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The computational complexity of the parallel knock-out problem☆

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Abstract

We consider computational complexity questions related to parallel knock-out schemes for graphs. In such schemes, in each round, each remaining vertex of a given graph eliminates exactly one of its neighbours. We show that the problem of whether, for a given bipartite graph, such a scheme can be found that eliminates every vertex is NP-complete. Moreover, we show that, for all fixed positive integers $k \geq 2$, the problem of whether a given bipartite graph admits a scheme in which all vertices are eliminated in at most (exactly) k rounds is NP-complete. For graphs with bounded tree-width, however, both of these problems are shown to be solvable in polynomial time. We also show that r-regular graphs with $r \geq 1$, factor-critical graphs and 1-tough graphs admit a scheme in which all vertices are eliminated in one round.

Keywords: Parallel knock-out; Graphs; Computational complexity

1. Introduction

In this paper, we consider *parallel knock-out schemes* for finite undirected simple graphs. These were introduced by Lampert and Slater [9]. Such a scheme proceeds in rounds: in the first round each vertex in the graph selects exactly one of its neighbours, and then all the selected vertices are eliminated simultaneously. In subsequent rounds this procedure is repeated in the subgraph induced by those vertices not yet eliminated. The scheme continues until there are no vertices left, or until an isolated vertex is obtained (since an isolated vertex will never be eliminated).

A graph is called *KO-reducible* or simply *reducible* if there exists a parallel knock-out scheme that eliminates the whole graph. The *parallel knock-out number* of a graph G, denoted by $\text{pko}(G)$, is the minimum number of rounds in a parallel knock-out scheme that eliminates every vertex of G. If G is not reducible, then $\text{pko}(G) = \infty$.

Our main motivation for studying the concept of reducibility is its intimate relation to well-studied concepts in structural and algorithmic graph theory, like matchings and cycles. To illustrate this, we note that a graph G with a

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perfect matching has $\text{pko}(G) = 1$, as each vertex can select the vertex it is matched with in a perfect matching of G. Similarly, a graph G with a hamiltonian cycle has $\text{pko}(G) = 1$, as each vertex can select its successor on a hamiltonian cycle of G with some fixed orientation. Whereas it is easy to check (i.e., by a polynomial algorithm) whether a graph admits a perfect matching, it is NP-complete to decide whether a graph has a hamiltonian cycle. What can be said about the complexity of deciding whether a graph G has a finite parallel knock-out number? Or about determining (an upper bound on) the value of $\text{pko}(G)$? These complexity questions are our main concern in this paper and will be answered in Section 4. We will also consider several structural properties related to reducibility, but only with some relation to complexity questions. Other structural properties related to reducibility can be found in [3] and [5].

1.1. Complexity questions related to reducibility

Consider the following decision problem.

PARALLEL KNOCK-OUT (PKO)
Instance: A graph G.
Question: Is G reducible?

In [9], which appeared in 1998, it was claimed that PKO is NP-complete even when restricted to the class of bipartite graphs. No proof was given; the reader was referred to a paper that was in preparation. Our attempts to obtain and verify this proof have been unsuccessful. We shall obtain the result as a corollary to a stronger theorem (Theorem 1 below) by considering a related problem, which is defined for each positive integer k.

PARALLEL KNOCK-OUT (k) (PKO(k))
Instance: A graph G.
Question: Is $\text{pko}(G) \le k$?

Our first result classifies the complexity of PKO(k), $k \ge 2$.

Theorem 1. *For $k \ge 2$, PKO(k) is NP-complete even if instances are restricted to the class of bipartite graphs.*

The proof is postponed to Section 4.

By using almost the same arguments, we will also show that deciding whether $\text{pko}(G) = k$ is polynomially solvable for $k = 1$ and NP-complete for any fixed $k \ge 2$ that is not part of the input.

As a matter of fact it is not difficult to show that a graph G has $\text{pko}(G) = 1$ if and only if G contains a $[1, 2]$-*factor*, i.e., a spanning subgraph in which every component is either a cycle or an edge: simply note that a vertex u that selects a vertex v is either selected by v or by a vertex $w \notin \{u, v\}$, and combine this observation with the fact that every vertex selects exactly one other vertex and that all the graphs we consider are finite.

The problem of deciding whether G contains a $[1, 2]$-factor is a folklore problem appearing in many standard books on combinatorial optimization. For convenience we shortly discuss it below.

Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Define a bipartite graph G' with vertex set $V(G') = \{u_1, u_2, \ldots, u_n, w_1, w_2, \ldots, w_n\}$ in which $u_i w_j \in E(G')$ and $u_j w_i \in E(G')$ if and only if $v_i v_j \in E(G)$. A $[1, 2]$-factor in G corresponds to a perfect matching in G'. Hence the related decision problem and consequently PKO(1), can be decided in polynomial time. This is also clear from the following polynomially solvable decision problem (see [7], problem GT13, page 193): given a directed graph D, decide whether $V(D)$ can be partitioned into disjoint sets of cardinality at least 2 such that each of the sets induces a subgraph with a directed hamiltonian cycle. To show the intimate relation with knock-out schemes, replace each edge of G by two oppositely directed arcs. Clearly G has a $[1, 2]$-factor if and only if the related directed graph has such a partition into hamiltonian cycles.

In [3], it was shown, using a dynamic programming approach, that the parallel knock-out number for trees can be computed in polynomial time. The authors presented an $O(n^{3.5} \log^2 n)$ algorithm for computing the parallel knock-out number of an n-vertex tree, and asked whether there exists a substantially faster algorithm for this problem, with a time complexity of, say, $O(n \log n)$ or $O(n^2)$? Our next result implies that there exists a linear time algorithm for this problem.

A key ingredient of the dynamic program for trees in [3] is the reduction to a number of polynomially solvable bipartite matching problems. For higher tree-widths, these bipartite matching problems have no natural polynomially solvable analogues. Therefore the dynamic program for trees does not carry over to the bounded tree-width classes.

In [3] it was asked whether one can avoid the computation of perfect matchings in auxiliary bipartite graphs while computing $\mathrm{pko}(T)$ for a tree T. And can one then generalize such a method to graphs of bounded tree-width? In our second result, we give an affirmative answer, although we do not provide an explicit algorithm.

Theorem 2. *The problem* $\mathrm{PKO}(k)$ *can be solved in linear time on graphs with bounded tree-width.*

We will also show that PKO can be solved in polynomial time on graphs with bounded tree-width.

1.2. Structural properties related to reducibility

As noted above, there is an intimate relationship between reducibility and other structural properties, like the existence of a $[1, 2]$-factor, a notion that is a common generalization of a perfect matching and a hamiltonian cycle. Apart from hamiltonian graphs and graphs that have a perfect matching, for example also all k-traversable graphs have been shown to have a $[1, 2]$-factor [4]. A graph is *k-traversable* if it admits a closed walk in which every vertex occurs exactly k times. These graphs were also studied in [8].

