# Approximability of Bounded Occurrence Max Ones 

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#### Abstract

We study the approximability of MAX Ones when the number of variable occurrences is bounded by a constant. For conservative constraint languages (i.e., when the unary relations are included) we give a complete classification when the number of occurrences is three or more and a partial classification when the bound is two. For the non-conservative case we prove that it is either trivial or equivalent to the corresponding conservative problem under polynomial-time many-one reductions.


Keywords: Approximability, Bounded occurrence, Constraint satisfaction problems, Matching, Max Ones

## 1 Introduction

Many combinatorial optimisation problems can be formulated as various variants of constraint satisfaction problems (CSPs). MAX OnES is a boolean CSP where we are not only interested in finding a solution but also the measure of the solution. In this paper we study a variant of MAX ONES when the number occurrences of each variable is bounded by a constant.

We denote the set of all $n$-tuples with elements from $\{0,1\}$ by $\{0,1\}^{n}$. A subset $R \subseteq\{0,1\}^{n}$ is a relation and $n$ is the arity of $R$. A constraint language is a finite set of relations. A constraint language is said to be conservative if every unary relation is included in the language. In the boolean case this means that the relations $\{(0)\}$ and $\{(1)\}$ are in the language. The constraint satisfaction problem over the constraint language $\Gamma$, denoted $\operatorname{CsP}(\Gamma)$, is defined to be the decision problem with instance $(V, C)$, where $V$ is a set of variables and $C$ is a set of constraints $\left\{C_{1}, \ldots, C_{q}\right\}$, in which each constraint $C_{i}$ is a pair $\left(R_{i}, s_{i}\right)$ with $s_{i}$ a list of variables of length $n_{i}$, called the constraint scope, and $R_{i}$ an $n_{i}$-ary relation over the set $\{0,1\}$, belonging to $\Gamma$, called the constraint relation. The question is whether there exists a solution to $(V, C)$ or not. A solution to $(V, C)$ is a function $s: V \rightarrow\{0,1\}$ such that, for each constraint $\left(R_{i},\left(v_{1}, v_{2}, \ldots, v_{n_{i}}\right)\right) \in C$, the image of the constraint scope is a member of the constraint relation, i.e., $\left(s\left(v_{1}\right), s\left(v_{2}\right), \ldots, s\left(v_{n_{i}}\right)\right) \in R_{i}$.

The optimisation problem W-MAX OnES can be defined as follows:
Definition 1 (W-Max Ones). W-Max Ones over the constraint language $\Gamma$ is defined to be the optimisation problem with

[^0]Instance: Tuple $(V, C, w)$, where $(V, C)$ is an instance of $\operatorname{Csp}(\Gamma)$ and $w: V \rightarrow \mathbb{N}$ is a function.
Solution: An assignment $f: V \rightarrow\{0,1\}$ to the variables which satisfies the $\operatorname{Csp}(\Gamma)$ instance ( $V, C$ ).
Measure: $\sum_{v \in V} w(v) \cdot f(v)$
The function $w: V \rightarrow \mathbb{N}$ is called a weight function. In the corresponding unweighted problem, denoted Max $\operatorname{ONES}(\Gamma)$, the weight function is restricted to map every variable to 1 . The approximability of (W-)MAx OnES has been completely classified by Khanna et al. [20]. Several well-known optimisation problems can be rephrased as (W-)Max Ones problems, in particular Independent Set. We will study WMax $\operatorname{OnES}(\Gamma)$ with a bounded number of variable occurrences, denoted by W-MAX Ones $(\Gamma)-k$ for an integer $k$. In this problem the instances are restricted to contain at most $k$ occurrences of each variable. The corresponding bounded occurrence variant of $\operatorname{Csp}(\Gamma)$ will be denoted by $\operatorname{Csp}(\Gamma)-k$.

Schaefer [26] classified the complexity of $\operatorname{CsP}(\Gamma)$ for every constraint language $\Gamma$. Depending on $\Gamma$, Schaefer proved that $\operatorname{Csp}(\Gamma)$ is either solvable in polynomial time or is NP-complete. The conservative bounded occurrence variant of $\operatorname{Csp}(\Gamma)$ has been studied by a number of authors 121415 16]. One result of that research is that the difficult case to classify is when the number of variable occurrences are restricted to two, in all other cases the bounded occurrence problem is no easier than the unrestricted problem. Kratochvíl et al. [21] have studied $k$-SAT-l, i.e., satisfiability where every clause have length $k$ and there are at most $l$ occurrences of each variable. $k$-SAT- $l$ is a non-conservative constraint satisfaction problem. The complexity classification seems to be significantly harder for such problems compared to the conservative ones. In particular, Kratochvíl et al [21] proves that there is a function $f$ such that $k$-SAT- $l$ is trivial if $l \leq f(k)$ (every instance has a solution) and NP-complete if $l \geq f(k)+1$. Some bounds of $f$ is given in [21], but the exact behaviour of $f$ is unknown.
$\operatorname{MAX} \operatorname{ONES}(\Gamma)-k$ can represent many well-known problems. For $k \geq 3$, we have for example, that INDEPENDENT SET in graphs of maximum degree $k$ is precisely MAX $\operatorname{OnES}(\{\{(0,0),(1,0),(0,1)\}\})-k$. However, the more interesting case is perhaps $k=2$ due to its connection to matching problems. (See [24] for definitions and more information about the matching problems mentioned below.) Ordinary weighted maximum matching in graphs is, for example, straightforward to formulate and we get certain generalisations "for free" (because they can be rephrased as ordinary matching problems), such as $f$-factors and capacitated $b$-matchings. The general factor problem can also be rephrased as a MAX ONES $(\cdot)-2$ problem. A dichotomy theorem for the existence problem of general factors has been proved by Cornuéjols [9]. Some research has also been done on the optimisation problem [8].

In this paper, we start the classification of bounded occurrence Max Ones. Our first result is a complete classification of W-Max $\operatorname{ONES}(\Gamma)-k$ when $k \geq 3$ and $\{(0)\}$ and $\{(1)\}$ are included in $\Gamma$. We show that, depending on $\Gamma$, this problem is either in PO, APX-complete or poly-APX-complete. Our second result is a partial classification of W-MAX ONES $(\Gamma)-2$. We also give hardness results for the non-conservative case.

The outline of the paper is as follows: in Section 2 we define our notation and present the tools we use. Section 3 and 4 contains our results for three or more occur-
rences and two occurrences, respectively. Section 5contains our results for the general case, i.e., when the constraint language is not necessarily conservative. Section 6 contains some concluding remarks. Due to lack of space most of the proofs can be found in the appendix.

## 2 Preliminaries

For an integer $n$ we will use $[n]$ to denote the set $\{1,2, \ldots, n\}$. The Hamming distance between two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ will be denoted by $d_{H}(\boldsymbol{x}, \boldsymbol{y})$. For a tuple or vector $\boldsymbol{x}$ the $n$ :th component will be denoted by $\boldsymbol{x}[n]$.

Unless explicitly stated otherwise we assume that the constraint languages we are working with are conservative, i.e., every unary relation is a member of the constraint language (in the boolean domain, which we are working with, this means that $\{(0)\}$ and $\{(1)\}$ are in the constraint language).

We define the following relations

$$
\begin{aligned}
& -N A N D^{m}=\left\{\left(x_{1}, \ldots, x_{m}\right) \mid x_{1}+\ldots+x_{m}<m\right\} \\
& -E Q^{m}=\left\{\left(x_{1}, \ldots, x_{m}\right) \mid x_{1}=x_{2}=\ldots=x_{m}\right\} \\
& -I M P L=\{(0,0),(0,1),(1,1)\}, c_{0}=\{(0)\}, c_{1}=\{(1)\}
\end{aligned}
$$

and the function $h_{n}\left(x_{1}, x_{2}, \ldots, x_{n+1}\right)=\bigvee_{i=1}^{n+1}\left(x_{1} \wedge \ldots \wedge x_{i-1} \wedge x_{i+1} \wedge \ldots \wedge x_{n+1}\right)$. For a relation $R$ of arity $r$, we will sometimes use the notation $R\left(x_{1}, \ldots, x_{r}\right)$ with the meaning $\left(x_{1}, \ldots, x_{r}\right) \in R$, i.e., $R\left(x_{1}, \ldots, x_{r}\right) \Longleftrightarrow\left(x_{1}, \ldots, x_{r}\right) \in R$. If $r$ is the arity of $R$ and $I=\left\{i_{1}, \ldots, i_{n}\right\} \subseteq[r], i_{1}<i_{2}<\ldots<i_{n}$, then we denote the projection of $R$ to $I$ by $\left.R\right|_{I}$, i.e., $\left.R\right|_{I}=\left\{\left(x_{i_{1}}, x_{i_{2}}, \ldots, x_{i_{n}}\right) \mid\left(x_{1}, x_{2}, \ldots, x_{r}\right) \in R\right\}$

Representations (sometimes called implementations) have been central in the study of constraint satisfaction problems. We need a notion of representability which is a bit stronger that the usual one, because we have to be careful with how many occurrences we use of each variable.

Definition 2 ( $k$-representable). An n-ary relation $R$ is $k$-representable by a set of relations $F$ if there is a collection of constraints $C_{1}, \ldots, C_{l}$ with constraint relations from $F$ over variables $\boldsymbol{x}=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ (called primary variables) and $\boldsymbol{y}=$ $\left(y_{1}, y_{2}, \ldots, y_{m}\right)$ (called auxiliary variables) such that,

- the primary variables occur at most once in the constraints,
- the auxiliary variables occur at most $k$ times in the constraints, and
- for every tuple $\boldsymbol{z}, \boldsymbol{z} \in R$ if and only if there is an assignment to $\boldsymbol{y}$ such that $\boldsymbol{x}=\boldsymbol{z}$ satisfies all of the constraints $C_{1}, C_{2}, \ldots, C_{l}$.

The intuition behind the definition is that if every relation in $\Gamma_{1}$ is $k$-representable by relations in $\Gamma_{2}$ then W-Max $\operatorname{Ones}\left(\Gamma_{2}\right)-k$ is no easier than W -Max $\operatorname{OnES}\left(\Gamma_{1}\right)-k$. This is formalised in Lemma6

### 2.1 Approximability, Reductions, and Completeness

A combinatorial optimisation problem is defined over a set of instances (admissible input data) $\mathcal{I}$; each instance $I \in \mathcal{I}$ has a finite set $\operatorname{SOL}(I)$ of feasible solutions associated with it. The objective is, given an instance $I$, to find a feasible solution of optimum value with respect to some measure function $m$ defined for pairs $(x, y)$ such that $x \in \mathcal{I}$ and $y \in \operatorname{SOL}(x)$. Every such pair is mapped to a non-negative integer by $m$. The optimal value is the largest one for maximisation problems and the smallest one for minimisation problems. A combinatorial optimisation problem is said to be an NPO problem if its instances and solutions can be recognised in polynomial time, the solutions are polynomially-bounded in the input size, and the objective function can be computed in polynomial time (see, e.g., [1]).

Definition 3 ( $r$-approximate). A solution $s \in \operatorname{SOL}(I)$ to an instance $I$ of an NPO problem $\Pi$ is $r$-approximate if $\max \left\{\frac{m(I, s)}{\operatorname{OPT}(I)}, \frac{\operatorname{OPT}(I)}{m(I, s)}\right\} \leq r$, where $\mathrm{OPT}(I)$ is the optimal value for a solution to $I$.

An approximation algorithm for an NPO problem $\Pi$ has performance ratio $\mathcal{R}(n)$ if, given any instance $I$ of $\Pi$ with $|I|=n$, it outputs an $\mathcal{R}(n)$-approximate solution.

Definition 4 (PO, APX, poly-APX). PO is the class of NPO problems that can be solved (to optimality) in polynomial time. An NPO problem $\Pi$ is in the class APX if there is a polynomial-time approximation algorithm for $\Pi$ whose performance ratio is bounded by a constant. Similarly, $\Pi$ is in the class poly-APX if there is a polynomialtime approximation algorithm for $\Pi$ whose performance ratio is bounded by a polynomial in the size of the input.

Completeness in APX and poly-APX is defined using $A P$-reductions [1]. However, we do not need $A P$-reductions in this paper, the simpler $L$ - and $S$-reductions are sufficient for us.

Definition 5 ( $L$-reduction). An NPO problem $\Pi_{1}$ is said to be $L$-reducible to an NPO problem $\Pi_{2}$, written $\Pi_{1} \leq_{L} \Pi_{2}$, if two polynomial-time computable functions $F$ and $G$ and positive constants $\beta$ and $\gamma$ exist such that

- given any instance $I$ of $\Pi_{1}$, algorithm $F$ produces an instance $I^{\prime}=F(I)$ of $\Pi_{2}$, such that $\mathrm{OPT}\left(I^{\prime}\right) \leq \beta \cdot \mathrm{OPT}(I)$.
- given $I^{\prime}=F(I)$, and any solution $s^{\prime}$ to $I^{\prime}$, algorithm $G$ produces a solution $s$ to $I$ such that $\left|m_{1}(I, s)-\mathrm{OPT}(I)\right| \leq \gamma \cdot\left|m_{2}\left(I^{\prime}, s^{\prime}\right)-\mathrm{OPT}\left(I^{\prime}\right)\right|$, where $m_{1}$ is the measure for $\Pi_{1}$ and $m_{2}$ is the measure for $\Pi_{2}$.

It is well-known (see, e.g., Lemma 8.2 in [1]) that, if $\Pi_{1}$ is $L$-reducible to $\Pi_{2}$ and $\Pi_{1} \in \mathbf{A P X}$ then there is an AP-reduction from $\Pi_{1}$ to $\Pi_{2}$.
$S$-reductions are similar to $L$-reductions but instead of the condition OPT $\left(I^{\prime}\right) \leq$ $\beta \cdot \operatorname{OPT}(I)$ we require that $\operatorname{OPT}\left(I^{\prime}\right)=\operatorname{OPT}(I)$ and instead of $\left|m_{1}(I, s)-\operatorname{OPT}(I)\right| \leq$ $\gamma \cdot\left|m_{2}\left(I^{\prime}, s^{\prime}\right)-\operatorname{OPT}\left(I^{\prime}\right)\right|$ we require that $m_{1}(I, s)=m_{2}\left(I^{\prime}, s^{\prime}\right)$. If there is an $S$ reduction from $\Pi_{1}$ to $\Pi_{2}$ (written as $\Pi_{1} \leq_{S} \Pi_{2}$ ) then there is an AP-reduction from $\Pi_{1}$ to $\Pi_{2}$. An NPO problem $\Pi$ is APX-hard (poly-APX-hard) if every problem in

APX (poly-APX) is $A P$-reducible to it. If, in addition, $\Pi$ is in APX (poly-APX), then $\Pi$ is called APX-complete (poly-APX-complete).

We will do several reductions from InDEPENDENT SET (hereafter denoted by MIS) which is poly-APX-complete [19]. We will also use the fact that for any $k \geq 3$, MIS restricted to graphs of degree at most $k$ is APX-complete [22]. We will denote the latter problem by MIS- $k$.

The following lemma shows the importance of $k$-representations in our work.
Lemma 6. For constraint languages $\Gamma_{1}$ and $\Gamma_{2}$ if every relation in $\Gamma_{1}$ can be $k$-represented by $\Gamma_{2}$ then W-MAX Ones $\left(\Gamma_{1}\right)-k \leq_{S} \mathrm{~W}-\mathrm{Max} \operatorname{OnEs}\left(\Gamma_{2}\right)-k$.

