Some Methodological Remarks about Categorical Equivalences in the Abstract Approach to Roughness. Part I. *

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Abstract. The categorical equivalence of three different approaches to roughness is discussed: the one based on the notion of abstract rough approximation spaces, the second one based on the abstract topological notions of interior and closure, and the third one based on a very weak form of BZ lattice.

Keywords: abstract approximations, topological operators, BZ lattice

1 Introduction

The main motivation of this paper is the unification of different abstract approaches to rough theory, under theoretical proofs of categorical equivalence of the involved structures. Indeed, in literature one can found at least three different points of view: the one based on the notion of *rough approximation space* [1], the second essentially based on the topological notion of *interior* and *closure operations* [2], and a third one based on two kinds of non usual complementations, the so–called *BZ approach* [3, 4]. We investigate under what conditions these three approaches can be considered equivalent, and so from the applicative point of view *indistinguishable*. For completeness let us quote another approach based on modal–like operators of *necessity* and *possibility* [5, 6, 7] which is not treated in the present paper and also rough mereology [8, 9] whose relationship with the present work wil be analyzed in a forthcoming paper.

Now, let us explain the role of equivalence between structures exemplifying the involved questions in the context of the well know Łukasiewicz approach to many–valued logic [10]. To this purpose, let us first consider the notion of Wajsberg algebra (W algebra for short) introduced in 1931 by Wajsberg [11] in order to give an algebraic axiomatization to Łukasiewicz many valued logic. In this axiomatization the primitive propositional connectives are *implication* $\to$ and *negation* $-$, giving rise to the structure $\langle A, \to, -, 1\rangle$. Several years later (1958), another algebraic approach to many–valued logic has been proposed by

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Chang in [12], with the notion of MV algebra $\langle A, \oplus, -, 0\rangle$ which has, as primitive operators, a *truncated sum* and a *negation*.

At a first glance these two seem to be quite different algebraic structures. However it is possible to prove (see [13]) that they are categorically equivalent: from any MV algebra it is possible to obtain a W algebra and vice versa.

This result assures that any theorem proved in one of the two structures can be "translated" as a theorem of the second one: the algebras are *categorical equivalent*. They are indistinguishable. It will be very misleading to "impose" one of them as "better" with respect to the second one. One can prefer the Wajsberg approach as the one nearer to the original *language* of Łukasiewicz logic, and this is a meta–theoretical (probably aesthetic) choice. But it is out of doubt that any result obtained in the context of the Chang approach to MV logic is also a result true in the Wajsberg–Łukasiewicz context, and vice versa. For instance, the *completeness* theorem given by Chang in the context of MV algebras is immediately translated as a completeness about Wajsberg algebras.

2 Equivalent Structures

2.1 Abstract Rough Approximation Spaces

The abstract approach to roughness introduced in [1] is based on a family of "approximable" concepts, with associated two well defined subfamilies describing "inner" and "outer" definable concepts respectively. In the formal description of this situation *imprecise (vague, unclassified) concepts*, with the associated *inner* and *outer (precise, crisp, sharp) knowledge* about them, are mathematically realized by *points* of an abstract set.

In this context some criteria must be given in order to "approximate" any vague concept by a pair consisting of a unique inner definable concept and a unique outer definable concept. Since we want that these approximations are the best possible inside the classes of corresponding definable concepts, it is necessary to have also a criterium to state how an approximation is sufficiently good. Abstractly, this is realized by a partial order relation $\leq$ on the set of all approximable elements which mathematical describes the fact that an element a is a *better approximation* of the element b, written $a \leq b$.

Definition 2.1. *An abstract* approximation space *is a system* $\mathfrak{A} := \langle \Sigma, \mathbb{L}(\Sigma), \mathbb{U}(\Sigma)\rangle$*, where:*

(1) $\langle \Sigma, \wedge, \vee, 0, 1\rangle$ *is a lattice with respect to the partial order relation* $a \leq b$ *iff* $a = a \wedge b$*, bounded by the least element* 0 *and the greatest element* 1*. Elements from* Σ *are interpreted as concepts, data, etc., and are said to be* approximable *elements;*

(2) $\mathbb{L}(\Sigma)$ *and* $\mathbb{U}(\Sigma)$ *are bounded subposet of* Σ *(and thus* $0, 1 \in \mathbb{L}(\Sigma), \mathbb{U}(\Sigma)$*) consisting, respectively, of all available* lower (inner) *and* upper (outer) definable *elements;*

This system must satisfy the following axioms:

(Ax1) *For any approximable element $a \in \Sigma$, there exists one element $i(a)$ s.t. $i(a)$ is an inner definable element ($i(a) \in \mathbb{L}(\Sigma)$); $i(a)$ is an inner definable* lower *approximation of a ($i(a) \leq a$); $i(a)$ is the best lower approximation of a by inner definable elements (let $e \in \mathbb{L}(\Sigma)$ be such that $e \leq a$, then $e \leq i(a)$).*

(Ax2) *For any approximable element $a \in \Sigma$, there exists one element $o(a)$ s.t. $o(a)$ is an outer definable element ($o(a) \in \mathbb{U}(X)$); $o(a)$ is an outer definable* upper *approximation of a ($a \leq o(a)$); $o(a)$ is the best upper approximation of a by outer definable elements (let $f \in \mathbb{U}(X)$ be such that $a \leq f$, then $o(a) \leq f$).*

It is easy to prove that, for any approximable element $a \in \Sigma$, the inner definable element $i(a) \in \mathbb{L}(\Sigma)$, whose existence is assured by (Ax1), is unique. Thus, it is possible to introduce the mapping $i : \Sigma \mapsto \mathbb{L}(\Sigma)$, called the *inner approximation mapping*, associating with any approximable element $a \in \Sigma$ its *lower* (or *inner*) approximation: $i(a) := \max\{\alpha \in \mathbb{L}(\Sigma) : \alpha \leq a\}$. Similarly, for any approximable element $a \in \Sigma$, the outer definable element $o(a) \in \mathbb{U}(\Sigma)$, whose existence is assured by (Ax2), is unique. Thus, it is possible to introduce the mapping $o : \Sigma \mapsto \mathbb{U}(\Sigma)$, called the *outer approximation mapping*, associating with any approximable element $a \in \Sigma$ its *upper* (or *outer*) approximation: $o(a) := \min\{\gamma \in \mathbb{U}(\Sigma) : a \leq \gamma\}$.

The *rough approximation* of any approximable element $a \in \Sigma$ is then the inner–outer pair $r(a) := (i(a), o(a))$, with $i(a) \leq a \leq o(a)$, which is the image of the element a under the *rough approximation mapping* $r : \Sigma \mapsto \mathbb{L}(\Sigma) \times \mathbb{U}(\Sigma)$.

We denote by $\mathbb{LU}(\Sigma) := \mathbb{L}(\Sigma) \cap \mathbb{U}(\Sigma)$ the set of all *innouter* (simultaneously inner and outer) definable elements. This set coincides with the collection of "*sharp*" (or "*crisp*", "*exact*"; also "*definable*," if one adopts the original Pawlak terminology) of Σ, that is, elements whose inner approximation is equal to the outer one, i.e., $i(x) = o(x)$. The rough approximation of any sharp element is therefore the trivial one $r(x) = (x, x)$.

2.2 Inner and outer approximation spaces

These being stated, in order to introduce the first categorical equivalence between two abstract approaches to rough theory, let us premise the following definitions.

Definition 2.2. *An* interior de Morgan lattice *is a system $\langle \Sigma, \wedge, \vee, {}', 0, 1\rangle$ where*

(IdM1) the structure $\langle \Sigma, \wedge, \vee, 0, 1\rangle$ is a lattice, bounded by the least element 0 and the greatest element 1. The mapping $' : \Sigma \to \Sigma$ is a unary operation on Σ, called de Morgan *complement, that satisfies the following conditions for arbitrary $a, b \in \Sigma$:*
(dM1) $a = a''$ *(dM2) $(a \vee b)' = a' \wedge b'$.*

(IdM2) The mapping $^o : \Sigma \to \Sigma$, that associates to any element a from Σ its interior *$a^o \in \Sigma$, is an interior operation, i.e., it satisfies the followings:*

$(I1)$	$1^o = 1$	*(normalized)*
$(I2)$	$a^o \leq a$	*(decreasing)*

$(I3)$	$a^o = a^{oo}$	*(idempotent)*
$(I4)$	$(a \wedge b)^o \leq a^o \wedge b^o$	*(sub–multiplicative)*

Given an interior operator, the subset of *open elements* is defined as the collection of elements which are equal to their interior $\mathbb{O}(\Sigma) = \{a \in \Sigma : a = a^o\}$.

