# Algebraic computation of some intersection D-modules 

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March, 2006


#### Abstract

Let $X$ be a complex analytic manifold, $D \subset X$ a locally quasi-homogeneous free divisor, $\mathcal{E}$ an integrable logarithmic connection with respect to $D$ and $\mathcal{L}$ the local system of the horizontal sections of $\mathcal{E}$ on $X-$ $D$. In this paper we give an algebraic description in terms of $\mathcal{E}$ of the regular holonomic $\mathcal{D}_{X}$-module whose de Rham complex is the intersection complex associated with $\mathcal{L}$. As an application, we perform some effective computations in the case of quasi-homogeneous plane curves.


## Introduction

On a complex analytic manifold, intersection complexes associated with irreducible local systems on a dense open regular subset of a closed analytic subspace are the simple pieces which form any perverse sheaf. The RiemannHilbert correspondence allows us to consider the regular holonomic D-modules which correspond to these intersection complexes, that we call "intersection D-modules". They are the simple pieces which form any regular holonomic D-module. Whereas intersection complexes are topological objects, intersection D-modules are algebraic: they are given by a system of partial linear differential equations with holomorphic coefficients.

Intersection complexes can be constructed by an important operation: the intermediate direct image. Its description in terms of Verdier duality and usual derived direct images can be algebraically interpreted in the category of holonomic regular D-modules by using the deep properties of the de Rham functor. We need to compute localizations and D-duals.

This can be effectively done, in principle, by using the general available algorithms in [25, 27, 26], but in the case of integrable logarithmic connections along a locally quasi-homogeneous free divisor, we exploit the logarithmic point of view [2, 4, 5, 8, 9, 30, 31] to previously obtain a general algebraic description of their associated intersection D-modules, from which we can easily derive effective computations.

The main ingredients we use are the duality theorem proved in 5 and the logarithmic comparison theorem for arbitrary integrable logarithmic connections proved in [6], both with respect to locally quasi-homogeneous free divisors.

[^0]The algorithmic treatment of the computations in this paper will be developed elsewhere.

Let us now comment on the content of this paper.
In section 1 we remind the reader of the basic notions and notations and we review our previous results on logarithmic $\mathcal{D}$-modules with respect to free divisors. We recall the logarithmic comparison theorem for arbitrary integrable logarithmic connections from [6], and we give the theorem describing the intersection D-module associated with an integrable logarithmic connection along a locally quasi-homogeneous free divisor.

In section 2 given a locally quasi-homogeneous free divisor $D$ with a reduced local equation $f=0$ and a cyclic integrable logarithmic connection $\mathcal{E}$ with respect to $D$, we explicitly describe a presentation of $\mathcal{D}[s] \cdot\left(\mathcal{E} f^{s}\right)$ over $\mathcal{D}[s]$ in terms of a presentation of $\mathcal{E}$ over the ring of logarithmic differential operators. This description will be useful in order to compute the Bernstein-Sato polynomials associated with $\mathcal{E}$.

In section 3 the general results of the previous section are explicitly written down in the case of a family of integrable logarithmic connections with respect to a quasi-homogeneous plane curves.

In section 4 we perform some explicit computations with respect to a cusp.
We wish to thank Hélène Esnault who, because of a question about our paper [5], drew our attention to computing intersection D-modules. We also thank Tristan Torrelli for helpful information about the Bernstein-Sato functional equations and for some comments on a previous version of this paper.

## 1 Logarithmic connections with respect to a free divisor: theoretical set-up

Let $X$ be a $n$-dimensional complex analytic manifold and $D \subset X$ a hypersurface, and let us denote by $j: U=X-D \hookrightarrow X$ the corresponding open inclusion.

We say that $D$ is a free divisor [28] if the $\mathcal{O}_{X}$-module $\operatorname{Der}(\log D)$ of $\operatorname{logarith-~}$ mic vector fields with respect to $D$ is locally free (of rank $n$ ), or equivalently if the $\mathcal{O}_{X}$-module $\Omega_{X}^{1}(\log D)$ of logarithmic 1-forms with respect to $D$ is locally free (of rank $n$ ).

Normal crossing divisors, plane curves, free hyperplane arrangements (e.g. the union of reflecting hyperplanes of a complex reflection group), discriminant of stable mappings or bifurcation sets are examples of free divisors.

We say that $D$ is quasi-homogeneous at $p \in D$ if there is a system of local coordinates $\underline{x}$ centered at $p$ such that the germ $(D, p)$ has a reduced weighted homogeneous defining equation (with strictly positive weights) with respect to $\underline{x}$. We say that $D$ is locally quasi-homogeneous if it is so at each point $p \in D$.

Let us denote by $\mathcal{D}_{X}(\log D)$ the 0 -term of the Malgrange-Kashiwara filtration with respect to $D$ on the sheaf $\mathcal{D}_{X}$ of linear differential operators on $X$. When $D$ is a free divisor, the first author has proved in [2] that $\mathcal{D}_{X}(\log D)$ is the universal enveloping algebra of the Lie algebroid $\operatorname{Der}(\log D)$, and then it is coherent and has noetherian stalks of finite global homological dimension. Locally, if $\left\{\delta_{1}, \ldots, \delta_{n}\right\}$ is a local basis of the logarithmic vector fields on an open set $V$, any differential operator in $\Gamma\left(V, \mathcal{D}_{X}(\log D)\right)$ can be written in a unique
way as a finite sum

$$
\sum_{\substack{\alpha \in \mathbb{N}^{n} \\|\alpha| \leq d}} a_{\alpha} \delta_{1}^{\alpha_{1}} \cdots \delta_{n}^{\alpha_{n}},
$$

where the $a_{\alpha}$ are holomorphic functions on $V$.
From now on, let us assume that $D$ is a free divisor.
We say that $D$ is a Koszul free divisor [2] at a point $p \in D$ if the symbols of any (some) local basis $\left\{\delta_{1}, \ldots, \delta_{n}\right\}$ of $\operatorname{Der}(\log D)_{p}$ form a regular sequence in $\operatorname{Gr} \mathcal{D}_{X, p}$. We say that $D$ is a Koszul free divisor if it is so at any point $p \in D$. Actually, as M. Schulze pointed out, Koszul freeness is equivalent to holonomicity in the sense of 28 .

Plane curves and locally quasi-homogeneous free divisors (e.g. free hyperplane arrangements or discriminant of stable mappings in Mather's "nice dimensions") are example of Koszul free divisors [3].

A logarithmic connection with respect to $D$ is a locally free $\mathcal{O}_{X}$-module $\mathcal{E}$ endowed with:
-) a $\mathbb{C}$-linear morphism (connection) $\nabla^{\prime}: \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{O}_{X}} \Omega_{X}^{1}(\log D)$, satisfying $\nabla^{\prime}(a e)=a \nabla^{\prime}(e)+e \otimes d a$, for any section $a$ of $\mathcal{O}_{X}$ and any section $e$ of $\mathcal{E}$, or equivalently, with
-) a left $\mathcal{O}_{X}$-linear morphism $\nabla: \operatorname{Der}(\log D) \rightarrow \operatorname{End}_{\mathbb{C}_{X}}(\mathcal{E})$ satisfying the Leibniz rule $\nabla(\delta)(a e)=a \nabla(\delta)(e)+\delta(a) e$, for any logarithmic vector field $\delta$, any section $a$ of $\mathcal{O}_{X}$ and any section $e$ of $\mathcal{E}$.

The integrability of $\nabla^{\prime}$ is equivalent to the fact that $\nabla$ preserve Lie brackets. Then, we know from [2] that giving an integrable logarithmic connection on a locally free $\mathcal{O}_{X}$-module $\mathcal{E}$ is equivalent to extending its original $\mathcal{O}_{X}$-module structure to a left $\mathcal{D}_{X}(\log D)$-module structure, and so integrable logarithmic connections are the same as left $\mathcal{D}_{X}(\log D)$-modules which are locally free of finite rank over $\mathcal{O}_{X}$.

Let us denote by $\mathcal{O}_{X}(\star D)$ the sheaf of meromorphic functions with poles along $D$. It is a holonomic left $\mathcal{D}_{X}$-module.

The first examples of integrable logarithmic connections (ILC for short) are the invertible $\mathcal{O}_{X}$-modules $\mathcal{O}_{X}(m D) \subset \mathcal{O}_{X}(\star D), m \in \mathbb{Z}$, formed by the meromorphic functions $h$ such that $\operatorname{div}(h)+m D \geq 0$.