Proof. Given an arbitrary instance $I=(V, C, w)$ of $\mathrm{W}-\operatorname{Max} \operatorname{Ones}\left(\Gamma_{1}\right)-k$, we will construct an instance $I^{\prime}=\left(V^{\prime}, C^{\prime}, w^{\prime}\right)$ of W-MAX $\operatorname{ONES}\left(\Gamma_{2}\right)-k$, in polynomial time. For each $c \in C$, add the $k$-representation of $c$ to $C^{\prime}$ and also add all variables which participate in the representation to $V^{\prime}$ in such a way that the auxiliary variables used in the representation are distinct from all other variables in $V^{\prime}$. Let $w^{\prime}(x)=w(x)$ for all $x \in V$ and $w(x)=0$ if $x \notin V$ (i.e., all auxiliary variables will have weight zero).

It is not hard to see that: (a) every variable in $I^{\prime}$ occurs at most $k$ times (b) $\operatorname{OPT}\left(I^{\prime}\right)=$ $\operatorname{OPT}(I)$, and (c) given a solution $s^{\prime}$ to $I^{\prime}$ we can easily construct a solution $s$ to $I$ (let $s(x)=s^{\prime}(x)$ for every $x \in V$ ) such that $m(I, s)=m\left(I^{\prime}, s^{\prime}\right)$. Hence, there is an $S$ reduction from W-Max $\operatorname{Ones}\left(\Gamma_{1}\right)-k$ to W-Max $\operatorname{Ones}\left(\Gamma_{2}\right)-k$.

### 2.2 Co-clones and Polymorphisms

Given an integer $k$, a function $f:\{0,1\}^{k} \rightarrow\{0,1\}$ can be extended to a function over tuples as follows: let $\boldsymbol{t}_{1}, \boldsymbol{t}_{2}, \ldots, \boldsymbol{t}_{\boldsymbol{k}}$ be $k$ tuples with $n$ elements each then $f\left(\boldsymbol{t}_{1}, \boldsymbol{t}_{2}, \ldots, \boldsymbol{t}_{\boldsymbol{k}}\right)$ is defined to be the tuple $\left(f\left(\boldsymbol{t}_{\mathbf{1}}[1], \boldsymbol{t}_{\mathbf{2}}[1], \ldots, \boldsymbol{t}_{\boldsymbol{k}}[1]\right), \ldots, f\left(\boldsymbol{t}_{\boldsymbol{1}}[n]\right.\right.$, $\left.\boldsymbol{t}_{\mathbf{2}}[n], \ldots, \boldsymbol{t}_{\boldsymbol{k}}[n]\right)$ ). Given a $n$-ary relation $R$ we say that $R$ is invariant (or, closed) under $f$ if $\boldsymbol{t}_{\mathbf{1}}, \boldsymbol{t}_{\mathbf{2}}, \ldots, \boldsymbol{t}_{\boldsymbol{k}} \in R \Rightarrow f\left(\boldsymbol{t}_{1}, \boldsymbol{t}_{\mathbf{2}}, \ldots, \boldsymbol{t}_{\boldsymbol{n}}\right) \in R$. Conversely, for a function $f$ and a relation $R, f$ is a polymorphism of $R$ if $R$ is invariant under $f$. For a constraint language $\Gamma$ we say that $\Gamma$ is invariant under $f$ if every relation in $\Gamma$ is invariant under $f$. We analogously extend the notion of polymorphisms to constraint languages, i.e., a function $f$ is a polymorphism of $\Gamma$ if $\Gamma$ is invariant under $f$. Those concepts has been very useful in the study of the complexity of various constraint satisfaction problems (see, e.g., [17]) and play an important role in this work, too.

The set of polymorphisms for a constraint language $\Gamma$ will be denoted by $\operatorname{Pol}(\Gamma)$, and for a set of functions $C$ the set of all relations which are invariant under $C$ will be denoted by $\operatorname{Inv}(B)$. The sets $\operatorname{Pol}(\Gamma)$ are clones in the sense of universal algebra. For a clone $C, \operatorname{Inv}(C)$ is called a relational clone or a co-clone. Over the boolean domain Emil Post has classified all such co-clones and their inclusion structure in [23].

For a set of relations $\Gamma$ we define a closure operator $\langle\Gamma\rangle$ as the set of relations that can be expressed with relations from $\Gamma$ using existential quantification and conjunction (note that we are only allowed to use the relations in $\Gamma$, hence equality is not necessarily allowed). Intuitively $\left\langle\Gamma \cup\left\{E Q^{2}\right\}\right\rangle$ is the set of relations which can be simulated by $\Gamma$ in $\operatorname{Csp}(\Gamma)$. An alternative classification of this set is $\left\langle\Gamma \cup\left\{E Q^{2}\right\}\right\rangle=\operatorname{Inv}(\operatorname{Pol}(\Gamma))$ [25]. These few paragraphs barely scratch the surface of the rich theory of clones and their
relation to the computational complexity of various constraint satisfaction problems, for a more thorough introduction see [5610].

We say that a set of relations $B$ is a plain basis for a constraint language $\Gamma$ if every relation in $\Gamma$ can be expressed with relations from $B$ using relations from $B \cup\{=\}$ and conjunction. Note that this differs from the definition of the closure operator $\langle\cdot\rangle$ as we do not allow existential quantification. See [11] for more information on plain bases.

We can not only study the co-clones when we try to classify $\operatorname{Max} \operatorname{Ones}(\Gamma)-k$ because the complexity of the problem do not only depend on the co-clone $\langle\Gamma\rangle$. However, the co-clone lattice with the corresponding plain bases and invariant functions will help us in our classification effort. Furthermore, as we mostly study the conservative constraint languages we can concentrate on the co-clones which contain $c_{0}$ and $c_{1}$. Figure 1 contains the conservative part of Post's lattice and Table 1 contains the plain bases for the relational clones which will be interesting to us (co-clones at and below $I V_{2}$ have been omitted as Max Ones is in PO there).


Table 1: Plain bases for some relational clones. The list of plain bases are from [11]. ${ }^{\ddagger}$

## 3 Three or More Occurrences

In this section we will prove a classification theorem for $\mathrm{W}-\operatorname{Max} \operatorname{OnES}(\Gamma)-k$ where $k \geq 3$. The main result of this section is the following theorem.

Theorem 7. Let $\Gamma$ be a conservative constraint language and $k \geq 3$,

[^1]1. If $\Gamma \subseteq I V_{2}$ then $\mathrm{W}-\operatorname{Max} \operatorname{Ones}(\Gamma)-k$ is in $\mathbf{P O}$.
2. Else if $I S_{12}^{2} \subseteq\langle\Gamma\rangle \subseteq I S_{12}$ then (W-)MAX ONES $(\Gamma)$ - $k$ is APX-complete if $E Q^{2}$ is not $k$-representable by $\Gamma$ and $\mathrm{W}-\operatorname{MAX} \operatorname{OnES}(\Gamma)$ - $k$ is poly-APX-complete otherwise.
3. Otherwise, W-Max $\operatorname{Ones}(\Gamma)$ and $\mathrm{W}-\operatorname{Max} \operatorname{Ones}(\Gamma)$ - $k$ are equivalent under $S$ reductions.

The first part of Theorem 7 follows from Khanna et al.'s results for Max Ones [20]. Intuitively the second part follows from the fact that W-MAX ONES $\left(\left\{N A N D^{2}\right\}\right)$ is equivalent to MIS, hence if we have access to the equality relation then the problem gets poly-APX-complete. On the other hand, if we do not have the equality relation then we essentially get MIS- $k$, for some $k$, which is APX-complete. The third part follows from Lemmas 8,10 and 11

Dalmau and Ford proved the following lemma in [12].
Lemma 8. If there is a relation $R$ in the constraint language $\Gamma$ such that $R \notin I E_{2}$, then either $x \vee y$ or $x \neq y$ can be 3 -represented by $\Gamma$. By duality, if there is a relation $R \in \Gamma$ such that $R \notin I V_{2}$, then either $N A N D^{2}$ or $x \neq y$ can be 3 -represented.

We can use the lemma above to get a 3-representation of either $E Q^{2}$ or $I M P L$. We will later, in Lemma 11 show that those relations makes the problem as hard as the unbounded occurrence variant.

Lemma 9. If there is a relation $R$ in the constraint language $\Gamma$ such that $R \notin I E_{2}$ and $R \notin I V_{2}$, then either $E Q^{2}$ or IMPL can be 3 -represented by $\Gamma$.

Proof. From Lemma 8 we know that either $x \neq y$ or both $x \vee y$ and $N A N D^{2}$ are 3representable. In the first case $\exists z: x \neq z \wedge z \neq y$ is a 3-representation of $E Q^{2}$. In the second case $\exists z: N A N D^{2}(x, z) \wedge(z \vee y)$ is a 3-representation of $\operatorname{IMPL}(x, y)$.

To get the desired hardness results for the $I S_{10}$ chain we need to prove that we can represent $E Q_{2}$ or $I M P L$ in that case too. To this end we have the following lemma.

Lemma 10. If there is a relation $R$ in the constraint language $\Gamma$ such that $R \in I E_{2}$ and $R \notin I S_{12}$, then either $E Q^{2}$ or IMPL can be 3 -represented by $\Gamma$.

Proof. Let $r$ be the arity of $R$ then, as $R \notin I S_{12}$, there exists a set of minimal cardinality $I \subseteq[r]$, such that $\left.R\right|_{I} \notin I S_{12}$.

As $g(x, y)=x \wedge y$ is a base of the clone which corresponds to $I E_{2},\left.R\right|_{I} \in I E_{2}$ implies that $g$ is a polymorphism of $\left.R\right|_{I}$. Furthermore, as $f(x, y, z)=x \wedge(y \vee \neg z)$ is a base of the clone which corresponds to $I S_{12},\left.R\right|_{I} \notin I S_{12}$ implies that $f$ is not a polymorphism of $\left.R\right|_{I}$. Hence, there exists tuples $\boldsymbol{t}_{\mathbf{1}}, \boldsymbol{t}_{\mathbf{2}},\left.\boldsymbol{t}_{\mathbf{3}} \in R\right|_{I}$ such that $f\left(\boldsymbol{t}_{\mathbf{1}}, \boldsymbol{t}_{\mathbf{2}}, \boldsymbol{t}_{\mathbf{3}}\right)=\left.\boldsymbol{t} \notin R\right|_{I}$.

There exists a coordinate $l_{1}, 1 \leq l_{1} \leq r$ such that $\left(\boldsymbol{t}_{\mathbf{1}}\left[l_{1}\right], \boldsymbol{t}_{\mathbf{2}}\left[l_{1}\right], \boldsymbol{t}_{\mathbf{3}}\left[l_{1}\right]\right)=(1,0,1)$, because otherwise $f\left(\boldsymbol{t}_{\mathbf{1}}, \boldsymbol{t}_{\mathbf{2}}, \boldsymbol{t}_{\mathbf{3}}\right)=\boldsymbol{t}_{\mathbf{1}}$. Similarly there exists a coordinate $l_{2}, 1 \leq l_{2} \leq r$ such that $\left(\boldsymbol{t}_{\mathbf{1}}\left[l_{2}\right], \boldsymbol{t}_{\mathbf{2}}\left[l_{2}\right], \boldsymbol{t}_{\mathbf{3}}\left[l_{2}\right]\right)$ is equal to one of $(0,1,0),(0,1,1)$ or $(1,0,0)$. Because otherwise $f\left(\boldsymbol{t}_{\mathbf{1}}, \boldsymbol{t}_{\mathbf{2}}, \boldsymbol{t}_{\mathbf{3}}\right)=\boldsymbol{t}_{\mathbf{2}}$. From now on, the case $\left(\boldsymbol{t}_{\mathbf{1}}\left[l_{2}\right], \boldsymbol{t}_{\mathbf{2}}\left[l_{2}\right], \boldsymbol{t}_{\mathbf{3}}\left[l_{2}\right]\right)=(1,0,0)$ will be denoted by $(*)$. Finally, there also exists a coordinate $l_{3}, 1 \leq l_{3} \leq r$ such that
$\left(\boldsymbol{t}_{\mathbf{1}}\left[l_{3}\right], \boldsymbol{t}_{\mathbf{2}}\left[l_{3}\right], \boldsymbol{t}_{\boldsymbol{3}}\left[l_{3}\right]\right)$ is equal to one of $(0,0,1),(0,1,1),(1,0,0),(1,0,1)$ or $(1,1,0)$, because otherwise $f\left(\boldsymbol{t}_{\mathbf{1}}, \boldsymbol{t}_{\mathbf{2}}, \boldsymbol{t}_{\mathbf{3}}\right)=\boldsymbol{t}_{\mathbf{3}}$. The case $\left(\boldsymbol{t}_{\mathbf{1}}\left[l_{3}\right], \boldsymbol{t}_{\mathbf{2}}\left[l_{3}\right], \boldsymbol{t}_{\mathbf{3}}\left[l_{3}\right]\right)=(1,0,0)$ will be denoted by ( ${ }^{* *}$ ).

As $\left.R\right|_{I}$ is invariant under $g$ we can place additional restrictions on $l_{1}, l_{2}$ and $l_{3}$. In particular, there has to be coordinates $l_{1}, l_{2}$ and $l_{3}$ such that we have at least one of the cases $\left({ }^{*}\right)$ or $\left({ }^{* *}\right)$, because otherwise $f\left(\boldsymbol{t}_{\mathbf{1}}, \boldsymbol{t}_{\mathbf{2}}, \boldsymbol{t}_{\mathbf{3}}\right)=g\left(\boldsymbol{t}_{\mathbf{1}}, \boldsymbol{t}_{\mathbf{2}}\right)$, which is in $\left.R\right|_{I}$ and we have assumed that $\left.f\left(\boldsymbol{t}_{\mathbf{1}}, \boldsymbol{t}_{\mathbf{2}}, \boldsymbol{t}_{\mathbf{3}}\right) \notin R\right|_{I}$. There is no problem in letting $l_{2}=l_{3}$ since we will then get both $\left({ }^{*}\right)$ and $\left({ }^{* *}\right)$. This will be assumed from now on. We can also assume, without loss of generality, that $l_{1}=1$ and $l_{2}=l_{3}=2$. We can then construct a 3-representation as $R_{\phi}(x, y) \Longleftrightarrow \exists z_{3} \ldots z_{r}:\left.R\right|_{I}\left(x, y, z_{3}, \ldots, z_{r}\right) \wedge c_{k_{3}}\left(z_{3}\right) \wedge$ $c_{k_{4}}\left(z_{4}\right) \wedge \ldots \wedge c_{k_{r}}\left(z_{r}\right)$ where $k_{i}=f\left(\boldsymbol{t}_{\mathbf{1}}[i], \boldsymbol{t}_{\mathbf{2}}[i], \boldsymbol{t}_{\mathbf{3}}[i]\right)$ for $3 \leq i \leq r$. We will now prove that $R_{\phi}$ is equal to one of the relations we are looking for.

