

On Herbrand's Theorem for Intuitionistic Logic^{*}

Alexander Lyaletski¹ and Boris Konev²

¹ Faculty of Cybernetics, Kiev National Taras Shevchenko University, Ukraine
lav@unicyb.kiev.ua

² Department of Computer Science, University of Liverpool, United Kingdom
B.Konev@csc.liv.ac.uk

Abstract. In this paper we reduce the question of validity of a first-order intuitionistic formula without equality to generating ground instances of this formula and then checking whether the instances are deducible in a propositional intuitionistic tableaux calculus, provided that the propositional proof is compatible with the way how the instances were generated. This result can be seen as a form of the Herbrand theorem, and so it provides grounds for further theoretical investigation of computer-oriented intuitionistic calculi.

1 Introduction

In its classical formulation, Herbrand's theorem [9] relates the question of validity of a first-order formula in Skolem prenex form, $\forall x_1 \dots \forall x_n \phi(x_1, \dots, x_n)$, with the question of validity of one of its *Herbrand extensions*: The formula $\forall x_1 \dots \forall x_n \phi(x_1, \dots, x_n)$ is valid if, and only if, $\bigwedge_i^m \phi(t_{i,1}, \dots, t_{i,n})$ is valid for some $m \geq 1$ and some collection of ground Herbrand terms $t_{i,j}$. Since every classical first-order formula can be reduced preserving satisfiability, through the Skolemisation, to this Skolem prenex form, Herbrand's theorem, essentially, provides a way to reduce the question of validity of first-order formulae to propositional logic. Even though the required Herbrand extension and the terms $t_{i,j}$ cannot be computed recursively (for otherwise first-order logic would be decidable), this result is particularly interesting for the automated reasoning community as it gives birth to a number of highly efficient proof methods such as resolution [21] and the inverse method [14]. Availability of similar results for other logics would also be of significant interest.

Yet, there is no general Herbrand-like theorem for intuitionistic logic, where formulae cannot in general be preprocessed into a prenex normal form, and the construction of a proof is often sensitive to the order in which the connectives and quantifiers are analysed. The biggest obstacle is that the Skolemisation does not preserve intuitionistic satisfiability. Consider, for example, formulae $\neg \forall x P(x) \supset \exists y \neg P(y)$ and $\exists x \neg P(x) \supset \exists y \neg P(y)$. The first of them is not intuitionistically valid while the other one, obviously, is; however, the Skolemised forms of the two coincide. These complications lead to the existence of limited forms of Herbrand's theorem for particular classes of intuitionistic formulae only [15, 16, 3].

While classical Herbrand's theorem is often proved semantically, it can also be obtained as a direct consequence of Gentzen's cut elimination theorem [7]: The question

^{*} Supported by the Nuffield foundation grant NAL/00841/G.

of the *deducibility* of a first-order formulae in Skolem prenex form can be reduced to the *deducibility* of a Herbrand extension, and then the necessary Herbrand terms can be extracted from the cut-free proof. In fact, a similar idea is used in free-variable tableau methods [8], where quantifiers are dealt with separately from dealing with the propositional proof skeleton. Since free-variable tableau techniques are also available for intuitionistic logic [22, 23], one can hope to obtain *deductive* forms of intuitionistic Herbrand’s theorem, in which the question of the deducibility of intuitionistic first-order formulae is reduced to the deducibility of a Herbrand extension.

We base the investigations presented in this paper on our earlier results in [10], where we introduce a tableau-based calculus without explicit rules dealing with quantifiers. Prior to proof search, we replace in a given formula bound variables with free variables and parameters depending on the polarity of the bounding quantifiers. Then, an *admissible substitution* suggests the correct order of quantifier rule applications, and a ground tableau proof of the given formula can be reconstructed, making it unnecessary to backtrack over different orders of quantifier rule applications. The soundness of the resulting calculus is provided by ordering restrictions in the way similar to the one considered in [20, 13, 22, 23] for calculi with quantifier rules.

The method presented in [10] is similar, to some extent, to the connection method for intuitionistic logic pioneered by Wallen [25] and developed further in [18, 19, 11, 24]. The key difference is in the way how we define admissibility of substitutions. In [25], the notion of admissibility is used to model proof search in a particular sequent intuitionistic calculus, and, therefore, this calculus is tightly integrated into the method. One has to search for an admissible substitution even for propositional intuitionistic formulae, and it is not easy to replace the chosen sequent calculus with a different proof system. In our approach, we try to separate dealing with quantifiers, impermutabilities, and propositional intuitionistic reasoning. To do that, we use admissibility to check eigenvariable conditions, we use a propositional proof system to check the deducibility, and, finally, we check that the propositional proof agrees with quantifiers. From the implementation point of view, some research in this direction was done in [17], where a tableau-like search for connections is implemented; however, they still use string unification in admissibility checks. Our admissibility checks are based on much simpler term unification; the price we have to pay is more complex proof search on the ground level.