If $f=0$ is a reduced local equation of $D$ at $p \in D$ and $\delta_{1}, \ldots, \delta_{n}$ is a local basis of $\operatorname{Der}(\log D)_{p}$ with $\delta_{i}(f)=\alpha_{i} f$, then $f^{-m}$ is a local basis of $\mathcal{O}_{X, p}(m D)$ over $\mathcal{O}_{X, p}$ and we have the following local presentation over $\mathcal{D}_{X, p}(\log D)([2]$, th. 2.1.4)

$$
\begin{equation*}
\mathcal{O}_{X, p}(m D) \simeq \mathcal{D}_{X, p}(\log D) / \mathcal{D}_{X, p}(\log D)\left(\delta_{1}+m \alpha_{1}, \ldots, \delta_{n}+m \alpha_{n}\right) \tag{1}
\end{equation*}
$$

(1.1) For any ILC $\mathcal{E}$ and any integer $m$, the locally free $\mathcal{O}_{X}$-modules $\mathcal{E}(m D):=$ $\mathcal{E} \otimes_{\mathcal{O}_{X}} \mathcal{O}_{X}(m D)$ and $\mathcal{E}^{*}:=\operatorname{Hom}_{\mathcal{O}_{X}}\left(\mathcal{E}, \mathcal{O}_{X}\right)$ are endowed with a natural structure of left $\mathcal{D}_{X}(\log D)$-module, where the action of logarithmic vector fields is given by

$$
\begin{equation*}
(\delta h)(e)=-h(\delta e)+\delta(h(e)), \quad \delta(e \otimes a)=(\delta e) \otimes a+e \otimes \delta(a) \tag{2}
\end{equation*}
$$

for any logarithmic vector field $\delta$, any local section $h$ of $\operatorname{Hom}_{\mathcal{O}_{X}}\left(\mathcal{E}, \mathcal{O}_{X}\right)$, any local section $e$ of $\mathcal{E}$ and any local section $a$ of $\mathcal{O}_{X}(m D)$ (cf. [5], §2). Then $\mathcal{E}(m D)$ and $\mathcal{E}^{*}$ are ILC again, and the usual isomorphisms

$$
\mathcal{E}(m D)\left(m^{\prime} D\right) \simeq \mathcal{E}\left(\left(m+m^{\prime}\right) D\right), \quad \mathcal{E}(m D)^{*} \simeq \mathcal{E}^{*}(-m D)
$$

are $\mathcal{D}_{X}(\log D)$-linear.
(1.2) If $D$ is Koszul free and $\mathcal{E}$ is an ILC, then the complex $\mathcal{D}_{X} \stackrel{L}{\otimes}_{\mathcal{D}_{X}(\log D)} \mathcal{E}$ is concentrated in degree 0 and its 0 -cohomology $\mathcal{D}_{X} \otimes_{\mathcal{D}_{X}(\log D)} \mathcal{E}$ is a holonomic $\mathcal{D}_{X}$-module (see [5], prop. 1.2.3).

If $\mathcal{E}$ is an ILC, then $\mathcal{E}(\star D)$ is a meromorphic connection (locally free of finite rank over $\mathcal{O}_{X}(\star D)$ ) and then it is a holonomic $\mathcal{D}_{X}$-module (cf. [20], th. 4.1.3). Actually, $\mathcal{E}(\star D)$ has regular singularities on the smooth part of $D$ (it has logarithmic poles! [10]) and then it is regular everywhere 19], cor. 4.3-14, which means that if $\mathcal{L}$ is the local system of horizontal sections of $\mathcal{E}$ on $U=X-D$, the canonical morphism

$$
\Omega_{X}^{\bullet}(\mathcal{E}(\star D)) \rightarrow R j_{*} \mathcal{L}
$$

is an isomorphism in the derived category.
For any $\operatorname{ILC} \mathcal{E}$, or even for any left $\mathcal{D}_{X}(\log D)$-module (without any finiteness property over $\mathcal{O}_{X}$ ), one can define its logarithmic de Rham complex $\Omega_{X}^{\bullet}(\log D)(\mathcal{E})$ in the classical way (cf. [10, def. I.2.15]), which is a subcomplex of $\Omega_{X}^{\bullet}(\mathcal{E}(\star D))$. It is clear that both complexes coincide on $U$.

For any ILC $\mathcal{E}$ and any integer $m, \mathcal{E}(m D)$ is a sub- $\mathcal{D}_{X}(\log D)$-module of the regular holonomic $\mathcal{D}_{X}$-module $\mathcal{E}(\star D)$, and then we have a canonical morphism in the derived category of left $\mathcal{D}_{X}$-modules

$$
\rho_{\mathcal{E}, m}: \mathcal{D}_{X} \stackrel{L}{\otimes}_{\mathcal{D}_{X}(\log D)} \mathcal{E}(m D) \rightarrow \mathcal{E}(\star D)
$$

given by $\rho_{\varepsilon, m}\left(P \otimes e^{\prime}\right)=P e^{\prime}$.
Since $\mathcal{E}\left(m^{\prime} D\right)(m D)=\mathcal{E}\left(\left(m+m^{\prime}\right) D\right)$ and $\mathcal{E}\left(m^{\prime} D\right)(\star D)=\mathcal{E}(\star D)$, we can identify morphisms $\rho_{\mathcal{E}\left(m^{\prime} D\right), m}$ and $\rho_{\mathcal{E}, m+m^{\prime}}$.

For any bounded complex $\mathcal{K}$ of sheaves of $\mathbb{C}$-vector spaces on $X$, let us denote by $\mathcal{K}^{\vee}=R \operatorname{Hom}_{\mathbb{C}_{X}}\left(\mathcal{K}, \mathbb{C}_{X}\right)$ its Verdier dual.

The dual local system $\mathcal{L}^{\vee}$ appears as the local system of the horizontal sections of the dual ILC $\mathcal{E}^{*}$.

We have the following theorem (see [5] th. 4.1] and [6, th. (2.1.1)]):
(1.3) Theorem. Let $\mathcal{E}$ be an ILC (with respect to the divisor $D$ ) and let $\mathcal{L}$ be the local system of its horizontal sections on $U=X-D$. The following properties are equivalent:

1) The canonical morphism $\Omega_{X}^{\bullet}(\log D)(\mathcal{E}) \rightarrow R j_{*} \mathcal{L}$ is an isomorphism in the derived category of complexes of sheaves of complex vector spaces.
2) The inclusion $\Omega_{X}^{\bullet}(\log D)(\mathcal{E}) \hookrightarrow \Omega_{X}^{\bullet}(\mathcal{E}(\star D))$ is a quasi-isomorphism.
3) The morphism $\rho_{\mathcal{E}, 1}: \mathcal{D}_{X} \stackrel{L}{\otimes}_{\mathcal{D}_{X}(\log D)} \mathcal{E}(D) \rightarrow \mathcal{E}(\star D)$ is an isomorphism in the derived category of left $\mathcal{D}_{X}$-modules.
4) The complex $\mathcal{D}_{X} \stackrel{L}{\otimes}_{\mathcal{D}_{X}(\log D)} \mathcal{E}(D)$ is concentrated in degree 0 and the $\mathcal{D}_{X^{-}}$ module $\mathcal{D}_{X} \otimes_{\mathcal{D}_{X}(\log D)} \mathcal{E}(D)$ is holonomic and isomorphic to its localization along $D$.

Moreover, if $D$ is a Koszul free divisor, the preceding properties are also equivalent to:
5) The canonical morphism $j!\mathcal{L}^{\vee} \rightarrow \Omega_{X}^{\bullet}(\log D)\left(\mathcal{E}^{*}(-D)\right)$ is an isomorphism in the derived category of complexes of sheaves of complex vector spaces.

For $D$ a locally quasi-homogeneous free divisor and $\mathcal{E}=\mathcal{O}_{X}$, the equivalent properties in theorem (1.3) hold: this is the so called "logarithmic comparison theorem" [7] (see also [5 th. 4.4] and [6] cor. (2.1.3)] for other proofs based on D-module theory).
(1.4) Let $\mathcal{E}$ be an ILC (with respect to $D$ ) and $p$ a point in $D$. Let $f \in$ $\mathcal{O}=\mathcal{O}_{X, p}$ be a reduced local equation of $D$ and let us write $\mathcal{D}=\mathcal{D}_{X, p}, \mathcal{V}_{0}=$ $\mathcal{D}_{X}(\log D)_{p}$ and $E=\mathcal{E}_{p}$. We know from [6] lemma (3.2.1)] that the ideal of polynomials $b(s) \in \mathbb{C}[s]$ such that

$$
b(s) E f^{s} \subset \mathcal{D}[s] \cdot\left(E f^{s+1}\right)\left(\subset E\left[f^{-1}, s\right] f^{s}\right)
$$

is generated by a non constant polynomial $b_{\varepsilon, p}(s)$. By the coherence of the involved objects we deduce that $b_{\mathcal{\varepsilon}, q}(s) \mid b_{\mathcal{\varepsilon}, p}(s)$ for $q \in D$ close to $p$.