If $(0,1) \in R_{\phi}$, then we would have $\left.t \in R\right|_{I}$, which is a contradiction, so $(0,1) \notin$ $R_{\phi}$. We will now show that $(0,0) \in R_{\phi}$. Assume that $(0,0) \notin R_{\phi}$. Then, $R^{*}=$ $\left.R\right|_{I \backslash\left\{l_{2}\right\}}$ is not in $I S_{12}$ which contradicts the minimality of $I$. To see this consider the following table of possible tuples in $\left.R\right|_{I}$,

|  | $1=l_{1} 2=l_{2}=l_{3}$ | 3 | 4 | $\cdots$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\boldsymbol{t}_{\mathbf{1}}$ | 1 | 1 | $\boldsymbol{t}_{\mathbf{1}}[3]$ | $\boldsymbol{t}_{\mathbf{1}}[4]$ | $\cdots$ |
| $\boldsymbol{t}_{\mathbf{2}}$ | 0 | 0 | $\boldsymbol{t}_{\mathbf{2}}[3]$ | $\boldsymbol{t}_{\mathbf{2}}[4]$ | $\cdots$ |
| $\boldsymbol{t}_{\mathbf{3}}$ | 1 | 0 | $\boldsymbol{t}_{\mathbf{3}}[3]$ | $\cdots$ |  |
| $\boldsymbol{a}$ | 0 | 1 | $f\left(\boldsymbol{t}_{\mathbf{1}}[3], \boldsymbol{t}_{\boldsymbol{2}}[3], \boldsymbol{t}_{\mathbf{3}}[3]\right)$ | $f\left(\boldsymbol{t}_{\mathbf{1}}[4], \boldsymbol{t}_{\mathbf{2}}[4], \boldsymbol{t}_{\mathbf{3}}[4]\right)$ | $\cdots$ |
| $\boldsymbol{b}$ | 0 | 0 | $f\left(\boldsymbol{t}_{\mathbf{1}}[3], \boldsymbol{t}_{\mathbf{2}}[3], \boldsymbol{t}_{\mathbf{3}}[3]\right) f\left(\boldsymbol{t}_{\mathbf{1}}[4], \boldsymbol{t}_{\mathbf{2}}[4], \boldsymbol{t}_{\mathbf{3}}[4]\right) \cdots$ |  |  |

We know that $\boldsymbol{t}_{\mathbf{1}}, \boldsymbol{t}_{\mathbf{2}},\left.\boldsymbol{t}_{\mathbf{3}} \in R\right|_{I}$ and we also know that $\left.\boldsymbol{a} \notin R\right|_{I}$. Furthermore, if $\left.\boldsymbol{b} \notin R\right|_{I}$, then $\left.f\left(\boldsymbol{t}_{\mathbf{1}}, \boldsymbol{t}_{\mathbf{2}}, \boldsymbol{t}_{\mathbf{3}}\right)\right|_{I \backslash\left\{l_{2}\right\}} \notin R^{*}$ which means that $I$ is not minimal. The conclusion is that we must have $(0,0) \in R_{\phi}$. In the same way it is possible to prove that unless $(1,1) \in R_{\phi}, I$ is not minimal.

To conclude, we have proved that $(0,0),(1,1) \in R_{\phi}$ and $(0,1) \notin R_{\phi}$, hence we either have $R_{\phi}=E Q^{2}$ or $R_{\phi}=\{(0,0),(1,0),(1,1)\}$.

It is now time to use our implementations of $E Q^{2}$ or $I M P L$ to prove hardness results. To this end we have the following lemma.

Lemma 11. If $E Q^{2}$ or IMPL is 3-representable by the constraint language $\Gamma$ then W-Max $\operatorname{OnEs}(\Gamma) \leq_{S}$ W-Max $\operatorname{OnES}(\Gamma)$-3.

The proof can be found in the appendix. As either $E Q^{2}$ or $I M P L$ is available we can construct a cycle of constraints among variables and such a cycle force every variable in the cycle to obtain the same value. Furthermore, each variable occurs only twice in such a cycle so we have one occurrence left for each variable.

## 4 Two Occurrences

In this section, we study $\mathrm{W}-\mathrm{Max} \operatorname{OnES}(\Gamma)-2$. We are not able to present a complete classification but a partial classification is achieved. We completely classify the co-
clones $I L_{2}$ and $I D_{2}$. For $\Gamma$ such that $\Gamma \nsubseteq I L_{2}, I D_{2}$ we show that if there is a relation which is not a $\Delta$-matroid relation (those are defined below) in $\Gamma$ then W-MAX Ones $(\Gamma)$-2 is APX-hard if W-Max Ones $(\Gamma)$ is not tractable.

### 4.1 Definitions and Results

Most of the research done on $\operatorname{Csp}(\Gamma)-k$ (e.g., in 141215) has used the theory of $\Delta$ matroids. Those objects are a generalisation of matroids and has been widely studied, cf. [43]. It turns out that the complexity of W-MAX $\operatorname{OnES}(\Gamma)-2$ depend to a large degree on if there is a relation which is not a $\Delta$-matroid relation in the constraint language. $\Delta$-matroid relations are defined as follows.

Definition 12 ( $\Delta$-matroid relation [12]). Let $R \subseteq\{0,1\}^{r}$ be a relation. If $\boldsymbol{x}, \boldsymbol{x}^{\prime} \in$ $\{0,1\}^{r}$, then $\boldsymbol{x}^{\prime}$ is a step from $\boldsymbol{x}$ to $\boldsymbol{y}$ if $d_{H}\left(\boldsymbol{x}, \boldsymbol{x}^{\prime}\right)=1$ and $d_{H}\left(\boldsymbol{x}, \boldsymbol{x}^{\prime}\right)+d_{H}\left(\boldsymbol{x}^{\prime}, \boldsymbol{y}\right)=$ $d_{H}(\boldsymbol{x}, \boldsymbol{y})$. $R$ is a $\Delta$-matroid relation if it satisfies the following two-step axiom: $\forall \boldsymbol{x}, \boldsymbol{y} \in$ $R$ and $\forall \boldsymbol{x}^{\prime}$ a step from $\boldsymbol{x}$ to $\boldsymbol{y}$, either $\boldsymbol{x}^{\prime} \in R$ or $\exists \boldsymbol{x}^{\prime \prime} \in R$ which is a step from $\boldsymbol{x}^{\prime}$ to $\boldsymbol{y}$.
As an example of a $\Delta$-matroid relation consider $N A N D^{3}$. It is not hard to see that $N A N D^{3}$ satisfies the two-step axiom for every pair of tuples as there is only one tuple which is absent from the relation. $E Q^{3}$ is the simplest example of a relation which is not a $\Delta$-matroid relation. The main theorem of this section is the following partial classification result for $\mathrm{W}-\operatorname{Max} \operatorname{ONES}(\Gamma)-2$. We say that a constraint language $\Gamma$ is $\Delta$-matroid if every relation in $\Gamma$ is a $\Delta$-matroid relation.

Theorem 13. Let $\Gamma$ be a conservative constraint language,

1. If $\Gamma \subseteq I V_{2}$ or $\Gamma \subseteq I D_{1}$ then $\mathrm{W}-\operatorname{Max} \operatorname{OnES}(\Gamma)-2$ is in $\mathbf{P O}$.
2. Else if $\Gamma \subseteq I L_{2}$ and,

- $\Gamma$ is not $\Delta$-matroid then, W-Max $\operatorname{OnES}(\Gamma)-2$ is APX-complete.
- otherwise, W-MAx Ones $(\Gamma)-2$ is in PO.

3. Else if $\Gamma \subseteq I D_{2}$ and,

- $\Gamma$ is not $\Delta$-matroid then, W-MAX Ones $(\Gamma)-2$ is poly-APX-complete.
- otherwise, W-Max Ones $(\Gamma)$ - 2 is in PO.

4. Else if $\Gamma \subseteq I E_{2}$ and $\Gamma$ is not $\Delta$-matroid then W -Max $\operatorname{OnES}(\Gamma)-2$ is APX-hard.
5. Else if $\Gamma$ is not $\Delta$-matroid then it is $\mathbf{N P}$-hard to find feasible solutions to W-MAX Ones $(\Gamma)-2$.

Part 1 of the theorem follows from the known results for W-Max Ones [1]. Part 4 follows from results for $\operatorname{Csp}(\Gamma)-2[14$ Theorem 4]. The other parts follows from the results in Sections 4.3 and 4.4 below.

### 4.2 Tractability Results for W-Max $\operatorname{Ones}(\boldsymbol{\Gamma})$-2

Edmonds and Johnson [13] has shown that the following integer linear programming problem is solvable in polynomial time: maximise $\boldsymbol{w} \boldsymbol{x}$ subject to the constraints $\mathbf{0} \leq$ $\boldsymbol{x} \leq \mathbf{1}, \boldsymbol{b}_{\mathbf{1}} \leq A \boldsymbol{x} \leq \boldsymbol{b}_{\mathbf{2}}$ and $\boldsymbol{x}$ is an integer vector. Here $A$ is a matrix with integer entries such that the sum of the absolute values of each column is at most $2 . \boldsymbol{b}_{\mathbf{1}}, \boldsymbol{b}_{\mathbf{2}}$ and $\boldsymbol{w}$ are arbitrary real vectors of appropriate dimensions. We will denote this problem by ILP-2. With the polynomial solvability of ILP-2 it is possible to prove the tractability of a number of W-MAX $\operatorname{OnES}(\Gamma)-2$ problems.

### 4.3 Classification of $I D_{2}$ and $I L_{2}$

When $\operatorname{Pol}(\Gamma)=\operatorname{Pol}\left(I D_{2}\right)$ or $\operatorname{Pol}(\Gamma)=\operatorname{Pol}\left(I L_{2}\right)$ we prove a complete classification result. We start with the hardness results for $I D_{2}$, which consists of the following lemma.

Lemma 14. Let $\Gamma$ be a constraint language such that $\operatorname{Pol}(\Gamma)=\operatorname{Pol}\left(I D_{2}\right)$. If there is a relation $R \in \Gamma$ which is not a $\Delta$-matroid relation, then $\mathrm{W}-\operatorname{Max} \operatorname{OnEs}(\Gamma)-2$ is poly-APX-complete.

The main observations used to prove the lemma is that since $\operatorname{Pol}(\Gamma)=\operatorname{Pol}\left(I D_{2}\right)$ we can 2-represent every two-literal clause. This has been proved by Feder in [14]. Furthermore, if we have access to every two-literal clause and also have a non- $\Delta$-matroid relation then it is possible to make variables participate in three clauses, which was also proved in [14]. The hardness result then follows with a reduction from MIS.

We will use some additional notation in the following proofs. For a tuple $\boldsymbol{x}=$ $\left(x_{1}, x_{2}, \ldots, x_{k}\right)$ and a set of coordinates $A \subseteq[k], \boldsymbol{x} \oplus A$ is defined to be the tuple $\left(y_{1}, y_{2}, \ldots, y_{k}\right)$ where $y_{i}=x_{i}$ if $i \notin A$ and $y_{i}=1-x_{i}$ otherwise. We extend this notation to relations: if $R \subseteq\{0,1\}^{n}$ and $A \subseteq[n]$ then $R \oplus A=\{\boldsymbol{t} \oplus A \mid \boldsymbol{t} \in R\}$.

We will now define a constraint language denoted by $\mathcal{Q}$. We will later prove that W-Max $\operatorname{Ones}(\mathcal{Q})-2$ is in PO. $\mathcal{Q}$ is the smallest constraint language such that:

- $\emptyset, c_{0}, c_{1}, E Q^{2}$ and $\{(0,1),(1,0)\}$ are in $\mathcal{Q}$.
- Every relation definable as $\left\{\boldsymbol{t} \mid d_{H}(\mathbf{0}, \boldsymbol{t}) \leq 1\right\}$ is in $\mathcal{Q}$.
- If $R, R^{\prime} \in \mathcal{Q}$ then their cartesian product $\left\{\left(\boldsymbol{t}, \boldsymbol{t}^{\prime}\right) \mid \boldsymbol{t} \in R, \boldsymbol{t}^{\prime} \in R^{\prime}\right\}$ is also in $\mathcal{Q}$.
- If $R \in \mathcal{Q}$ and $n$ is the arity of $R$ then $R \oplus A \in \mathcal{Q}$ for every $A \subseteq[n]$.
- If $R \in \mathcal{Q}, n$ is the arity of $R$ and $f:[n] \rightarrow[n]$ is a permutation on $[n]$ then $\left\{\left(t_{f(1)}, t_{f(2)}, \ldots, t_{f(n)}\right) \mid \boldsymbol{t} \in R\right\}$ is in $\mathcal{Q}$.

The relation between $\mathcal{Q}$ and the $\Delta$-matroid relations in $I D_{2}$ is given by the following lemma.

Lemma 15. If $R \in I D_{2}$ is a $\Delta$-matroid relation, then $R \in \mathcal{Q}$.
As for the tractability part we have the following lemma.
Lemma 16. Let $\Gamma$ be a constraint language such that $\Gamma \subseteq I D_{2}$, if all relations in $\Gamma$ are $\Delta$-matroid relations then $\mathrm{W}-\operatorname{Max} \operatorname{OnEs}(\Gamma)-2$ is in $\mathbf{P O}$.

The idea behind the proof is that $\mathrm{W}-\operatorname{MAX} \operatorname{OnES}(\mathcal{Q})-2$ can be seen as an ILP-2 problem and is therefore solvable in polynomial time.

As for $I L_{2}$ the result is the same, non $\Delta$-matroids give rise to APX-complete problems and absence of such relations makes the problem tractable. Also in this case the tractability follows from a reduction to ILP-2.

## 4.4 $\quad I E_{2}, I S_{12}$ and $I S_{10}$

The structure of the $\Delta$-matroids do not seem to be as simple in $I S_{12}$ and $I S_{10}$ as they are in $I D_{2}$ and $I L_{2}$. There exists relations in $I S_{12}$ which are $\Delta$-matroid relations
but for which we do not know of any polynomial time algorithm. One such relation is $R(x, y, z, w) \Longleftrightarrow N A N D^{3}(y, z, w) \wedge N A N D^{3}(x, z, w) \wedge N A N D^{2}(x, y)$. However, we get tractability results for some relations with the algorithm for ILP-2. In particular if the constraint language is a subset of $\left\{N A N D^{m} \mid m \in \mathbb{N}\right\} \cup\{I M P L\}$ then W-MAX Ones $(\cdot)-2$ is in PO.

We manage to prove hardness results for every non- $\Delta$-matroid relation contained in those co-clones. The main part of our hardness results for the non- $\Delta$-matroid relations is the following lemma.

Lemma 17. Let $R\left(x_{1}, x_{2}, x_{3}\right) \quad \Longleftrightarrow \quad N A N D^{2}\left(x_{1}, x_{2}\right) \wedge N A N D^{2}\left(x_{2}, x_{3}\right)$, then W-Max Ones $\left(\left\{c_{0}, c_{1}, R\right\}\right)-2$ is APX-complete.

Note that $R$ is not a $\Delta$-matroid relation. With Lemma 17 and a careful enumeration of the types of non- $\Delta$-matroid relations that exists in $I E_{2}$, we can deduce the desired result: if there is a non- $\Delta$-matroid relation in the constraint language, then W-MAX ONES ( $\cdot$ )-2 is APX-hard. The proof builds upon the work in [14182].

## 5 Non-conservative Constraint Languages

In this section we will take a look at the non-conservative case, i.e., we will look at constraint languages which do not necessarily contain $c_{0}$ and $c_{1}$. A relation $R$ is said to be 1 -valid if it contains the all ones tuple, i.e., $R$ is 1 -valid if $(1,1, \ldots, 1) \in R$. A constraint language is said to be 1 -valid if every relation in the language is 1 -valid.