We establish several other results in Section 6 that explore the relationship between reducible graphs and properties related to the existence of 'near' perfect matchings or hamiltonian cycles. For example we show that all factor-critical and all 1-tough graphs have a $[1, 2]$-factor, i.e., have parallel knock-out number 1. We refer to Section 6 for definitions. We also show there that r-regular graphs with $r \geq 1$ have a $[1, 2]$-factor, and we have a closer look at 'almost' regular bipartite graphs, i.e., bipartite graphs in which all vertices in the same bipartition class have the same degree.

1.3. Organization of the paper

In Sections 2 and 3 we introduce a number of definitions and preliminary observations. In Sections 4 and 5 are the proofs and corollaries of Theorems 1 and 2, respectively. Section 6 deals with factor-critical graphs, 1-tough graphs, r-regular graphs and 'almost' regular bipartite graphs.

2. Preliminaries

Graphs in this paper are denoted by $G = (V, E)$. An edge joining vertices u and v is denoted uv. For graph terminology not defined below, we refer to [2]. For convenience we allow graphs to have an empty vertex set. We say that $G = (V, E)$ is the *null graph* if $V = E = \emptyset$.

For a vertex $u \in V$ we denote its *neighbourhood*, that is, the set of adjacent vertices, by $N(u) = \{v \mid uv \in E\}$. The *degree* $d_G(v)$ of a vertex v in G is the number of edges incident with it, or, equivalently, the size of its neighbourhood. A maximal connected subgraph of a graph G is called a *component* of G.

Adopting the terminology and notation from [3], for a graph G, a *KO-selection* is a function $f : V \to V$ with $f(v) \in N(v)$ for all $v \in V$. If $f(v) = u$, we say that vertex v *fires at* vertex u, or that vertex u *is knocked out* by vertex v.

For a KO-selection f, we define the corresponding *KO-successor* of G as the subgraph of G that is induced by the vertices in $V \setminus f(V)$; if H is the KO-successor of G we write $G \rightsquigarrow H$. Note that every graph without isolated vertices except for the null graph has at least one KO-successor. A graph G is called *KO-reducible*, if there exists a finite sequence

$$G \rightsquigarrow G_1 \rightsquigarrow G_2 \rightsquigarrow \cdots \rightsquigarrow G_r,$$

where G_r is the null graph. If no such sequence exists, then $\mathrm{pko}(G) = \infty$. Otherwise, the parallel knock-out number $\mathrm{pko}(G)$ of G is the smallest number r for which such a sequence exists. A sequence of KO-selections that transform G into the null graph is called a *KO-reduction scheme*. A single step in this sequence is called a *round* of the KO-reduction scheme. A subset of V is *knocked out* in a certain round if every vertex in the subset is knocked out in that round.

We make some simple observations that we will use later on.

Observation 3. *Let $G = (V, E)$ be a KO-reducible graph, and let $V_1 = \{v \in V \mid d(v) = 1\}$. Then in the first round of any KO-reduction scheme each vertex of V_1 is knocked out by its unique neighbour in G.*

Proof. This is clear, since otherwise some vertex $v \in V_1$ will be an isolated vertex after the first round, as the neighbour of v is knocked out by v in the first round. ■

Observation 4. *Let G be a graph on at least three vertices. If G contains two vertices of degree* 1 *that share the same neighbour, then G is not KO-reducible.*

Proof. Suppose G is KO-reducible. Then by Observation 3, the shared neighbour knocks out both vertices of degree 1, a contradiction. ■

Observation 5. *Let u_1, u_2, u_3, u_4 be four vertices of a KO-reducible graph G such that $N(u_2) = \{u_1, u_3\}$, $N(u_3) = \{u_2, u_4\}$ and $N(u_4) = \{u_3\}$. If u_1 is knocked out in the first round of a KO-reduction scheme, then u_1 fires at u_2 in the first round.*

Proof. By Observation 3, u_3 and u_4 knock each other out in the first round, so u_3 does not knock out u_2. If u_1 is knocked out in the first round of a KO-reduction scheme, then u_1 fires at u_2 in the first round; otherwise u_2 will be an isolated vertex after the first round. ■

An odd path $u_1u_2 \dots u_{2k+1}$ is called a *centred path* of G with *centrevertex* u_{k+1} if $G - \{u_{k+1}\}$ contains as components the path $u_1u_2 \dots u_k$ and the path $u_{k+2}u_{k+3} \dots u_{2k+1}$.

Observation 6. *Let $P = u_1u_2 \dots u_7$ be a centred path of a KO-reducible graph G. In the first round of any KO-reduction scheme u_1 and u_2 fire at each other, u_3 fires at u_2, u_6 and u_7 fire at each other, u_5 fires at u_6, u_4 fires at u_3 or u_5, and u_4 will not be knocked out. In the second round of any KO-reduction scheme u_4 and its remaining neighbour in P fire at each other.*

Proof. By Observation 3, u_1 and u_2, and u_6 and u_7 knock each other out in the first round. Suppose u_3 fires at u_4 in the first round. Then u_4 has to fire at u_3; otherwise u_3 will be an isolated vertex after the first round. But now u_5 will be an isolated vertex after the first round. Hence u_3 fires at u_2, and similarly u_5 fires at u_6. So at least one of u_3 and u_5 survives the first round. This implies that u_4 has to survive the first round as well. The result now follows by applying Observations 3 and 4 to the KO-successor of G. ■

3. NP-complete problems

In this section, we consider two NP-complete problems that will play a key role in our proof of Theorem 1. We refer to [7,10] for further details.

The first problem concerns dominating sets. A set $S \subseteq V$ is a *dominating set* of a graph $G = (V, E)$ if every vertex of G is in S or adjacent to a vertex in S.

We will make use of the following NP-complete decision problem.

DOMINATING SET (DS)
Instance: A graph $G = (V, E)$ and a positive integer p.
Question: Does G have a dominating set of cardinality at most p?

The second problem concerns hypergraph 2-colourings. A *hypergraph* $J = (Q, \mathcal{S})$ is a pair of sets where $Q = \{q_1, \dots, q_m\}$ is the vertex set and $\mathcal{S} = \{S_1, \dots, S_n\}$ is the set of *hyperedges*. Each member S_j of $\mathcal{S}$ is a subset of Q. A 2-*colouring* of $J = (Q, \mathcal{S})$ is a partition of Q into sets B and W such that, for each $S \in \mathcal{S}$, $B \cap S \neq \emptyset$ and $W \cap S \neq \emptyset$.

We will also make use of the following NP-complete decision problem.

HYPERGRAPH 2-COLOURABILITY (H2C)
Instance: A hypergraph $J = (Q, \mathcal{S})$.
Question: Is there a 2-*colouring* of $J = (Q, \mathcal{S})$?

Before we turn to our proofs of the complexity results in Section 4, we need a few more definitions.

The *incidence graph* I of a hypergraph $J = (Q, \mathcal{S})$ is a bipartite graph with vertex set $Q \cup \mathcal{S}$ where (q, S) forms an edge if and only if $q \in S$.