If $b_{\mathcal{\varepsilon}, p}(s)$ has some integer root, let us call $\kappa(\mathcal{E}, p)$ the minimum of those roots. If not, let us write $\kappa(\mathcal{E}, p)=+\infty$.

Let us call

$$
\kappa(\mathcal{E})=\inf \{\kappa(\mathcal{E}, p) \mid p \in D\} \in \mathbb{Z} \cup\{ \pm \infty\}
$$

From now on let us suppose that $D$ is a locally quasi-homogeneous free divisor.
(1.5) ThEOREM. Under the above hypothesis, if $\kappa(\mathcal{E})>-\infty$, then the morphism

$$
\begin{equation*}
\rho_{\mathcal{E}, k}: \mathcal{D}_{X} \stackrel{L}{\otimes}_{\mathcal{D}_{X}(\log D)} \mathcal{E}(k D) \rightarrow \mathcal{E}(\star D) \tag{3}
\end{equation*}
$$

is an isomorphism in the derived category of left $\mathcal{D}_{X}$-modules, for all $k \geq-\kappa(\mathcal{E})$.

Proof. It is a straightforward consequence of [3, 4] th. 5.6] and theorem (3.2.6) of [6] and its proof.
Q.E.D.

Let us note that the hypothesis $\kappa(\mathcal{E})>-\infty$ in theorem (1.5) holds locally on $X$.

In the situation of theorem (1.5) if $\mathcal{L}$ is the local system of the horizontal sections of $\mathcal{E}$ on $U=X-D$, then the derived direct image $R j_{*} \mathcal{L}$ is canonically isomorphic (in the derived category) to the de Rham complex of the holonomic $\mathcal{D}_{X}$-module $\mathcal{D}_{X} \otimes_{\mathcal{D}_{X}(\log D)} \mathcal{E}(k D)$ :

$$
\begin{gathered}
\operatorname{DR}\left(\mathcal{D}_{X} \otimes_{\mathcal{D}_{X}(\log D)} \mathcal{E}(k D)\right)=\mathrm{DR}\left(\mathcal{D}_{X} \stackrel{L}{\otimes_{\mathcal{D}_{X}(\log D)}} \mathcal{E}(k D)\right) \simeq \\
\operatorname{DR} \mathcal{E}(\star D) \simeq \Omega_{X}^{\bullet}(\mathcal{E}(\star D)) \simeq R j_{*} \mathcal{L} .
\end{gathered}
$$

Proceeding as above for the dual ILC $\mathcal{E}^{*}$, we find that if $\kappa\left(\mathcal{E}^{*}\right)>-\infty$, then we have that the canonical morphism

$$
\operatorname{DR}\left(\mathcal{D}_{X} \otimes_{\mathcal{D}_{X}(\log D)} \mathcal{E}^{*}\left(k^{\prime} D\right)\right) \rightarrow R j_{*} \mathcal{L}^{\vee}
$$

is an isomorphism in the derived category for $k^{\prime} \geq-\kappa\left(\mathcal{E}^{*}\right)$.

Let us denote by

$$
\begin{equation*}
\varrho_{\mathcal{E}, k, k^{\prime}}: \mathcal{D}_{X} \otimes_{\mathcal{D}_{X}(\log D)} \mathcal{E}\left(\left(1-k^{\prime}\right) D\right) \rightarrow \mathcal{D}_{X} \otimes_{\mathcal{D}_{X}(\log D)} \mathcal{E}(k D) \tag{4}
\end{equation*}
$$

the $\mathcal{D}_{X}$-linear morphism induced by the inclusion $\mathcal{E}\left(\left(1-k^{\prime}\right) D\right) \subset \mathcal{E}(k D), 1-k^{\prime} \leq$ $k$, and by $\operatorname{IC}_{X}(\mathcal{L})$ the intersection complex of Deligne-Goresky-MacPherson associated with $\mathcal{L}$, which is described as the intermediate direct image $j_{!*} \mathcal{L}$, i.e. the image of $j!\mathcal{L} \rightarrow R j_{*} \mathcal{L}$ in the category of perverse sheaves (cf. [1], def. 1.4.22).

The following theorem describes the "intersection $\mathcal{D}_{X}$-module" corresponding to $\mathrm{IC}_{X}(\mathcal{L})$ by the Riemann-Hilbert correspondence of Mebkhout-Kashiwara [13, 16, 17.
(1.6) THEOREM. Under the above hypothesis, we have a canonical isomorphism in the category of perverse sheaves on $X$,

$$
\mathrm{IC}_{X}(\mathcal{L}) \simeq \operatorname{DR}\left(\operatorname{Im} \varrho_{\varepsilon, k, k^{\prime}}\right)
$$

for $k \geq-\kappa(\mathcal{E}), k^{\prime} \geq-\kappa\left(\mathcal{E}^{*}\right)$ and $1-k^{\prime} \leq k$.
Proof. Using our duality results in [5, §3], the Local Duality Theorem for holonomic $\mathcal{D}_{X}$-modules ([18], ch. I, th. (4.3.1); see also [22]) and theorem (1.5) we obtain

$$
\begin{gathered}
\operatorname{DR}\left(\mathcal{D}_{X} \otimes_{\mathcal{D}_{X}(\log D)} \mathcal{E}\left(\left(1-k^{\prime}\right) D\right)\right) \simeq \operatorname{DR}\left(\mathcal{D}_{X} \otimes_{\mathcal{D}_{X}(\log D)} \mathcal{E}^{*}\left(k^{\prime} D\right)^{*}(D)\right) \simeq \\
\operatorname{DR}\left(\mathbb{D}_{\mathcal{D}_{X}}\left(\mathcal{D}_{X} \otimes_{\mathcal{D}_{X}(\log D)} \mathcal{E}^{*}\left(k^{\prime} D\right)\right)\right) \simeq\left[\operatorname{DR}\left(\mathcal{D}_{X} \otimes_{\mathcal{D}_{X}(\log D)} \mathcal{E}^{*}\left(k^{\prime} D\right)\right)\right]^{\vee} \simeq \\
{\left[R j_{*} \mathcal{L}^{\vee}\right]^{\vee} \simeq j!\mathcal{L} .}
\end{gathered}
$$

On the other hand, the canonical morphism $j!\mathcal{L} \rightarrow R j_{*} \mathcal{L}$ corresponds, through the de Rham functor, to the $\mathcal{D}_{X}$-linear morphism $\varrho_{\varepsilon, k, k^{\prime}}$, and the theorem is a consequence of the Riemann-Hilbert correspondence which says that the de Rham functor establishes an equivalence of abelian categories between the category of regular holonomic $\mathcal{D}_{X}$-modules and the category of perverse sheaves on $X$.
Q.E.D.
(1.7) Remark. For $\mathcal{E}=\mathcal{O}_{X}$, one has $\mathcal{E}^{*}=\mathcal{O}_{X}$ and there are examples where morphisms $\rho_{\mathcal{O}_{X}, k}$ in (3) are never isomorphisms ([5], ex. 5.3). Nevertheless, for $k=k^{\prime}=1$ the image of the morphism

$$
\varrho_{\mathcal{O}_{X}, 1,1}: \mathcal{D}_{X} \otimes_{\mathcal{D}_{X}(\log D)} \mathcal{O}_{X} \rightarrow \mathcal{D}_{X} \otimes_{\mathcal{D}_{X}(\log D)} \mathcal{O}_{X}(D)
$$

is always (canonically isomorphic to) $\mathcal{O}_{X}$, which is the regular holonomic $\mathcal{D}_{X^{-}}$ module corresponding by the Riemann-Hilbert correspondence to $\mathrm{IC}_{X}\left(\mathbb{C}_{U}\right)=$ $\mathbb{C}_{X}$, where $\mathbb{C}_{U}$ is the local system of horizontal sections of $\mathcal{O}_{X}$ on $U$. To see this, let us work locally as in (11). Then, morphism $\varrho_{\mathcal{O}_{x}, 1,1}$ is given at point $p$ by

$$
\bar{P} \in \mathcal{D}_{X, p} / \mathcal{D}_{X, p}\left(\delta_{1}, \ldots, \delta_{n}\right) \mapsto \overline{P f} \in \mathcal{D}_{X, p} / \mathcal{D}_{X, p}\left(\delta_{1}+\alpha_{1}, \ldots, \delta_{n}+\alpha_{n}\right)
$$

and the stalk at $p$ of $\operatorname{Im} \varrho_{\mathcal{O}_{X}, 1,1}$ is given by $\mathcal{D}_{X, p} / J$ where $J$ is the left ideal

$$
J=\left\{P \in \mathcal{D}_{X, p} \mid P f \in \mathcal{D}_{X, p}\left(\delta_{1}+\alpha_{1}, \ldots, \delta_{n}+\alpha_{n}\right)\right\}
$$