Theorem 18. For any constraint language $\Gamma$ which is not 1 -valid, if $\mathrm{W}-\mathrm{MAx} \operatorname{OnES}(\Gamma \cup$ $\left.\left\{c_{0}, c_{1}\right\}\right)$ - $k$ is NP-hard for some integer $k$ then so is W-Max ONES $(\Gamma)-k$.

Note that for constraint languages $\Gamma$ which are 1 -valid W-Max $\operatorname{ONES}(\Gamma)$ is trivial: the all-ones solution is optimal. The idea in the proof is that we can simulate $c_{1}$ constraints by giving the variable a large weight. Furthermore, if there are relations which are not 1 -valid then we can represent $c_{0}$ constraints when we have access to $c_{1}$ constraints. It fairly easy to see why this fails to give us any inapproximability results: due to the large weight used to simulate $c_{1}$ any feasible solution is a good approximate solution.

## 6 Conclusions

We have started the study of the approximability properties of bounded occurrence Max Ones. We have presented a complete classification for the weighted conservative case when three or more variable occurrences are allowed. Furthermore, a partial classification of the two occurrence case has been presented. In the latter case we have proved that non- $\Delta$-matroid relations give rise to problems which are APX-hard if the unbounded occurrence variant is not tractable. We have also given complete classifications for the $I L_{2}$ and $I D_{2}$ co-clones.

There are still lots of open questions in this area. For example, what happens with the complexity if the weights are removed? Many constraint satisfaction problems such as Max Ones and Max Csp do not get any harder when weights are added. Such
results are usually proved by scaling and replicating variables and constraints a suitable number of times. However, such techniques do not work in the bounded occurrence setting and we do not know of any substitute which is equally general.

Except for the $I S_{12}$ and $I S_{10}$ chains the open questions in the two occurrence case are certain constraint languages $\Gamma$ such that $\Gamma$ only contains $\Delta$-matroid relations and $\operatorname{Pol}(\Gamma)=\operatorname{Pol}(B R)$. It would be very interesting to find out the complexity of W-MAX ONES $(\cdot)-2$ for some of the classes of $\Delta$-matroid relations which have been proved to be tractable for $\operatorname{Csp}(\cdot)-2$ in 14121615. Instead of trying to classify the entire $I S_{12}$ or $I S_{10}$ chain one could start with $I S_{12}^{3}$ or $I S_{10}^{3}$. The approximability of the nonconservative case is also mostly open. In light of [21] the computational structure of those problems seems to be quite complex.

## Appendix

## Proofs for Results in Section 3

Proof (Of Lemma 11). Let $I=(V, C, w)$ be an instance of W-Max Ones $(\Gamma)$. We will start with the case when $I M P L$ is 3-representable.

If $I M P L$ is 3-representable we can reduce $I$ to an instance $I^{\prime}=\left(V^{\prime}, C^{\prime}, w^{\prime}\right)$ of WMax $\operatorname{OnES}(\Gamma)-2$ as follows: for each variable $v_{i} \in V$, let $o_{i}$ be the number of occurrences of $v_{i}$ in $I$, we introduce the variables $v_{i}^{1}, \ldots, v_{i}^{o_{i}}$ in $V^{\prime}$. We let $w^{\prime}\left(v_{i}^{1}\right)=w\left(v_{i}\right)$ and $w^{\prime}\left(v_{i}^{j}\right)=0$ for $j \neq 1$. We also introduce the constraints $\operatorname{IMPL}\left(v_{i}^{k}, v_{i}^{k+1}\right)$ for $k, 1 \leq k \leq o_{i}-1$ and $\operatorname{IMPL}\left(v_{i}^{o_{i}}, v_{i}^{1}\right)$ into $C^{\prime}$. For every $i, 1 \leq i \leq|V|$ those constraints makes the variables $v_{i}^{1}, \ldots, v_{i}^{o_{i}}$ have the same value in every feasible solution of $I^{\prime}$.

For every constraint $c=(R, s) \in C$ the constraint scope $s=\left(v_{l_{1}}, \ldots, v_{l_{m}}\right)$ is replaced by $s^{\prime}=\left(v_{l_{1}}^{k_{1}}, \ldots, v_{l_{m}}^{k_{m}}\right)$ and $\left(R, s^{\prime}\right)$ is added to $C^{\prime}$. The numbers $k_{1}, \ldots, k_{m}$ are chosen in such a way that every variable in $V^{\prime}$ occur exactly three times in $I^{\prime}$. This is possible since there are $o_{i}$ variables in $V^{\prime}$ for every $v_{i} \in V$.

It is clear that the procedure described above is an $S$-reduction from W-MAX $\operatorname{Ones}(\Gamma)$ to W-Max $\operatorname{Ones}(\Gamma)$-3.
$I$ can easily be $S$-reduced to an instance $I^{\prime}$ of W-Max $\operatorname{OnEs}\left(\Gamma \cup\left\{E Q^{2}\right\}\right)$-3. And as $E Q^{2}$ is 3 -representable by $\Gamma$ we are done, as every constraint involving $E Q^{2}$ can be replaced by the 3 -representation of $E Q^{2}$ and any auxiliary variables used in the representation can be assigned the weight zero.

We need a couple of lemmas before we can state the proof of the classification theorem (Theorem 7. The following lemma will be used in several places to prove hardness results.

Lemma 19. Let $\Gamma$ be a constraint language such that $\operatorname{Pol}(\Gamma)=\operatorname{Pol}\left(I S_{1 \alpha}^{m}\right)$ for some integer $m$ and $\alpha \in\{0,2\}$, then $N A N D^{m}$ can be 2 -represented by $\Gamma$.

Proof. As $\operatorname{Pol}(\Gamma)=\operatorname{Pol}\left(I S_{1 \alpha}^{m}\right), \Gamma$ is invariant under $h_{m}$ and not invariant under $h_{m-1}$. Let $r$ be the arity of $R$ and let $X \subseteq[r]$ be a set of minimal cardinality such that there exist tuples $\boldsymbol{x}_{\mathbf{1}}, \boldsymbol{x}_{\boldsymbol{2}}, \ldots,\left.\boldsymbol{x}_{\boldsymbol{m}} \in R\right|_{X}$ which satisfies $h_{m-1}\left(\boldsymbol{x}_{\mathbf{1}}, \boldsymbol{x}_{\boldsymbol{2}}, \ldots, \boldsymbol{x}_{\boldsymbol{m}}\right)=$ $\left.\boldsymbol{z} \notin R\right|_{X}$. If there is a coordinate $i \in X$ such that $\boldsymbol{x}_{\mathbf{1}}[i]=\boldsymbol{x}_{\mathbf{2}}[i]=\ldots=\boldsymbol{x}_{\boldsymbol{m}}[i]$ then $\boldsymbol{z}[i]=\boldsymbol{x}_{\mathbf{1}}[i]$ and as $X$ is minimal we must have $\left.\boldsymbol{z} \oplus i \in R\right|_{X}$. However, this means that $h_{m}\left(\boldsymbol{x}_{\mathbf{1}}, \boldsymbol{x}_{\mathbf{2}}, \ldots, \boldsymbol{x}_{\boldsymbol{m}}, \boldsymbol{z} \oplus i\right)=\left.\boldsymbol{z} \notin R\right|_{X}$ which is a contradiction with the assumption that $R$ is invariant under $h_{m}$. We conclude that no coordinate is constant in every $\boldsymbol{x}_{\mathbf{1}}, \boldsymbol{x}_{\mathbf{2}}, \ldots, \boldsymbol{x}_{\boldsymbol{m}}$.

Now assume that there is a coordinate $j \in X$ such that $\boldsymbol{z}[j]=0$, then for $X$ to be minimal we must have $\left.\boldsymbol{z} \oplus j \in R\right|_{X}$. However, $h_{m}\left(\boldsymbol{x}_{\mathbf{1}}, \boldsymbol{x}_{\mathbf{2}}, \ldots, \boldsymbol{x}_{\boldsymbol{m}}, \boldsymbol{z} \oplus j\right)=\left.\boldsymbol{z} \notin R\right|_{X}$, a contradiction, hence there is no $j \in X$ such that $\boldsymbol{z}[j]=0$.

We can assume that $|X| \geq m$ because every relation of arity less than $m$ which is invariant under $h_{m}$ is also invariant under $h_{m-1}$ [7] Proposition 3.6].

We do now know three things, no coordinate in $X$ is constant in every $\boldsymbol{x}_{\mathbf{1}}, \boldsymbol{x}_{\mathbf{2}}, \ldots, \boldsymbol{x}_{\boldsymbol{m}}$, $\boldsymbol{z}=(1,1, \ldots, 1)$ and $|X| \geq m$. As $\boldsymbol{z}=(1,1, \ldots, 1)$ there is at most one zero for every given coordinate $i \in X$ among $\boldsymbol{x}_{\mathbf{1}}[i], \boldsymbol{x}_{\boldsymbol{2}}[i], \ldots, \boldsymbol{x}_{\boldsymbol{m}}[i]$, however as there is no constant
coordinate and $|X| \geq m$ we must have at least one zero in every $\boldsymbol{x}_{\mathbf{1}}, \boldsymbol{x}_{\mathbf{2}}, \ldots, \boldsymbol{x}_{\boldsymbol{m}}$. We can in fact assume that there is exactly one zero entry, because if it is two distinct coordinates $i, j \in X$ such that $\boldsymbol{x}_{\mathbf{1}}[i]=\boldsymbol{x}_{\mathbf{1}}[j]=0$ then as $\boldsymbol{z}=(1,1, \ldots, 1)$ no other tuple can have $\boldsymbol{x}_{\boldsymbol{k}}[i]=0$ or $\boldsymbol{x}_{\boldsymbol{k}}[j]=0$. The conclusion is that $\left.R\right|_{X \backslash\{j\}}$ is not invariant under $h_{m-1}$ either.

This implies that $\boldsymbol{x}_{\boldsymbol{i}}=(1,1, \ldots, 1) \oplus i$. It is not hard to see that by using the fact that $R$ is invariant under and we can get any tuple $\boldsymbol{y}=\left(y_{1}, y_{2}, \ldots, y_{m}\right)$ such that $y_{1}+y_{2}+\ldots+y_{m}<m$ by applying and to the $\boldsymbol{x}_{\boldsymbol{i}} \mathrm{s}$ an appropriate number of times. Hence, we must have $\left.R\right|_{X}=N A N D^{m}$.

Lemma 20. If $\operatorname{Pol}(\{R\})=\operatorname{Pol}\left(I S_{12}^{m}\right)$ for some $m \geq 2$ and $R$ cannot represent $E Q^{2}$, then $\left\langle\left\{R, c_{0}, c_{1}\right\}\right\rangle=\left\langle\left\{N A N D^{m}, c_{1}\right\}\right\rangle$.

Proof. We will denote $N A N D^{m}$ by $N$. Let $r$ be the arity of $R$ then $B=\left\{N, E Q^{2}, c_{1}\right\}$ is a plain basis for $I S_{12}^{m}$ (see Table As $B$ is a plain basis for $R$ there is an implementation of R on the following form,

$$
\begin{aligned}
R\left(x_{1}, \ldots, x_{r}\right) \Longleftrightarrow & N\left(x_{k_{1}^{1}}, x_{k_{1}^{2}}, \ldots, x_{k_{1}^{m}}\right) \wedge \ldots \wedge N\left(x_{k_{n}^{1}}, \ldots, x_{k_{n}^{m}}\right) \wedge \\
& E Q^{2}\left(x_{l_{1}^{1}}, x_{l_{1}^{2}}\right) \wedge \ldots \wedge E Q^{2}\left(x_{l_{c}^{1}}, x_{l_{c}^{2}}\right) \\
& c_{1}\left(x_{c_{1}}\right) \wedge \ldots \wedge c_{1}\left(x_{c_{w}}\right)
\end{aligned}
$$

for some $n, c$ and $w$ such that $k_{i}^{j} \in[r], l_{i}^{j} \in[r]$ and $c_{i} \in[r]$.
Assume that the representation above is minimal in the sense that it contains a minimal number of constraints. Hence, the only equalities that are possible are of the form $E Q^{2}\left(x_{i}, x_{j}\right)$ for $i \neq j$. If there is such an equality there are a number of cases to consider,

1. $\left.R\right|_{\{i, j\}}=\{(0,0),(1,1)\}$,
2. $\left.R\right|_{\{i, j\}}=\{(1,1)\}$, and
3. $\left.R\right|_{\{i, j\}}=\{(0,0)\}$.

We cannot have equalities of type 1 because then $E Q^{2}$ would be representable by $R$. Furthermore, equalities of type 2 and 3 can be replaced by constraints of the form $c_{1}\left(x_{i}\right) \wedge c_{1}\left(x_{j}\right)$ and $N\left(x_{i}, \ldots, x_{i}\right) \wedge N\left(x_{j}, \ldots, x_{j}\right)$, respectively.

The conclusion is that $R$ can be represented without $E Q^{2}$ and hence it is representable by $\left\{N, c_{1}\right\}$ alone. We have thus proved that $\left\langle\left\{R, c_{0}, c_{1}\right\}\right\rangle \subseteq\left\langle\left\{N, c_{1}\right\}\right\rangle$. The other inclusion, $\left\langle\left\{N, c_{1}\right\}\right\rangle \subseteq\left\langle\left\{R, c_{0}, c_{1}\right\}\right\rangle$, is given by Lemma 19

As for the containment we have the following lemma.
Lemma 21. Let $\Gamma$ be a constraint language if $\Gamma \subseteq I S_{12}^{m}$ for some $m$ and $\Gamma$ cannot represent $E Q^{2}$ then $\mathrm{W}-\operatorname{Max} \operatorname{OnEs}(\Gamma)-k$ is in APX.

Proof. Lemma 20 tells us that $\langle\Gamma\rangle=\left\langle\left\{N A N D^{m}, c_{1}\right\}\right\rangle$, hence an instance $J$ of WMax Ones $(\Gamma)-k$ can be reduced to an instance $J^{\prime}$ of W-Max Ones $\left(\left\{N A N D^{m}, c_{1}\right\}\right)$ $k^{\prime}$ for some constant $k^{\prime}$. To prove the lemma it is therefore sufficient show that W-MAX $\operatorname{OnES}\left(\left\{N A N D^{m}, c_{1}\right\}\right)-l$ is in APX for every fixed $l$.

Let $I=(V, C, w)$ be an arbitrary instance of W-MAx ONES $\left(\left\{N A N D^{m}, c_{1}\right\}\right)-l$, for some $l$, and assume that $V=\left\{x_{1}, \ldots, x_{n}\right\}$. By Schaefer's result [26] we can decide in polynomial time whether $I$ have a solution or not. Hence, we can safely assume that $I$ has a solution. If a variable occurs in a constant constraint, say $c_{1}(x)$, then $x$ must have the same value in every model of $I$. Thus, we can eliminate all such variables and assume that $I$ only contains constraints of the type $N A N D^{m}\left(x_{1}, \ldots, x_{m}\right)$.

We will give a polynomial-time algorithm that creates a satisfying assignment $s$ to $I$ with measure at least $\frac{1}{l+1}$ OPT $(I)$. Hence we have a $\frac{1}{l+1}$-approximate algorithm proving that W-Max Ones $\left(I S_{12}\right)$ - $l$ is in APX.

The algorithm is as follows: Repeatedly delete from $I$ any variable $x_{i}$ having maximum weight and all variables that appear together with $x_{i}$ in a clause of size two. In $s$ we assign 1 to $x_{i}$ and 0 to all variables appearing together with $x_{i}$ in a clause of size two.

