), (m_n, v_n), y_n) = M_n(y_n - m_n) + V_n((y_n - m_n)^2 - v_n).$$

Intuitively, Forecaster is asked to forecast  $y_n$  with a number  $m_n$  and also forecast the accuracy  $(y_n - m_n)^2$  of his first forecast with a number  $v_n$ . This protocol, although usually without the restriction  $y_n \in [A, B]$ , is used extensively in [9] (e.g., in Chaps. 4 and 5).

An equivalent representation of this protocol is

$$\mathbf{Y} = \{(t, t^2) \mid t \in [A, B]\}, \quad \mathbf{F} = \mathbf{S} = \mathbb{R}^2$$

and

$$\lambda(s_n, f_n, y_n) = \lambda((s'_n, s''_n), (f'_n, f''_n), (t_n, t_n^2)) = s'_n(t_n - f'_n) + s''_n(t_n^2 - f''_n).$$

The equivalence of the two representations can be seen as follows: Reality's move  $(x_n, t_n)$  in the first representation corresponds to  $(x_n, y_n) = (x_n, (t_n, t_n^2))$  in the second representation, Forecaster's move  $(m_n, v_n)$  in the first representation corresponds to  $(f'_n, f''_n) = (m_n, v_n + m_n^2)$  in the second representation, and Skeptic's move  $(s'_n, s''_n)$  in the second representation corresponds to  $(M_n, V_n) = (s'_n + 2m_n s''_n, s''_n)$  in the first representation. This establishes a bijection between Reality's move spaces, a bijection between Forecaster's move spaces, and a bijection between Skeptic's move spaces in the two representations; Skeptic's gains are also the same in the two representations:

$$\begin{aligned} & s'_n(t_n - f'_n) + s''_n(t_n^2 - f''_n) \\ &= s'_n(t_n - m_n) + s''_n((t_n - m_n)^2 + 2(t_n - m_n)m_n + m_n^2) - (v_n + m_n^2) \\ &= (s'_n + 2m_n s''_n)(t_n - m_n) + s''_n((t_n - m_n)^2 - v_n). \end{aligned}$$

### 3 Linear protocol

Forecasting Game 1 is too general to derive results of the kind we are interested in. In this subsection we will introduce a narrower protocol which will still be wide enough to cover all special cases considered so far.

All move spaces are now subsets of a Euclidean space  $\mathbf{L}$  (i.e.,  $\mathbf{L} = \mathbb{R}^m$  for some positive integer  $m$ ), equipped with the usual dot product “ $\cdot$ ”. Reality's move space is a non-empty bounded subset  $\mathbf{Y} \subset \mathbf{L}$ , Forecaster's move space  $\mathbf{F}$  is the whole of  $\mathbf{L}$ , and Skeptic's move space  $\mathbf{S}$  is also the whole of  $\mathbf{L}$ . Skeptic's gain function is

$$\lambda(s_n, f_n, y_n) = s_n \cdot (y_n - f_n).$$

Therefore, we consider the following perfect-information game:

FORECASTING GAME 2

**Players:** Reality, Forecaster, Skeptic

**Parameters:**  $\mathbf{L}$  (Euclidean space),  $\mathbf{X}, \mathbf{Y}$  (a non-empty bounded subset of  $\mathbf{L}$ )

**Protocol:**

$\mathcal{K}_0$  is set to a positive number.

FOR  $n = 1, 2, \dots$ :

Reality announces  $x_n \in \mathbf{X}$ .

Forecaster announces  $f_n \in \mathbf{L}$ .

Skeptic announces  $s_n \in \mathbf{L}$ .

Reality announces  $y_n \in \mathbf{Y}$ .

$\mathcal{K}_n := \mathcal{K}_{n-1} + s_n \cdot (y_n - f_n)$ . (2)

END FOR

**Restriction on Skeptic:** Skeptic must choose the  $s_n$  so that his capital is always nonnegative no matter how the other players move.

Let us check that the specific protocols considered in the previous section are covered by this *linear protocol*. At first sight, even the binary forecasting protocol is not covered, as Forecaster's move space is  $[0, 1]$  rather than  $\mathbb{R}$ . It is easy to

see, however, that Forecaster's move  $f_n \notin \overline{\text{co}} \mathbf{Y}$  outside the convex closure  $\overline{\text{co}} \mathbf{Y}$  of Reality's move space (the convex closure  $\overline{\text{co}} A$  of a set  $A$  is defined to be the intersection of all convex closed sets containing  $A$ ) is always inadmissible, in the sense that there exists Skeptic's reply  $s_n$  making him arbitrarily rich regardless of Reality's move, and so we can as well choose  $\mathbf{F} := \overline{\text{co}} \mathbf{Y}$ . Indeed, suppose that  $f_n \notin \overline{\text{co}} \mathbf{Y}$  in the linear protocol. Then  $\overline{\text{co}} \mathbf{Y} - f_n$  is a compact convex set not containing the origin. By the hyperplane separation theorem, there exists a vector  $s_n \in \mathbf{L}$  such that

$$s_n \cdot (y_n - f_n) > 0, \quad \forall y_n \in \overline{\text{co}} \mathbf{Y}.$$