By Saito's criterion [28] we can suppose

$$
\left(\begin{array}{c}
\delta_{1} \\
\vdots \\
\delta_{n}
\end{array}\right)=A\left(\begin{array}{c}
\frac{\partial}{\partial x_{1}} \\
\vdots \\
\frac{\partial}{\partial x_{n}}
\end{array}\right)
$$

where $A$ is a $n \times n$ matrix with entries in $\mathcal{O}_{X, p}$ and $\operatorname{det} A=f$. Writing $B=$ $\operatorname{adj}(A)^{t}$ we obtain

$$
B\left(\begin{array}{c}
\delta_{1} \\
\vdots \\
\delta_{n}
\end{array}\right)=f\left(\begin{array}{c}
\frac{\partial}{\partial x_{1}} \\
\vdots \\
\frac{\partial}{\partial x_{n}}
\end{array}\right) \quad \underset{\sim}{\text { eval. on } f} \quad B\left(\begin{array}{c}
\alpha_{1} \\
\vdots \\
\alpha_{n}
\end{array}\right)=\left(\begin{array}{c}
\frac{\partial f}{\partial x_{1}} \\
\vdots \\
\frac{\partial f}{\partial x_{n}}
\end{array}\right) .
$$

Then

$$
\left(\begin{array}{c}
\frac{\partial}{\partial x_{1}} \\
\vdots \\
\frac{\partial}{\partial x_{n}}
\end{array}\right) f=f\left(\begin{array}{c}
\frac{\partial}{\partial x_{1}} \\
\vdots \\
\frac{\partial}{\partial x_{n}}
\end{array}\right)+\left(\begin{array}{c}
\frac{\partial f}{\partial x_{1}} \\
\vdots \\
\frac{\partial f}{\partial x_{n}}
\end{array}\right)=\cdots=B\left(\begin{array}{c}
\delta_{1}+\alpha_{1} \\
\vdots \\
\delta_{n}+\alpha_{n}
\end{array}\right)
$$

and $\frac{\partial}{\partial x_{i}} \in J$ for $i=1, \ldots, n$. Since $J$ is is not the total ideal, we deduce by maximality that $J$ is the ideal generated by the $\frac{\partial}{\partial x_{i}}$ and $\mathcal{D}_{X, p} / J \simeq \mathcal{O}_{X, p}$. To conclude, one easily sees, from the fact that morphism $\varrho_{\mathcal{O}_{X}, 1,1}$ factors through

$$
a \in \mathcal{O}_{X} \mapsto 1 \otimes a \in \mathcal{D}_{X} \otimes_{\mathcal{D}_{X}(\log D)} \mathcal{O}_{X}(D)
$$

[it is $\mathcal{D}_{X}$-linear since, for any derivation $\delta$ and any holomorphic function $a, \delta(1 \otimes$ $\left.a)=\delta \otimes a=\delta \otimes\left(f f^{-1} a\right)=(\delta f) \otimes\left(f^{-1} a\right)=1 \otimes(\delta f)\left(f^{-1} a\right)=1 \otimes(\delta a)\right]$ that the isomorphisms above at different $p$ glue together and give a global isomorphism $\operatorname{Im} \varrho_{\mathcal{O}_{X, 1,1}} \simeq \mathcal{O}_{X}$.

This example suggests studying the comparison between $\operatorname{DR}\left(\operatorname{Im} \varrho_{\varepsilon, k, k^{\prime}}\right)$, $k, k^{\prime} \gg 0$, and $\operatorname{IC}_{X}(\mathcal{L})$ in theorem (1.6) independent of the fact that $\rho_{\varepsilon, k}$ and $\rho_{\mathcal{E}^{*}, k^{\prime}}$ are isomorphisms or not.

## 2 Bernstein-Sato polynomials for cyclic integrable logarithmic connections

In the situation of (1.4) let us assume that $E$ is a cyclic $\mathcal{V}_{0}$-module generated by an element $e \in E$. The following result is proved in [6] prop. (3.2.3)].
(2.1) Proposition. Under the above conditions, the polynomial $b_{\mathcal{E}, p}(s)$ coincides with the Bernstein-Sato polynomial $b_{e}(s)$ of e with respect to $f$, where $e$ is considered to be an element of the holonomic $\mathcal{D}$-module $E\left[f^{-1}\right]$ (cf. [12]).
(2.2) Let $\Theta_{f, s} \subset \mathcal{D}[s]$ be the set of operators in $\operatorname{ann}_{\mathcal{D}[s]} f^{s}$ of total order (in $s$ and in the derivatives $) \leq 1$. The elements of $\Theta_{f, s}$ are of the form $\delta-\alpha s$ with $\delta \in \operatorname{Der}_{\mathbb{C}}(\mathcal{O}), \alpha \in \mathcal{O}$ and $\delta(f)=\alpha f$. In particular $\Theta_{f, s} \subset \mathcal{V}_{0}[s]$.

The $\mathcal{O}$-linear map

$$
\delta \in \operatorname{Der}(\log D)_{p} \mapsto \delta-\frac{\delta(f)}{f} s \in \Theta_{f, s}
$$

is an isomorphism of Lie-Rinehart algebras over $(\mathbb{C}, \mathcal{O})$ and extends to a unique ring isomorphism $\Phi: \mathcal{V}_{0}[s] \rightarrow \mathcal{V}_{0}[s]$ with $\Phi(s)=s$ and $\Phi(a)=a$ for all $a \in \mathcal{O}$. Let us note that $\Phi^{-1}(\delta)=\delta+\frac{\delta(f)}{f} s$ for each $\delta \in \operatorname{Der}(\log D)_{p}$.

It is clear that $E[s] f^{s}$ is a sub- $\mathcal{V}_{0}[s]$-module of $E\left[s, f^{-1}\right] f^{s}$ and that for any $P \in \mathcal{V}_{0}[s]$ and any $e^{\prime} \in E[s]$, the following relation holds

$$
\begin{equation*}
\left(P e^{\prime}\right) f^{s}=\Phi(P)\left(e^{\prime} f^{s}\right) \tag{5}
\end{equation*}
$$

(2.3) Proposition. Under the above conditions, the following relation holds

$$
\operatorname{ann}_{\mathcal{V}_{0}[s]}\left(e f^{s}\right)=\mathcal{V}_{0}[s] \cdot \Phi\left(\operatorname{ann}_{\mathcal{V}_{0}} e\right)
$$

Proof. The inclusion $\supset$ comes from (5). For the other inclusion, let $Q \in$ $\operatorname{ann}_{\mathcal{V}_{0}[s]}\left(e f^{s}\right)$ and let us write $\Phi^{-1}(Q)=\sum_{i=1}^{d} P_{i} s^{i}$ with $P_{i} \in \mathcal{V}_{0}$. We have

$$
0=Q\left(e f^{s}\right)=\left(\Phi^{-1}(Q) e\right) f^{s}=\left(\sum_{i=1}^{d}\left(P_{i} e\right) s^{i}\right) f^{s}
$$

and then $P_{i} \in \operatorname{ann}_{\mathcal{V}_{0}} e$. Therefore

$$
Q=\Phi\left(\sum_{i=1}^{d} P_{i} s^{i}\right)=\sum_{i=1}^{d} \Phi\left(P_{i}\right) s^{i} \in \mathcal{V}_{0}[s] \cdot \Phi\left(\operatorname{ann}_{\mathcal{V}_{0}} e\right)
$$

Q.E.D.
(2.4) Proposition. Under the above conditions, if $D$ is a locally quasihomogeneous free divisor, then

$$
\operatorname{ann}_{\mathcal{D}[s]}\left(e f^{s}\right)=\mathcal{D}[s] \cdot \operatorname{ann}_{\mathcal{V}_{0}[s]}\left(e f^{s}\right)
$$

Proof. From (5) we know that $E[s] f^{s}=\mathcal{V}_{0}[s] \cdot\left(e f^{s}\right)$, and from [6] cor. (3.1.2)] we know that the morphism

$$
\rho_{E, s}: P \otimes\left(e^{\prime} f^{s}\right) \in \mathcal{D}[s] \otimes_{\mathcal{V}_{0}[s]} E[s] f^{s} \mapsto P\left(e^{\prime} f^{s}\right) \in \mathcal{D}[s] \cdot\left(E[s] f^{s}\right)=\mathcal{D}[s] \cdot\left(e f^{s}\right)
$$

is an isomorphism of left $\mathcal{D}[s]$-modules. Therefore

$$
\operatorname{ann}_{\mathcal{D}[s]}\left(e f^{s}\right)=\mathcal{D}[s] \cdot \operatorname{ann}_{\mathcal{V}_{0}[s]}\left(e f^{s}\right)
$$