For simplicity, assume that the algorithm chooses variables $x_{1}, x_{2}, \ldots, x_{t}$ before it stops. If the algorithm at some stage choose a variable $x$ with weight $w(x)$, then, in the worst case, it is forced to set $l$ (remember that no variable occurs more than $l$ times in $I$ ) variables to 0 and each of these variables have weight $w(x)$. This implies that $(l+1) \cdot \sum_{i=1}^{t} w\left(x_{i}\right) \geq \sum_{i=1}^{n} w\left(x_{i}\right)$ and

$$
m(I, s)=\sum_{i=1}^{t} w\left(x_{i}\right) \geq \frac{1}{l+1} \sum_{i=1}^{n} w\left(x_{i}\right) \geq \frac{\operatorname{OPT}(I)}{l+1}
$$

Lemma 22. Let $\Gamma$ be a constraint language such that $I S_{12}^{2} \subseteq\langle\Gamma\rangle \subseteq I S_{12}$ then W Max Ones $(\Gamma)$ - $k$ is APX-hard for $k \geq 3$.

Proof. Note that MIS-3 is exactly the same as Max $\operatorname{OnES}\left(\left\{N A N D^{2}\right\}\right)-3$. The lemma then follows from the fact that MIS-3 is APX-hard, Lemma 19 and Lemma6.

We are now ready to give the proof of the classification theorem for three or more occurrences.

Proof (Of Theorem 7 part 1). Follows directly from Khanna et al's results for MAX Ones [20].

Proof (Of Theorem 7 part 2). The APX-hardness follows from Lemma 22 Containment in APX follows from Lemma 21 If $E Q^{2}$ is $k$-representable by $\Gamma$ then the result follows from Lemma 11] and Khanna et al's results for MAX Ones [20].

Proof (Of Theorem 7 part 3). There are two possibilities, the first one is that $\Gamma \nsubseteq I E_{2}$ and $\Gamma \nsubseteq I V_{2}$, the second case is that $\Gamma \subseteq I E_{2}$ and $\Gamma \nsubseteq I S_{12}$.

In the first case we can use the 3-representation of $E Q^{2}$ or $I M P L$ from Lemma 9 The result then follows from Lemma 11 In the second case the result follows from Lemma 10 and Lemma 11

## Proofs for Results in Section 4

We will start with the case when $\operatorname{Pol}(\Gamma)=\operatorname{Pol}\left(I D_{2}\right)$. We need the following lemma before we can give the proof of Lemma 14

Lemma 23. Let $\Gamma$ be a constraint language such that $\operatorname{Pol}(\Gamma)=\operatorname{Pol}\left(I D_{2}\right)$ then $x \vee y$, $I M P L$ and $N A N D^{2}$ are 2-representable by $\Gamma$.

Proof. A part of the proof of Theorem 3 in [14] is the following: let $F$ be a constraint language such that there are relations $R_{1}, R_{2}, R_{3} \in F$ with the following properties:

- $R_{1}$ is not closed under $f(x, y)=x \vee y$.
- $R_{2}$ is not closed under $g(x, y)=x \wedge y$.
- $R_{3}$ is not closed under $h(x, y, z)=x+y+z(\bmod 2)$.
then $F$ can 2-represent every two-literal clause. As we have assumed that $\operatorname{Pol}(\Gamma)=$ $\operatorname{Pol}\left(I D_{2}\right)$ there are relations in $\Gamma$ which full fills the conditions above. The lemma follows.

Proof (Of Lemma 14). We will do an $S$-reduction from the poly-APX-complete problem MIS, which is precisely Max $\operatorname{OnEs}\left(\left\{N A N D^{2}\right\}\right)$. Let $I=(V, C)$ be an arbitrary instance of Max Ones $\left(\left\{N A N D^{2}\right\}\right)$. We will construct an instance $I^{\prime}=\left(V^{\prime}, C^{\prime}, w\right)$ of W-Max $\operatorname{OnES}(\Gamma)-2$. From Lemma 23 we know that we can 2-represent every twoliteral clause. It is easy to modify $I$ so that each variable occur at most three times. For a variable $x \in V$ which occur $k$ times, introduce $k$ fresh variables $y_{1}, y_{2}, \ldots, y_{k}$ and add the constraints $I M P L\left(y_{1}, y_{2}\right), I M P L\left(y_{2}, y_{3}\right), \ldots, I M P L\left(y_{k}, y_{1}\right)$. Each occurrence of $x$ is then replaced with one of the $y_{i}$ variables. In every solution each of the $y_{i}$ variables will obtain the same value, furthermore they occur three times each. Hence, if we can create a construction which allows us to let a variable participate in three clauses we are done with our reduction.

In Theorem 4 in [14] it is shown that given a relation which is not a $\Delta$-matroid we can make variables participate in three clauses if we have access to all clauses.

If we assign appropriate weights to the variables in $V^{\prime}$ it is clear that $\operatorname{OPT}(I)=$ $\mathrm{OPT}\left(I^{\prime}\right)$ and each solution to $I^{\prime}$ corresponds to a solution of $I$ with the same measure. Hence, we get an $S$-reduction.

We will now give the proof of Lemma 15 which describes the structure of the $\Delta$ matroid relations in $I D_{2}$.

For a relation $R \in \mathcal{Q}$ if $R$ can be decomposed (possibly after a permutation of the coordinates of $R$ ) into a cartesian product of other relations, $P_{1}, P_{2}, \ldots, P_{n} \in \mathcal{Q}$ then $P_{1}, P_{2}, \ldots, P_{n}$ will be called the factors of $R$.

Proof (Of Lemma 15). In this proof we will denote the majority function by $m$, i.e., $m(x, y, z)=(x \wedge y) \vee(y \wedge z) \vee(x \wedge z)$. Note that every relation in $I D_{2}$ is invariant under $m$. Let $R$ be a relation which contradicts the lemma, i.e., $R \in I D_{2}, R$ is a $\Delta$ matroid and $R \notin \mathcal{Q}$. Let $n$ be the arity of $R$. We can assume without loss of generality that $R$ consists of one factor, i.e., it is not possible to decompose $R$ into a cartesian product of other relations. In particular, $R$ do not contain any coordinate which has the same value in all tuples.

As every relation of arity less than or equal to two is in $\mathcal{Q}$ we can assume that $n \geq 3$. If for every pair of tuples $\boldsymbol{t}, \boldsymbol{t}^{\prime} \in R$ we have $d_{H}\left(\boldsymbol{t}, \boldsymbol{t}^{\prime}\right) \leq 2$ then $R \in \mathcal{Q}$ which is a contradiction. To see this let $\boldsymbol{t}_{\mathbf{1}}, \boldsymbol{t}_{\mathbf{2}}, \boldsymbol{t}_{\mathbf{3}}$ be three distinct tuples in $R$ (if there are less than three tuples in $R$ then either $R$ is not a $\Delta$-matroid relation or there is some coordinate which is constant in all tuples). Then $\boldsymbol{t}_{\mathbf{2}}=\boldsymbol{t}_{\mathbf{1}} \oplus A, \boldsymbol{t}_{\mathbf{3}}=\boldsymbol{t}_{\mathbf{1}} \oplus B$ for some $A, B \subseteq[n]$ such that $|A|,|B| \leq 2$ and $|A \cap B| \leq 1$. If $|A \cup B| \leq 2$ for all such sets then $R$ is either of arity 2 or there is a coordinate in $R$ which is constant. Hence, assume that $|A \cup B|=3$, which implies $|A|=|B|=2$. Let $\boldsymbol{t}=m\left(\boldsymbol{t}_{\mathbf{1}}, \boldsymbol{t}_{\mathbf{2}}, \boldsymbol{t}_{\mathbf{3}}\right)$. We will prove that for every tuple $\boldsymbol{t}^{\prime} \in R$ we have $d_{H}\left(\boldsymbol{t}^{\prime}, \boldsymbol{t}\right) \leq 1$. To this end, let $\boldsymbol{t}^{\prime}=\boldsymbol{t} \oplus C$, with $|C|=2(|C| \leq 1$ implies $d_{H}\left(\boldsymbol{t}^{\prime}, \boldsymbol{t}\right) \leq 1$ ), be an arbitrary tuple in $R$. If $|A \cap C|=0$ (or, $|B \cap C|=0$ ) then $d_{H}\left(\boldsymbol{t}_{\mathbf{2}}, \boldsymbol{t}^{\prime}\right) \geq 3\left(d_{H}\left(\boldsymbol{t}_{\mathbf{3}}, \boldsymbol{t}^{\prime}\right) \geq 3\right)$. Hence, we must have $|A \cap C|,|B \cap C| \geq 1$ but this implies $d_{H}\left(\boldsymbol{t}, \boldsymbol{t}^{\prime}\right) \leq 1$ or $d_{H}\left(\boldsymbol{t}, \boldsymbol{t}^{\prime}\right) \geq 3$, but the latter is not possible. We conclude that for every tuple $\boldsymbol{t}^{\prime} \in R$ we have $d_{H}\left(\boldsymbol{t}, \boldsymbol{t}^{\prime}\right) \leq 1$, hence $R \in \mathcal{Q}$ which contradicts our assumption that $R \notin \mathcal{Q}$.

Hence, there exists tuples $\boldsymbol{t}, \boldsymbol{t}^{\prime} \in R$ such that $d_{H}\left(\boldsymbol{t}, \boldsymbol{t}^{\prime}\right) \geq 3$. If for every pair of such tuples it is the case that every step, $s$, from $t$ to $t^{\prime}$ we have $s \in R$, then as no coordinate is constant, we must have $(0,0, \ldots, 0) \in R$ and $(1,1, \ldots, 1) \in R$. However, if $(0,0, \ldots, 0),(1,1, \ldots, 1) \in R$ and every step from the former to the latter is in $R$ then every tuple with one coordinate set to 1 is in $R$, too. We can continue this way and get every tuple with two coordinates set to one and then every tuple with $k$ coordinates set to 1 for $k \in[n]$. Hence, we must have $R=\{0,1\}^{n} \in \mathcal{Q}$.

We can therefore assume that there exists an coordinate $l$ such that the step $s=$ $\boldsymbol{t} \oplus l$ from $\boldsymbol{t}$ to $\boldsymbol{t}^{\prime}$ is not in $R$. Then, as $R$ is a $\Delta$-matroid relation, there exist another coordinate $K$ such that $s \oplus\{K\} \in R$ is a step from $\boldsymbol{s}$ to $\boldsymbol{t}^{\prime}$. Let $X$ denote the set of coordinates $i$ such that $\boldsymbol{t} \oplus i \notin R$ but $\boldsymbol{t} \oplus\{K, i\} \in R$, furthermore choose $\boldsymbol{t}$ and $K$ such that $|X|$ is maximised and let $X^{\prime}=X \cup\{K\}$.

Our goal in the rest of the proof is to show that if $X^{\prime}=[n]$ then $R \in \mathcal{Q}$ and otherwise it is possible to decompose $R$ into a cartesian product with $\left.R\right|_{X^{\prime}}$ in one factor and $\left.R\right|_{[n] \backslash X^{\prime}}$ in the other factor. As we have assumed that $R$ cannot be decomposed into a cartesian product we get a contradiction and hence the relation $R$ cannot exist.

Case 1: $\left|X^{\prime}\right|=2$
We will start with the case when $\left|X^{\prime}\right|=2$. Assume, without loss of generality, that $X^{\prime}=\{x, K\}$ then $\boldsymbol{t}, \boldsymbol{t} \oplus\{x, K\} \in R$ and $\boldsymbol{t} \oplus x \notin R$. We will now prove that we cannot have any tuples $\boldsymbol{v}$ in $R$ such that $\left.\boldsymbol{v}\right|_{X^{\prime}}=\left.(\boldsymbol{t} \oplus x)\right|_{X^{\prime}}$. If we had such a tuple then $m(\boldsymbol{v}, \boldsymbol{t}, \boldsymbol{t} \oplus\{x, K\})=\boldsymbol{w} \in R$ due to the fact that $R \in I D_{2}$ and $m$ is a polymorphism of $I D_{2}$. Furthermore, $\boldsymbol{w}$ must have the same value as $t$ on every coordinate except for possibly $x$ and $K$, this follows from the fact that $\boldsymbol{t}$ has the same value as $\boldsymbol{t} \oplus\{x, K\}$ on every coordinate except for $x$ and $K$. Hence, the only coordinates for which we do not know the value of $\boldsymbol{w}$ are $x$ and $K$. However, $\boldsymbol{v}[K]=\boldsymbol{t}[K]$ (due do the construction of $\boldsymbol{v}$ and the fact that $K \in X^{\prime}$ ). Hence we must get $\boldsymbol{w}[K]=\boldsymbol{t}[K]$. For $\boldsymbol{w}[x]$ note that $\boldsymbol{v}[x]=(\boldsymbol{t} \oplus\{x, K\})[x]$, hence $\boldsymbol{w}[x]=(\boldsymbol{t} \oplus x)[x]$. We can finally conclude $\boldsymbol{w}=\boldsymbol{t} \oplus x$ which is a contradiction with the construction of $X^{\prime}$.

Similar arguments as the above will be used repeatedly in this proof. However, the presentation will not be as detailed as the one above.

We split the remaining part of case 1 into two subcases, when $\boldsymbol{t} \oplus K \notin R$ (subcase 1a) and $\boldsymbol{t} \oplus K \in R$ (subcase 1b).

Subcase 1a: $\boldsymbol{t} \oplus \boldsymbol{K} \notin \boldsymbol{R}$ Assume that $\boldsymbol{t} \oplus K \notin R$, then $\left.\left.(\boldsymbol{t} \oplus K)\right|_{X^{\prime}} \notin R\right|_{X^{\prime}}$, because given a tuple $\boldsymbol{v}$ such that $\left.\boldsymbol{v}\right|_{X^{\prime}}=\left.(\boldsymbol{t} \oplus K)\right|_{X^{\prime}}$ then $m(\boldsymbol{t}, \boldsymbol{t} \oplus\{x, K\}, \boldsymbol{v})=\boldsymbol{t} \oplus K$, which is not in $R$ by the assumption we made.

Furthermore, for any tuple $\boldsymbol{v} \in R, \boldsymbol{v} \oplus x$ is a step from $\boldsymbol{v}$ to either $\boldsymbol{t}$ or $\boldsymbol{t} \oplus\{x, K\}$, but $\boldsymbol{v} \oplus x \notin R$ (because either $\left.\boldsymbol{v}\right|_{X^{\prime}}=\left.\boldsymbol{t}\right|_{X^{\prime}}$ which would imply $\boldsymbol{v} \oplus x \notin R$, or $\left.\boldsymbol{v}\right|_{X^{\prime}}=\left.(\boldsymbol{t} \oplus\{x, K\})\right|_{X^{\prime}}$ which implies $\left.\left.(\boldsymbol{v} \oplus x)\right|_{X^{\prime}}=\left.\left.(\boldsymbol{t} \oplus K)\right|_{X^{\prime}} \notin R\right|_{X^{\prime}}\right)$.

The only way to get from $\boldsymbol{v} \oplus x$ to something which is in $R$ is by flipping coordinate $K$, hence $\boldsymbol{v} \oplus\{x, K\} \in R$. This is the end of the case when $\boldsymbol{t} \oplus K \notin R$, because what we have proved above is that $R$ can be decomposed into a cartesian product with the coordinates $X^{\prime}$ in one factor and $[n] \backslash X^{\prime}$ in the other factor.