With a hypergraph $J = (Q, \mathcal{S})$ we can associate another hypergraph $J' = (X, \mathcal{Z})$ called the *triple* of J; triples of hypergraphs will play a crucial role in our NP-completeness proofs in Section 4. It requires a little effort to define the vertex set X and hyperedge set $\mathcal{Z}$ of the triple of J.

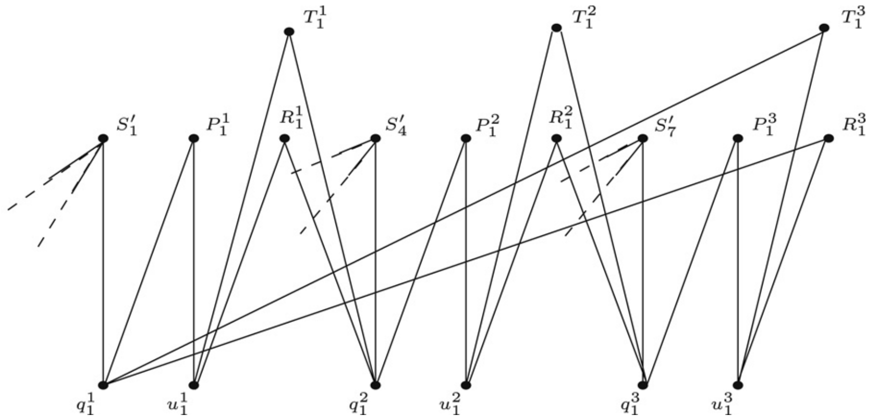

Fig. 1. Part of the incidence graph of the triple of a hypergraph.

Recall that $Q = \{q_1, \ldots, q_m\}$ and $\mathcal{S} = \{S_1, \ldots, S_n\}$. For $1 \le i \le m$, let $\ell(i)$ be the number of hyperedges in $\mathcal{S}$ that contain q_i, let $Q_i = \{q_i^1, \ldots, q_i^{\ell(i)}\}$ and let $U_i = \{u_i^1, \ldots, u_i^{\ell(i)}\}$. The union of all such sets is the vertex set of J', that is

$$X = \bigcup_{i=1}^{m} (Q_i \cup U_i).$$

Now the hyperedges.

Let us first define the following sets:

- for $1 \le i \le m$, for $1 \le k \le \ell(i)$, let $P_i^k = \{q_i^k, u_i^k\}$,
- for $1 \le i \le m$, for $1 \le k \le \ell(i) - 1$, let $R_i^k = T_i^k = \{u_i^k, q_i^{k+1}\}$, and
- for $1 \le i \le m$, let $R_i^{\ell(i)} = T_i^{\ell(i)} = \{u_i^{\ell(i)}, q_i^1\}$.

Let $\mathcal{P}_i = \{P_i^1, \ldots, P_i^{\ell(i)}\}$, $\mathcal{R}_i = \{R_i^1, \ldots, R_i^{\ell(i)}\}$, and $\mathcal{T}_i = \{T_i^1, \ldots, T_i^{\ell(i)}\}$, and let

$$\mathcal{P} = \bigcup_{i=1}^{m} \mathcal{P}_i, \qquad \mathcal{R} = \bigcup_{i=1}^{m} \mathcal{R}_i, \qquad \mathcal{T} = \bigcup_{i=1}^{m} \mathcal{T}_i.$$

For $1 \le j \le n$, let us also define a set S_j'. If in J, S_j contains q_i, then in J', S_j' contains a vertex of Q_i. In particular, if S_j is the kth hyperedge that contains q_i in J, then S_j' contains q_i^k. For example, if q_1 is in S_1, S_4 and S_7 (only) in J, then $\ell(1) = 3$ and in J' there are vertices q_1^1, q_1^2, q_1^3 with $q_1^1 \in S_1'$, $q_1^2 \in S_4'$, and $q_1^3 \in S_7'$.

Let $\mathcal{S}' = \{S_1', \ldots, S_n'\}$. The set of hyperedges for J' is

$$\mathcal{Z} = \mathcal{S}' \cup \mathcal{P} \cup \mathcal{R} \cup \mathcal{T}.$$

We denote the incidence graph of the triple J' of J by I'. See Fig. 1 for an example that illustrates the case where q_1 belongs to S_1, S_4 and S_7.

Proposition 7. *The hypergraph $J = (Q, \mathcal{S})$ has a 2-colouring $B \cup W$ if and only if its triple $J' = (X, \mathcal{Z})$ has a 2-colouring $B' \cup W'$ such that for each $1 \le i \le m$ either $Q_i \subseteq B'$ and $U_i \subseteq W'$, or $Q_i \subseteq W'$ and $U_i \subseteq B'$.*

Proof. Suppose $B \cup W$ is a 2-colouring of J. Define a partition $B' \cup W'$ of X as follows. If q_i is in B, then each q_i^k is in B' and each u_i^k is in W'. If q_i is in W, then each q_i^k is in W' and each u_i^k is in B'. Obviously, $B' \cup W'$ is a 2-colouring of J' with the desired property.

Suppose we have a 2-colouring $B' \cup W'$ of J' such that for each $1 \le i \le m$ either $Q_i \subseteq B'$ and $U_i \subseteq W'$, or $Q_i \subseteq W'$ and $U_i \subseteq B'$. Then let $q_i \in B$ if and only if $Q_i \subseteq B'$, and let $W = Q \setminus B$. Clearly, if S_j contains only elements from B (respectively W), then S_j' would contain only elements from B' (respectively W'). Hence $B \cup W$ is a 2-colouring of J. ∎

4. Complexity classification

We now have all the ingredients to prove our main complexity result. We repeat it here for convenience.

Theorem 8. *For $k \geq 2$, PKO(k) is NP-complete even if instances are restricted to the class of bipartite graphs.*

Proof. It is clear that PKO(k) is in NP. The rest of the proof is in two cases. We give separate proofs for the cases $k = 2$ and $k \geq 3$.

Case 1. $k = 2$. We use reduction from DS. Given $G = (V, E)$ and a positive integer $p \leq |V|$, we shall complete the proof by constructing a bipartite graph B such that $\text{pko}(B) = 2$ if and only if G has a dominating set D with $|D| \leq p$.

Let the vertex set of B be the disjoint union of $V = \{v_1, \ldots, v_n\}$, $V' = \{v'_1, \ldots, v'_n\}$ and $W = \{w_1, \ldots, w_{n-p}\}$. Let the edge set of B consist of

- $v_i v'_i$, $1 \leq i \leq n$,
- $v_i v'_j$ and $v'_i v_j$, for each edge $v_i v_j \in E$, and
- $v_i w_h$, $1 \leq i \leq n$, $1 \leq h \leq n - p$.

Suppose that G has a dominating set $D = \{v_1, \ldots, v_d\}$ where $d \leq p$. Note that every vertex in V' is adjacent to a vertex of D in B. We shall describe a 2-round KO-reduction scheme for B. In the first round

- for $1 \leq i \leq n$, v_i fires at v'_i,
- for $1 \leq j \leq p$, v'_j fires at v_j,
- for $p + 1 \leq j \leq n$, v'_j fires at a vertex in D, and
- for $1 \leq h \leq n - p$, w_h fires at a vertex in D.