Q.E.D.
(2.5) Corollary. Under the above conditions, if $D$ is a locally quasi-homogeneous free divisor, then

$$
\operatorname{ann}_{\mathcal{D}[s]}\left(e f^{s}\right)=\mathcal{D}[s] \cdot \Phi\left(\operatorname{ann}_{\mathcal{V}_{0}} e\right)
$$

Proof. It follows from propositions (2.3) and (2.4)
Q.E.D.
(2.6) REmaRk. Theorems (1.5) and (1.6) proposition (2.4) and corollary (2.5) remain true if we only assume that our divisor $D$ is Koszul free and of commutative linear type, i.e. its jacobian ideal is of linear type (see [6 §3]).
(2.7) Remark. As we shall see in sections 3 and 4 theorem (1.6) proposition (2.1) and corollary (2.5) provide an effective method of computing the intersection $\mathcal{D}_{X}$-module corresponding to $\operatorname{IC}_{X}(\mathcal{L})$ in terms of the ILC $\mathcal{E}$, at least if $D$ is a locally quasi-homogeneous free divisor, or more generally, if $D$ is Koszul free and of commutative linear type (see remark (2.6).
(2.8) Remark. In the particular case of $\mathcal{E}=\mathcal{O}_{X}$ and $E=\mathcal{O}$, corollary (2.5) says that

$$
\operatorname{ann}_{\mathcal{D}[s]}\left(f^{s}\right)=\mathcal{D}[s] \cdot\left(\delta_{1}-\alpha_{1} s, \ldots, \delta_{n}-\alpha_{n} s\right)
$$

where $\delta_{1}, \ldots, \delta_{n}$ is a local basis of $\operatorname{Der}(\log D)_{p}$ and $\delta_{i}(f)=\alpha_{i} f$ (see corollary 5.8 , (b) in [4]).
(2.9) Example. Let us suppose that $D \subset X$ is a non-necessarily free divisor and let $f=0$ be a reduced local equation of $D$ at a point $p \in D$. Let $\left\{\delta_{1}, \ldots, \delta_{m}\right\}$ a system of generators of $\operatorname{Der}(\log D)_{p}$ and let us write $\delta_{i}(f)=\alpha_{i} f$.

Let us call $\operatorname{ann}_{\mathcal{D}[s]}^{(1)}\left(f^{s}\right)$ the ideal of $\mathcal{D}[s]$ generated by $\Theta_{f, s}$ (see (2.2)p:

$$
\operatorname{ann}_{\mathcal{D}[s]}^{(1)}\left(f^{s}\right)=\mathcal{D}[s] \cdot\left(\delta_{1}-\alpha_{1} s, \ldots, \delta_{m}-\alpha_{m} s\right) \subset \operatorname{ann}_{\mathcal{D}[s]}\left(f^{s}\right)
$$

The Bernstein functional equation for $f$

$$
b(s) f^{s}=P(s) f^{s+1}
$$

means that the operator $b(s)-P(s) f$ belongs to the annihilator of $f^{s}$ over $\mathcal{D}[s]$. Then, an explicit knowledge of the ideal $\operatorname{ann}_{\mathcal{D}[s]}\left(f^{s}\right)$ allows us to find $b(s)$ by computing the ideal

$$
\mathbb{C}[s] \cap\left(\mathcal{D}[s] \cdot f+\operatorname{ann}_{\mathcal{D}[s]}\left(f^{s}\right)\right)
$$

(see [25]). However, the ideal $\operatorname{ann}_{\mathcal{D}[s]}\left(f^{s}\right)$ is in general difficult to compute.
When $D$ is a locally quasi-homogeneous free divisor, or more generally, a divisor of differential linear type ([6], def. (1.4.5)), $\operatorname{ann}_{\mathcal{D}[s]}\left(f^{s}\right)=\operatorname{ann}_{\mathcal{D}[s]}^{(1)}\left(f^{s}\right)$ and the computation of $b(s)$ is in principle easier.

But there are other examples where the Bernstein polynomial $b(s)$ belongs to

$$
\mathbb{C}[s] \cap\left(\mathcal{D}[s] \cdot f+\operatorname{ann}_{\mathcal{D}[s]}^{(1)}\left(f^{s}\right)\right)
$$

even if $\operatorname{ann}_{\mathcal{D}[s]}\left(f^{s}\right) \neq \operatorname{ann}_{\mathcal{D}[s]}^{(1)}\left(f^{s}\right)$. For instance, when $X=\mathbb{C}^{3}$ and $f=$ $x_{1} x_{2}\left(x_{1}+x_{2}\right)\left(x_{1}+x_{2} x_{3}\right)$ (see example 6.2 in [4) or in any of the examples in page 445 of [9]. In all this examples the divisor is free and satisfies the logarithmic comparison theorem.

## 3 Integrable logarithmic connections along quasihomogeneous plane curves

Let $D \subset X=\mathbb{C}^{2}$ be a divisor defined by a reduced polynomial equation $h\left(x_{1}, x_{2}\right)$, which is quasi-homogeneous with respect to the strictly positive integer weights $\omega_{1}, \omega_{2}$ of the variables $x_{1}, x_{2}$. We denote by $\omega(f)$ the weight of a quasi-homogeneous polynomial $f\left(x_{1}, x_{2}\right)$. The divisor $D$ is free, a global basis of $\operatorname{Der}(\log D)$ is $\left\{\delta_{1}, \delta_{2}\right\}$, where

$$
\binom{\delta_{1}}{\delta_{2}}=\left(\begin{array}{cc}
\omega_{1} x_{1} & \omega_{2} x_{2} \\
-h_{x_{2}} & h_{x_{1}}
\end{array}\right)\binom{\frac{\partial}{\partial x_{1}}}{\frac{\partial}{\partial x_{2}}} .
$$

We have:
-) $\delta_{1}(h)=\omega(h) h, \quad \delta_{2}(h)=0$,
-) the determinant of the coefficient matrix is equal to $\omega(h) h$,
-) $\left[\delta_{1}, \delta_{2}\right]=c \delta_{2}$, with $c=\omega(h)-\omega_{1}-\omega_{2}$.
We consider a logarithmic connection $\mathcal{E}=\oplus_{i=1}^{n} \mathcal{O}_{X} e_{i}$ given by actions:

$$
\delta_{1} \cdot\left(\begin{array}{c}
e_{1} \\
\vdots \\
e_{n}
\end{array}\right)=A_{1}\left(\begin{array}{c}
e_{1} \\
\vdots \\
e_{n}
\end{array}\right), \quad \delta_{2} \cdot\left(\begin{array}{c}
e_{1} \\
\vdots \\
e_{n}
\end{array}\right)=A_{2}\left(\begin{array}{c}
e_{1} \\
\vdots \\
e_{n}
\end{array}\right)
$$

For $\mathcal{E}$ to be integrable, the following integrability condition

$$
\begin{equation*}
\delta_{1}\left(A_{2}\right)-\delta_{2}\left(A_{1}\right)+\left[A_{2}, A_{1}\right]=c A_{2} \tag{6}
\end{equation*}
$$

must hold.
(3.1) We shall focus on the case where $A_{1}, A_{2}$ are $n \times n$ matrices satisfying (6) and of the form:

$$
\begin{gathered}
A_{1}=\left(\begin{array}{ccccc}
-a & 0 & 0 & \cdots & 0 \\
-\delta_{2}(a) & -a+c & 0 & \cdots & 0 \\
-\delta_{2}^{2}(a) & -\binom{2}{1} \delta_{2}(a) & -a+2 c & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & 0 \\
-\delta_{2}^{n-2}(a) & -\binom{n-2}{1} \delta_{2}^{n-3}(a) & -\binom{n-2}{2} \delta_{2}^{n-4}(a) & \cdots & -a+(n-2) c \\
-\delta_{2}^{n-1}(a) & -\binom{n-1}{1} \delta_{2}^{n-2}(a) & -\binom{n-1}{2} & \delta_{2}^{n-3}(a) & \cdots \\
0 & -\binom{n-1}{n-2} \delta_{2}(a) & -a+(n-1) c
\end{array}\right) \\
A_{2}=\left(\begin{array}{ccccc}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & 0 \\
0 & 0 & 0 & \cdots & 1 \\
-b_{0} & -b_{1} & -b_{2} & \cdots & -b_{n-1}
\end{array}\right)
\end{gathered}
$$