By the compactness of  $\overline{\text{co}} \mathbf{Y}$ ,

$$\inf_{y_n \in \mathbf{Y}} s_n \cdot (y_n - f_n) \geq \min_{y_n \in \overline{\text{co}} \mathbf{Y}} s_n \cdot (y_n - f_n) > 0.$$

Skeptic's move  $Cs_n$  can make him as rich as he wishes as  $C$  can be arbitrarily large. In what follows, we will always assume that Forecaster's move space is  $\overline{\text{co}} \mathbf{Y}$  and use  $\mathbf{F}$  as a shorthand for  $\overline{\text{co}} \mathbf{Y}$ .

Now it is obvious that the binary forecasting, bounded regression, and bounded mean-variance forecasting (in its second representation) protocols are special cases of the linear protocol (with  $\mathbf{F} = \overline{\text{co}} \mathbf{Y}$ ). For the multi-class forecasting protocol, we should represent  $\mathbf{Y}$  as the vertices

$$y^1 := (1, 0, 0, \dots, 0), \quad y^2 := (0, 1, 0, \dots, 0), \dots, \quad y^m := (0, 0, 0, \dots, 1)$$

of the standard simplex in  $\mathbb{R}^m$ , where  $m$  is the size of  $\mathbf{Y}$ , represent the probability distributions  $f$  on  $\mathbf{Y}$  as vectors  $(f\{y^1\}, \dots, f\{y^m\})$  in  $\mathbb{R}^m$ , and represent the real-valued functions  $s$  on  $\mathbf{Y}$  as vectors  $(s(y^1), \dots, s(y^m))$  in  $\mathbb{R}^m$ .

## 4 Meta-theorem

In this section we prove the main mathematical result of this paper: for any continuous strategy for Skeptic there exists a strategy for Forecaster that does not allow Skeptic's capital to grow, regardless of what Reality is doing. As in [14], we make Skeptic announce his strategy at the outset of each round rather than at the beginning of the game, and we drop all restrictions on Skeptic. Forecaster's move space is restricted to  $\mathbf{F} = \overline{\text{co}} \mathbf{Y}$ . The resulting perfect-information game is:

FORECASTING GAME 3

**Players:** Reality, Forecaster, Skeptic

**Parameters:**  $\mathbf{L}$  (Euclidean space),  $\mathbf{X}, \mathbf{Y} \subset \mathbf{L}$  (non-empty and bounded)

**Protocol:**

$\mathcal{K}_0$  is set to a real number.

FOR  $n = 1, 2, \dots$ :

Reality announces  $x_n \in \mathbf{X}$ .

Skeptic announces continuous  $S_n : \overline{\text{co}} \mathbf{Y} \rightarrow \mathbf{L}$ .  
 Forecaster announces  $f_n \in \overline{\text{co}} \mathbf{Y}$ .  
 Reality announces  $y_n \in \mathbf{Y}$ .  
 $\mathcal{K}_n := \mathcal{K}_{n-1} + S_n(f_n) \cdot (y_n - f_n)$ .  
 END FOR

**Theorem 1** *Forecaster has a strategy in Forecasting Game 3 that ensures  $\mathcal{K}_0 \geq \mathcal{K}_1 \geq \mathcal{K}_2 \geq \dots$ .*

## 5 A weak law of large numbers in the feature space

Unfortunately, the usual law of large numbers is not useful for the purpose of designing forecasting strategies (see the discussion in [14]). Therefore, we state a generalized law of large numbers; at the end of this section we will explain connections with the usual law of large numbers. In this section we consider Forecasting Game 2 without the requirement  $\mathcal{K}_0 > 0$  and with the restriction on Skeptic dropped. If we fix a strategy for Skeptic and Skeptic's initial capital  $\mathcal{K}_0$  (not necessarily a positive number),  $\mathcal{K}_n$  defined by (2) becomes a function of Reality's and Forecaster's moves. Such functions will be called *capital processes*.

Let  $\Phi : \mathbf{F} \times \mathbf{X} \rightarrow \mathbf{H}$  (as usual,  $\mathbf{F} = \overline{\text{co}} \mathbf{Y}$ ) be a *feature mapping* into an inner product (typically Hilbert) space  $\mathbf{H}$ . The next theorem uses the notion of tensor product; the relevant definitions and facts can be found in Appendix C.