Thus each vertex in $\{v_1, \ldots, v_p\}$ and V' is eliminated in the first round, and each vertex in $V \setminus \{v_1, \ldots, v_p\}$ and W survives to round 2. As the surviving vertices induce the balanced complete bipartite graph $K_{n-p,n-p}$ in B, it is clear that every surviving vertex can be eliminated in one further round.

Now suppose that B has a 2-round KO-reduction scheme. Let D be the subset of V containing vertices that are fired at in round 1. As every vertex in V' fires at – and so is adjacent to – a vertex in D, D is a dominating set in G (since each vertex in V' is joined only to copies of itself and its neighbours). We complete the proof of Case 1 by showing that $|D| \leq p$. Let $V_S = V \setminus D$ and $V'_S \subset V' \cup W$ be the sets of vertices that survive round 1. As round 2 is the final round,

$$|V_S| = |V'_S|. \tag{1}$$

As $|V' \cup W| = 2n - p$ and at most n vertices in $V' \cup W$ are fired at in round 1, $|V'_S| \geq n - p$. Thus, by (1), $|V_S| \geq n - p$. Therefore

$$\begin{aligned} |D| &= |V| - |V_S| \\ &\leq n - (n - p) \\ &= p. \end{aligned}$$

Case 2. $k \geq 3$. We use reduction from H2C. Let $J = (Q, \mathcal{S})$ be an instance of H2C. Let I' be the incidence graph of its triple $J' = (X, \mathcal{Z})$. Recall that $\mathcal{Z} = \mathcal{S}' \cup \mathcal{P} \cup \mathcal{R} \cup \mathcal{T}$. From I', we obtain another bipartite graph G by adding $|X| + |\mathcal{Z}|$ mutually vertex-disjoint paths and connecting each vertex of I' with one of these added paths as follows:

- For each vertex x in X, add a path $H^x = y^x_1 y^x_2 y^x_3$ and join x to y^x_1.
- For each vertex R in $\mathcal{R}$, add a path $H^R = y^R_1 \ldots y^R_4$ and join R to y^R_1.
- For each vertex T in $\mathcal{T}$, add a path $H^T = y^T_1 \ldots y^T_4$ and join T to y^T_1.
- For each vertex P in $\mathcal{P}$, add a path $H^P = y^P_1 \ldots y^P_7$ and join P to the centrevertex y^P_4.
- For each vertex S' in $\mathcal{S}'$, add a path $H^{S'} = y^{S'}_1 \ldots y^{S'}_7$ and join S' to the centrevertex $y^{S'}_4$.

Fig. 2 illustrates G.

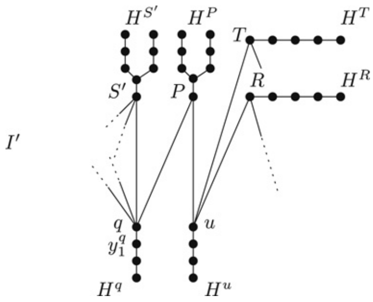

Fig. 2. The graph G in Case 2.

We complete the proof by showing that J is 2-colourable if and only if $\text{pko}(G) \leq k$. Throughout the proof, G_1 and G_2 denote the graphs induced by the surviving vertices of G after, respectively, one and two rounds of a KO-reduction scheme.

Suppose $B \cup W$ is a 2-colouring of J. By Proposition 7, J' has a 2-colouring $B' \cup W'$. We define a three-round KO-reduction scheme for G, so we show that in this case $\text{pko}(G) \leq 3 \leq k$.

Round 1. Vertices of degree 1 and their neighbours fire at each other. Each H^P with $P \in \mathcal{P}$ and each $H^{S'}$ with $S' \in \mathcal{S}'$ is a centred path of G, and the vertices fire as in Observation 6. For each $z \in \mathcal{R} \cup \mathcal{T}$, vertex y_1^z fires at y_2^z and y_2^z fires at y_3^z. Each vertex in $\mathcal{Z}$ fires at one of its neighbours in B'. Each vertex x in X fires at its neighbour y_1^x in H^x. Each y_1^x with $x \in B'$ fires at x. Each y_1^x with $x \in W'$ fires at y_2^x.

Thus every vertex in W' and no vertex in B' survives the first round. Also every vertex in $\mathcal{Z}$ survives the first round. After the first round, each vertex $z \in \mathcal{R} \cup \mathcal{T}$ is adjacent to a vertex y_1^z of degree 1, and each vertex $z \in \mathcal{S}' \cup \mathcal{P}$ is adjacent to a vertex y_4^z whose only other neighbour is a vertex y_3^z (or y_5^z) of degree 1.

Round 2. Because $B' \cup W'$ is a 2-colouring of $J = (X, \mathcal{Z})$, every vertex in $\mathcal{Z}$ has a neighbour in W' in G_1. For each $S'_j \in \mathcal{S}'$ we choose one neighbour in W' and let W'' be the set of selected vertices. Since no two vertices in $\mathcal{S}'$ have a common neighbour in X, $|W''| = n$. The vertices in G_1 fire as follows. Vertices of degree 1 and their neighbours fire at each other. Each vertex $P \in \mathcal{P}$ with a neighbour in $W' \backslash W''$ fires at this neighbour. Otherwise P fires at y_4^P. Each $x \in X$ fires at its neighbour in $\mathcal{P}$. Each $S' \in \mathcal{S}'$ fires at $y_4^{S'}$.

Thus the vertex set of G_2 is $W'' \cup \mathcal{S}'$.

Round 3. Each $S' \in \mathcal{S}'$ and its unique neighbour in W'' fire at each other, which leaves us with the null graph.

Now we suppose that $\text{pko}(G) \leq k$. We assume that a particular KO-reduction scheme for G is given and prove that J has a 2-colouring. We start with the following useful property.

Claim 1. If a vertex of a set Q_i is knocked out in the first round, then all the vertices of Q_i are knocked out in the first round.

Proof of Claim 1. Suppose that a vertex $q_i^k \in Q_i$ is knocked out in the first round. We prove the claim by showing that q_i^{k+1} (with $q_i^{\ell(i)+1} = q_i^1$) is also knocked out in the first round.

If $q_i^k \in Q_i$ is knocked out in the first round, then, by Observation 5, q_i^k fires at $y_1^{q_i^k}$. Suppose q_i^{k+1} is *not* knocked out in the first round. Observation 6 implies that P_i^{k+1} must fire at u_i^{k+1}, and P_i^k must fire at either q_i^k or u_i^k. If P_i^k fires at u_i^k, then, by Observation 5, u_i^k fires at $y_1^{q_i^k}$. Since vertices in $H^{P_i^k}$ must fire as in Observation 6, this means that G_1 contains a component isomorphic to a path on three vertices. By Observation 4, G_1 is not KO-reducible. Hence, P_i^k fires at q_i^k.

