Symmetric powers of elliptic curve L-functions

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Abstract. The conjectures of Deligne, Beĭlinson, and Bloch-Kato assert that there should be relations between the arithmetic of algebro-geometric objects and the special values of their L-functions. We make a numerical study for symmetric power L-functions of elliptic curves, obtaining data about the validity of their functional equations, frequency of vanishing of central values, and divisibility of Bloch-Kato quotients.

1 Introduction and Motivation

There are many conjectures that relate special values of L-functions to the arithmetic of algebro-geometric objects. The celebrated result $\zeta(2) = \pi^2/6$ of Euler [19, §XV] can be reinterpreted as such, but Dirichlet's class number formula [15, §5] is better seen to be the primordial example. Modern examples run the gamut, from conjectures of Stark [40] on Artin L-functions and class field theory, to that of Birch and Swinnerton-Dyer [2] for elliptic curves, to those of Beĭlinson [1,32] related to K-theory, with a passel of others we do not mention. For maximal generality the language of motives is usually used (see [20, §1-4]).

One key consideration is where the special value is taken. The L-function can only vanish inside the critical strip or at trivial zeros; indeed, central values (at the center of symmetry of the functional equation) are the most interesting ones that can vanish, and the order of vanishing is likely related to the rank of a geometric object (note that orders of trivial zeros can be similarly interpreted).

We have chosen to explore a specific family of examples, namely symmetric power L-functions for rational elliptic curves. The impetus for this work was largely a result in the thesis [29] of the first author, whose computation of Euler factors in the difficult case of additive primes greatly reduced the amount of hassle needed to do large-scale computations. Previous theoretical work includes that of Coates and Schmidt [7] on the symmetric square and Buhler, Schoen, and Top on the symmetric cube [6]; this second paper also contains a lot of computational evidence, while Watkins has provided much data [44] in the symmetric square case. In some cases the L-functions we use are not known to possess the properties that would be required to justify that our computations produce numbers of any validity whatsoever — in these cases, the "numerical coincidence" in our computations can be seen as evidence for the relevant conjectures.

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1.1 Acknowledgements

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2 L-functions

We define the symmetric power L-functions of an elliptic curve $E/\mathbf{Q}$ via computing an Euler factor at every prime p. This Euler factor is computed by a process that essentially just takes the symmetric power representation of the standard 2-dimensional Galois representation associated to E, and thus our method is a generalisation of that used by Coates and Schmidt [7] for the symmetric square,[1] following the original description of Serre [36]. We briefly review the theoretical framework, and give explicit formulae for the Euler factors in a later section.

For every prime p choose an auxiliary prime $l \neq 2, p$ and fix an embedding of $\mathbf{Q}_l$ into $\mathbf{C}$. Let E_t denote the t-torsion of E, and $T_l(E) = \varprojlim E_{l^n}$ be the l-adic Tate module of E (we fix a basis). The module $V_l(E) = T_l(E) \otimes_{\mathbf{Z}_l} \mathbf{Q}_l$ has dimension 2 over $\mathbf{Q}_l$ and has a natural action of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ [indeed one of of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$], and from this we get a representation $\rho_l : \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p) \to \mathrm{Aut}(V_l)$. We write $H_l^1(E) = \mathrm{Hom}_{\mathbf{Q}_l}(V_l(E), \mathbf{Q}_l)$, and take the mth symmetric power of the contragredient of ρ_l, getting

$$\rho_l^m : \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p) \to \mathrm{Aut}\bigl(\mathrm{Sym}^m\bigl(H_l^1(E)\bigr)\bigr) \subset GL_{m+1}(\mathbf{C}).$$

We write $D_p = \mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$, let I_p be the inertia group of this extension, and let Frob_p be the element of $D_p/I_p \cong \mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$ given by $x \to x^p$. With all of this, we have

$$L(\mathrm{Sym}^m E, s) = \prod_p \det\Bigl[\mathrm{Id}_{m+1} - \rho_l^m(\mathrm{Frob}_p^{-1})p^{-s} \Bigm| \bigl(\mathrm{Sym}^m\bigl(H_l^1(E)\bigr)\bigr)^{I_p}\Bigr]^{-1}.$$

For brevity, we write $L_m(E, s) = L(\mathrm{Sym}^m E, s)$, and denote the factors on the right side by $U_m(p; s)$. As mentioned by Coates and Schmidt [7, p. 106], it can be shown that $U_m(p; s)$ is independent of our choices. The analytic theory and conjectures concerning these symmetric power L-functions are described in [39]. In particular, the above Euler product converges in a half-plane, and is conjectured to have a meromorphic continuation to the whole complex plane.

We also need the conductor N_m of this symmetric power representation. We have $N_m = \prod_p p^{f_m(p)}$ where $f_m(p) = \epsilon_m(I_p) + \delta_m(p)$. Here $\epsilon_m(I_p)$ is the codimension of $\bigl(\mathrm{Sym}^m\bigl(H_l^1(E)\bigr)\bigr)^{I_p}$ in $\mathrm{Sym}^m\bigl(H_l^1(E)\bigr)$; we shall see that it can be computed via a character-theoretic argument. The wild conductor $\delta_m(p)$ is 0 unless $p = 2, 3$, when it can be computed as in [36, §2.1] or the appendix of [7].

[1] Note that Buhler, Schoen, and Top [6] phrase their definition of Euler factors differently, as they emphasise that conjecturally the L-function is related to a motive or higher-dimensional variety; however, their definition is really the same as ours.