with $a, b_{0}, \ldots, b_{n-1}$ polynomials. Let us call $\mathcal{E}_{a, \underline{b}}$ the corresponding ILC.
(3.2) Lemma. The $\mathcal{D}_{X}(\log D)$-module $\mathcal{E}_{a, \underline{b}}$ is generated by $e_{1}$ (so it is cyclic) and the $\mathcal{D}_{X}(\log D)$-annihilator of $e_{1}$ is the left ideal $J_{a, \underline{b}}$ generated by $\delta_{1}+a$ and
$\delta_{2}^{n}+b_{n-1} \delta_{2}^{n-1}+\cdots+b_{1} \delta_{2}+b_{0}$. So, the $\mathcal{D}_{X}(\log D)$-module $\mathcal{E}_{a, \underline{b}}$ is isomorphic to $\mathcal{D}_{X}(\log D) / J_{a, \underline{b}}$.
Proof. The first part is clear since $\delta_{2} \cdot e_{i}=e_{i+1}$ for $i=1, \ldots, n-1$. For the second part, the inclusion $J_{a, \underline{b}} \subset \operatorname{ann}_{\mathcal{D}_{X}(\log D)}\left(e_{1}\right)$ is also clear. To prove the opposite inclusion, we use the fact that any germ of logarithmic differential operator $P$ has a unique expression as a sum $P=\sum_{i, j} a_{i, j} \delta_{1}^{i} \delta_{2}^{j}$, where the $a_{i, j}$ are germs of holomorphic functions ([2], th. 2.1.4) and a division argument. Q.E.D.
(3.3) Remark. Theorem 2.1.4 in [2] says that $\mathcal{D}_{X}(\log D)=\mathcal{O}_{X}\left[\delta_{1}, \delta_{2}\right]$ with relations:

$$
\left[\delta_{1}, f\right]=\delta_{1}(f),\left[\delta_{2}, f\right]=\delta_{2}(f),\left[\delta_{1}, \delta_{2}\right]=c \delta_{2}, \quad f \in \mathcal{O}_{X}
$$

In particular, we can define the support and the exponent of any germ of logarithmic differential operator $P$ (or of any polynomial logarithmic differential operator in the Weyl algebra) by using the (unique) expression $P=\sum_{i, j} a_{i, j} \delta_{1}^{i} \delta_{2}^{j}$, and we obtain a division theorem and a notion of Gröbner basis for ideals. Under this scope, the integrability condition (6) reads out as the fact that the generators

$$
g_{1}=\delta_{1}+a, \quad g_{2}=\delta_{2}^{n}+b_{n-1} \delta_{2}^{n-1}+\cdots+b_{0}
$$

of $J_{a, \underline{b}}$ satisfy Buchberger's criterion, i.e. that $\delta_{2}^{n} g_{1}-\delta_{1} g_{2}$ has a vanishing remainder with respect to the division by $g_{1}, g_{2}$, and then they form a Gröbner basis of $J_{a, \underline{b}}$.
(3.4) Corollary. The $\mathcal{D}_{X}$-module $\mathcal{D}_{X} \otimes_{\mathcal{D}_{X}(\log D)} \mathcal{E}_{a, \underline{b}}$ is isomorphic to $\mathcal{D}_{X} / I_{a, \underline{b}}$, where $I_{a, \underline{b}}=\mathcal{D}_{X}\left(\delta_{1}+a, \delta_{2}^{n}+b_{n-1} \delta_{2}^{n-1}+\cdots+b_{0}\right)$.

For any integer $k$, we can consider the logarithmic connections $\mathcal{E}_{a, \underline{b}}(k D)$ and $\mathcal{E}_{a, \underline{b}}^{*}$ (see section (1.1).
(3.5) Lemma. With the above notations, the $I L C \mathcal{E}_{a, \underline{b}}(k D)$ and $\mathcal{E}_{a+\omega(h) k, \underline{b}}$ are isomorphic.
Proof. An $\mathcal{O}_{X}$-basis of $\mathcal{E}_{a, \underline{b}}(k D)$ is $\left\{e_{i}^{k}=e_{i} \otimes h^{-k}\right\}_{i=1}^{n}$ and the action of $\operatorname{Der}(\log D)$ over this basis is given by (see (21)):

$$
\delta_{1} \cdot e_{i}^{k}=\left(\delta_{1} \cdot e_{i}\right) \otimes h^{-k}+e_{i} \otimes\left(-\omega(h) k h^{-k}\right), \quad \delta_{2} \cdot e_{i}^{k}=\left(\delta_{2} \cdot e_{i}\right) \otimes h^{-k}
$$

Then, the isomorphism of $\mathcal{O}_{X}$-modules

$$
\sum_{i=1}^{n} b_{i} e_{i} \in \mathcal{E}_{a+\omega(h) k, \underline{b}} \mapsto \sum_{i=1}^{n} b_{i} e_{i}^{k} \in \mathcal{E}_{a, \underline{b}}(k D)
$$

is clearly $\mathcal{D}_{X}(\log D)$-linear.
Q.E.D.

The proof of the following proposition is clear.
(3.6) Proposition. The morphism
defined in (4), corresponds, through the isomorphisms in corollary (3.4) and lemma (3.5), to the morphism

$$
\varrho_{\mathcal{E}_{a, \underline{b},}, k, k^{\prime}}^{\prime}: \bar{P} \in \mathcal{D}_{X} / I_{a+\omega(h)\left(1-k^{\prime}\right), \underline{b}} \mapsto \overline{P h^{k+k^{\prime}-1}} \in \mathcal{D}_{X} / I_{a+\omega(h) k}
$$

For the dual connection $\mathcal{E}_{a, \underline{b}}^{*}$, in order to simplify, let us concentrate on case $n=2$, where the integrability condition (6) reduces to:

$$
\begin{equation*}
\left(\delta_{1}-c\right)\left(b_{1}\right)=2 \delta_{2}(a), \quad\left(\delta_{1}-2 c\right)\left(b_{0}\right)=\delta_{2}^{2}(a)+b_{1} \delta_{2}(a) \tag{7}
\end{equation*}
$$

(3.7) Lemma. With the above notations, the $\operatorname{ILC} \mathcal{E}_{a, \underline{b}}^{*}$ and $\mathcal{E}_{c-a, \underline{b}^{*}}$, with $\underline{b}=$ $\left(b_{1}, b_{0}\right)$ and $\underline{b}^{*}=\left(-b_{1}, b_{0}-\delta_{2}\left(b_{1}\right)\right)$, are isomorphic.
Proof. The action of $\operatorname{Der}(\log D)$ over the dual basis $\left\{e_{1}^{*}, e_{2}^{*}\right\}$ in $\mathcal{E}_{a, \underline{b}}^{*}$ is given by:

$$
\left(\delta_{i} \cdot e_{j}^{*}\right)\left(e_{k}\right)=\delta_{i}\left(e_{j}^{*}\left(e_{k}\right)\right)-e_{j}^{*}\left(\delta_{i} e_{k}\right)=-e_{j}^{*}\left(\delta_{i} e_{k}\right)
$$

for $i=1,2$ and $j, k=1,2$ (see (2)). Then

$$
\delta_{1}\binom{e_{1}^{*}}{e_{2}^{*}}=-A_{1}^{t}\binom{e_{1}^{*}}{e_{2}^{*}}, \quad \delta_{2}\binom{e_{1}^{*}}{e_{2}^{*}}=-A_{2}^{t}\binom{e_{1}^{*}}{e_{2}^{*}}
$$

Choosing the new basis $\left\{w_{1}=e_{2}^{*}, w_{2}=-e_{1}^{*}+b_{1} e_{2}^{*}\right\}$ of $\mathcal{E}_{a, \underline{b}}^{*}$, we obtain

$$
\begin{gathered}
\delta_{1}\binom{w_{1}}{w_{2}}=\cdots=\left(\begin{array}{cc}
a-c & 0 \\
\delta_{2}(a) & a
\end{array}\right)\binom{w_{1}}{w_{2}}, \\
\delta_{2}\binom{w_{1}}{w_{2}}=\cdots=\left(\begin{array}{cc}
0 & 1 \\
\delta_{2}\left(b_{1}\right)-b_{0} & b_{1}
\end{array}\right)\binom{w_{1}}{w_{2}}
\end{gathered}
$$

and the isomorphism of $\mathcal{O}_{X}$-modules

$$
\sum_{i=1}^{2} b_{i} w_{i} \in \mathcal{E}_{a, \underline{b}}^{*} \mapsto \sum_{i=1}^{2} b_{i} e_{i} \in \mathcal{E}_{c-a, \underline{b}^{*}}
$$

is clearly $\mathcal{D}_{X}(\log D)$-linear.
Q.E.D.