Subcase 1b: $\boldsymbol{t} \oplus \boldsymbol{K} \in \boldsymbol{R}$ We know that $\left.\left.(\boldsymbol{t} \oplus x)\right|_{X^{\prime}} \notin R\right|_{X^{\prime}}$. We will now show that for any $\boldsymbol{v} \in R$ such that, $\left.\boldsymbol{v}\right|_{X^{\prime}}$ is either $\left.\boldsymbol{t}\right|_{X^{\prime}}$ or $\left.(\boldsymbol{t} \oplus\{x, K\})\right|_{X^{\prime}}$, we have $\boldsymbol{v} \oplus\{x, K\} \in R$.

To this end, let $\boldsymbol{v}$ be an arbitrary tuple in $R$ satisfying one of the conditions above. We will consider the two possible cases separately.

- If $\left.\boldsymbol{v}\right|_{X^{\prime}}=\left.\boldsymbol{t}\right|_{X^{\prime}}$ then $\boldsymbol{v} \oplus x$ is a step from $\boldsymbol{v}$ to $\boldsymbol{t} \oplus\{x, K\}$ and $\boldsymbol{v} \oplus x \notin R$. Furthermore, the only way to get into $R$ is by flipping $K$ hence $\boldsymbol{v} \oplus\{x, K\} \in R$.
- If $\left.\boldsymbol{v}\right|_{X^{\prime}}=\left.(\boldsymbol{t} \oplus\{x, K\})\right|_{X^{\prime}}$ then $\boldsymbol{v} \oplus K$ is a step from $\boldsymbol{v}$ to $\boldsymbol{t}$ and $\boldsymbol{v} \oplus K \notin R$. Furthermore, the only way to get into $R$ is by flipping $x$ hence $\boldsymbol{v} \oplus\{x, K\} \in R$.

Now, let $\boldsymbol{v}$ be an arbitrary tuple in $R$ such that $\left.\boldsymbol{v}\right|_{X^{\prime}}=\left.(\boldsymbol{t} \oplus K)\right|_{X^{\prime}}$ then $\boldsymbol{v} \oplus K$ is a step from $\boldsymbol{v}$ to $\boldsymbol{t}$. If $\boldsymbol{v} \oplus K \in R$ or $\boldsymbol{v} \oplus x \in R$ then we are done with this step, so assume that $\boldsymbol{v} \oplus K, \boldsymbol{v} \oplus x \notin R$. However, as $R$ is a $\Delta$-matroid relation there has to exist a coordinate $l$ such that $\boldsymbol{v} \oplus\{K, l\} \in R$. Then we get, $\left.(\boldsymbol{v} \oplus\{K, l\})\right|_{X^{\prime}}=\left.\boldsymbol{t}\right|_{X^{\prime}}$ which implies $\boldsymbol{v} \oplus\{x, l\} \in R$ by the argument above. However, this means that $|X|$ is not maximal we could have chosen $\boldsymbol{v}, l$ and $X^{\prime}$ instead of $\boldsymbol{t}, K$ and $X$. We conclude that $\boldsymbol{v} \oplus K \in R$.

Finally, let $\boldsymbol{v}$ be an arbitrary tuple in $R$ such that $\left.\boldsymbol{v}\right|_{X^{\prime}}=\left.\boldsymbol{t}\right|_{X^{\prime}}$ then $\boldsymbol{v} \oplus K \in R$. To see this note that $m(\boldsymbol{t} \oplus K, \boldsymbol{v}, \boldsymbol{v} \oplus\{x, K\})=\boldsymbol{v} \oplus K$.

We have now proved that $R$ can be decomposed into a cartesian product with the coordinates $X^{\prime}$ in one factor and $[n] \backslash X^{\prime}$ in the other factor for this case too.

As we have assumed that the arity of $R$ is strictly greater than two we have $X^{\prime} \neq[n]$. Hence, $[n] \backslash X^{\prime} \neq \emptyset$.

## Case 2: $\left|X^{\prime}\right|>2$

The rest of the proof will deal with the case when $\left|X^{\prime}\right|>2$. We will begin with establishing a number of claims of $R$. Assuming that $X^{\prime} \neq[n]$, our main goal is still to show that $R$ can be decomposed into a cartesian product with $X^{\prime}$ in one factor and $[n] \backslash X^{\prime}$ in one factor. If $X^{\prime}=[n]$ we will show that $R \in \mathcal{Q}$.

Claim 1: if $d_{H}\left(\left.\boldsymbol{x}\right|_{X},\left.\boldsymbol{t}\right|_{X}\right)=1$ and $\boldsymbol{x}[K]=\boldsymbol{t}[K]$ then $\boldsymbol{x} \notin R$

Let $\boldsymbol{x}$ be a tuple which satisfies the precondition in the claim, assume that $\boldsymbol{x} \in R$, and let $i \in X$ be a coordinate where $\boldsymbol{x}$ differs from $\boldsymbol{t}$. By the construction of $X$ we have that $\boldsymbol{t} \oplus\{K, i\} \in R$, hence we get $m(\boldsymbol{t}, \boldsymbol{t} \oplus\{K, i\}, \boldsymbol{x})=\boldsymbol{t} \oplus\{i\} \in R$, which is a contradiction.

Claim 2: if $d_{H}\left(\left.\boldsymbol{x}\right|_{X},\left.\boldsymbol{t}\right|_{X}\right)=m$, for any $m$ such that $2 \leq m \leq|X|$, then $\boldsymbol{x} \notin R$
We will prove this claim by induction on $m$. For the base case, let $m=2$. Let $x \in X$ be some coordinate such that $\boldsymbol{x}[x] \neq \boldsymbol{t}[x]$, if $\boldsymbol{x} \in R$ and $\boldsymbol{x}[K]=\boldsymbol{t}[K]$ then $m(\boldsymbol{t}, \boldsymbol{x}, \boldsymbol{t} \oplus\{x, K\})=\boldsymbol{t} \oplus x \notin R$. Hence $\boldsymbol{x}[K]=\boldsymbol{t}[K]$ is not possible.

On the other hand if $\boldsymbol{x}[K] \neq \boldsymbol{t}[K]$ then $\boldsymbol{x} \oplus K$ is a step from $\boldsymbol{x}$ to $\boldsymbol{t}$. By the argument in the preceding paragraph we get $\boldsymbol{x} \oplus K \notin R$ (note that $K \notin X$ hence we have $\left.d_{H}\left(\left.(\boldsymbol{x} \oplus K)\right|_{X},\left.\boldsymbol{t}\right|_{X}\right)=m\right)$. Furthermore as $R$ is a $\Delta$-matroid we can flip some coordinate $l \in X$ such that $\boldsymbol{t}[l] \neq \boldsymbol{x}[l]$ to get a tuple which is in $R(l \notin X$ will not work as the argument in the preceding paragraph still applies in that case). However, $d_{H}\left(\left.(\boldsymbol{x} \oplus\{K, l\})\right|_{X},\left.\boldsymbol{t}\right|_{X}\right)=1$ hence by claim 1 we get a contradiction.

Now, assume that claim 2 holds for $m=m^{\prime}$. We will prove that it also holds for $m=m^{\prime}+1$ such that $2<m \leq|X|$. Note that we can use exactly the same argument as above except for the very last sentence in which we appeal to claim 2 with $m=m^{\prime}$ instead of using claim 1 . As we have assumed that claim 2 holds for $m=m^{\prime}$ we are done.

Claim 3: there is a tuple $\left.z \in R\right|_{X^{\prime}}$ such that for any tuple $\left.\boldsymbol{x} \in R\right|_{X^{\prime}}$ we have $d_{H}(\boldsymbol{z}, \boldsymbol{x}) \leq 1$

If $\left|X^{\prime}\right|>2$, then there are tuples $\boldsymbol{t} \oplus\{i, K\}$ and $\boldsymbol{t} \oplus\{j, K\}$ for distinct $i, j, K \in X^{\prime}$ in $R$. Hence, the tuple $\boldsymbol{z}^{\prime}=m(\boldsymbol{t}, \boldsymbol{t} \oplus\{i, K\}, \boldsymbol{t} \oplus\{j, K\})=\boldsymbol{t} \oplus K \in R$. Let $\boldsymbol{z}=\left.\boldsymbol{z}^{\prime}\right|_{X^{\prime}}$. We will now show that $d_{H}\left(\boldsymbol{z},\left.\boldsymbol{x}\right|_{X^{\prime}}\right) \leq 1$ for every tuple $\boldsymbol{x}$ in $R$. To this end, let $\boldsymbol{x}$ be an arbitrary tuple in $R$. By claim 2 we must have $d_{H}\left(\left.\boldsymbol{x}\right|_{X},\left.\boldsymbol{t}\right|_{X}\right) \leq 1$, furthermore if $\boldsymbol{x}[K]=\boldsymbol{z}^{\prime}[K] \neq \boldsymbol{t}[K]$ then we are done as $d_{H}\left(\left.\boldsymbol{x}\right|_{X^{\prime}},\left.\boldsymbol{z}^{\prime}\right|_{X^{\prime}}\right)=1$ in this case. On the other hand, if $\boldsymbol{x}[K]=\boldsymbol{t}[K]$ then claim 1 and claim 2 tells us that we must have $d_{H}\left(\left.\boldsymbol{x}\right|_{X},\left.\boldsymbol{t}\right|_{X}\right)=0$ in which case claim 3 follows.

Claim 4: if $\boldsymbol{x} \in R$ and $\left.\boldsymbol{x}\right|_{X^{\prime}}=\boldsymbol{z} \oplus\{i\}$ for some $i \in X^{\prime}$, then $\boldsymbol{x} \oplus\{i, j\} \in R$ for every $j \in X^{\prime}$.

Given $j \in X^{\prime}, j \neq i$, there is at least one tuple $\boldsymbol{v} \in R$ such that $\boldsymbol{v}[j] \neq \boldsymbol{x}[j]$ since otherwise the coordinate $j$ would be constant and $R$ could be decomposed into a cartesian product. Hence, $\boldsymbol{x}^{\prime}=\boldsymbol{x} \oplus j$ is a step from $\boldsymbol{x}$ to $\boldsymbol{v}$, but claim 3 tells us that $\boldsymbol{x}^{\prime} \notin R$ and the only way to full fill the two-step axiom is if $\boldsymbol{x} \oplus\{i, j\} \in R$ (due to claim 3 we cannot have $d_{H}(\boldsymbol{x}, \boldsymbol{v})=1$ ).

We will now prove that $R$ can be decomposed into cartesian product where the coordinates $X^{\prime}$ make up one factor and $[n] \backslash X^{\prime}$ make up the other factor. Let $P=\left.R\right|_{X^{\prime}}$. Our goal is to show that for any $\boldsymbol{p} \in P$ and $\left.\boldsymbol{v} \in R\right|_{[n] \backslash X^{\prime}}$ we have $(\boldsymbol{p}, \boldsymbol{v}) \in R$ (we have assumed that $X^{\prime}=\left\{1,2,3, \ldots,\left|X^{\prime}\right|\right\}$ here).

To this end, let $\boldsymbol{v}$ and $\boldsymbol{v}^{\prime}$ be arbitrary tuples in $R$. By claim 3 there either is a coordinate $i \in X^{\prime}$ such that $\left.(\boldsymbol{v} \oplus i)\right|_{X^{\prime}}=\boldsymbol{z}$ or $\left.\boldsymbol{v}\right|_{X^{\prime}}=\boldsymbol{z}$. The same is true for $\boldsymbol{v}^{\prime}$; either there is an coordinate $i^{\prime} \in X^{\prime}$ such that $\left.\left(\boldsymbol{v}^{\prime} \oplus i^{\prime}\right)\right|_{X^{\prime}}=\boldsymbol{z}$ or $\left.\boldsymbol{v}^{\prime}\right|_{X^{\prime}}=\boldsymbol{z}$,

If $\left.\boldsymbol{v}^{\prime}\right|_{X^{\prime}}=\left.\boldsymbol{v}\right|_{X^{\prime}}$ or $\left.\boldsymbol{v}^{\prime}\right|_{[n] \backslash X^{\prime}}=\left.\boldsymbol{v}^{\prime}\right|_{[n] \backslash X^{\prime}}$ then we are done, so assume that neither holds.

If $\left.\boldsymbol{v}^{\prime}\right|_{X^{\prime}} \neq \boldsymbol{z}$ and $\left.\boldsymbol{v}\right|_{X^{\prime}} \neq \boldsymbol{z}$ then $\boldsymbol{s}=\boldsymbol{v} \oplus i^{\prime}$ is a step from $\boldsymbol{v}$ to $\boldsymbol{v}^{\prime}$ but by claim 3 $\boldsymbol{s} \notin R$ and the only way to go a step from $\boldsymbol{s}$ to $\boldsymbol{v}^{\prime}$ and get into $R$ is $\boldsymbol{s}^{\prime}=\boldsymbol{s} \oplus i$, hence $s^{\prime} \in R$.

For the other case, when $\left.\boldsymbol{v}^{\prime}\right|_{X^{\prime}}=\boldsymbol{z}$ and $\left.\boldsymbol{v}\right|_{X^{\prime}} \neq \boldsymbol{z}$, if there is a coordinate $j \in X^{\prime}$ such that $\boldsymbol{v}^{\prime} \oplus j \in R$ then we are back to the previous case, so assume that such a $j$ do not exist. As $R$ is a $\Delta$-matroid there must be a coordinate $x \notin X^{\prime}$ such that $\boldsymbol{s}=\boldsymbol{v}^{\boldsymbol{\prime}} \oplus\{i, x\} \in R$, because for some appropriate $x, \boldsymbol{s}$ is a step from $\boldsymbol{v}^{\prime} \oplus i$ to $\boldsymbol{v}$. Due to claim 4 we will then have $\boldsymbol{v}^{\prime} \oplus\{x, y\} \in R$ for every $y \in X^{\prime}$. However, this contradicts the maximality of $X$ since we could have chosen $\boldsymbol{v}^{\prime}, x$, and $X^{\prime}$ instead of $\boldsymbol{t}, K$, and $X$. The conclusion is that if $X^{\prime} \neq[n]$, then $R$ can be decomposed into a cartesian product. On the other hand, if $X^{\prime}=[n]$, then we can easily deduce from claim 3 that $R \in \mathcal{Q}$.

Proof (Of Lemma 16). Let $I$ be an arbitrary instance of W-MAx OnEs $(\Gamma)-2$. We will show that the problem is in PO by reducing it to an instance $I^{\prime}$ of ILP-2. For any relation $R \in \Gamma$ of arity $n$ we know, from Lemma 15, that $R \in \mathcal{Q}$. We can assume that $R$ is not the cartesian product of any other two relations, because if it is then every use of $R$ can be substituted by the factors in the cartesian product. If $R$ is unary we can replace $R(x)$ by $x=0$ or $x=1$. If $R=E Q^{2}$ then we can replace $R(x, y)$ by $x=y$ and if $R(x, y) \Longleftrightarrow x \neq y$ then we replace $R(x, y)$ by $x=y-1$.