2.1 Critical values

The work of Deligne [14, Prop. 7.7ff] tells us when and where to expect critical values; these are a subset of the more-general special values, and are the easiest to consider.[2] When $m = 2v$ with v odd there is a critical value $L_m(E, v+1)$ at the edge of the critical strip, and when $m = 2u-1$ is odd there is a critical central value $L_m(E, u)$. We let Ω_+, Ω_- be the real/imaginary periods of E for $m \equiv 1, 2 \pmod 4$, and vice-versa for $m \equiv 3 \pmod 4$. In the respective cases of m even/odd we expect rationality (likely with small denominator) of either

$$\frac{L_m(E, v+1)}{(2\pi)^{v+1}} \left(\frac{2\pi N}{\Omega_+ \Omega_-} \right)^{v(v+1)/2} \qquad \text{or} \qquad \frac{L_m(E, u)(2\pi N)^{u(u-1)/2}}{\Omega_+^{u(u+1)/2} \Omega_-^{u(u-1)/2}}. \tag{1}$$

When m is odd, the order of $L_m(E, s)$ at $s = u$ should equal the rank of an associated geometric object. The Bloch-Kato conjecture [4] relates the quotients in (1) to H^0-groups, Tamagawa numbers, and generalised Shafarevich-Tate groups.[3]

3 Computation of Euler factors and local conductors

We first consider multiplicative and potentially multiplicative reduction for a given prime p; these cases can easily be detected since $v_p(j_E)$, the valuation of the j-invariant, is negative, with the reduction being potentially multiplicative when $p|c_4$. When E has multiplicative reduction, the filtration of [6, §8] implies the local tame conductor ϵ_m is m and $\delta_m(2) = \delta_m(3) = 0$ for all m. The Euler factor is $U_m(p; s) = (1 - a_p^m/p^s)^{-1}$, where $a_p = \pm 1$ is the trace of Frobenius. In the case of potentially multiplicative reduction, for m odd we have $\epsilon_m = m+1$, and so $U_m(p; s) \equiv 1$, while with m even, we have that $\epsilon_m = m$ and compute that $U_m(p; s) = (1 - 1/p^s)^{-1}$. The wild conductor at $p = 2$ is $\delta_m(2) = \frac{m+1}{2}\delta_1(2)$ for odd m and is zero for even m, while $\delta_m(3) = 0$ for all m.

3.1 Good and additive reduction — tame conductors

Let E have good or potentially good reduction at a prime p, and choose an auxiliary prime $l \neq 2, p$. The inertia group I_p acts on $V_l(E)$ by a finite quotient in this case. Let $\mathbf{G}_p = \mathrm{Gal}\big(\mathbf{Q}_p(E_l)/\mathbf{Q}_p\big)$ and Φ_p be the inertia group of this extension.[4] The work of Serre [37] lists the possibilities for Φ_p. It can be a cyclic group C_d with $d = 1, 2, 3, 4, 6$; additionally, when $p = 2$ it can be Q_8 or $SL_2(\mathbf{F}_3)$, and when $p = 3$ it can be $C_3 \rtimes C_4$. For each group there is a unique faithful 2-dimensional representation Ψ_Φ of determinant 1 over $\mathbf{C}$, which determines $\overline{\rho}_l$.

[2] Critical values conjecturally only depend on periods (which are local objects), while the more-general special values can also depend on (global) regulators from K-theory.

[3] See [18, §7] for an explicit example; note his imaginary period is twice that of our normalisation (and the formula is out by a power-of-2 in any case), and the conductor enters the formula in a different place (this doesn't matter for semistable curves).

[4] The group Φ_p is independent of the choice of l (see [37, p. 312]), while only whether $\mathbf{G}_p$ is abelian matters, and this independence follows as in [7, Lemmata 1.4 & 1.5].

Our result now only depends on Φ; for a representation Ψ we have the trace relation (which is related to Chebyshev polynomials of the second kind)

$$\operatorname{tr}(\operatorname{Sym}^m\Psi) = \sum_{k=0}^{m/2} \binom{m-k}{k} \operatorname{tr}(\Psi)^{m-2k}(-\det\Psi)^k, \tag{2}$$

and from taking the inner product of $\operatorname{tr}(\operatorname{Sym}^m\Psi_\Phi)$ with the trivial character we find the dimension of the Φ-fixed subspace of $\operatorname{Sym}^m\big(H_l^1(E)\big)$, which we denote by $\beta_m(\Phi)$. Upon carrying out this calculation, we obtain Table 1, which lists values for $\beta_m(\Phi)$, from which we get the tame conductor $\epsilon_m(\Phi) = m+1-\beta_m(\Phi)$. The wild conductors $\delta_m(p)$ are 0 for $p \geq 5$, and for $p = 2, 3$ are described below.

Table 1. Values of $\beta_m(\Phi)$ for various inertia groups; here $\tilde{m}$ is m modulo 12.

$\tilde{m}$	C_2	C_3	C_4	C_6	Q_8	$C_3 \rtimes C_4$	$SL_2(\mathbf{F}_3)$
0	$m+1$	$(m+3)/3$	$(m+2)/2$	$(m+3)/3$	$(m+4)/4$	$(m+6)/6$	$(m+12)/12$
1	0	$(m-1)/3$	0	0	0	0	0
2	$m+1$	$(m+1)/3$	$m/2$	$(m+1)/3$	$(m-2)/4$	$(m-2)/6$	$(m-2)/12$
3	0	$(m+3)/3$	0	0	0	0	0
4	$m+1$	$(m-1)/3$	$(m+2)/2$	$(m-1)/3$	$(m+4)/4$	$(m+2)/6$	$(m-4)/12$
5	0	$(m+1)/3$	0	0	0	0	0
6	$m+1$	$(m+3)/3$	$m/2$	$(m+3)/3$	$(m-2)/4$	$m/6$	$(m+6)/12$
7	0	$(m-1)/3$	0	0	0	0	0
8	$m+1$	$(m+1)/3$	$(m+2)/2$	$(m+1)/3$	$(m+4)/4$	$(m+4)/6$	$(m+4)/12$
9	0	$(m+3)/3$	0	0	0	0	0
10	$m+1$	$(m-1)/3$	$m/2$	$(m-1)/3$	$(m-2)/4$	$(m-4)/6$	$(m-10)/12$
11	0	$(m+1)/3$	0	0	0	0	0

3.2 Good and additive reduction — Euler factors for $p \geq 5$

When $p \geq 5$, a result of Serre [37] tells us that the inertia group is $\Phi = C_d$ where $d = 12/\gcd\big(12, v_p(\Delta_E)\big)$. Note that this gives $d = 1$ when p is a prime of good reduction, which we naturally include in the results of this part. We summarise the results of Martin's dissertation [29] concerning the Euler factors. Note that the result of Dąbrowski [10, Lemma 1.2.3] appears to be erroneous.