## 4 Some explicit examples

In this section we consider the case where $D \subset X=\mathbb{C}^{2}$ is defined by the reduced equation $h=x_{1}^{2}-x_{2}^{3}$, and then $\omega\left(x_{1}\right)=3, \omega\left(x_{2}\right)=2, \omega(h)=6$ and the basis of $\operatorname{Der}(\log D)$ is $\left\{\delta_{1}, \delta_{2}\right\}$, with

$$
\binom{\delta_{1}}{\delta_{2}}=\left(\begin{array}{ll}
3 x_{1} & 2 x_{2} \\
3 x_{2}^{2} & 2 x_{1}
\end{array}\right)\binom{\frac{\partial}{\partial x_{1}}}{\frac{\partial}{\partial x_{2}}}
$$

-) $\delta_{1}(h)=6 h, \quad \delta_{2}(h)=0$,
-) the determinant of the coefficient matrix is equal to $6 h$,
-) $\left[\delta_{1}, \delta_{2}\right]=\delta_{2}(c=1)$.
(4.1) Since the $\operatorname{ILC} \mathcal{E}_{a, \underline{b}}$ and the ideals $I_{a, \underline{b}}$ in corollary (3.4) are defined globally by differential operators with polynomial coefficients and $D$ has a global polynomial equation, the study of morphism

$$
\rho_{\mathcal{E}_{a, \underline{b}, k}}: \mathcal{D}_{X} \stackrel{L}{\otimes}_{\mathcal{D}_{X}(\log D)} \mathcal{E}_{a, \underline{b}}(k D) \rightarrow \mathcal{E}_{a, \underline{, b}}(\star D)
$$

can be done globally at the level of the Weyl algebra $\mathbb{W}_{2}=\mathbb{C}\left[x_{1}, x_{2}, \frac{\partial}{\partial x_{1}}, \frac{\partial}{\partial x_{2}}\right]$.
The integrability conditions in (7) (for $n=2$ ) become in our case

$$
\begin{equation*}
\left(\delta_{1}-1\right)\left(b_{1}\right)=2 \delta_{2}(a), \quad\left(\delta_{1}-2\right)\left(b_{0}\right)=\delta_{2}^{2}(a)+b_{1} \delta_{2}(a) \tag{8}
\end{equation*}
$$

Once $a$ is fixed, it allows us to determine, uniquely, $b_{1}$ (the operator $\delta_{1}-1$ is injective), and to also determine $b_{0}$ up to a term $e x_{2}, e \in \mathbb{C}$ (the kernel of the operator $\delta_{1}-2$ is generated by $x_{2}$ ). In order to simplify, let us take

$$
a=\lambda+m x_{1}+n x_{2}
$$

where $\underline{\mu}=(\lambda, m, n)$ are complex parameters, and then

$$
b_{1}=2 m x_{2}^{2}+2 n x_{1}
$$

and

$$
b_{0}=e x_{2}+3 n x_{2}^{2}+4 m x_{1} x_{2}+n^{2} x_{1}^{2}+2 m n x_{1} x_{2}^{2}+m^{2} x_{2}^{4}
$$

with $e$ another complex parameter. For convenience (see the rational factorization of $B(s)$ below), let us consider another complex parameter $\nu$ and make $e=\nu-\nu^{2}$.

Let us define the family of ILC of rank two, $\mathcal{F}_{\nu, \underline{\mu}}:=\mathcal{E}_{a, \underline{b}}$ (see (3.1), with $a, b_{0}, b_{1}$ as above. We have $\mathcal{F}_{\nu, \underline{\mu}}=\mathcal{D}_{X}(\log D) \cdot \bar{e}_{1}$ and $\operatorname{ann}_{\mathcal{D}_{X}(\log D)} e_{1}=$ $\mathcal{D}_{X}(\log D)\left(g_{1}, g_{2}\right)$, with $g_{1}=\delta_{1}+a$ and $g_{2}=\delta_{2}^{2}+b_{1} \delta_{2}+b_{0}$ (see lemma (3.2). It is clear that $\mathcal{F}_{\nu, \underline{\mu}}=\mathcal{F}_{1-\nu, \underline{\mu}}$.

The conclusion of corollary (2.5) can be globalized and we obtain

$$
\operatorname{ann}_{\mathcal{D}_{X}[s]}\left(e_{1} h^{s}\right)=\mathcal{D}_{X}[s]\left(\Phi\left(g_{1}\right), \Phi\left(g_{2}\right)\right)=\mathcal{D}_{X}[s]\left(\delta_{1}+a-6 s, g_{2}\right)
$$

and

$$
\operatorname{ann}_{\mathbb{W}_{2}[s]}\left(e_{1} h^{s}\right)=\mathbb{W}_{2}[s]\left(\delta_{1}+a-6 s, g_{2}\right)
$$

Let us consider the Weyl algebra with parameters

$$
\mathbb{W}^{\prime}=\mathbb{C}\left[\lambda, m, n, \nu, x_{1}, x_{2}, \frac{\partial}{\partial x_{1}}, \frac{\partial}{\partial x_{2}}\right][s]
$$

and the left ideal $I$ generated by

$$
h, \quad \delta_{1}+a-6 s, \quad \delta_{2}^{2}+b_{1} \delta_{2}+b_{0}
$$

By a Gröbner basis computation with an elimination order, for example, with the help of [14], we compute the generator $B(s)$ of the ideal $I \cap \mathbb{C}[s]$ and operators $P(s), C(s), D(s) \in \mathbb{W}^{\prime}$ such that

$$
B(s)=P(s) h+C(s)\left(\delta_{1}+a-6 s\right)+D(s)\left(\delta_{2}^{2}+b_{1} \delta_{2}+b_{0}\right)
$$

We find

$$
B(s)=\left(s-\frac{\lambda-5}{6}\right)\left(s-\frac{\lambda-8}{6}\right)\left(s-\frac{\lambda-\nu-6}{6}\right)\left(s-\frac{\lambda+\nu-7}{6}\right) .
$$

For $\lambda, \nu \in \mathbb{C}$, let us call $B_{\lambda, \nu}(s) \in \mathbb{C}[s]$ the polynomial obtained from $B(s)$ in the obvious way. We obtain then for each $\nu, \lambda, m, n \in \mathbb{C}$ the global BernsteinSato functional equation

$$
\begin{equation*}
B_{\lambda, \nu}(s) e_{1} h^{s}=P(s)\left(e_{1} h^{s+1}\right) \tag{9}
\end{equation*}
$$

in $\mathcal{F}_{\nu, \underline{\mu}}\left[h^{-1}, s\right] h^{s}$. Therefore, $b_{\mathcal{F}_{\nu, \underline{\mu}}, p}(s) \mid B_{\lambda, \nu}(s)$ (see prop. (2.1) for any $p \in D^{1}$ and

$$
\kappa\left(\mathcal{F}_{\nu, \underline{\mu}}\right) \geq \tau(\lambda, \nu):=\min \left\{\text { integer roots of } B_{\lambda, \nu}(s)\right\} \in \mathbb{Z} \cup\{+\infty\}
$$

We can apply theorem (1.5) to deduce that morphism

$$
\rho_{\mathcal{F}_{\nu, \underline{\mu}}, k}: \mathcal{D}_{X} \otimes_{\mathcal{D}_{X}(\log D)} \mathcal{F}_{\nu, \underline{\mu}}(k D) \rightarrow \mathcal{F}_{\nu, \underline{\mu}}(\star D)
$$

is an isomorphism for all $k \geq-\tau(\lambda, \nu)$. On the other hand, from lemma (3.7) we know that $\left(\mathcal{F}_{\nu, \lambda, m, n}\right)^{*}=\mathcal{F}_{\nu, 1-\lambda,-m,-n}$ and then morphism

$$
\rho_{\mathcal{F}_{\nu, \underline{\mu}}^{*}, k^{\prime}}: \mathcal{D}_{X} \otimes_{\mathcal{D}_{X}(\log D)} \mathcal{F}_{\nu, \underline{\mu}}^{*}\left(k^{\prime} D\right) \rightarrow \mathcal{F}_{\nu, \underline{\mu}}^{*}(\star D)
$$

is an isomorphism for all $k^{\prime} \geq-\tau(1-\lambda, \nu)$.
The above results can be rephrased in the following way:

1) Morphism

$$
\rho_{\mathcal{F}_{\nu, \mu}, k}: \mathcal{D}_{X} \otimes_{\mathcal{D}_{X}(\log D)} \mathcal{F}_{\nu, \underline{\mu}}(k D) \rightarrow \mathcal{F}_{\nu, \underline{\mu}}(\star D)
$$

is an isomorphism if the four following conditions hold:
$\lambda+6 k \neq-1,-7,-13,-19, \ldots$
$\lambda+6 k \neq 2,-4,-10,-16, \ldots$
$\lambda+6 k-\nu \neq 0,-6,-12,-18, \ldots$
$\lambda+6 k+\nu \neq 1,-5,-11,-17, \ldots$
2) Morphism

$$
\rho_{\mathcal{F}_{\nu, \underline{\mu}}^{*}, k^{\prime}}: \mathcal{D}_{X} \otimes_{\mathcal{D}_{X}(\log D)} \mathcal{F}_{\nu, \underline{\mu}}^{*}\left(k^{\prime} D\right) \rightarrow \mathcal{F}_{\nu, \underline{\mu}}^{*}(\star D)
$$

is an isomorphism if the four following conditions hold:

$$
\begin{aligned}
& 1-\lambda+6 k^{\prime} \neq-1,-7,-13,-19, \ldots \\
& 1-\lambda+6 k^{\prime} \neq 2,-4,-10,-16, \ldots \\
& 1-\lambda+6 k^{\prime}-\nu \neq 0,-6,-12,-18, \ldots \\
& 1-\lambda+6 k^{\prime}+\nu \neq 1,-5,-11,-17, \ldots
\end{aligned}
$$

or equivalently, if the four following conditions hold:

$$
\begin{aligned}
& \lambda-6 k^{\prime} \neq 2,8,14,20, \ldots \\
& \lambda-6 k^{\prime} \neq-1,5,11,17, \ldots \\
& \lambda+\nu-6 k^{\prime} \neq 1,7,13,19, \ldots \\
& \lambda-\nu-6 k^{\prime} \neq 1,-5,-11,-17, \ldots
\end{aligned}
$$