For the same reason R_i^{k+1} or T_i^{k+1} cannot fire at u_i^k, and consequently they fire at $y_1^{R_i^{k+1}}$ and $y_1^{T_i^{k+1}}$, respectively. Due to Observation 5 this implies that $y_1^{R_i^{k+1}}$ fires at $y_2^{R_i^{k+1}}$, and $y_1^{T_i^{k+1}}$ fires at $y_2^{T_i^{k+1}}$.

In G_1, T_i^k and R_i^k have exactly the same neighbours, namely u_i^k and q_i^{k+1}. If T_i^k and R_i^k fire at a different neighbour in the second round, then due to Observation 5 both will be isolated vertices in G_2. Suppose T_i^k and R_i^k fire at the same neighbour. Then in all possible schemes G_2 will contain two vertices of degree 1 having the same neighbour.

Observation 4 implies that G_2 is not KO-reducible. We conclude that q_i^{k+1} must be knocked out in the first round as well.
Using the same arguments, we get the following claim.

Claim 2. If a vertex in a set U_i is knocked out in the first round, then all vertices in U_i are knocked out in the first round.

By Claims 1 and 2 we may define a set $B' \subseteq X$ as follows. All vertices of a set Q_i or U_i are in B' if and only if the set is knocked out in the first round. Let $W' = X \backslash B'$.

We need one more claim.

Claim 3. For all $1 \leq i \leq m$, either $Q_i \subseteq B'$ and $U_i \subseteq W'$, or $Q_i \subseteq W'$ and $U_i \subseteq B'$.

Proof of Claim 3. Let $1 \leq i \leq m$. By Observation 6, each vertex $P_i^k \in \mathcal{P}_i$ must fire at either q_i^k or u_i^k in the first round. The previous two claims imply that Q_i or U_i is knocked out in the first round. Suppose both sets are knocked out in the first round. Then, by Observation 5, u_i^1 fires at $y_1^{u_i^1}$, and q_i^1 fires at $y_1^{q_i^1}$. Then, by Observation 6, P_i^1 will not be knocked out in any round. The claim is proved.

By Claim 3, all vertices in $\mathcal{Z} \backslash \mathcal{S}'$ have one neighbour in B' and one neighbour in W'. Let S_j' be a vertex in $\mathcal{S}$. By Observation 6, S_j' fires at a neighbour in $\bigcup_{i=1}^m Q_i$. By definition, this neighbour is in B'. By Observations 5 and 6, S_j' is knocked out by a neighbour in $\bigcup_{i=1}^m Q_i$ that is not knocked out in the first round. By definition, this neighbour is in W'. It is now clear that $B' \cup W'$ is a 2-colouring of J' such that for each $1 \leq i \leq m$ either $Q_i \subseteq B'$ and $U_i \subseteq W'$, or $Q_i \subseteq W'$ and $U_i \subseteq B'$. Hence, by Proposition 7, the hypergraph J also has a 2-colouring. This completes the proof of Theorem 1. ∎

Theorem 1 has the following two easy consequences.

Corollary 9. *The problem* PKO *is* NP*-complete, even if instances are restricted to the class of bipartite graphs.*

Proof. The problem PKO is clearly in NP. We use reduction from H2C. From an instance $J = (Q, \mathcal{S})$ we construct the graph G as in the proof of Theorem 1. We claim that J is 2-colourable if and only if G is KO-reducible.

Suppose that J is 2-colourable. As we have seen in the proof of Theorem 1 this implies that $\text{pko}(G) \leq 3$. Hence G is KO-reducible.

Suppose that G is KO-reducible. We copy the proof of Case 2 of Theorem 1. ∎

The second corollary of Theorem 1 involves the following decision problem.

EXACT PARALLEL KNOCK-OUT (k) (EPKO(k))
Instance: A graph G.
Question: Is $\text{pko}(G) = k$?

Corollary 10. *The problem* EPKO(k) *is polynomially solvable for* $k = 1$ *and is* NP*-complete for* $k \geq 2$*, even if instances are restricted to the class of bipartite graphs.*

Proof. We already observed in Section 1 that EPKO(1) is polynomially solvable. This implies that EPKO(2) is NP-complete since PKO(2) is NP-complete.

For the case $k \geq 3$ we make use of a family of trees Y_ℓ with $\text{pko}(Y_\ell) = \ell$ that have been constructed in [3]. For convenience, we recall the recursive definition of two sequences $\langle Y_1, Y_2, \ldots \rangle$ and $\langle Z_1, Z_2, \ldots \rangle$ of rooted trees:

- The tree Y_1 consists of a root with one child (Y_1 is a rooted P_2).
- The tree Z_1 consists of a root with one child and one grandchild (Z_1 is a rooted P_3).
- For $\ell \geq 2$, the tree Y_ℓ consists of a root r and ℓ disjoint subtrees. The first $\ell - 2$ of these subtrees are copies of the rooted trees $Z_1, \ldots, Z_{\ell-2}$; the last two of these subtrees are copies of $Z_{\ell-1}$; r is adjacent to the roots of the ℓ subtrees.
- For $\ell \geq 2$, the tree Z_ℓ consists of a root r and ℓ subtrees. These subtrees are copies of the rooted trees $Y_1, \ldots, Y_\ell$; r is adjacent to the roots of the ℓ subtrees.

We add a disjoint copy of the tree Y_k to the graph G constructed in the proof of Case 2 in Theorem 1. The new graph G' has $\mathrm{pko}(G') = k$ if and only if $\mathrm{pko}(G) \leq k$.

Note that the size of a tree Y_k only depends on k and not on the size of our input graph G (so we do not need the exact description of this family). We can even make the instance graph connected by adding an edge between the neighbour of a leaf in Y_k and the neighbour of a degree-one vertex in G. Note that H2C remains NP-complete for connected hypergraphs. Also note that by Observation 3, in any KO-reduction scheme of the new graph a degree-one vertex and its neighbour knock each other out in the first round, so the added edges do not change the KO-reducibility properties of the graph. ■

5. Bounded tree-width

In this section we use *monadic second-order logic*; that is, that fragment of second-order logic where quantified relation symbols must have arity 1. For example, the following sentence, which expresses that a graph (whose edges are given by the binary relation E) can be 3-coloured, is a sentence of monadic second-order logic:

$$\exists R \exists W \exists B \left\{ \forall x \left((R(x) \vee W(x) \vee B(x)) \wedge \neg(R(x) \wedge W(x)) \right.\right.$$
$$\left. \wedge \neg(R(x) \wedge B(x)) \wedge \neg(W(x) \wedge B(x)) \right) \wedge \forall x \forall y \left(E(x, y) \Rightarrow \right.$$
$$\left.\left. (\neg(R(x) \wedge R(y)) \wedge \neg(W(x) \wedge W(y)) \wedge \neg(B(x) \wedge B(y))) \right) \right\}$$

(the quantified unary relation symbols are R, W and B, and should be read as sets of 'red', 'white' and 'blue' vertices, respectively). Thus, in particular, there exist NP-complete problems that can be defined in monadic second-order logic.