There are two different cases for the behaviour of the Euler factor, depending on whether the decomposition group $\mathbf{G}_p = \operatorname{Gal}\big(\mathbf{Q}_p(E_l)/\mathbf{Q}_p\big)$ is abelian. From [33, Prop. 2.2] or [44, Th. 2.1], we get that this decomposition group is abelian precisely when $p \equiv 1 \pmod d$. When $\mathbf{G}_p$ is nonabelian we have

$$U_m(p;s) = (1-(-p)^{m/2}/p^s)^{-A_m}(1+(-p)^{m/2}/p^s)^{-B_m}, \tag{3}$$

where $A_m + B_m = \beta_m$ and A_m is the dimension of $\big(\operatorname{Sym}^m\big(H_l^1(E)\big)\big)^{\mathbf{G}_p}$. Using $\mathbf{G}_p/\Phi_p \cong C_2$ and $\det\big(\Psi_\Phi(x)\big) = -1$ for $x \in \mathbf{G}_p \backslash \Phi_p$, more character calculations tell us this dimension is $(\beta_m+1)/2$ when β_m is odd and is $\beta_m/2$ when β_m is even. This also holds for the non-cyclic Φ when $p = 2, 3$, for which $\mathbf{G}_p$ is automatically nonabelian. When $\Phi = C_3$ and m is odd, we have $U_m(p;s) = (1+p^m/p^{2s})^{-\beta_m/2}$.

When $\mathbf{G}_p$ is abelian, we need to compute a Frobenius eigenvalue α_p (whose existence follows from [38, p. 499]). In the case of good reduction, this comes from counting points mod p on the elliptic curve; we have $\alpha_p = (a_p/2) \pm i\sqrt{p - a_p^2/4}$ where $p + 1 - a_p$ is the number of (projective) points on E modulo p. And when $\Phi = C_2$ we count points on the pth quadratic twist of E. In general, we need to re-scale the coefficients of our curve by some power of p that depends on the valuations v_p of the coefficients. Since $p \geq 5$, we can write our curve as $y^2 = x^3 + Ax + B$, and then re-scale by a factor $t = p^{\min(v_p(A)/2, v_p(B)/3)}$ to get a new curve $E^t : y^2 = x^3 + Ax/t^2 + B/t^3$, possibly defined over some larger field. Because of our choice of t, at least one of A/t^2 and B/t^3 will have v_p equal to 0. The reduction $\tilde{E}^t$ modulo some (fractional) power of p is then well-defined and non-singular, and we get α_p from counting points on $\tilde{E}^t$; it turns out that choices of roots of unity will not matter when we take various symmetric powers. Returning back to $U_m(p; s)$, we get that when $\mathbf{Q}_p(E_l)/\mathbf{Q}_p$ is abelian this Euler factor is

$$U_m(p; s) = \prod_{\substack{0 \leq i \leq m \\ d|(2i-m)}} (1 - \alpha_p^{m-i} \bar{\alpha}_p^i / p^s)^{-1}. \tag{4}$$

3.3 Considerations when $p = 3$

Next we consider good and additive reduction for $p = 3$. We first determine the inertia group, using the 3-valuation of the conductor as our main guide. In the case that $v_3(N) = 0$ we have good reduction, while when $v_3(N) = 2$ and $v_3(\Delta)$ is even we have $\Phi = C_2$. Since $\mathbf{G}_3$ is abelian here, the Euler factor is given by (4), while the wild conductor is 0 and tame conductor is obtained from Table 1. When $v_3(N) = 2$ and $v_3(\Delta)$ is odd we have that $\Phi = C_4$ and $\mathbf{G}_3$ is nonabelian. The wild conductor $\delta_m(3)$ is 0, and the Euler factor is given by (3).

When $v_3(N) = 4$ we get $\Phi = C_3$ or C_6, the former case when $4|v_3(\Delta)$. For these inertia groups, the question of whether $\mathbf{G}_3$ is abelian can be resolved as follows (see [44, Th. 2.4]). Let $\hat{c}_4$ and $\hat{c}_6$ be the invariants of the minimal twist of E at 3. In the case that $\hat{c}_4 \equiv 9 \pmod{27}$, we have that $\mathbf{G}_3$ is abelian when $\hat{c}_6 \equiv \pm 108 \pmod{243}$ while if $3^3|\hat{c}_4$ then $\mathbf{G}_3$ is abelian when $\hat{c}_4 \equiv 27 \pmod{81}$. In the abelian case we have $\alpha_3 = \zeta_{12}\sqrt{3}$ up to sixth roots, which is sufficient. The Euler factor is then given by either (3) or (4), the tame conductor can be obtained from Table 1, and the wild conductor (computed as in the appendix of [7]) from Table 3. When $v_3(N) = 3, 5$ we have that $\Phi = C_3 \rtimes C_4$. The Euler factor is given by (3) and the wild conductor can be obtained from Table 3, with the first $C_3 \rtimes C_4$ corresponding to $v_3(N) = 3$, and the second to $v_3(N) = 5$.