It is this separation of propositional proof search and the search for admissible substitutions (with further check that the two agree), what allows us to formulate an analog of Herbrand’s theorem. We reduce the question of the deducibility of an intuitionistic first-order formula to the deducibility of an analog of the Herbrand extension in an intuitionistic propositional calculus. We solve the problems with impermutabilities by imposing restrictions on derivations in the propositional calculus.

2 Preliminaries

We use the standard terminology of first-order logic without equality. The first order language is constructed over a *signature* Sig containing a finite (possibly empty) set of functional symbols, and a finite (nonempty) set of predicate symbols, the logical

connectives: the universal quantifier symbol \forall , the existential quantifier symbol \exists , and the propositional connectives for the implication (\supset), disjunction (\vee), conjunction (\wedge), and negation (\neg).

As for the set of variables Vr , we assume that Vr consists of two disjoint countable sets: mVr (original variables) and iVr (indexed variables) so that $Vr = mVr \cup iVr$, where the following holds: for any $v \in mVr$ and any positive integer (index) k ($k = 1, 2, \dots$), iVr contains the indexed variable ${}^k v$.

Additionally, we extend the signature Sig in the following way: for any natural number (index) k ($k = 1, 2, \dots$) and any symbol s from Sig , we add the indexed symbol ${}^k s$ to Sig denoting the constructed extension by $eSig$. For example, ${}^1\forall$, ${}^3\supset$, and ${}^5\forall$ are symbols of the extended signature. These left upper indices are used to distinguish connectives in different copies of the same formula, stemming from multiplicities, to encode impermutabilities.

The notions of terms, atomic formulae, literals, formulae, free and bound variables, and scopes of quantifiers over both Sig and $eSig$ are defined in the usual way [8] and assumed to be known to the reader. We assume that no two quantifiers in any formula have a common variable, which can be achieved by renaming bound variables.

If the formula F' is constructed by renaming (some or all) bound variables in a formula F , we call F' a *variant* of F .

An *equation* is an unordered pair of terms s and t written as $s \approx t$. Assume L is a literal of the form $R(t_1, \dots, t_n)$ (or $\neg R(t_1, \dots, t_n)$) and M is a literal of the form $R(s_1, \dots, s_n)$ (or $\neg R(s_1, \dots, s_n)$, respectively), where R is a predicate symbol and $t_1, \dots, t_n, s_1, \dots, s_n$ are terms. Then $\Sigma(L, M)$ denotes the set of equations $\{t_1 \approx s_1, \dots, t_n \approx s_n\}$. In this case, L and M are said to be *equal modulo* $\Sigma(L, M)$ ($L \approx M$ modulo $\Sigma(L, M)$).

A *substitution*, σ , is a finite mapping from variables to terms denoted by $\sigma = \{x_1 \mapsto t_1, \dots, x_n \mapsto t_n\}$, where variables x_1, \dots, x_n are pairwise different and $x_i \neq t_i$ for all $i = 1 \dots n$. For an expression Ex and a substitution σ , the result of the application of σ to Ex is denoted by $Ex \cdot \sigma$. For any set Ξ of expressions, $\Xi \cdot \sigma$ denotes the set obtained by the application of σ to every expression in Ξ . If Ξ is a set of (at least two) expressions and $\Xi \cdot \sigma$ is a singleton, then σ is called a *unifier* of Ξ .

Expressions of the form $\mathbf{T}\phi$ or $\mathbf{F}\phi$, where ϕ is a formula, are termed *signed formulae*, and \mathbf{T} and \mathbf{F} are called *signs*. A *sequent* is a non-empty multiset of signed formulae having no common bound variables in pairs. Capital Greek letters Γ, Δ, \dots denote multisets of signed formulae, and we write $\mathbf{T}\Gamma$ (or $\mathbf{F}\Delta$) to express the fact that all formulae in Γ (in Δ) are of the form $\mathbf{T}\phi$ (of the form $\mathbf{F}\psi$, respectively). We denote by $\text{sf}(\Gamma)$ the multiset of all *sign free* formulae obtained from the formulae in Γ by deleting signs. For example, $\text{sf}(\{\mathbf{T}p, \mathbf{F}q\}) = \{p, q\}$.

We say that an occurrence of a subformula ϕ in ψ is

- *positive* if ϕ is ψ ;
- *positive (negative)* if ψ is of the form $(\chi \wedge \xi)$, $(\xi \wedge \chi)$, $(\chi \vee \xi)$, $(\xi \vee \chi)$, $(\chi \supset \xi)$, $\forall x\xi$, or $\exists x\xi$ and ϕ is positive (negative) in ξ ;
- *negative (positive)* if ψ is of the form $(\chi \supset \xi)$ or $\neg\chi$ and ϕ is positive (negative) in χ .