**Theorem 2** *The function*

$$\mathcal{K}_n := \left\| \sum_{i=1}^n (y_i - f_i) \otimes \Phi(f_i, x_i) \right\|^2 - \sum_{i=1}^n \|(y_i - f_i) \otimes \Phi(f_i, x_i)\|^2 \quad (3)$$

*is a capital process (not necessarily non-negative) of some strategy for Skeptic.*

**Proof** We start by noticing that

$$\begin{aligned}
 \mathcal{K}_n - \mathcal{K}_{n-1} &= \left\| \sum_{i=1}^{n-1} (y_i - f_i) \otimes \Phi(f_i, x_i) + (y_n - f_n) \otimes \Phi(f_n, x_n) \right\|^2 \\
 &\quad - \left\| \sum_{i=1}^{n-1} (y_i - f_i) \otimes \Phi(f_i, x_i) \right\|^2 - \|(y_n - f_n) \otimes \Phi(f_n, x_n)\|^2 \\
 &= 2 \left( \sum_{i=1}^{n-1} (y_i - f_i) \otimes \Phi(f_i, x_i) \right) \cdot ((y_n - f_n) \otimes \Phi(f_n, x_n)) \\
 &= 2 \sum_{i=1}^{n-1} ((y_i - f_i) \cdot (y_n - f_n)) (\Phi(f_i, x_i) \cdot \Phi(f_n, x_n))
 \end{aligned}$$

(in the last equality we used Lemma 2). Introducing the notation

$$K((f, x), (f', x')) := \Phi(f, x) \cdot \Phi(f', x'), \quad (4)$$

where  $(f, x), (f', x') \in \mathbf{F} \times \mathbf{X}$ , we can rewrite the expression for  $\mathcal{K}_n - \mathcal{K}_{n-1}$  as

$$\left( 2 \sum_{i=1}^{n-1} K((f_i, x_i), (f_n, x_n))(y_i - f_i) \right) \cdot (y_n - f_n).$$

Therefore,  $\mathcal{K}_n$  is the capital process corresponding to Skeptic's strategy

$$2 \sum_{i=1}^{n-1} K((f_i, x_i), (f_n, x_n))(y_i - f_i), \quad (5)$$

which completes the proof. ■

### More standard statements of the weak law

In the rest of this section we explain connections of Theorem 2 with more standard statements of the weak law of large numbers; in this part of the paper we will use some notions introduced in [9]. The rest of the paper does not depend on this material, and the reader may wish to skip the rest of this section.

Let us assume that

$$C := \sup_{(f, x) \in \mathbf{F} \times \mathbf{X}} \|\Phi(f, x)\| < \infty.$$

We will use the notation

$$\text{diam}(\mathbf{Y}) := \sup_{y, y' \in \mathbf{Y}} \text{dist}(y, y'),$$

where  $\text{dist}(y, y') = \|y - y'\|$  stands for the Euclidean distance in  $\mathbf{L}$ .

For any initial capital  $\mathcal{K}_0$ , the function

$$\begin{aligned} \mathcal{K}_n &:= \mathcal{K}_0 + \left\| \sum_{i=1}^n (y_i - f_i) \otimes \Phi(f_i, x_i) \right\|^2 - \sum_{i=1}^n \|(y_i - f_i) \otimes \Phi(f_i, x_i)\|^2 \\ &= \mathcal{K}_0 + \left\| \sum_{i=1}^n (y_i - f_i) \otimes \Phi(f_i, x_i) \right\|^2 - \sum_{i=1}^n \|y_i - f_i\|^2 \|\Phi(f_i, x_i)\|^2 \end{aligned}$$

is the capital process of some strategy for Skeptic. Suppose a positive integer  $N$  (the duration of the game, or the *horizon*) is given in advance and  $\mathcal{K}_0 := \text{diam}^2(\mathbf{Y})C^2N$ . Then, in the game lasting  $N$  rounds,  $\mathcal{K}_n$  is never negative and

$$\mathcal{K}_N \geq \left\| \sum_{i=1}^N (y_i - f_i) \otimes \Phi(f_i, x_i) \right\|^2.$$



If we do not believe that Skeptic can increase his capital  $1/\delta$ -fold for a small  $\delta > 0$  without risking bankruptcy, we should believe that

$$\left\| \sum_{i=1}^N (y_i - f_i) \otimes \Phi(f_i, x_i) \right\|^2 \leq \text{diam}^2(\mathbf{Y}) C^2 N / \delta,$$

which can be rewritten

$$\left\| \frac{1}{N} \left( \sum_{i=1}^N (y_i - f_i) \otimes \Phi(f_i, x_i) \right) \right\| \leq \text{diam}(\mathbf{Y}) C (N\delta)^{-1/2}. \quad (6)$$

In the terminology of [9], the game-theoretic lower probability of the event (6) is at least  $1 - \delta$ .