3.4 Considerations when $p = 2$

Finally we consider $p = 2$, where first we determine the inertia group. Let M be the conductor of the minimal twist F of E at 2, recalling [44, § 2.1] that in general we need to check four curves to determine this twist. Table 4 then gives the inertia group. The appendix of [7] omits a few of these cases; see [44]. When

$\Phi = C_1, C_2$ we can always determine α_p via counting points modulo p on E or a quadratic twist, and $\mathbf{G}_2$ is always abelian. The Euler factor is then as in (4). For $\Phi = C_3, C_6$ the group $\mathbf{G}_2$ is always nonabelian, and the Euler factor is as in (3). For the case of $\Phi = C_4$ and $p = 2$, the question of whether $\mathbf{G}_2$ is abelian comes down [44, Th. 2.3] to whether the c_4 invariant of F is 32 or 96 modulo 128, it being abelian in the latter case, where we have $\alpha_2 = \zeta_8\sqrt{2}$ up to fourth roots. The Euler factors for this and the two cases of noncyclic Φ are obtained from (3) or (4), while the wild conductors $\delta_m(2)$ are given in Table 2, with the appropriate line being determinable from the conductor of the first symmetric power.

Table 2. Values for $\delta_m(2)$.

Φ_2	$m = 1$	formula
C_2, C_6	2	$\epsilon_m(C_2)$
C_2, C_6	4	$2\epsilon_m(C_2)$
C_4	6	$2\epsilon_m(C_4) + \epsilon_m(C_2)$
Q_8	3	$\epsilon_m(Q_8) + \frac{1}{2}\epsilon_m(C_2)$
Q_8	4	$\epsilon_m(Q_8) + \epsilon_m(C_2)$
Q_8	6	$\epsilon_m(Q_8)+\epsilon_m(C_4)+\epsilon_m(C_2)$
$SL_2(\mathbf{F}_3)$	1	$\frac{1}{3}\epsilon_m(Q_8) + \frac{1}{6}\epsilon_m(C_2)$
$SL_2(\mathbf{F}_3)$	2	$\frac{1}{3}\epsilon_m(Q_8) + \frac{2}{3}\epsilon_m(C_2)$
$SL_2(\mathbf{F}_3)$	4	$\frac{1}{3}\epsilon_m(Q_8) + \frac{5}{3}\epsilon_m(C_2)$
$SL_2(\mathbf{F}_3)$	5	$\frac{5}{3}\epsilon_m(Q_8) + \frac{5}{6}\epsilon_m(C_2)$

Table 3. Values for $\delta_m(3)$.

Φ_3	$m = 1$	formula
C_3, C_6	2	$\epsilon_m(C_3)$
$C_3 \rtimes C_4$	1	$\frac{1}{2}\epsilon_m(C_3)$
$C_3 \rtimes C_4$	3	$\frac{3}{2}\epsilon_m(C_3)$

Table 4. Values of Φ_2.

$v_2(M)$	Φ_2
0	C_1 if $v_2(N) = 0$ else C_2
2	C_3 if $v_2(N) = 2$ else C_6
3,7	$SL_2(\mathbf{F}_3)$
5	Q_8
8	Q_8 if $2^9 \mid c_6(F)$ else C_4

3.5 The case of complex multiplication

When E has complex multiplication by an order of some imaginary quadratic field K, the situation simplifies since we have $L(E, s) = L(\psi, s - 1/2)$ for some[5] Hecke Grössencharacter ψ. For the symmetric powers we have the factorisation

$$L(\mathrm{Sym}^m E, s) = \prod_{i=0}^{m/2} L(\psi^{m-2i}, s - m/2), \tag{5}$$

where ψ^0 is the ζ-function when $4|m$, and when $2\|m$ it is $L(\theta_K, s)$ for the quadratic character θ_K of the field K. Note that the local conductors and Euler factors for each $L(\psi^j, s)$ can be computed iteratively from (5) since this information is known for the left side from the previous subsections. This factorisation reduces the computational complexity significantly, as the individual conductors will be smaller than their product; however, since there are more theoretical results in this case, the data obtained will often lack novelty. The factorisation (5) also implies that $L_{2u-1}(E, s)$ should vanish to high degree at $s = u$, since each term has about a 50% chance of having odd functional equation. We found some examples where $L(\psi^3, s)$, $L(\psi^5, s)$, or $L(\psi^7, s)$ has a double zero at the central point, but we know of no such triple zeros.

[5] This is defined on ideals coprime to the conductor by $\psi(z) = \chi(|z|)(z/|z|)$ where z is the primary generator of the ideal and χ is generally a quadratic Dirichlet character, but possibly cubic or sextic if $K = \mathbf{Q}(\sqrt{-3})$, or quartic if $K = \mathbf{Q}(\sqrt{-1})$. When taking powers, we take χ^j to be the primitive Dirichlet character which induces χ^j.

4 Global considerations and computational techniqus

We now give our method for computing special values of the symmetric power L-functions defined above. To do this, we complete the L-function with a Γ-factor corresponding to the prime at infinity, and then use the (conjectural) functional equation in conjunction with the method of Lavrik [28] to write the special value as a "rapidly-converging" series whose summands involve inverse Mellin transforms related to the Γ-factor. First we digress on poles of our L-functions.

4.1 Poles of L-functions

It is conjectured that $L_m(E,s)$ has an entire continuation, except when $4|m$ and E has complex multiplication (CM) there is a pole at $s=1+m/2$, which is the edge of the critical strip.[6] We give an explanation of this expectation from the standpoint of analytic number theory; it is likely that a different argument could be given via representation theory. We write each Euler factor as $U_m(p;s) = \bigl(1-b_m(p)/p^s+\cdots\bigr)^{-1}$ and as $s\to 1+m/2$ we have $\log L_m(s)\sim\sum_p b_m(p)/p^s$. We will now compute that the conjectural Sato-Tate distribution [41] implies that the average value of $b_m(p)$ is 0, while for CM curves the Hecke distribution [23] will yield an average value for $b_m(p)$ of $p^{m/2}$ when $4|m$.