In particular, if the four following conditions:

[^1](i) $\lambda \not \equiv 2(\bmod 6)$ or $\lambda=2$
(ii) $\lambda \not \equiv 5(\bmod 6)$ or $\lambda=-1$
(iii) $\lambda+\nu \not \equiv 1(\bmod 6)$ or $\lambda+\nu=1$
(iv) $\lambda-\nu \not \equiv 0(\bmod 6)$ or $\lambda-\nu=0$
hold, both morphisms
\[

$$
\begin{aligned}
& \rho_{\mathcal{F}_{\nu, \underline{\mu}}, 1}: \mathcal{D}_{X} \otimes_{\mathcal{D}_{X}(\log D)} \mathcal{F}_{\nu, \underline{\mu}}(D) \rightarrow \mathcal{F}_{\nu, \underline{\mu}}(\star D), \\
& \rho_{\mathcal{F}_{\nu, \underline{,}}^{*}, 1}: \mathcal{D}_{X} \otimes_{\mathcal{D}_{X}(\log D)} \mathcal{F}_{\nu, \underline{\mu}}^{*}(D) \rightarrow \mathcal{F}_{\nu, \underline{\mu}}^{*}(\star D)
\end{aligned}
$$
\]

are isomorphisms.
Let us denote by $\mathcal{L}_{\nu, \mu}$ the local system over $X-D$ of the horizontal sections of $\mathcal{F}_{\nu, \underline{\mu}}$. By theorem (1.6) we have

$$
\operatorname{IC}_{X}\left(\mathcal{L}_{\nu, \underline{\mu}}\right) \simeq \operatorname{DR}\left(\operatorname{Im} \varrho_{\mathcal{F}_{\nu, \underline{\mu}}, 1,1}\right)
$$

provided that conditions (i)-(iv) are satisfied.
Proposition (3.6) and (4.1) reduce the computation of $\operatorname{Im} \varrho_{\mathcal{F}_{\nu, \underline{\mu}, 1,1}}$ to the computation of the image of the map

$$
\theta_{\nu, \underline{\mu}}: \bar{L} \in \mathbb{W}_{2} / \mathbb{W}_{2}\left(g_{1}, g_{2}\right) \mapsto \overline{L h} \in \mathbb{W}_{2} / \mathbb{W}_{2}\left(g_{1}+6, g_{2}\right)
$$

but $\operatorname{Im} \theta_{\nu, \underline{\mu}}=\mathbb{W}_{2} / K_{\nu, \underline{\mu}}$ where

$$
K_{\nu, \underline{\mu}}=\left\{R \in \mathbb{W}_{2} \mid R h \in \mathbb{W}_{2}\left(g_{1}+6, g_{2}\right)\right\}
$$

Now, in order to compute generators of $K_{\nu, \underline{\mu}}$, we proceed as follows. Since $\left[g_{1}, g_{2}\right]=2 g_{2}($ for any $\nu, \underline{\mu})$ and the symbols $\sigma\left(\overline{g_{1}}\right)=\sigma\left(\delta_{1}\right), \sigma\left(g_{2}\right)=\sigma\left(\delta_{2}\right)^{2}$ form a regular sequence ( $D$ is Koszul free!), we deduce that

$$
\sigma\left(\mathbb{W}_{2}\left(g_{1}+6, g_{2}\right)\right)=\left(\sigma\left(\delta_{1}\right), \sigma\left(\delta_{2}\right)^{2}\right)
$$

and consequently $\sigma\left(K_{\nu, \underline{\mu}}\right) \subset\left(\sigma\left(\delta_{1}\right), \sigma\left(\delta_{2}\right)^{2}\right): h$. A straightforward (commutative) computation shows that

$$
\left(\sigma\left(\delta_{1}\right), \sigma\left(\delta_{2}\right)^{2}\right): h=\left(\sigma\left(\delta_{1}\right), \sigma\left(Q_{0}\right)\right)
$$

with $Q_{0}=9 x_{2} \frac{\partial^{2}}{\partial x_{1}^{2}}-4 \frac{\partial^{2}}{\partial x_{2}^{2}}$, and

$$
\begin{equation*}
\sigma\left(Q_{0}\right) h=x_{2} \sigma\left(\delta_{1}\right)^{2}-\sigma\left(\delta_{2}\right)^{2}=x_{2} \sigma\left(\delta_{1}\right) \sigma\left(g_{1}+6\right)-\sigma\left(g_{2}\right) \tag{10}
\end{equation*}
$$

Searching to lift the relation (10) to $\mathbb{W}_{2}$, we find

$$
Q h=x_{2}\left(\delta_{1}+m x_{1}+n x_{2}+7-\lambda\right)\left(g_{1}+6\right)-g_{2}+\left(\lambda^{2}-\lambda+\nu-\nu^{2}\right) x_{2}
$$

with $Q=Q_{0}+6 m x_{2} \frac{\partial}{\partial x_{1}}-4 n \frac{\partial}{\partial x_{2}}+m^{2} x_{2}-n^{2}$. In particular, if condition

$$
\begin{equation*}
\lambda^{2}-\lambda+\nu-\nu^{2}=0 \quad(\Leftrightarrow \lambda-\nu=0 \quad \text { or } \quad \lambda+\nu=1) \tag{11}
\end{equation*}
$$

holds, then $Q \in K_{\nu, \underline{\mu}}$.
Actually, by using the equality $\left[Q, g_{1}\right]=4 Q$ and the fact that $\sigma(Q)=\sigma\left(Q_{0}\right)$ and $\sigma\left(g_{1}\right)=\sigma\left(\delta_{1}\right)$ also form a regular sequence in $\mathrm{Gr} \mathbb{W}_{2}$, condition (11) implies that

$$
K_{\nu, \underline{\mu}}=\mathbb{W}_{2}\left(g_{1}, Q\right), \quad \sigma\left(K_{\nu, \underline{\mu}}\right)=\left(\sigma\left(\delta_{1}\right), \sigma\left(Q_{0}\right)\right) .
$$

On the other hand, since $\sigma\left(Q_{0}\right)$ is not contained in the ideal $\left(x_{1}, x_{2}\right)$, we finally deduce the following result:

If parameters $\nu, \mu=(\lambda, m, n)$ satisfy conditons (i)-(iv) and (11), then the conormal of the origin $T_{0}^{*}(X)$ does not appear as an irreducible component of the characteristic variety of $\operatorname{Im} \theta_{\nu, \underline{\mu}}=\mathbb{W}_{2} / K_{\nu, \underline{\mu}}$, and consequently

$$
\operatorname{Ch}\left(\operatorname{IC}_{X}\left(\mathcal{L}_{\nu, \underline{\mu}}\right)\right)=\operatorname{Ch}\left(\mathbb{W}_{2} / K_{\nu, \underline{\mu}}\right)=\left\{\sigma\left(\delta_{1}\right)=\sigma\left(Q_{0}\right)=0\right\}=T_{X}^{*}(X) \cup T_{D}^{*}(X)
$$

The existence of such an example has been suggested by 21], example (3.4), but the question on the values of the parameters $\nu, \underline{\mu}$ for which the local system $\mathcal{L}_{\nu, \underline{\mu}}$ is irreducible will be treated elsewhere.

If condition (11) does not hold, it is not clear that there exists a general expression for a system of generators of $K_{\nu, \underline{\mu}}$ as before.
(4.2) Remark. The relationship between the preceding results and examples and the hypergeometric local systems (cf. [23, 24, 29]) is interesting and possibly deserves further work.

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[^0]:    *The authors are partially supported by MTM2004-07203-C02-01 and FEDER.

[^1]:    ${ }^{1}$ In fact it is possible to show that $b_{\mathcal{F}_{\nu, \underline{\mu}}, 0}(s)=B_{\lambda, \nu}(s)$.