Definition 2.3. *A structure* $\langle \Sigma, \wedge, \vee, ', ^*, 0, 1\rangle$ *is a* closure de Morgan lattice *iff*

(CdM1) $\langle \Sigma, \wedge, \vee, ' 0, 1\rangle$ *is a De Morgan lattice;*

(CdM2) The mapping $^* : \Sigma \to \Sigma$*, that associates to any element a from* Σ *its* closure $a^* \in \Sigma$*, is a closure operation, that is, it satisfies the properties:*

$(C1)$	$0^* = 0$	*(normalized)*
$(C2)$	$a \leq a^*$	*(increasing)*
$(C3)$	$a^* = a^{**}$	*(idempotent)*
$(C4)$	$a^* \vee b^* \leq (a \vee b)^*$	*(sub–additive)*

In a closure de Morgan lattice, the subset of *closed elements* is defined as the collection of elements which are equal to their closure $\mathbb{O}(\Sigma) = \{a \in \Sigma : a = a^*\}$. Both the set of open and closed elements are not empty, since $0, 1$ are at the same time open and closed.

The notions of interior de Morgan lattice and closure de Morgan lattice are strictly linked, since in any interior de Morgan lattice it is possible to define a closure operator by the law $\forall a \in \Sigma : \; a^* := ((a')^o)'$. Vice versa in any closure de Morgan lattice an interior operator can be naturally induced by the law $\forall a \in \Sigma : \; a^o := ((a')^*)'$. Hence the de Morgan complement determines a duality relation between the closure and the interior of any element a.

Theorem 2.1. *(i) Suppose a rough approximation space* $\mathcal{A} = \langle \Sigma, \mathbb{L}(\Sigma), \mathbb{U}(\Sigma)\rangle$ *and for arbitrary* $a \in \Sigma$ *let us define* $a^o := i(a)$ *and* $a^* := o(a)$*. Then,* $\mathcal{A}^{\blacktriangle} := \langle \Sigma, ^o, ^* \rangle$ *is a lattice equipped with an interior and a closure operations such that* $\mathbb{O}(\Sigma) = \mathbb{L}(\Sigma)$ *and* $\mathbb{C}(\Sigma) = \mathbb{U}(\Sigma)$*.*

(ii) Suppose a lattice equipped with an interior and a closure operations $\mathcal{A} = \langle \Sigma, ^o, ^* \rangle$ *and let us define* $\mathbb{L}(\Sigma) := \mathbb{O}(\Sigma)$ *and* $\mathbb{U}(\Sigma) := \mathbb{C}(\Sigma)$*. Then,* $\mathcal{A}^{\blacktriangledown} := \langle \Sigma, \mathbb{L}(\Sigma), \mathbb{U}(\Sigma)\rangle$ *is a rough approximation space in which for arbitrary a it is* $i(a) = a^o$ *and* $o(a) = a^*$*.*

(iii) Let $\mathcal{A} = \langle \Sigma, \mathbb{L}(\Sigma), \mathbb{U}(\Sigma)\rangle$ *be a rough approximation space. Then:* $\mathcal{A}^{\blacktriangle\blacktriangledown} = \mathcal{A}$*.*

(iv) Let $\mathcal{A} = \langle \Sigma, ^o, ^* \rangle$ *be a lattice equipped with an interior and a closure operator. Then:* $\mathcal{A}^{\blacktriangledown\blacktriangle} = \mathcal{A}$*.*

In this way we have shown the indistinguishability between the structure $\langle \Sigma, \mathbb{L}(\Sigma), \mathbb{U}(\Sigma)\rangle$ of rough approximation space based on the lattice Σ and satisfying axioms (Ax1) and (Ax2), and the structure $\langle \Sigma, ^o, ^* \rangle$ based on the same lattice Σ and equipped with an interior and a closure operation, satisfying conditions (I1)-(I4) and (C1)–(C4) respectively. Clearly, the set of definable elements $\mathbb{LU}(\Sigma)$ of a rough approximation space conincide with the set of *clopen* elements, i.e., elements which are both closed and open.

Finally, let us note that in any interior (equiv., closure) de Morgan lattice, we have both an interior and a closure operator, thus applying Theorem 2.1, it is possible to define an equivalent rough approximation space.

2.3 Pre–Brouwer Zadeh lattice and interior–closure spaces

In this section, we want to investigate another structure based on two weak form of negations and which turns out to be categorically equivalent to closure de Morgan lattices (and hence to rough approximation spaces).