Similar to (2), for a good prime p we have $b_m(p)=\sum_{i=0}^{m/2}\binom{m-i}{i}a_p^{m-2i}(-p)^i$. The Sato-Tate and Hecke distributions imply that the average values of the kth power of a_p are given by

$$\langle a_p^k\rangle = (2\sqrt{p})^k\,\frac{\int_0^\pi(\cos\theta)^k\,(\sin\theta)^2\,d\theta}{\int_0^\pi(\sin\theta)^2\,d\theta}\quad\text{and}\quad \langle a_p^k\rangle_{CM}=(2\sqrt{p})^k\,\frac{\int_0^\pi(\cos\theta)^k\,d\theta}{2\int_0^\pi d\theta}.$$

We have $\langle a_p^k\rangle=0$ for k odd; for even k the Wallis formula [43] implies

$$\int_0^\pi(\cos\theta)^k(\sin\theta)^2\,d\theta=\frac{\pi(k-1)!!}{k!!}-\frac{\pi(k+1)!!}{(k+2)!!}=\frac{\pi(k-1)!!}{(k+2)!!},$$

so that $\langle a_p^k\rangle$ is $(2\sqrt{p})^k\frac{2(k-1)!!}{(k+2)!!}$. An induction exercise shows that this implies $\langle b_m(p)\rangle=0$ when E does not have CM. We also have $\langle a_p^k\rangle_{CM}=(2\sqrt{p})^k\frac{(k-1)!!}{2\cdot k!!}$ for even k, and again an inductive calculation shows that $\langle b_m(p)\rangle_{CM}=p^{m/2}$ when $4|m$ and is zero otherwise. This behaviour immediately implies the aforementioned conjecture about the poles of $L_m(E,s)$ at $s=1+m/2$.

4.2 Global considerations

Let $\Lambda_m(E,s)=C_m^s\gamma_m(s)L_m(E,s)$, where $C_m^2=N_m/(2\pi)^{m+1}$ for m odd and is twice this for m even. For m odd we write $m=2u-1$ and for m even we write $m=2v$; then from [14, §5.3] we have respectively either

$$\gamma_m(s)=\prod_{i=0}^{u-1}\Gamma(s-i)\qquad\text{or}\qquad\gamma_m(s)=\Gamma\bigl(s/2-\lfloor v/2\rfloor\bigr)\prod_{i=0}^{v-1}\Gamma(s-i).$$

[6] The case of $m=4$ follows as a corollary of work of Kim [25, Corollary 7.3.4].

When $4|m$ and E has CM, we multiply $\gamma_m(s)$ by $(s-v)(s-v-1)$. We expect $\Lambda_m(E,s)$ to have an entire continuation and satisfy a functional equation $\Lambda_m(E,s) = w_m \Lambda_m(E, m+1-s)$ for some $w_m = \pm 1$. The works of Kim and Shahidi [26] establish parts of this conjecture.[7] We can find w_m via experiment as described in Section 4.4, but we can also try to determine w_m theoretically.

4.3 Digression on local root numbers

The sign w_m can theoretically be determined via local computations as in [12], but this is non-trivial to implement algorithmically, especially when $p = 2, 3$. We expect to have a factorisation $w_m = \prod_p w_m(p)$ where the product is over bad primes p including infinity. For m even, the very general work of Saito [35] can then be used to show[9] that $w_m = +1$, so we assume that m is odd. From [14, §5.3] we have $w_m(\infty) = -\left(\frac{-2}{m}\right)$; combined with the relation $w_m(p) = w_1(p)^m$ for primes p of multiplicative reduction, this gives the right sign for semistable curves. The potentially multiplicative case has $w_m(p) = w_1(p)^{(m+1)/2}$.

In the additive cases, Martin [29] has used the work of Rohrlich [33] to compute the sign for $p \geq 5$. We get that[10] $w_m(p) = w_1(p)^{\epsilon_m(\Phi_p)/2}$, and $w_1(p)$ is listed in [33]. For $p = 2, 3$ the value of $w_1(p)$ is given[11] by Halberstadt [22], and our experiments for higher (odd) powers indicate that $w_m(2) = \eta_2 w_1(2)^{\epsilon_m(\Phi_p)/2}$ where $\eta_2 = -1$ if $v_2(N)$ is odd and $m \equiv 3 \pmod 8$ and else $\eta_2 = +1$, while the expected values of $w_m(3)$ are given in Table 5.

Table 5. Experimental values for $w_m(3)$ (periodic mod 12 in m.)

Φ_3	1	3	5	7	9	11	Φ_3	1	3	5	7	9	11	Φ_3	1	3	5	7	9	11
C_3, C_4	+	+	+	+	+	+	C_6	+	−	−	−	+	+	$C_3 \rtimes C_4$	+	+	−	+	+	+
C_2	−	+	−	+	−	+	C_6	−	+	−	+	−	+	$C_3 \rtimes C_4$	−	−	−	−	−	+

4.4 Computations

From [28], [8, Appendix B], or [16], the assumption of the functional equation $\Lambda_m(E,s) = w_m \Lambda_m(E, m+1-s)$ allows us to compute (to a given precision) any value/derivative $\Lambda_m^{(d)}(E,s)$ in time proportional to $C_m \approx \sqrt{N_m}$. Additionally, numerical tests on the functional equation arise naturally from the method.