A seminal result of Courcelle [6] is that on any class of graphs of bounded tree-width, every problem definable in monadic second-order logic can be solved in time linear in the number of vertices of the graph. Moreover, Courcelle's result holds not just when graphs are given in terms of their edge relation, as in the example above, but also when the domain of a structure encoding a graph G consists of the disjoint union of the set of vertices and the set of edges, as well as unary relations V and E to distinguish the vertices and the edges, respectively, and also a binary incidence relation I which denotes when a particular vertex is incident with a particular edge (thus, $I \subseteq V \times E$). The reader is referred to [6] for more details and also for the definition of tree-width which is not required here. To prove Theorem 2, we need only prove the following proposition.

Proposition 11. *For $k \geq 1$, PKO(k) can be defined in monadic second-order logic.*

Proof. Recall that a parallel knock-out scheme for a graph $G = (V, E)$ is a sequence of graphs

$$G \rightsquigarrow G_1 \rightsquigarrow G_2 \rightsquigarrow \cdots \rightsquigarrow G_r,$$

where G_r is the null graph. Let $W_0 = V$ and, for $1 \leq i \leq r$, let W_i be the vertex set of G_i. If we can write a formula $\Phi(W_i, W_{i+1})$ of monadic second-order logic that says

there exists a KO-selection f_i on W_i such that the vertex set of the KO-successor is W_{i+1},

then we could prove the proposition with the following sentence Ω_k which is satisfied if and only if G is in PKO(k):

$$\begin{aligned}&\exists W_0 \exists W_1 \cdots \exists W_k \, \{\forall v (W_0(v) \Leftrightarrow V(v)) \\ &\quad \wedge \Phi(W_0, W_1) \wedge \Phi(W_1, W_2) \wedge \cdots \wedge \Phi(W_{k-1}, W_k) \\ &\quad \wedge (\forall v (\neg W_k(v) \Leftrightarrow V(v)))\}\end{aligned}$$

(Here and elsewhere we have presupposed that each W_i is a set of vertices; we could easily include additional clauses to check this explicitly.)

The following claim will help us write $\Phi(W_i, W_{i+1})$.

Claim 4. There is a KO-selection f_i on W_i such that W_{i+1} is the vertex set of the KO-successor if and only if there is a partition V_1, V_2, V_3 of W_i and subsets E_1, E_2, E_3 of E such that

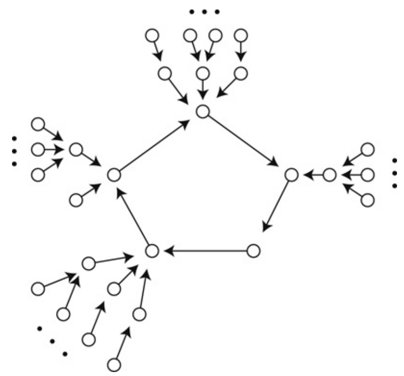

Fig. 3. A representation of vertices firing.

(a) for $j = 1, 2, 3$, each vertex in V_j is incident with exactly one edge of E_j, this edge joins it to a vertex in $W_i \setminus V_j$, and this accounts for every edge in E_j (so $|V_j| = |E_j|$).
(b) $W_{i+1} \subset W_i$ and, for $j = 1, 2, 3$, $W_{i+1} \cap V_j$ is the set of vertices in V_j not incident with edges in $E_{j'}$ for any $j' \neq j$.

We will prove the claim later. First we use it to write $\Phi(W_i, W_{i+1})$.

The following formula $\psi(V_1, E_1, V_2, E_2, V_3, E_3, W_i)$ checks that the sets V_1, V_2 and V_3 partition W_i, that the sets E_1, E_2, E_3 are edges in the graph, and that (a) is satisfied.

$$\begin{aligned}
&\forall v((V_1(v) \vee V_2(v) \vee V_3(v)) \Leftrightarrow W_i(v)) \wedge \forall v(\neg(V_1(v) \wedge V_2(v))\\
&\quad \wedge\neg(V_1(v) \wedge V_3(v)) \wedge \neg(V_2(v) \wedge V_3(v)))\\
&\quad \wedge\forall x((E_1(x) \vee E_2(x) \vee E_3(x)) \Rightarrow E(x))\\
&\quad \wedge\forall x(E_1(x) \Rightarrow \exists u \exists v(V_1(u) \wedge (V_2(v) \vee V_3(v)) \wedge I(u, x) \wedge I(v, x)))\\
&\quad \wedge\forall x(E_2(x) \Rightarrow \exists u \exists v(V_2(u) \wedge (V_1(v) \vee V_3(v)) \wedge I(u, x) \wedge I(v, x)))\\
&\quad \wedge\forall x(E_3(x) \Rightarrow \exists u \exists v(V_3(u) \wedge (V_1(v) \vee V_2(v)) \wedge I(u, x) \wedge I(v, x)))\\
&\quad \wedge\forall v(V_1(v) \Rightarrow \exists! x(I(v, x) \wedge E_1(x)))\\
&\quad \wedge\forall v(V_2(v) \Rightarrow \exists! x(I(v, x) \wedge E_2(x)))\\
&\quad \wedge\forall v(V_3(v) \Rightarrow \exists! x(I(v, x) \wedge E_3(x))).
\end{aligned}$$

(The semantics of $\exists!$ is 'there exists exactly one'; clearly, this abbreviates a more complex though routine first-order formula.) The following formula checks that (b) is satisfied and is denoted $\chi(V_1, E_1, V_2, E_2, V_3, E_3, W_i, W_{i+1})$.

$$\begin{aligned}
&\forall v(W_{i+1}(v) \Leftrightarrow (W_i(v) \wedge ((V_1(v) \wedge \neg\exists x((E_2(x) \vee E_3(x)) \wedge I(v, x)))\\
&\quad \vee(V_2(v) \wedge \neg\exists x((E_1(x) \vee E_3(x)) \wedge I(v, x)))\\
&\quad \vee(V_3(v) \wedge \neg\exists x((E_1(x) \vee E_2(x)) \wedge I(v, x)))))).
\end{aligned}$$

And now we can write $\Phi(W_i, W_{i+1})$:

$$\begin{aligned}
&\exists V_1 \exists E_1 \exists V_2 \exists E_2 \exists V_3 \exists E_3 (\psi(V_1, E_1, V_2, E_2, V_3, E_3, W_i)\\
&\quad \wedge\chi(V_1, E_1, V_2, E_2, V_3, E_3, W_i, W_{i+1})).
\end{aligned}$$

It only remains to prove Claim 4. Suppose that we have sets V_1, V_2, V_3, E_1, E_2 and E_3 that satisfy the conditions of the claim. Then to define the KO-selection f_i, for $j = 1, 2, 3$, for each vertex $v \in V_j$, let v fire at the unique neighbour joined to v by an edge in E_j. It is easy to check that W_{i+1} is the vertex set of the KO-successor.

Now suppose that we have a KO-selection f_i. Let H_i be the spanning subgraph of G_i with edge set $\{vf_i(v) \mid v \in W_i\}$. The firing can be represented as an orientation of H: orient each edge from v to $f_i(v)$ (some edges may be oriented in both directions). As each vertex has exactly one edge oriented away from it, each component of the oriented graph contains one directed cycle, of length at least 2, with a pendant in-tree attached to each vertex of the cycle; see Fig. 3.