Definition 2.4. *A system* $\langle \Sigma, \wedge, \vee, ', {}^\sim, 0, 1\rangle$ *is a* pre Brouwer Zadeh (pBZ) lattice *iff*

(BZ1) the substructure $\langle \Sigma, \wedge, \vee, ', 0, 1\rangle$ *is a de Morgan lattice;*
(BZ2) the unary operation ${}^\sim$ *satisfies the properties:*
  (i) $1 = 0^\sim$
  (ii) if $a \leq b$ *then* $b^\sim \leq a^\sim$ *(contraposition)*
(BZ3) the two complementations are linked by the following interconnection rules*:*
  (i) $a^\sim \leq a'$ *(minimal interconnection)*
  (ii) $a'^\sim \leq a'^{\sim\prime\sim}$ *(weak interconnection)*

Note that $1^\sim = 0$, indeed by minimal interconnection $1^\sim \leq 1' = 0$.

The properties of pre Brouwer Zadeh lattices allow one to define an interior and a closure operator on a lattice structure. Indeed, we can see that any pre-BZ lattice is equivalent to a closure (resp., interior) de Morgan lattice.

Theorem 2.2.

(i) *Let* $\mathcal{T} = \langle \Sigma, \wedge, \vee, ', {}^\sim, 0, 1\rangle$ *be a pre BZ lattice. Let us introduce the mapping* ${}^* : \Sigma \mapsto \Sigma$ *defined for every* $a \in \Sigma$ *as* $a^* := a^{\sim\prime}$*, then the structure* $\mathcal{T}^{\mathcal{C}} = \langle \Sigma, \wedge, \vee, ', {}^*, 0, 1\rangle$ *is a closure de Morgan lattice.*
(ii) *Let* $\mathcal{T} = \langle \Sigma, \wedge, \vee, ', {}^*, 0, 1\rangle$ *be a closure de Morgan lattice. Let us introduce the mapping* ${}^\sim : \Sigma \mapsto \Sigma$ *defined for every* $a \in \Sigma$ *as* $a^\sim := a^{*\prime}$ *then the structure* $\mathcal{T}^{\mathcal{B}} = \langle \Sigma, \wedge, \vee, ', {}^\sim, 0, 1\rangle$ *is a pre BZ lattice.*
(iii) *If* $\mathcal{T} = \langle \Sigma, \wedge, \vee, ', {}^\sim, 0, 1\rangle$ *is a pre BZ lattice, then* $\mathcal{T} = \mathcal{T}^{\mathcal{CB}}$.
(iv) *If* $\mathcal{T} = \langle \Sigma, \wedge, \vee, ', {}^*, 0, 1\rangle$ *is a closure de Morgan lattice, then* $\mathcal{T} = \mathcal{T}^{\mathcal{BC}}$.

By the result (i) of this theorem, and considering the equivalence between interior and closure de Morgan lattices, in a pre BZ lattice the closure and interior operator are defined for every element $a \in \Sigma$ as $a^* = a^{\sim\prime}$ and $a^o = a'^\sim$ (with $a^{\sim\prime} \leq a \leq a'^\sim$). Thus, we have that pre BZ lattices are the weakest lattice structure in which we are able to define an interior operator, and a closure operator and consequently a rough approximation space.

Definition 2.5. *A closure de Morgan lattice is said to be* topological *iff the closure operator satisfies the* additive *property:* $a^* \vee b^* = (a \vee b)^*$*. Dually, an interior de Morgan lattice is said to be* topological *iff the interior operator satisfies the* multiplicative *property:* $a^o \wedge b^o = (a \wedge b)^o$.

The following three structures are equivalent among them

(1) pre–BZ lattice satisfying also the join de Morgan property $(a \vee b)^\sim = a^\sim \wedge b^\sim$;
(2) topological closure de Morgan lattices;
(3) topological interior de Morgan lattices.

3 Conclusion

We have shown a categorical equivalence among rough approximation spaces, interior–closure spaces and preBZ lattices. The Pawlak approach to rough set theory is a concrete example of these structures. Indeed, given a universe X equipped with an equivalence relation $\mathcal{R}$, one can obtain the rough approximation space $\langle \mathcal{P}(X), \mathcal{E}(X), \mathcal{E}(X) \rangle$ where the power set of X, $\mathcal{P}(X)$, is the collection of approximable elements and the exact elements $\mathcal{E}(X)$ are all subsets of X which are set theoretical union of equivalence classes with respect to $\mathcal{R}$, plus the empty set. Trivially, axioms (Ax1) and (Ax2) are satisfied by the triple $\langle \mathcal{P}(X), \mathcal{E}(X), \mathcal{E}(X) \rangle$ which in this way turns out to be a concrete model of rough approximation space. Hence, all the results one can derive from the abstract environment sketched in section 2 are immediately true in the particular Pawlak environment. Thus, we hope to have clarified that all the approaches of section 2 are equivalent among them, and can play the same role in the abstract approach to roughness.

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