We follow [6, § 7, p. 119ff]. Suppose we have $\Lambda_m(s) = w_m \Lambda(m+1-s)$, and the dth derivative is the first one that is nonzero at $s = \kappa$. Our main interest is in $\kappa = u$ for $m = 2u-1$ and $\kappa = v+1$ for $m = 2v$, and we note that $d = 0$ for even m. Via Cauchy's residue theorem, for every real $A > 0$ we have

$$\frac{\Lambda_m^{(d)}(\kappa)}{d!} = \frac{1}{2\pi i}\left(\int_{(\delta)} - \int_{(-\delta)}\right) \frac{\Lambda_m(z+\kappa)}{z^{d+1}} \frac{dz}{A^z},$$

[7] The full conjecture[8]follows from Langlands functoriality [27]. In the CM case, the functional equation follows from the factorisation (5) and the work of Hecke [23].

[8] Added in proof: A preprint [42] on Taylor's webpage proves this for curves with $j \notin \mathbf{Z}$.

[9] The work of Fröhlich and Queyrut [21] and Deligne [13] might give a direct argument.

[10] Since we are assuming that m is odd, the exponent is just $(m+1)/2$ unless $\Phi_p = C_3$.

[11] Note the third case in Table 1 of [22] needs a *Condition spéciale* of $c_4' \equiv 3 \pmod 4$.

where δ is small and positive and $\int_{(\sigma)}$ is the integral along $\Re z = \sigma$. In the second integral we change variables $z \to -z$ and apply the functional equation. Then we write $\kappa + \lambda = m + 1$, move both contours sufficiently far to right (say $\Re z = 2m$) and expand Λ_m in terms of the L-function to get

$$\frac{\Lambda_m^{(d)}(\kappa)}{d!} = \int_{(2m)} C_m^{z+\kappa}\gamma_m(z+\kappa)\sum_{n=1}^{\infty}\frac{b_m(n)}{n^{z+\kappa}}\frac{1}{z^{d+1}}\frac{dz}{2\pi i\, A^z}$$
$$+ (-1)^d w_m \int_{(2m)} C_m^{z+\lambda}\gamma_m(z+\lambda)\sum_{n=1}^{\infty}\frac{b_m(n)}{n^{z+\lambda}}\frac{1}{z^{d+1}}\frac{A^z\, dz}{2\pi i}.$$

Thus we get that

$$\frac{\Lambda_m^{(d)}(\kappa)}{d!} = C_m^{\kappa}\sum_{n=1}^{\infty}\frac{b_m(n)}{n^{\kappa}}F_m^d\big(\kappa; \tfrac{n}{AC_m}\big) + (-1)^d w_m C_m^{\lambda}\sum_{n=1}^{\infty}\frac{b_m(n)}{n^{\lambda}}F_m^d\big(\lambda; \tfrac{nA}{C_m}\big),$$

where

$$F_m^d(\mu; x) = \int_{(2m)} \frac{\gamma_m(z+\mu)}{z^{d+1}x^z}\frac{dz}{2\pi i}.$$

The $F_m^d(\mu; x)$-functions are "rapidly decreasing" inverse Mellin transforms. Note that we have $L_m^{(d)}(\kappa) = \Lambda_m^{(d)}(\kappa)/\gamma_m(\kappa)C_m^{\kappa}$, and so can recover the L-value as desired. The parameter A allows us to test the functional equation; if we compute $\Lambda_m^{(d)}(\kappa)$ to a given precision for $A = 1$ and $A = 9/8$, we expect disparate answers if we have the wrong Euler factors or sign w_m.

We compute $F_m^d(\mu; x)$ as a sum of residues at poles in the left half-plane, the first pole being at $z = 0$, following [11]. We need to calculate Laurent series expansions of the Γ-factors about the poles.[12] We let $\zeta(1)$ denote Euler's constant $\gamma \approx 0.577$, and define $H_1(n) = 1$ for all n, and $H_k(1) = \sum_{i=1}^{k} 1/i$ for all k, and recursively define $H_k(n) = H_{k-1}(n) + H_k(n-1)/k$ for $n, k \geq 2$. At a pole $z = -k$ for k a nonnegative integer, we have the Laurent expansion

$$\Gamma(z) = \frac{(-1)^k}{k!(z+k)}\left(1 + \sum_{n=1}^{\infty} H_k(n)(z+k)^n\right)\exp\left(\sum_{n=1}^{\infty}\frac{(-1)^n\zeta(n)}{n}(z+k)^n\right),$$

and for k a negative integer (these only occur for a few cases) we can use the relation $z\Gamma(z) = \Gamma(z+1)$ to shift. To expand $\Gamma(z/2)$ around an odd integer $z = -k$, we use the duplication formula $\Gamma(z) = \Gamma(z/2)\Gamma\big(\frac{z+1}{2}\big)\frac{\sqrt{\pi}}{2^{z-1}}$ to replace $\Gamma(z/2)$ by a quotient of Γ-factors that can each be expanded as above. The trick works in reverse to expand $\Gamma\big(\frac{z+1}{2}\big)$ about an even integer $z = -k$. We also have the series expansions for 2^z and $1/z$ about $z = -k$ given by

$$2^z = 2^{-k}\sum_{n=0}^{\infty}\frac{(\log 2)^n}{n!}(z+k)^n \quad \text{and} \quad \frac{1}{z} = -\frac{1}{k} - \sum_{n=1}^{\infty}\frac{(z+k)^n}{k^{n+1}} \ \text{ (for } k \neq 0\text{)}.$$

[12] When $\Lambda_m(E, s)$ has a pole the factor $\gamma_m(s)$ has two additional linear factors (which are easily handled). But in this case it is better to use the factorisation (5).