The game-theoretic version of Bernoulli's law of large numbers is a special case of (6) corresponding to  $\Phi(f, x) = 1$ , for all  $f$  and  $x$ ,  $\mathbf{Y} = \{0, 1\}$ , and  $|\mathbf{X}| = 1$  (the last two conditions mean that we are considering the binary forecasting protocol with the  $x$ s absent); as usual, we assume that  $f_i$  are chosen from  $[0, 1]$ . As explained in [9], in combination with the measurability of Skeptic's strategy guaranteeing (6), this implies that the measure-theoretic probability of the event (6) is at least  $1 - \delta$ , assuming that the  $y_i$  are generated by a probability distribution and that each  $f_i$  is the conditional probability that  $y_i = 1$  given  $y_1, \dots, y_{i-1}$ . This measure-theoretic result was proved by Kolmogorov in 1929 (see [5]) and is the origin of the name "K29 strategy".

We will see in the next section that the feature-space version (6) of the weak law of large numbers is much more useful than the standard version for the purpose of forecasting; in particular, we will see that K29 guarantees (6) with  $\delta = 1$ .

## 6 The K29 strategy and its properties

According to Theorem 1, under the continuity assumption there is a strategy for Forecaster that does not allow  $\mathcal{K}_n$  to grow, where  $\mathcal{K}_n$  is defined by (3). Fortunately (but not unusually), this strategy depends on the feature mapping  $\Phi$  only via the corresponding *Mercer kernel*  $K$  defined by (4). The continuity assumption needed is that  $K((f, x), (f', x'))$  is continuous in  $f$ ; such kernels will be called *forecast-continuous*. According to (5), the corresponding forecasting strategy, which we will call the *K29 strategy* with parameter  $K$ , is to output, on the  $n$ th round, a forecast  $f_n$  satisfying

$$S(f_n) := \sum_{i=1}^{n-1} K((f_i, x_i), (f_n, x_n)) (y_i - f_i) = 0$$

(or, if such  $f_n$  does not exist, the forecast is chosen to be a point  $f_n \in \partial \mathbf{F}$  where  $S(f_n)$  is normal and directed exteriorly to  $\mathbf{F}$ ).

The protocol of this section is essentially that of Forecasting Game 3; as Skeptic ceases to be an active player, it simplifies to:

FOR  $n = 1, 2, \dots$ :  
 Reality announces  $x_n \in \mathbf{X}$ .  
 Forecaster announces  $f_n \in \overline{\text{co}} \mathbf{Y}$ .  
 Reality announces  $y_n \in \mathbf{Y}$ .  
 END FOR

**Theorem 3** *The K29 strategy guarantees that always*

$$\left\| \sum_{i=1}^n (y_i - f_i) \otimes \Phi(f_i, x_i) \right\| \leq \text{diam}(\mathbf{Y}) C n^{1/2}, \quad (7)$$

where  $C := \sup_{(f,x) \in \mathbf{F} \times \mathbf{X}} \|\Phi(f, x)\|$  is assumed to be finite.

**Proof** The K29 strategy ensures that (3) never increases; therefore,

$$\begin{aligned} \left\| \sum_{i=1}^n (y_i - f_i) \otimes \Phi(f_i, x_i) \right\|^2 &\leq \sum_{i=1}^n \|(y_i - f_i) \otimes \Phi(f_i, x_i)\|^2 \\ &= \sum_{i=1}^n \|y_i - f_i\|^2 \|\Phi(f_i, x_i)\|^2 \\ &\leq \text{diam}^2(\mathbf{Y}) C^2 n \end{aligned}$$

(we have used Lemma 2). ■

**Remark** The property (7) is a special case of (6) corresponding to  $\delta = 1$ ; we gave an independent derivation to make our exposition self-contained and to avoid the extra assumptions used in the derivation of (6), such as the horizon being finite and known in advance.

## Calibration and resolution

Two important properties of a forecasting strategy are its calibration and resolution, which we introduce informally. Our discussion in this section extends the discussion in [13], §5, to the case of linear protocols (in particular, to the case of multi-class forecasting). Forecaster's move space is assumed to be  $\mathbf{F} = \overline{\text{co}} \mathbf{Y}$ .