Now, assume that none of the cases above occur. We will show that

$$
\begin{equation*}
R\left(t_{1}, t_{2}, \ldots, t_{n}\right) \Longleftrightarrow \sum_{i=1}^{n} a_{i} t_{i} \leq b \tag{1}
\end{equation*}
$$

for some $a_{i} \in\{-1,1\}$ and integer $b$. Let $N$ be set of negated coordinates of $R$, i.e., let $N \subseteq[n]$ such that

$$
R=\left\{\left(f\left(t_{1}, 1\right), f\left(t_{2}, 2\right), \ldots, f\left(t_{n}, n\right)\right) \mid d_{H}(\mathbf{0}, \boldsymbol{t}) \leq 1\right\}
$$

where $f:\{0,1\} \times[n] \rightarrow\{0,1\}$ and $f(x, i)=\neg x$ if $i \in N$ and $f(x, i)=x$ otherwise. According to the definition of $\mathcal{Q}, R$ can be written on this form. Let $a_{i}=-1$ if $i \in N$ and $a_{i}=1$ otherwise. Furthermore, let $b=1-|N|$. It is now easy to verify that (1) holds.

As every variable occur at most twice in $I$ every variable occur at most twice in $I^{\prime}$ too. Furthermore, the coefficient in front of any variable in $I^{\prime}$ is either $-1,0$ or 1 , hence the sum of the absolute values in any column in $I^{\prime}$ is bounded by $2 . I^{\prime}$ is therefore an instance of ILP-2. If we let the weight function of $I^{\prime}$ be the same as the weight function in $I$ it easily seen that any solution $s$ to $I^{\prime}$ is also a solution to $I$ with the same measure.

As we are done with $I D_{2}$ we will continue with $I L_{2}$. A linear system of equations over $\operatorname{GF}(2)$ with $n$ equations and $m$ variables can be represented by a matrix $A$, a constant column vector $\boldsymbol{b}$ and a column vector $\boldsymbol{x}=\left(x_{1}, \ldots, x_{m}\right)$ of variables. The system of equations is then given by $A \boldsymbol{x}=\boldsymbol{b}$. Assuming that the rows of $A$ are linearly independent the set of solutions to $\boldsymbol{A} \boldsymbol{x}=\boldsymbol{b}$ are

$$
\left\{\left(\boldsymbol{x}^{\prime}, \boldsymbol{x}^{\prime \prime}\right) \mid \boldsymbol{x}^{\prime \prime} \in \mathbb{Z}_{2}^{n-m} \text { and } \boldsymbol{x}^{\prime}=A^{\prime} \boldsymbol{x}^{\prime \prime}+\boldsymbol{b}^{\prime}\right\}
$$

where $\boldsymbol{x}^{\prime}=\left(x_{1}, \ldots, x_{m}\right), \boldsymbol{x}^{\prime \prime}=\left(x_{m+1}, \ldots, x_{n}\right)$ and $A^{\prime}$ and $\boldsymbol{b}^{\prime}$ are suitably chosen.
If there is a column in $A^{\prime}$ with more than one entry which is equal to 1 (or, equivalently more than one non-zero entry), then we say that the system of equations is coupled.

Lemma 24. Let $\Gamma$ be a conservative constraint language such that $\Gamma \subseteq I L_{2}$. If there is a relation $R \in \Gamma$ such that $R$ is the set of solutions to a coupled linear system of equations over $G F(2)$ then W-Max $\operatorname{OnES}(\Gamma) \leq_{L}$ W-Max $\operatorname{OnES}(\Gamma)-2$, otherwise W-Max Ones $(\Gamma)-2$ is in $\mathbf{P O}$.

Proof. First note that every relation $R \in I L_{2}$ is the set of solutions to a linear system of equations over GF(2) [11.

We will start with the hardness proof. To this end we will construct a 2-representation of $E Q^{3}$. Let $R \in \Gamma$ be defined by

$$
R=\left\{\left(\boldsymbol{x}^{\prime}, \boldsymbol{x}^{\prime \prime}\right) \mid \boldsymbol{x}^{\prime \prime} \in \mathbb{Z}_{2}^{n-m} \text { and } \boldsymbol{x}^{\prime}=A^{\prime} \boldsymbol{x}^{\prime \prime}+\boldsymbol{b}^{\prime}\right\}
$$

for some $n, m, A^{\prime}$ and $\boldsymbol{b}^{\prime}$. Furthermore, we can assume that there is one column (say $j$ ) in $A^{\prime}$ with more than one entry equal to 1 (say $i$ and $i^{\prime}$ ). Hence, $A_{i j}^{\prime}$ and $A_{i^{\prime} j}^{\prime}$ are equal to 1 . Our first implementation consists of $R$ and a number of $c_{0}$ constraints,

$$
Q\left(x_{j+m}, x_{i}, x_{i^{\prime}}\right) \Longleftrightarrow R\left(x_{1}, \ldots, x_{n}\right) \bigwedge_{\substack{k: m+1 \leq k \leq n \\ k \notin\left\{j+m, i, i^{\prime}\right\}}} c_{0}\left(x_{k}\right)
$$

This implementation leaves us with three cases, the first one is $Q=E Q^{3}$, in which case we are done. The other two cases are $Q=\{(0,0,1),(1,1,0)\}$ and $Q=\{(0,1,0),(1,0,1)\}$. We will give an implementation of $E Q^{3}$ with the first case, the other one is similar. Note that

$$
E Q^{3}\left(y_{1}, y_{2}, y_{3}\right) \Longleftrightarrow \exists z, z^{\prime}: Q\left(y_{1}, y_{2}, z\right) \wedge Q\left(y_{3}, z^{\prime}, z\right)
$$

For the containment proof note that every relation in $\Gamma$ is the set of solutions to some non-coupled linear system of equations over GF(2). The set of feasible solutions to an instance of W-MAX $\operatorname{ONES}(\Gamma)-2$ is therefore the set of solutions to a linear system of equations over $\mathrm{GF}(2)$ with the property that every variable occurs at most twice. This problem is solvable by Edmonds and Johnson's method [13].

Corollary 25. Let $\Gamma$ be a conservative constraint language such that $\operatorname{Pol}(\Gamma)=\operatorname{Pol}\left(I L_{2}\right)$ if there is a relation $R \in \Gamma$ such that $R$ is not a $\Delta$-matroid relation then W-MAX Ones $(\Gamma)-2$ is APX-complete, otherwise W-MAX Ones $(\Gamma)-2$ is in PO.

Proof. Given a constraint language $\Gamma$ such that $\operatorname{Pol}(\Gamma)=\operatorname{Pol}\left(I L_{2}\right)$ then W-MAX Ones $(\Gamma)$ is APX-complete [20].

It is not hard to see that for a relation $R \in \Gamma, R$ is not a $\Delta$-matroid relation if and only if $R$ is the set of solutions to a coupled system of equations. (The "if"-part follows directly from the representation of relation $Q$ in Lemma 24)

Hence, if there is a relation $R \in \Gamma$ such that $R$ is not a $\Delta$-matroid relation then we get APX-completeness for $\mathrm{W}-\mathrm{Max} \operatorname{OnES}(\Gamma)-2$ from Lemma 24 On the other hand, if there is no non- $\Delta$-matroid relation in $\Gamma$ then no relation is the set of solutions to a coupled system of equations and hence we get tractability from Lemma 24

The final sub-case is $\Gamma \subseteq I E_{2}$.
Proof (Of Lemma 17). For the containment note that the algorithm in Lemma 21 can be used, as an instance of W-Max $\operatorname{OnES}\left(\left\{c_{0}, c_{1}, R\right\}\right)-2$ can easily be reduced to an instance of W-MAx OnEs $\left(\left\{c_{0}, c_{1}, N A N D^{2}\right\}\right)-4$.

We will do a reduction from MAx 2 SAT-3 (i.e., MAX 2 SAT where every variable occurs at most three times), which is APX-complete [1. Chap. 8]. The reduction is based on Theorem 1 in [2], which in turn is based on some of Viggo Kann's work on 3-dimensional matching [18].

We will do the reduction in two steps, we will first reduce MAX 2SAT-3, to a restricted variant of MIS-3. More precisely the graphs produced by the reduction will have maximum degree three and it will be possible to "cover" the graphs with $R$ (we will come back to this soon).

Let $I=(V, C)$ be an arbitrary instance of MAX 2 SAT-3. We will construct an instance $I^{\prime}=(G, w)$, where $G=\left(V^{\prime}, E^{\prime}\right)$ and $w: V^{\prime} \rightarrow \mathbb{Q}$, of weighted maximum independent set. We can assume, without loss of generality, that each variable in $I$ occurs at least once unnegated and at least once negated. For a node $v \in V$ construct four paths with three nodes each. Sequentially label the nodes in path number $x$ by $p_{x 1}, p_{x 2}, p_{x 3}$. Construct three complete binary trees with four leaves each and label the roots of the trees with $v_{1}, \neg v_{1}, v_{2}$ (or $v_{1}, \neg v_{1}, \neg v_{2}$ if $v$ occurs once unnegated and twice negated). Finally, identify the leaves of each of the trees with the nodes in the paths with similar labels, where two labels $p_{x y}$ and $p_{u v}$ are similar if $y=v$. Figure 2 contains this gadget for our example variable, $v$.


Fig. 2: The graph gadget for the variable $v$ which occurs three times, two times unnegated and one time negated.

Let the $w$ be defined as follows, $w\left(p_{12}\right)=w\left(p_{22}\right)=w\left(p_{32}\right)=w\left(p_{42}\right)=2.25$, $w\left(x_{21}\right)=w\left(x_{22}\right)=2$ and $w(\cdot)=1$ otherwise.

Denote the disjoint union of those paths and trees for all variables by $X$. A solution $S$ for the independent set problem for $X$ will be called consistent if for each variable, $v$, (which occurs twice unnegated and once negated) we have $v_{1}, v_{2} \in S$ and $\neg v_{1} \notin S$ or
vice versa (i.e., $v_{1}, v_{2} \notin S$ and $\neg v_{1} \in S$ ). It is not hard to verify (e.g., with a computer assisted search) that the optimal solutions to $X$ are consistent. Furthermore, for each consistent solution there is a solution which is optimal and includes or excludes the $v_{i}$ 's and $\neg v_{i}$ 's in the same way.

For each clause $c \in C$, containing the literals $l_{1}$ and $l_{2}$, add two fresh nodes $l_{1}$ and $l_{2}$ to $G^{\prime}$. Connect $l_{1}$ and $l_{2}$ with an edge and connect $l_{1}$ with the node which is labelled with this literal (one of the roots of the trees). Do the same thing for $l_{2}$.

We deduce that given a solution to $I^{\prime}$ it is possible to construct a consistent solution with a measure which is greater than or equal to the measure of the original solution. The only case we have to be careful about is when we are given a solution $S$ where $v_{1}, v_{2}, \neg v_{1} \notin S$. In this case the measure of the gadget is strictly less than the locally optimal solution. Hence, we can add $\neg v_{1}$ which, in the worst case, will force us to remove one node which was attached to $\neg v_{1}$ due to the clause which $\neg v_{1}$ is in. However, this loss will be made up for as we can assign an optimal solution to the gadget.

We have $\operatorname{OPT}\left(I^{\prime}\right) \leq|V| K+\operatorname{OPT}(I)$ where $K=14$ is the optimum value for our gadget. As $\operatorname{OPt}(I) \geq|C| / 2$ and $|V| \leq 3|C|$ we get OPT $\left(I^{\prime}\right) \leq 3 K|C|+\operatorname{OPT}(I) \leq$ $(6 K+1) \mathrm{OPT}(I)$, hence $\beta=6 K+1$ is an appropriate parameter for an $L$-reduction.

For any consistent solution $S^{\prime}$ to $I^{\prime}$ we can construct a solution $s$ to $I$ as follows, for each variable $v \in V$ let $s(v)=$ TRUE if $v_{i} \notin S^{\prime}$ and $s(v)=$ FALSE otherwise. We will then have $|\operatorname{OPT}(I)-m(I, s)|=\left|\operatorname{OPT}\left(I^{\prime}\right)-m\left(I^{\prime}, S^{\prime}\right)\right|$. Hence, $\gamma=1$ is an appropriate parameter for the $L$-reduction.

Using $c_{0}$ and $R$ it is possible to 2 -represent $N A N D^{2}(x, y)$. To reduce $I^{\prime}$ to an instance of W-Max Ones $(\{R\})-2$ note that we can "cover" each variable gadget with $R$ and $N A N D^{2}$, see Figure 3 how this is done. Furthermore, in the covering we have only used $v_{1}, \neg v_{1}$ and $v_{2}$ once so it wont be any problems with connecting the gadgets to each other with $N A N D^{2}$ constraints.

We need the following result which has been proved by Feder [14 Theorem 3, fact 1].

Lemma 26. Given a relation $R$ which is not closed under $f(x, y)=x \vee y$, then $R$ can 2 -represent either $N A N D^{2}$ or $x \neq y$.

Corollary 27. Given a relation $R \in I E_{2}$ which is not closed under $f(x, y)=x \vee y$, then $R$ can 2 -represent $N A N D^{2}$.

Proof. From Lemma 26 we deduce that $R$ can 2-represent either $N A N D^{2}$ or $x \neq y$, but the latter is not contained in $I E_{2}$, hence we must have the former.

Lemma 28. Let $\Gamma$ be a conservative constraint language, if $I S_{12}^{2} \subseteq\langle\Gamma\rangle \subseteq I E_{2}$, and there is a relation $R \in \Gamma$ such that $R$ is not a $\Delta$-matroid relation, then W-MAX Ones $(\Gamma)$ - 2 is APX-hard.

Some parts of the following proof is similar to Feder's proof in [14] that non- $\Delta$ matroids causes $\operatorname{Csp}(\cdot)-2$ to be no easier than $\operatorname{Csp}(\cdot)$.

Proof. As $R$ is not a $\Delta$-matroid relation there exists tuples $\boldsymbol{t}, \boldsymbol{t}^{\prime} \in R$ such that $d_{H}\left(\boldsymbol{t}, \boldsymbol{t}^{\prime}\right) \geq$ 3 and a step $\boldsymbol{s} \notin R$ from $\boldsymbol{t}$ to $\boldsymbol{t}^{\prime}$ such that no step from $\boldsymbol{s}$ to $\boldsymbol{t}^{\prime}$ is contained in $R$.


Part of graph|Constraints

| Paths | $R\left(p_{11}, p_{12}, p_{13}\right), R\left(p_{21}, p_{22}, p_{23}\right)$ |
| :--- | :--- |
|  | $R\left(p_{31}, p_{32}, p_{33}\right), R\left(p_{41}, p_{42}, p_{43}\right)$ |
| Tree for $v_{1}$ | $R\left(p_{11}, x_{11}, p_{21}\right), R\left(p_{31}, x_{12}, p_{41}\right), R\left(x_{11}, v_{1}, x_{12}\right)$ |
| Tree for $\neg v_{1}$ | $R\left(p_{12}, x_{21}, p_{22}\right), R\left(p_{32}, x_{22}, p_{42}\right), R\left(x_{21}, \neg v_{1}, x_{22}\right)$ |
| Tree for $v_{2}$ | $R\left(p_{13}, x_{31}, p_{23}\right), R\left(p_{33}, x_{32}, p_{43}\right), R\left(x_{31}, v_{2}, x_{32}\right)$ |

Fig. 3: The gadget for the variable $v$ covered by the relation $R$. Note that each variable occurs at most twice and that $v_{1}, v_{2}$ and $\neg v_{1}$ occurs once. Constraints with overlapping nodes are represented by two different line styles in the graph: solid and dotted.

Let $n$ be the arity of $R$ and let $X \subseteq[n]$ be the set of coordinates where $t$ differs from $\boldsymbol{t}^{\prime}$, i.e., $\boldsymbol{t}=\boldsymbol{t}^{\prime} \oplus X$. Furthermore, let $k \in[n]$ be the coordinate where $\boldsymbol{s}$ differs from $t$.