Since these $F_m^d(\mu; x)$ functions are (except for CM) independent of the curve, we pre-computed a large mesh of values and derivatives of these functions, and then in our programme we compute via local power series. Thus, unlike the setting of Dokchitser [16], we are not worried too much about the cost of computing $F_m^d(\mu; x)$ for large x via a massively-cancelling series expansion, since we only do this in our pre-computations. For each implemented function we have its value and first 35 derivatives for all $x = i2^k/32$ for $32 \le i \le 63$ for k in some range, such as $-3 \le k \le 19$. For sufficiently small x we just use the log-power-series expansion. The choice of 35 derivatives combined with the maximal radius of $x/64$ for expansions about x implies that our maximal precision is around $35 \times 6 = 210$ bits. When working to a lower precision, we need not sum so many terms in the local power series. Note that $F_m^d(\mu; x)$ dies off roughly like $\exp(-x^{2/(m+1)})$, and thus it is difficult to do high precision calculations for m large.

To compute the meshes of inverse Mellin transforms described above, we used PARI/GP [31], which can compute to arbitrary precision. However, PARI/GP was too slow to use when actually computing the L-values; instead we used a C-based adaption of Bailey's quad-double package [24], which provides up to 212 bits of precision while remaining fairly fast.[13]

5 Results

We tested the functional equation (via the above method of comparing the computed values for $A = 1$ and $A = 9/8$) for odd symmetric powers $m = 2u - 1$ at the central point $\kappa = u$, and for even symmetric powers $m = 2v$ at the edge of the critical strip $\kappa = v + 1$. We did this for all non-CM isogeny classes in Cremona's database [9] with conductor less than 130000; this took about 3 months on a cluster of 48 computers (each running at about 1 Ghz).

We computed as many as 10^8 terms of the various L-series for each curve, which was always sufficient to check the functional equation of the third symmetric power to about six decimal digits.[14] In all cases, we found the expected functional equation to hold to the precision of our calculation. The results for the order of vanishing (at the central point) for odd powers appear in the left half of Table 6. The right half lists how many tests[15] we did for other[16] symmetric powers (again to six digits of precision). There are less data for higher symmetric powers due to our imposed limit of 10^8 terms in the L-series computations, but since the symmetric power conductors for curves with exotic inertia groups often do not grow so rapidly, we can still test quite high powers in some cases.

[13] The SYMPOW package can be obtained from `www.maths.bris.ac.uk/~mamjw`

[14] In about 0.3% of the cases, the computations for both the zeroth and first derivatives showed no discrepancy with $A = 1$ and $A = 9/8$; this coincidence is to be expected on probabilistic grounds, and for these cases we computed to higher precision to get an experimental confirmation of the sign of the functional equation.

[15] We need not compute even powers when there is a lack of quadratic-twist-minimality.

[16] We did not test the fourth symmetric power, as the work of Kim [25] proves the validity of the functional equation in this case. Since there is no critical value, a calculation would do little more than verify that our claimed Euler factors are correct.

Table 6. Test-counts (right) and data for order of vanishing (non-CM isogeny classes)

m	Tested	Order 0	Order 1	2	3	4
1	567735	216912	288128	61787	908	0
3	567735	262751	287281	16782	905	16
5	46105	22448	23076	569	12	0
7	3573	1931	1616	25	1	0
9	947	542	400	5	0	0
11	134	51	82	1	0	0

m	# tests
6	4953
8	1259
10	190
12	142
13	5 even
13	30 odd

m	# tests
14	26
15	1 even
15	16 odd
16	8
17	3 odd
18	2

Buhler, Schoen, and Top [6] already listed 2379b1 and 31605ba1 as 2 examples of (suspected) 4th order zeros for the symmetric cube. We found 14 more, but no examples of 5th order zeros. For higher powers, we found examples of 3rd order zeros for the 5th and 7th powers, and 2nd order zeros for the 9th, 11th, and 13th powers, though as noted above, we cannot obtain as much data for higher powers.[17] We list the Cremona labels for the isogeny classes in Table 7.

Table 7. Experimentally observed high order vanishings (non-CM isogeny classes)

ord	format is **power**:label(s)
4th	**3**:2379b 5423a 10336d 29862s 31605ba 37352d 46035a 48807b 55053a
	3:59885g 64728a 82215d 91827a 97448a 104160bm 115830a
3rd	**5**:816b 2340i 2432d 3776h 5248a 6480t 7950w 8640bl 16698s 16848r
	5:18816n 57024du **7**:176a
2nd	**7**:128b 160a 192a 198b 200e 320b 360b 425a 576b 726g 756b 1440a
	7:1568i 1600b 2304g 3267f 3600h 3600j 3600n 3888e 4225m 6272d
	7:11552r 15876f 21168g **9**:40a 96a 162b 324d 338b **11**:162b **13**:324c

We also looked at extra vanishings of the 3rd symmetric power in a quadratic twist family. We took E as 11a3:$[0,-1,1,0,0]$ and computed the twisted central value $L_3(E_d,2)$ or central derivative $L_3'(E_d,2)$ for fundamental discriminants $|d|<5000$. We found 58 double zeros (to 9 digits) and one triple zero ($d=3720$). A larger experiment (for $|d|<10^5$) for 10 different CM curves found (proportionately) fewer double zeros and no triple zeros.

Finally, we used higher-precision calculations to obtain the Bloch-Kato numbers of equation (1) for various symmetric powers of some non-CM curves of small conductor (see Table 8). More on the arithmetic significance of these quotients will appear elsewhere. In some cases, we were able to lessen the precision because it was known that a large power of a small prime divided the numerator.

5.1 Other directions

In this work, we looked at symmetric powers for weight 2 modular forms. Delaunay has done some computations [11] for modular forms of higher weight; in that case, the work of Deligne again tells us where to expect critical values, and the experiments confirm that we do indeed get small-denominator rationals after proper normalisation. We looked at critical values at the edge and center of

[17] Given that we only computed the L-value of the 13th symmetric power for five curves of even sign, to find one that has a double-order zero is rather surprising. Higher-order zeros were checked to 12 digits; the smallest "nonzero" value was $\approx 2.9\cdot 10^{-8}$.