We say that the forecasts  $f_n$  are *properly calibrated* if, for any  $f^* \in \mathbf{F}$ ,

$$\frac{\sum_{i=1, \dots, n: f_i \approx f^*} y_i}{\sum_{i=1, \dots, n: f_i \approx f^*} 1} \approx f^*$$

provided  $\sum_{i=1, \dots, n: f_i \approx f^*} 1$  is not too small. (We shorten  $(1/c)v$  to  $v/c$ , where  $v$  is a vector and  $c \neq 0$  is a number.) Proper calibration is only a necessary but far from sufficient condition for good forecasts: for example, a forecaster who ignores the objects  $x_n$  can be perfectly calibrated, no matter how much useful information  $x_n$  contain. (Cf. the discussion in [2].)

We say that the forecasts  $f_n$  are *properly calibrated and resolved* if, for any  $(f^*, x^*) \in \mathbf{F} \times \mathbf{X}$ ,

$$\frac{\sum_{i=1, \dots, n: (f_i, x_i) \approx (f^*, x^*)} y_i}{\sum_{i=1, \dots, n: (f_i, x_i) \approx (f^*, x^*)} 1} \approx f^* \quad (8)$$

provided  $\sum_{i=1, \dots, n: (f_i, x_i) \approx (f^*, x^*)} 1$  is not too small.

Instead of “crisp” points  $(f^*, x^*) \in \mathbf{F} \times \mathbf{X}$  we will consider “fuzzy points”  $I : \mathbf{L} \rightarrow [0, 1]$  such that  $I(f^*, x^*) = 1$  and  $I(f, x) = 0$  for all  $(f, x)$  outside a small neighborhood of  $(f^*, x^*)$ . A standard choice would be something like  $I := \mathbb{I}_E$ , where  $E \subset \mathbf{L}$  is a small neighborhood of  $(f^*, x^*)$  and  $\mathbb{I}_E$  is its indicator function, but we will want  $I$  to be continuous (it can, however, be arbitrarily close to  $\mathbb{I}_E$ ).

Let  $(f^*, x^*)$  be a point in  $\mathbf{F} \times \mathbf{X}$ ; we would like the average of  $y_i$ ,  $i = 1, \dots, n$ , such that  $(f_i, x_i)$  is close to  $(f^*, x^*)$  to be close to  $f^*$ . (Cf. (8).) Fix a forecast-continuous Mercer kernel  $K : (\mathbf{F} \times \mathbf{X})^2 \rightarrow \mathbb{R}$  and consider the “soft neighborhood”

$$I_{(f^*, x^*)}(f, x) := K((f^*, x^*), (f, x)) \quad (9)$$

of the point  $(f^*, x^*)$ . The following is an easy corollary of Theorem 3.

**Corollary 1** *The K29 strategy with parameter  $K \geq 0$  ensures*

$$\left\| \sum_{i=1}^n (y_i - f_i) I_{(f^*, x^*)}(f_i, x_i) \right\| \leq \text{diam}(\mathbf{Y}) C^2 \sqrt{n} \quad (10)$$

for each point  $(f^*, x^*) \in (\mathbf{F} \times \mathbf{X})$ , where  $I$  is defined by (9) and  $C$  is defined as in Theorem 3.

We can rewrite (10) as

$$\left\| \frac{\sum_{i=1}^n (y_i - f_i) I_{(f^*, x^*)}(f_i, x_i)}{\sum_{i=1}^n I_{(f^*, x^*)}(f_i, x_i)} \right\| \leq \frac{\text{diam}(\mathbf{Y}) C^2 \sqrt{n}}{\sum_{i=1}^n I_{(f^*, x^*)}(f_i, x_i)} \quad (11)$$

(assuming the denominator  $\sum_{i=1}^n I_{(f^*, x^*)}(f_i, x_i)$  is positive); therefore, we can expect proper calibration and resolution in the soft neighborhood of  $(f^*, x^*)$  when

$$\sum_{i=1}^n I_{(f^*, x^*)}(f_i, x_i) \gg \sqrt{n}. \quad (12)$$

In conclusion, we will illustrate (11) on a simple example. Choose the scale  $\sigma > 0$  at which calibration and resolution are sought, and consider the Gaussian kernel (obviously forecast-continuous)

$$K((f, x), (f', x')) := \exp \left( -\frac{\|(f, x) - (f', x')\|^2}{2\sigma^2} \right); \quad (13)$$

the corresponding soft neighborhoods  $I_{(f^*, x^*)}$  will be Gaussian bells of size  $\sigma$ . Fix  $(f^*, x^*) \in \mathbf{F} \times \mathbf{X}$ . If  $n$  is large enough, we expect (12) to hold (indeed, the left-hand side of (12) typically grows as  $\Theta(n)$  as  $n \rightarrow \infty$ ), and so we expect proper calibration and resolution at  $(f^*, x^*)$ .