We find the sets $V_1, V_2, V_3, E_1, E_2, E_3$; the edge sets contain only edges of H_i. We may assume that H_i is connected (else we can find the sets componentwise). Let the vertices of the unique cycle in the orientation be $v_1, \dots, v_c$ where the edges are $v_l v_{l+1}$, $1 \le l \le c-1$, and $v_c v_1$. So H_i contains vertices $v_1, \dots, v_c$ with a pendant tree (possibly trivial) attached to each.

For $1 \le l \le c$, let U_e^l be the set of vertices in the pendant tree attached to v_l whose distance from v_l is even (but not zero), and let U_o^l be the vertices in the tree at odd distance from v_l. Let

$$\begin{aligned} V_1 &= \bigcup_{l \text{ odd}} U_o^l \cup \bigcup_{l \text{ even}} U_e^l \cup \{v_l : l \text{ is even}, l \neq c\}, \\ V_2 &= \bigcup_{l \text{ odd}} U_e^l \cup \bigcup_{l \text{ even}} U_o^l \cup \{v_l : l \text{ is odd}, l \neq c\}, \text{ and} \\ V_3 &= \{v_c\}, \end{aligned}$$

and, for $i = 1, 2, 3$, let E_i contain $v f_i(v)$ for each $v \in V_i$. It is clear that the sets we have chosen satisfy the conditions of the claim.

This completes the proof of the claim and of the proposition. ■

Theorem 2 follows from the proposition. And, noting that EPKO(k) is defined by the monadic second-order sentence $\Omega_k \wedge \neg\Omega_{k-1}$, we have the following result.

Corollary 12. *For $k \geq 1$, EPKO(k) is solvable in linear time on any class of graphs with bounded tree-width.*

In particular, we obtain the following result for trees, answering an open question in [3].

Corollary 13. *For $k \geq 1$, EPKO(k) is solvable in linear time for trees.*

Finally, we note that to check whether a graph G is reducible it is sufficient to check whether $\text{pko}(G) = k$, for $1 \le k \le \Delta$, where Δ is the maximum degree of G. Thus G is reducible if and only if the sentence $\Omega_\Delta \vee \Omega_{\Delta-1} \vee \cdots \vee \Omega_1$ is satisfied. This gives us our last result of this section.

Corollary 14. *On any class of graphs with bounded tree-width,* PKO *can be solved in polynomial time.*

6. Graphs with a small parallel knock-out number

As we noted in the introduction, graphs with a $[1,2]$-factor, or more particularly with a perfect matching or with a hamiltonian cycle have parallel knock-out number 1. We start this section by studying the related classes of *factor-critical* graphs and of *1-tough* graphs. A graph G is said to be factor-critical if $G - v$ has a perfect matching for every vertex v of G. A graph $G = (V, E)$ is called 1-tough if $\omega(G - S) \le |S|$ for every nonempty subset S of V, where $\omega(G - S)$ denotes the number of components of the graph $G - S$. Clearly, every hamiltonian graph is 1-tough and every factor-critical graph has a matching leaving only one vertex unmatched. A natural question is whether factor-critical graphs and 1-tough graphs have a small parallel knock-out number. The next results show that these graphs in fact have a $[1,2]$-factor, i.e., have parallel knock-out number 1 (unless they are trivial, i.e., contain only one vertex).

We start with factor-critical graphs.

Theorem 15. *Let G be a nontrivial factor-critical graph and $v \in V(G)$. Then G has a $[1,2]$-factor consisting of an odd cycle C containing v and a perfect matching in $G - V(C)$.*

Proof. Let M be a perfect matching in $G - v$. If some neighbours of v are matched by M, we immediately find the desired odd cycle (triangle) C and perfect matching in $G - V(C)$, and we are done. Suppose this is not the case. Then we take an arbitrary neighbour x of v, and note that there exists a perfect matching M^* in $G - x$. Clearly x is matched to a vertex $y \neq v$ under M and v is matched to a vertex p under M^*. By our assumption we may assume that $p \neq y$; otherwise we find a triangle C with the desired properties. Since both M and M^* saturate all vertices except for v and x, respectively, there exists an (M^*, M)-alternating path P between v and y beginning and ending with an edge of M^*. Now P together with the edges between x and v and x and y forms an odd cycle C, and the remaining edges of M^* form a perfect matching in $G - V(C)$. ■

The above result also implies that nontrivial 1-tough graphs on an odd number of vertices have a $[1, 2]$-factor. In order to prove this, we need the following well-known result of Tutte [11].

Let $\omega_o(G)$ denote the number of *odd components* of a graph G, i.e., the number of components containing an odd number of vertices.

Theorem 16 *([11]). A graph G has a perfect matching if and only if $\omega_o(G - S) \le |S|$ for all $S \subseteq V(G)$.*

This theorem has the following consequence.

Corollary 17. *If G is a 1-tough graph on an odd number of vertices, then G is factor-critical.*

Proof. Suppose G is 1-tough on an odd number of vertices, but not factor-critical. Then there exists a vertex $v \in V(G)$ such that $G' = G - v$ has no perfect matching. Thus by Theorem 16 there exists a set $X' \subseteq V(G')$ with $\omega_o(G' - X') = |X'| + 1 + k$, for some integer $k \ge 0$. Setting $X = X' \cup \{v\}$, and letting ω_e denote the number of even components, we have

$$\begin{aligned}\omega(G - X) &= \omega_o(G - X) + \omega_e(G - X)\\ &= \omega_o(G' - X') + \omega_e(G' - X')\\ &= |X'| + 1 + k + \omega_e(G' - X')\\ &= |X| + k + \omega_e(G' - X')\\ &\ge |X| \ge 1.\end{aligned}$$

Since G is 1-tough, $k = 0$ and $\omega_e(G' - X') = 0$; otherwise $\omega(G - X) > |X| \ge 1$, a contradiction. Let $H_1, \dots, H_{|X'|+1}$ denote the odd components of $G' - X'$. Then $|V(G)| = 1 + |X'| + \sum_{i=1}^{|X'|+1} |V(H_i)| = \sum_{i=1}^{|X'|+1} (|V(H_i)| + 1)$. Since all $|V(H_i)|$ are odd, $|V(G)|$ is even, a contradiction. ■

Corollary 18. *Every nontrivial 1-tough graph has a $[1, 2]$-factor.*

Proof. Consider a nontrivial 1-tough graph G on n vertices. If n is odd the result follows by combining Theorem 15 and Corollary 17. If n is even G has a perfect matching by Theorem 16. ■

We now turn to regular graphs and 'almost' regular bipartite graphs. First we note that the trick introduced in Section 1 immediately implies that every r-regular graph G with $r \ge 1$ has a $[1, 2]$-factor, i.e., $\text{pko}(G) = 1$.