By using projections and the $c_{0}$ and $c_{1}$ constraints together with $R$ we can 2represent a new relation, $P$, which is not a $\Delta$-matroid relation and has arity 3 . To do this, choose a subset $X^{\prime} \subset X$ of minimal cardinality such that $k \in X^{\prime}$ and $\boldsymbol{t} \oplus X^{\prime} \in R$. Note that $\left|X^{\prime}\right| \geq 3$. Let $a$ and $b$ be two distinct coordinates in $X^{\prime}$ which differs from $k$. Construct $P$ as follows:

$$
P\left(x_{k}, x_{a}, x_{b}\right) \Longleftrightarrow R\left(x_{1}, x_{2}, \ldots, x_{n}\right) \bigwedge_{\substack{l \in[n] \backslash X^{\prime} \\ t[l]=1}} c_{1}\left(x_{l}\right) \bigwedge_{\substack{l \in[n] \backslash X^{\prime} \\ \boldsymbol{t}[l]=0}} c_{0}\left(x_{l}\right)
$$

Furthermore, let $\boldsymbol{v}=\left.\boldsymbol{t}\right|_{\{k, a, b\}}$ and $\boldsymbol{v}^{\prime}=\left.\boldsymbol{t}^{\prime}\right|_{\{k, a, b\}}$ we then have $\boldsymbol{v}, \boldsymbol{v}^{\prime} \in P$ and $\boldsymbol{v} \oplus 1, \boldsymbol{v} \oplus$ $\{1,2\}, \boldsymbol{v} \oplus\{1,3\} \notin P$. Hence, depending on $\boldsymbol{v}$ and which other tuples that are in $P$ we get a number of possibilities. We will use the following notation: $\boldsymbol{a}=\boldsymbol{v} \oplus 2, \boldsymbol{b}=\boldsymbol{v} \oplus 3$ and $\boldsymbol{c}=\boldsymbol{v} \oplus\{2,3\}$. Zero or more of $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ may be contained in $P$. Tables 4-7 list the possible relations we can get, up to permutations of the coordinates. Note that $\boldsymbol{a} \in P, \boldsymbol{b}, \boldsymbol{c} \notin P$ and $\boldsymbol{b} \in P, \boldsymbol{a}, \boldsymbol{c} \notin P$ are equivalent if we disregard permutations of the coordinates. Similarly $\boldsymbol{a}, \boldsymbol{c} \in P, \boldsymbol{b} \notin P$ and $\boldsymbol{b}, \boldsymbol{c} \in P, \boldsymbol{a} \notin P$ are equivalent.

Some of the relations are not in $I E_{2}$ and can therefore be omitted from further consideration (it is clear that if $P$ is not in $I E_{2}$ then $R$ is not in $I E_{2}$ either, which is a contradiction with the assumptions in the lemma). Others can 2-represent $E Q^{3}$, or can
do so together with $N A N D^{2}$. As an example consider $A 5$, then

$$
\begin{aligned}
& \exists y_{1}, y_{2}, y_{3}, z_{1}, z_{2}, z_{3}: A 5\left(y_{1}, x_{1}, z_{1}\right) \wedge N A N D^{2}\left(z_{1}, y_{2}\right) \wedge \\
& A 5\left(y_{2}, x_{2}, z_{2}\right) \wedge N A N D^{2}\left(z_{2}, y_{3}\right) \wedge \\
& A 5\left(y_{3}, x_{3}, z_{3}\right) \wedge N A N D^{2}\left(z_{3}, y_{1}\right)
\end{aligned}
$$

is a 2-representation of $E Q^{3}\left(x_{1}, x_{2}, x_{3}\right)$. Similar constructions works for some of the other relations. If we can 2 -represent $E Q^{3}$ then we get poly-APX-hardness due to the construction in Lemma 19 Lemma 6 and a simple reduction from MIS. Information about which relations this applies to is contained in Table2

Furthermore, some of the relations can 2-represent other relations in the table, see Table 3 for those. This implies that the only relation that is left to prove APX-hardness for is $A B C 1$. We will do this with a reduction from MIS-3. Let $G=(V, E)$ be an instance of MIS-3, we will construct an instance $I^{\prime}=\left(V^{\prime}, C^{\prime}, w^{\prime}\right)$ of W-MAX $\operatorname{OnEs}(\Gamma)-2$ with the assumption that $A B C 1 \in \Gamma$. Furthermore, due to Lemma 19 and Corollary 27we are free to assume that $N A N D^{2} \in \Gamma$. For every variable $v \in V$, if there are three occurrences of $v$ in $I$ add one fresh variable for each occurrence of $v$ in $I$ to $V^{\prime}$, name those fresh variables $v_{1}, v_{2}$ and $v_{3}$. If there are less than three occurrences add $v$ to $V^{\prime}$. Furthermore, for each edge $(v, x)$ for some $x \in V$ add a $N A N D^{2}\left(v_{i}, x_{j}\right)$ constraint to $C^{\prime}$. So far $I^{\prime}$ is an instance where each variable occurs at most twice and the variables which corresponds to nodes in $G$ with degree three occurs once in $I^{\prime}$.

For each node $v \in V$ with degree three add the constraint $A B C 1\left(v_{1}, v_{2}, v_{3}\right)$ to $C^{\prime}$. Finally, let $w(x)=1$ for every $x \in V$ with degree less than three and $w\left(v_{1}\right)=1$ and $w\left(v_{2}\right)=w\left(v_{3}\right)=0$ for every $v \in V$ with degree three. For every solution $s$ to $I^{\prime}$ we can construct a solution $S$ to $I$ such that $m\left(I^{\prime}, s\right)=m(I, S)$ to see this note that if $s(x)=1$ for some variable $x$ then due to the $A B C 1$ constraints the other occurrences of $x$ also have the value 1 . On the other hand, if $s(x)=0$ then we can set the other occurrences of $x$ to 0 without changing the measure of the solution and without conflicts with any constraints. This implies that there is an $S$-reduction from MIS-3 to W-MAX Ones $(\Gamma)-2$.

The results obtained in Lemma 28 is not optimal for all non- $\Delta$-matroids. It is noted in the proof that we get poly-APX-hardness results for some of the relations, but we do not get this for all of them. In particular we do not get this for $A 3, A B 1, B C 4, A B C 1$, $A B C 3, A B C 5$ and $A B C 6$. However, $A B C 5$ is contained in APX by Lemma 17

We are now finally ready to state the proof of the classification theorem for two variable occurrences.

Proof (OfTheorem[13] part 1). Follows from Khanna et al's results on W-Max Ones [20].

Proof (Of Theorem 13 part 2). Follows from Corollary 25 Lemma 14 and Lemma 16

Proof (Of Theorem 13 part 3). Follows from Lemma 28
Proof (Of Theorem [3] part 3). Follows from [14, Theorem 4].

| Relation Implementation or <br> comment |  |  | Relation Implementation or |
| :--- | :--- | :--- | :--- |
| comment |  |  |  |

Table 2: Non- $\Delta$-matroid relations in Lemma 28 If there is a relation in the "Implementation or comment" column then this relation can 2-represent $E Q^{3}$ together with the noted relation. If this second relation is $E Q^{2}$ then the relation can in fact 2-represent $E Q^{3}$ on its own, $E Q^{2}$ is not needed.

Relation Implements Implementation

| A3 | $A B C 5$ | $\exists x^{\prime}: A 3\left(x_{1}, x^{\prime}, x_{3}\right) \wedge N A N D^{2}\left(x^{\prime}, x_{2}\right)$ |
| :--- | :--- | :--- |
| AB1 | $A B C 5$ | $\exists x^{\prime}, x^{\prime \prime}: A B 1\left(x_{1}, x^{\prime}, x^{\prime \prime}\right) \wedge N A N D^{2}\left(x^{\prime}, x_{2}\right) \wedge N A N D^{2}\left(x^{\prime \prime}, x_{3}\right)$ |
| BC4 | $A B C 5$ | $\exists x^{\prime}: B C 4\left(x_{1}, x^{\prime}, x_{2}\right) \wedge N A N D^{2}\left(x^{\prime}, x_{3}\right)$ |
| ABC3 | $A B C 5$ | $\exists x^{\prime}: A B C 3\left(x_{1}, x^{\prime}, x_{3}\right) \wedge N A N D^{2}\left(x^{\prime}, x_{2}\right)$ |
| ABC6 | $A B C 1$ | $\exists x^{\prime}: A B C 6\left(x^{\prime}, x_{2}, x_{3}\right) \wedge N A N D^{2}\left(x^{\prime}, x_{1}\right)$ |
|  |  | Table 3: Implementations in Lemma 28 |


| 111 | 110 | 000 | 100 | 010 | 110 | 101 | 111 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $\frac{000}{1}$ | $\frac{001}{2}$ | $\frac{010}{\text { A1 }}$ | $\frac{110}{\text { A2 }}$ | $\frac{000}{\mathrm{~A} 3}$ | $\frac{100}{\mathrm{~A} 4}$ | $\frac{111}{\mathrm{~A} 5}$ | $\frac{101}{\mathrm{~A} 6}$ |

Table 4: Non- $\Delta$-matroid relations where $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \notin P$ followed by the relations where $\boldsymbol{a} \in P$ and $\boldsymbol{b}, \boldsymbol{c} \notin P$.

| 000 | 100 | 010 | 110 | 011 | 111 |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 111 | 011 | 101 | 001 | 100 | 000 |  |  |
| $\frac{011}{\mathrm{C} 1}$ | $\frac{111}{\mathrm{C} 2}$ | $\frac{001}{\mathrm{C} 3}$ | $\frac{101}{\mathrm{C} 4}$ | $\frac{011}{\mathrm{C} 5}$ | $\frac{100}{\mathrm{C} 6}$ |  |  |
|  |  | Table 5: |  |  |  |  | Non- $\Delta$-matroid relations where only $\boldsymbol{c} \in P$. |


| 000 | 100 | 010 | 001 | 000 | 100 | 010 | 110 | 011 | 111 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 111 | 011 | 101 | 110 | 111 | 011 | 101 | 001 | 100 | 000 |
| 001 | 101 | 011 | 000 | 010 | 110 | 000 | 100 | 001 | 101 |
| $\frac{011}{\overline{\mathrm{BC} 1}}$ | $\frac{111}{\mathrm{BC} 2}$ | $\overline{\mathrm{BC} 3}$ | $\frac{0010}{\mathrm{BC} 4}$ | $\frac{001}{\mathrm{AB} 1}$ | $\frac{101}{\mathrm{AB} 2}$ | $\frac{011}{\mathrm{AB} 3}$ | $\frac{111}{\mathrm{AB} 4}$ | $\frac{010}{\mathrm{AB} 5}$ | $\frac{110}{\mathrm{AB} 6}$ |

Table 6: Non- $\Delta$-matroid relations where $\boldsymbol{b}, \boldsymbol{c} \in P$ and $\boldsymbol{a} \notin P$ followed by relations where $\boldsymbol{a}, \boldsymbol{b} \in P$ and $\boldsymbol{c} \notin P$.

| 000 | 100 | 010 | 110 | 011 | 111 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 111 | 011 | 101 | 001 | 100 | 000 |
| 010 | 110 | 000 | 100 | 001 | 101 |
| 001 | 101 | 011 | 111 | 010 | 110 |
| $\frac{011}{\overline{\mathrm{ABC}} 1}$ | $\frac{111}{\mathrm{ABC}} 2$ | $\frac{001}{\mathrm{ABC} 3}$ | $\frac{101}{\mathrm{ABC}} 4$ | $\overline{\mathrm{ABC}} 5$ | $\overline{\mathrm{ABC}} 6$ |

Table 7: Non- $\Delta$-matroid relations where $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in P$.

## Proofs for Results in Section 5

Proof (Of Theorem 18). Let $\Gamma$ be a non-1-valid constraint language and $k$ an integer such that W-Max $\operatorname{ONES}\left(\Gamma \cup\left\{c_{0}, c_{1}\right\}\right)$ - $k$ (this problem will hereafter be denoted by $\Pi_{01}$ ) is NP-hard. We will prove the theorem with a reduction from $\Pi_{01}$ to W-MAX Ones $(\Gamma)-k$ (hereafter denoted by $\Pi$ ).

As $\Gamma$ is not 1 -valid there exists a relation $R \in \Gamma$ such that $(1, \ldots, 1) \notin R$. Let $r$ be the arity of $R$ and let $t$ be the tuple in $R$ with the maximum number of ones. Assume, without loss of generality, that $\boldsymbol{t}=(0,1, \ldots, 1)$.

The assumption in the theorem implies that it is $\mathbf{N P}$-hard to decide the following question: given an instance $I=(V, C, w)$ of $\Pi_{01}$ and an integer $K$ is $\operatorname{OPT}(I) \geq K$ ?

Let $I=(V, C, w), K$ be an arbitrary instance of the decision variant of $\Pi_{01}$. We will transform $I$ into an instance $I^{\prime}=\left(V^{\prime}, C^{\prime}, w^{\prime}\right), K^{\prime}$ of the decision variant of $\Pi$ by first removing constraint applications using $c_{0}$ and then removing constraint applications using $c_{1}$.

At the start of the reduction let $V^{\prime}=V$ and $C^{\prime}=C$. For each constraint $\left(c_{0},(v)\right) \in$ $C^{\prime}$ replace this constraint with $\left(R,\left(v, v_{1}, \ldots, v_{r-1}\right)\right)$ where $v_{1}, \ldots, v_{r-1}$ are fresh variables, furthermore add the constraint $\left(c_{1},\left(v_{k}\right)\right)$ for $k=1, \ldots, r-1$ to $C^{\prime}$.

Let $c$ be the number of variables which are involved in $c_{1}$ constraints. For each constraint using $c_{1},\left(c_{1},(v)\right) \in C^{\prime}$, remove this constraint and set $w^{\prime}(v)=L+w(v)$, where $L$ is a sufficiently large integer $\left(L=1+\sum_{v \in V} w(v)\right.$ is enough). For every variable $v$ which is not involved in a $c_{1}$ constraint let $w^{\prime}(v)=w(v)$.

Finally let $K^{\prime}=K+c K$. Given a solution $s^{\prime}$ to $I^{\prime}$ such that $m\left(I^{\prime}, s^{\prime}\right) \geq K^{\prime}$ it is clear that this solution also is a solution to $I$ such that $m\left(I, s^{\prime}\right) \geq K$. Furthermore, if there is a solution $s$ to $I$ such that $m(I, s) \geq K$ then $s$ is a solution $I^{\prime}$ such that $m\left(I^{\prime}, s\right) \geq K^{\prime}$.

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[^0]:    * Supported by the National Graduate School in Computer Science (CUGS), Sweden.

[^1]:    ${ }^{\ddagger}$ In [11] the listed plain basis for $I S_{12}^{m}$ is $\left\{E Q^{2}, c_{1}\right\} \cup\left\{N_{k} \mid k \leq m\right\}$ however, if we have $N_{m}$ then $N_{m-1}$ can be represented without auxiliary variables by $N_{m-1}\left(x_{1}, x_{2}, \ldots, x_{m-1}\right) \Longleftrightarrow$ $N_{m}\left(x_{1}, x_{1}, x_{2}, x_{3}, \ldots, x_{m-1}\right)$, hence the set of relations listed in Table $\square$ is a plain basis for $I S_{12}^{m}$. The same modification has been done to $I S_{10}^{m}$.