## 7 Further research

The main result of this paper is an existence theorem: we did not show how to compute Forecaster’s strategy ensuring  $\mathcal{K}_0 \geq \mathcal{K}_1 \geq \dots$ . (The latter was easy in the case of binary forecasting considered in [14].) It is important to develop computationally efficient ways to find zeros of vector fields. There are several popular methods for finding zeros, such as the Newton–Raphson method (see, e.g., [7], Chap. 9), but it would be ideal to have efficient methods that are guaranteed to find a zero (or a near zero) in a prespecified time.

In this paper we considered only the case where  $\mathbf{Y}$  is a subset of a finite-dimensional space  $\mathbf{L}$ . There are important protocols (such as the one in [9], p. 360) in which  $\mathbf{Y}$ ,  $\mathbf{F}$ , and  $\mathbf{S}$  are subsets of, e.g., a Banach space. The proof techniques used in this paper, however, depend on the assumption that  $\mathbf{L}$  is finite-dimensional in an essential way.

Finally, it is interesting to study performance guarantees for K29 when used in conjunction with universal kernels [10]. The disadvantage of kernel (13), for example, is that it is not clear how to choose  $\sigma$ : too large  $\sigma$  are useless ((12) holds but calibration and resolution are not useful at a crude scale) and too small  $\sigma$  are not achievable ((12) does not hold). In the binary case, this work is started in [13].

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## Appendix A Proof of Theorem 1

Fix a round  $n$  and Skeptic's move  $S_n : \mathbf{F} \rightarrow \mathbf{L}$  (we will refer to  $S_n$  as a vector field in  $\mathbf{F}$ ). Our task is to prove the existence of a point  $f_n \in \mathbf{F}$  such that, for all  $y \in \mathbf{Y}$ ,  $S_n(f_n)(y - f_n) \leq 0$ .

If for some  $f \in \partial\mathbf{F}$  (we use  $\partial A$  to denote the boundary of  $A \subseteq \mathbf{L}$ ) the vector  $S_n(f)$  is normal and directed exteriorly to  $\mathbf{F}$  (in the sense that  $S_n(f) \cdot (y - f) \leq 0$  for all  $y \in \mathbf{F}$ ), we can take such  $f$  as  $f_n$ . Therefore, we assume, without loss of generality, that  $S_n$  is never normal and directed exteriorly on  $\partial\mathbf{F}$ . Then by Lemma 1 below there exists  $f$  such that  $S_n(f) = 0$ , and we can take such  $f$  as  $f_n$ .

**Remark** Notice that Theorem 1 will not become weaker if the first move by Reality (choosing  $x_n$ ) is removed from each round of the protocol.

### Zeros of vector fields

The following lemma is the main component of the proof of Theorem 1.

**Lemma 1** *Let  $\mathbf{F}$  be a compact convex set in  $\mathbf{L}$  and  $S : \mathbf{F} \rightarrow \mathbf{L}$  be a continuous vector field on  $\mathbf{F}$ . If at no point of the boundary  $\partial\mathbf{F}$  the vector field  $S$  is normal and directed exteriorly then there exists  $f \in \mathbf{F}$  such that  $S(f) = 0$ .*

**Proof** If the boundary  $\partial\mathbf{F}$  were assumed to be smooth, the lemma would follow from [6], Theorem  $A_0$  on p. 170; without smoothness, we will have to give an independent proof, starting with a modification of a simple trick from [6].

For each  $\epsilon > 0$  we define

$$\mathbf{F}_\epsilon := \{z \mid \text{dist}(z, \mathbf{F}) \leq \epsilon\},$$

where, as usual,  $\text{dist}(z, \mathbf{F}) := \inf_{f \in \mathbf{F}} \text{dist}(z, f)$ ;  $\mathbf{F}_\epsilon$  is called the *tube of radius  $\epsilon$  around  $\mathbf{F}$*  ([4]) or the local parallel set of radius  $\epsilon$  around  $\mathbf{F}$  ([8], §4). Note that  $\mathbf{F}_\epsilon$  can be written as the Minkowski sum  $\mathbf{F}_\epsilon = \mathbf{F} + \epsilon U$  of  $\mathbf{F}$  and the ball  $\epsilon U$  of radius  $\epsilon$ . Therefore, the convexity of  $\mathbf{F}$  implies that  $\mathbf{F}_\epsilon$  is also convex.