Table 8. Selected Bloch-Kato numbers for various powers and curves

5th powers	
20a2	2^{9}
37a1	2^{9}
43a1	$2^{7}5$
44a1	2^{17}

6th powers	
11a3	$2^{4}5$
14a4	$2^{9}3$
15a8	2^{10}
17a4	2^{12}
19a3	$2^{4}3^{3}5^{2}$
20a2	$2^{17}/3$
24a4	$2^{17}/3$
26a3	$2^{7}3 \cdot 5 \cdot 23$
26b1	$2^{7}3 \cdot 7^{3} \cdot 23$
30a1	$2^{15}3^{3}7$
33a2	$2^{17}3 \cdot 5 \cdot 7$
34a1	$2^{13}3^{3}59$
35a3	$2^{8}3 \cdot 7^{2}31$
37a1	$2^{9}3^{4}7$
37b3	$2^{7}3^{4}467$
38a3	$2^{7}3^{4}5 \cdot 11 \cdot 137$
38b1	$2^{7}3^{2}5^{2}13 \cdot 31$
39a1	$2^{20}3^{2}7$
40a3	$2^{20}7/3$
42a1	$2^{19}3^{2}7 \cdot 19$
43a1	$2^{6}3^{2}1697$
44a1	$2^{21}5 \cdot 31/3$
46a1	$2^{9}5 \cdot 23 \cdot 30661$
50a1	$2^{3}5^{11}/3$
51a1	$2^{9}3^{3}4517$

7th powers	
24a4	$2^{23}7/3$
37a1	$2^{13}3 \cdot 5$
43a1	$2^{17}3 \cdot 5$

9th powers	
11a3	2^{12}
14a4	$2^{14}3^{4}5$
15a8	2^{16}
17a4	$2^{16}3^{6}5$
19a3	$2^{19}3^{2}5$
21a4	$2^{20}5 \cdot 59^{2}$
24a4	$2^{38}/9$
26a3	$2^{11}3^{4}5 \cdot 7^{4}$
26b1	$2^{11}3^{2}5 \cdot 7^{3}1933^{2}$
30a1	$2^{16}3^{5}5^{5}37^{2}$
33a2	$2^{24}5 \cdot 107^{2}167^{2}$
34a1	$2^{23}3^{5}5 \cdot 7^{2}53^{2}$
35a3	$2^{25}3^{4}5$
37b3	$2^{20}3^{2}5 \cdot 7^{2}53^{2}$
38a3	$2^{11}3^{14}5 \cdot 19^{2}$
38b1	$2^{11}5^{8}109^{2}$
39a1	$2^{40}3^{2}57^{4}$
40a3	0
42a1	$2^{25}5 \cdot 223^{2}241^{2}$
44a1	$2^{47}3$
45a1	$2^{16}3^{19}5^{2}7 \cdot 13^{2}$
46a1	$2^{14}3^{10}5^{3}14071^{2}$
48a4	$2^{43}5$
50a1	$2^{5}3 \cdot 5^{22}$
54a3	$2^{9}3^{24}$
54b1	$2^{7}3^{25}5$

10th powers	
11a3	$2^{14}5 \cdot 22453/3$
14a4	$2^{16}3^{3}5 \cdot 6691$
15a8	$2^{26}5 \cdot 541$
17a4	$2^{23}3^{2}7 \cdot 11 \cdot 227$
19a3	$2^{14}3^{2}47 \cdot 179 \cdot 5023$
20a2	$2^{44}53/3$
21a4	$2^{28}3^{7}5^{2}29$
24a4	$2^{49}13/9$
26a3	$2^{19}3^{5}7 \cdot 47 \cdot 1787$
26b1	$2^{19}3^{3}5^{2}7^{3}127 \cdot 2102831$
40a3	$2^{54}5 \cdot 683$
44a1	$2^{56}5 \cdot 11 \cdot 215447/3$
50a1	$2^{11}5^{28}7/3$
52a2	$2^{44}3^{3}5 \cdot 7 \cdot 19 \cdot 279751$
54b1	$2^{14}3^{35}7$
56a1	$2^{66}3^{2}5 \cdot 11 \cdot 71$
75c1	$2^{14}5^{28} \cdot 31 \cdot 41 \cdot 61/9$
96b1	$2^{84}197/3$
99a1	$2^{18}3^{31}5^{3}7 \cdot 1367$

11th powers	
11a3	$2^{26}5^{4}/3$
14a4	$2^{23}3^{5}5^{2}7^{2}11^{2}$
15a8	$2^{29}11^{2}23^{2}/3$
17a4	$2^{26}3^{11}$
21a4	$2^{36}11^{2}211^{2}/3$
24a4	$2^{57}13^{2}/45$
48a4	$2^{70}11^{2}/3$
54b1	$2^{20}3^{41}$
56a1	$2^{74}11^{2}/5$
72a1	$2^{58}3^{28}59^{2}/5$

the critical strip, whereas we expect L-functions evaluated at other integers to take special values related to K-theory; see [3,30,17,45] for examples. The programmes written for this paper are readily modifiable to compute other special values. The main advantage that our methods have over those of Dokchitser [16] is that we fixed the Γ-factors and the L-values of interest, which then allowed a large pre-computation for the inverse Mellin transforms; if we wanted (say) to compute zeros of L-functions (as with [34]), our method would not be as useful.

Finally, the thesis of Booker [5] takes another approach to some of the questions we considered. The scope is much more broad, as it considers not only numerical tests of modularity, but also tests of GRH (§3.4), recovery of unknown Euler factors possibly using twists (§5.1), and also high symmetric powers (§7.2).

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