Proposition 19. *Every r-regular graph G with $r \ge 1$ has a $[1, 2]$-factor.*

Proof. Let G be an r-regular graph with $r \ge 1$ and with $V(G) = \{v_1, v_2, \dots, v_n\}$. Define a bipartite graph G' with vertex set $\{u_1, u_2, \dots, u_n, w_1, w_2, \dots, w_n\}$ in which $u_i w_j \in E(G')$ and $u_j w_i \in E(G')$ if and only if $v_i v_j \in E(G)$. Then G' is an r-regular bipartite graph and has a perfect matching (See, e.g., [2] Exercise 5.2.3(a)). This matching corresponds to a $[1, 2]$-factor in G. ■

The above result also immediately implies the following statement for graphs that contain a *k-factor*, i.e., a spanning k-regular subgraph.

Corollary 20. *Every nontrivial graph with a k-factor has a $[1, 2]$-factor.*

Our complexity results for bipartite graphs motivated us to consider bipartite graphs that are 'almost' regular in the following sense. A bipartite graph G is called *(r, s)-regular* if all vertices in one class of the bipartition have degree r and all other vertices have degree s. By Proposition 19 any (r, r)-regular bipartite graph G with $r \ge 1$ has $\text{pko}(G) = 1$, and one easily checks that any $(1, s)$-regular bipartite graph G with $s \ge 2$ has $\text{pko}(G) = \infty$. With the next result we characterize all reducible $(2, s)$-regular bipartite graphs, noting that $(2, s)$-regular graphs with $s \ne 2, 3$ are not reducible.

Let G be a $(2, 3)$-regular bipartite graph, and let L denote the vertices with degree 2 (*left* vertices) and R the vertices with degree 3 (*right* vertices). Then $|E(G)| = 2|L| = 3|R|$, so $|R| = 2k$ and $|L| = 3k$ for some positive integer k. We call a subset A of R with k vertices that has the whole set L as its neighbourhood a *k-star cover* of G. Clearly this implies that all vertices of A have mutually disjoint neighbours in L. We will also need the notion of an f-factor and a result due to Ore. If f is an integer-valued function on the set $V(G)$ such that $0 \le f(v) \le d_G(v)$ for each vertex

$v \in V(G)$, then a spanning subgraph F of G is called an *f-factor* of G if $d_F(v) = f(v)$ for each vertex $v \in V(G)$. The following theorem of Ore (see, e.g., [1], Theorem 7.2.2) characterizes bipartite graphs with an f-factor. In this theorem $E(U, y)$ denotes the set of edges between a vertex set U and a vertex y, and $f(U) = \sum_{v \in U} f(v)$.

Theorem 21. *Let G be a bipartite graph with bipartition (L, R). Then G has an f-factor if and only if $f(L) = f(R)$ and for any set U of R: $f(U) \leq \sum_{y \in L} \min(f(y), |E(U, y)|)$.*

With the adoption of the above conventions and result we can prove the following result for (2, 3)-regular bipartite graphs.

Theorem 22. *Let G be a $(2, 3)$-regular bipartite graph. Then G is reducible if and only if pko$(G) = 2$ if and only if G has a k-star cover. Moreover, we can determine pko(G) in polynomial time.*

Proof. Let G be a (2, 3)-regular bipartite graph, and let L denote the vertices with degree 2 and R the vertices with degree 3. If G is reducible, then pko$(G) = 2$: it cannot be 1 since $|L| > |R|$, and it cannot be larger than 2 since in every round the degree of the vertices decreases by at least 1 and the vertices in R cannot eliminate each other since R is an independent set. This clearly proves the first equivalence of the statement.

For the same reasons, if pko$(G) = 2$, then in the second round the remaining vertices of L have degree 1, and those vertices and the remaining vertices of R eliminate each other along a perfect matching M. We now show that $|M| = k$. First of all, $|M| \geq k$ since $|R| = 2k$ and hence the vertices of R can eliminate at most $2k$ vertices of L in the first round; secondly, $|M| \leq k$ since otherwise all $3k$ vertices in L eliminate fewer than k from R in round 1, contradicting that every vertex of R has degree 3 in G. Since the remaining k vertices of R are saved in round 1, all $3k$ left vertices eliminate together only $|R| - k = k$ right vertices. Since all vertices in R have degree 3, this scheme corresponds to a k-star cover of G.

Conversely, suppose G has a k-star cover A in R. Then $B = R \setminus A$ is also a k-star cover of G. Now we set $f(x) = 1$ for all $x \in A \cup L$, $f(x) = 2$ for all $x \in B$. Then $f(L) = 3k$ and $f(R) = k + 2k = 3k$, so $f(L) = f(R)$. For a set $U \subseteq R$, $f(U) = |U \cap A| + 2|U \cap B| \leq 3 \max(|U \cap A|, |U \cap B|)$. Since A is a k-star cover, all neighbours of the vertices of $U \cap A$ in L are distinct, and the same holds for B and $U \cap B$. Any neighbour $y \in L$ of each of the vertices from $U \cap A$ or $U \cap B$ clearly has $\min(f(y), |E(U, y)|) \geq 1$ since $f(y) = 1$ and y is a neighbour of a vertex of U. So $\sum_{y \in L} \min(f(y), |E(U, y)|) \geq 3 \max(|U \cap A|, |U \cap B|)$. Using Theorem 21, we conclude that G has an f-factor. Consider the following KO-scheme for G: in round 1, all vertices in L fire at A, while vertices in A fire via matching edges of the f-factor and vertices in B fire along one of the edges of the f-factor. After round 1, all remaining vertices at the right side are precisely the set B. Because of the f-factor in G and the firing in the first round, the vertices of B form a perfect matching M with the remaining vertices in L. We use M to eliminate all remaining vertices in the second round. This completes the proof of the second equivalence of the statement.

Determining whether G has a k-star cover is a problem that can be solved in polynomial time. Since both A and B must be k-star covers, one can start by putting one arbitrary vertex v of R in A, and then putting all vertices of R that have a common neighbour with v in B, and so on, until all vertices have been allocated or a conflict occurs. ■

The cases of reducible (r, s)-regular bipartite graphs for other values of r and s do not seem to admit a similar characterization. We leave them as interesting open problems.

7. Conclusions

In this paper we have studied the computational complexity of problems related to the parallel knock-out number pko(G) of a graph G. We have shown that determining whether pko$(G) = 1$ is polynomially solvable, whereas determining whether pko$(G) \leq k$ (or pko$(G) = k$) is NP-complete for any fixed $k \geq 2$ that is not part of the input, even when restricted to the class of bipartite graphs. We also showed that the latter problems restricted to graphs with bounded tree-width are solvable in linear time, by formulating them in monadic second-order logic. Moreover, we studied some special graph classes with small parallel knock-out numbers.

An interesting open problem is the computational complexity of both the decision problems when restricted to planar graphs. Since outer-planar graphs have bounded tree-width, both problems can be solved in linear time when restricted to outer-planar graphs. Since 4-connected planar graphs are hamiltonian, pko$(G) = 1$ for a 4-connected planar graph G. From a result in [3] we can easily deduce that pko$(G) \leq 20 \log n$ for any reducible planar graph G on n vertices.

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