Following [6], we extend the vector field  $S$  from  $\mathbf{F}$  to  $\mathbf{F}_1$  ( $\mathbf{F}_\epsilon$  with  $\epsilon = 1$ ) as follows: for any point  $f \in \partial \mathbf{F}_1$  and any  $t \in [0, 1)$ , set

$$S(ty + (1 - t)f) := tS(y) + (1 - t)S(f), \quad (14)$$

where  $y \in \mathbf{F}$  is the unique closest point to  $f$  and  $S(f)$  is defined as  $y - f$ .

Let us first prove that this extension is well defined, i.e., that each point  $p \in \mathbf{F}_1 \setminus \mathbf{F}$  has a unique representation in the form  $ty + (1 - t)f$ , as above. Such a representation exists since we can take  $y$  to be the closest point of  $\mathbf{F}$  to  $p$  and  $f$  to be the point lying on the straight line connecting  $y$  and  $p$  at a distance of 1 from  $y$  in the direction of  $p$ ; it is clear that  $y$  is the closest point to  $f$  in  $\mathbf{F}$  and that  $f \in \partial \mathbf{F}_1$ . Such a representation is unique since  $y$  is uniquely determined as the closest to  $p$  point of  $\mathbf{F}$ ,  $f$  is uniquely determined as the point lying on the straight line connecting  $y$  and  $p$  at a distance of 1 from  $y$  in the direction of  $p$ , and  $t$  is uniquely determined as the distance between  $f$  and  $p$ .

Let us now prove that the extension of  $S$  to  $\mathbf{F}_1$  is continuous. Let

$$p \in (\mathbf{F}_1 \setminus \mathbf{F}) \cup \partial \mathbf{F}, \quad p_k \in \mathbf{F}_1 \setminus \mathbf{F}, \quad k = 1, 2, \dots,$$

be such that  $p_k \rightarrow p$  as  $k \rightarrow \infty$ ; we are required to prove that  $S(p_k) \rightarrow S(p)$ . Represent each  $p_k$  and  $p$  in the form  $p_k = t_k y_k + (1 - t_k) f_k$  and  $p = ty + (1 - t)f$ , as above (i.e.,  $f$  and  $f_k$  are in  $\partial \mathbf{F}_1$  and  $y$  and  $y_k$  are the corresponding closest points in  $\mathbf{F}$ ). It is easy to check that  $y_k \rightarrow y$  (as  $y_k$  and  $y$  are the closest points in  $\mathbf{F}$  to  $p_k$  and  $p$ , respectively), then to check that  $f_k \rightarrow f$  (provided  $f \notin \partial \mathbf{F}$ ), and finally to check that  $t_k \rightarrow t$  (this is true even if  $f \in \partial \mathbf{F}$ , in which case  $t = 1$  and  $t_k \rightarrow 1$ ). This immediately implies  $S(p_k) \rightarrow S(p)$ .

Since  $S$  is never normal and directed exteriorly on  $\partial \mathbf{F}$ ,  $S$  will have no zeros inside  $\mathbf{F}_1 \setminus \mathbf{F}$ . Since the vector field  $S$  is interiorly directed on  $\partial \mathbf{F}_1$  (we will never need a formal definition of “interiorly directed” in this paper), our task would be accomplished if we assumed that the boundary of  $\mathbf{F}_1$  is smooth: we would apply the Poincaré–Hopf theorem to deduce that  $S$  has at least one zero in  $\mathbf{F}_1$  and, therefore, at least one zero in  $\mathbf{F}$ . We will, however, give an argument that does not depend on any smoothness assumptions.

The proof will be complete if we show that the continuous vector field  $S$  in the closed convex set  $\mathbf{F}_1$ , which is normal and interiorly directed on  $\partial \mathbf{F}_1$ , always has at least one zero. We consider the tube  $\mathbf{F}_2$  of radius 2 around  $\mathbf{F}$  and extend the vector field  $S$  to  $\mathbf{F}_2 \setminus \mathbf{F}_1$  by

$$S(ty + (1 - t)f) := S(f)$$

in the notation of (14) but with  $t \in [-1, 0)$  (as before,  $f$  ranges over  $\partial\mathbf{F}_1$  and  $y \in \mathbf{F}$  is the closest point to  $f$ ). Again, it is easy to check that this extension is well defined and continuous.

By the compactness of  $\mathbf{F}_2$ ,

$$C := \max \left( \sup_{f \in \mathbf{F}_2} \|S(f)\|, 1 \right) < \infty.$$

Notice that  $f + tS(f) \in \mathbf{F}_2$  for all  $f \in \mathbf{F}_2$  and all  $t \in [0, 1/C)$  (for  $f \in \mathbf{F}_2 \setminus \mathbf{F}_1$  this follows from the fact that  $f + tS(f)$  lies between the closest points to  $f$  in  $\mathbf{F}$  and in  $\mathbf{F}_1$ , and for  $f \in \mathbf{F}_1$  this follows from the fact that the distance between  $f$  and the complement of  $\mathbf{F}_2$  is at least 1). The function

$$\begin{aligned} G : \mathbf{F}_2 &\rightarrow \mathbf{F}_2 \\ f &\mapsto f + \frac{1}{2C}S(f) \end{aligned}$$

is continuous and so, by Schauder's fixed point theorem (see, e.g., [1], Chap. 4), has a fixed point; it is clear that such a fixed point will be a zero of  $S$ .  $\blacksquare$

## Appendix B Proof of Corollary 10

Let  $\Phi : \mathbf{F} \times \mathbf{X} \rightarrow \mathbf{H}$  be a function taking values in an inner product space  $\mathbf{H}$  and satisfying (4). Theorem 3 then implies

$$\begin{aligned} \left\| \sum_{i=1}^n (y_i - f_i) I_{(f^*, x^*)}(f_i, x_i) \right\| &= \left\| \sum_{i=1}^n (y_i - f_i) (\Phi(f_i, x_i) \cdot \Phi(f^*, x^*)) \right\| \\ &= \left\| \sum_{i=1}^n ((y_i - f_i) \otimes \Phi(f_i, x_i)) \Phi(f^*, x^*) \right\| \\ &\leq \left\| \sum_{i=1}^n (y_i - f_i) \otimes \Phi(f_i, x_i) \right\| \|\Phi(f^*, x^*)\| \\ &\leq \text{diam}(\mathbf{Y}) C^2 \sqrt{n} \end{aligned}$$

(the second equality follows from Lemma 4 and the first inequality from Lemma 3).

## Appendix C Tensor product

In this appendix we list several definitions and simple facts about tensor products, in the form used in this paper.

The *tensor product*  $\mathbf{L} \otimes \mathbf{H}$  of  $\mathbf{L} = \mathbb{R}^m$  and  $\mathbf{H}$  (an inner product space, perhaps infinite-dimensional) is the vector space  $\mathbf{H}^m$  with the addition and

scalar multiplication defined component-wise,

$$\begin{aligned}(a_1, \dots, a_m) + (b_1, \dots, b_m) &:= (a_1 + b_1, \dots, a_m + b_m), \\ c(a_1, \dots, a_m) &:= (ca_1, \dots, ca_m),\end{aligned}$$

and the inner product

$$(a_1, \dots, a_m) \cdot (b_1, \dots, b_m) := a_1 \cdot b_1 + \dots + a_m \cdot b_m.$$

The tensor product of  $(t_1, \dots, t_m) \in \mathbf{L}$  and  $h \in \mathbf{H}$  is defined to be

$$(t_1, \dots, t_m) \otimes h := (t_1 h, \dots, t_m h).$$

**Lemma 2** *For any  $t_1, t_2 \in \mathbf{L}$  and  $h_1, h_2 \in \mathbf{H}$ ,*

$$(t_1 \otimes h_1) \cdot (t_2 \otimes h_2) = (t_1 \cdot t_2)(h_1 \cdot h_2).$$

**Proof** Immediate from the definition. ■

If  $v \in \mathbf{L} \otimes \mathbf{H}$  and  $h \in \mathbf{H}$ , we define the *product*  $vh \in \mathbf{L}$  by the equality

$$(v_1, \dots, v_m)h := (v_1 \cdot h, \dots, v_m \cdot h),$$

where  $(v_1, \dots, v_m) := v$ . The following lemma generalizes (and is an easy implication of) the Cauchy–Schwarz inequality.

**Lemma 3** *For any  $v \in \mathbf{L} \otimes \mathbf{H}$  and  $h \in \mathbf{H}$ ,*

$$\|vh\| \leq \|v\|\|h\|.$$

**Proof** Our goal is to prove

$$\|(v_1 \cdot h, \dots, v_m \cdot h)\| \leq \|(v_1, \dots, v_m)\|\|h\|,$$

which is equivalent to

$$(v_1 \cdot h)^2 + \dots + (v_m \cdot h)^2 \leq \|v_1\|^2 \|h\|^2 + \dots + \|v_m\|^2 \|h\|^2;$$

the last inequality follows from  $(v_i \cdot h)^2 \leq \|v_i\|^2 \|h\|^2$  (a special case of the Cauchy–Schwarz inequality). ■

**Lemma 4** *For any  $t \in \mathbf{L}$  and  $a, b \in \mathbf{H}$ ,*

$$(a \cdot b)t = (t \otimes a)b. \tag{15}$$

**Proof** If  $t = (t_1, \dots, t_m)$ , both sides of (15) equal  $(t_1(a \cdot b), \dots, t_m(a \cdot b))$ . ■