The polarity of an occurrence of a subformula ϕ in a sequent $S = \mathbf{T}\Gamma, \mathbf{F}\Delta$ is determined by the polarity of the corresponding occurrence of ϕ in the formula $(\bigwedge \text{sf}(\mathbf{T}\Gamma)) \supset$

$$\begin{array}{c}
\frac{}{\Gamma, \mathbf{T}A, \mathbf{F}A} (Ax) \\
\frac{\Gamma, \mathbf{T}A \wedge B}{\Gamma, \mathbf{T}A, \mathbf{T}B} (\mathbf{T}\wedge) \quad \frac{\Gamma, \mathbf{F}A \wedge B}{\Gamma, \mathbf{F}A \quad \Gamma, \mathbf{F}B} (\mathbf{F}\wedge) \\
\frac{\Gamma, \mathbf{T}A \vee B}{\Gamma, \mathbf{T}A \quad \Gamma, \mathbf{T}B} (\mathbf{T}\vee) \quad \frac{\Gamma, \mathbf{F}A \vee B}{\Gamma, \mathbf{F}A} (\mathbf{F}\vee_1) \quad \frac{\Gamma, \mathbf{F}A \vee B}{\Gamma, \mathbf{F}B} (\mathbf{F}\vee_2) \\
\frac{\Gamma, \mathbf{T}A \supset B, \mathbf{F}\phi}{\Gamma, \mathbf{T}A \supset B, \mathbf{F}A \quad \Gamma, \mathbf{T}B, \mathbf{F}\phi} (\mathbf{T}\supset) \quad \frac{\Gamma, \mathbf{F}A \supset B}{\Gamma, \mathbf{T}A, \mathbf{F}B} (\mathbf{F}\supset) \\
\frac{\Gamma, \mathbf{T}\neg A, \mathbf{F}\phi}{\Gamma, \mathbf{T}\neg A, \mathbf{F}A} (\mathbf{T}\neg) \quad \frac{\Gamma, \mathbf{F}\neg A}{\Gamma, \mathbf{T}A} (\mathbf{F}\neg) \\
\frac{\Gamma, \mathbf{T}\forall xA(x)}{\Gamma, \mathbf{T}\forall xA(x), \mathbf{T}A(t)} (\mathbf{T}\forall) \quad \frac{\Gamma, \mathbf{F}\forall xA(x)}{\Gamma, \mathbf{F}A(y)} (\mathbf{F}\forall) \\
\frac{\Gamma, \mathbf{T}\exists xA(x)}{\Gamma, \mathbf{T}A(y)} (\mathbf{T}\exists) \quad \frac{\Gamma, \mathbf{F}\exists xA(x)}{\Gamma, \mathbf{F}A(t)} (\mathbf{F}\exists)
\end{array}$$

No sequent contains more than one formula of the form $\mathbf{F}\xi$. In the $\mathbf{T}\supset$ and $\mathbf{T}\neg$ rules, the expression $\mathbf{F}\phi$ might be empty (that is, the sequent contains no formula of the form $\mathbf{F}\psi$). In the rule (Ax) , A is an atomic formula. In the rules $(\mathbf{F}\forall)$ and $(\mathbf{T}\exists)$ the variable y has no free occurrences in the conclusions of the rule.

Fig. 1. Tableau calculus TJ for intuitionistic logic

$(\forall \text{sf}(\mathbf{F}\Delta))$: If a subformula ϕ occurs positively in $\text{sf}(\mathbf{F}\Delta)$ or negatively in $\text{sf}(\mathbf{T}\Gamma)$, we say that the occurrence of ϕ is *positive* in S , otherwise, the occurrence of ϕ is *negative*.

If an occurrence of a subformula $\forall x\psi$ is positive (or an occurrence of $\exists x\psi$ is negative) in a formula ϕ (or in a sequent S), we say that the quantifier $\forall x$ (respectively, $\exists x$) is *strong* in the formula ϕ (in the sequent S); otherwise, the quantifier $\forall x$ (respectively, $\exists x$) is *weak* in the formula ϕ (in the sequent S). If a quantifier Qx , where Q is \forall or \exists , is *strong* (weak) in a sequent S , the variable x is called *strong* (weak) in S .

An indexed variable $^k v$ can be a *free variable* or *parameter* depending on v being weak or strong, respectively in a sequent S . For technical reasons only, if v is a weak (strong) variable, then $^k \underline{v}$ ($^k \bar{v}$) denotes its free variable (parameter) ‘copy’.

A formula ϕ is intuitionistically valid if, and only if, the sequent $\mathbf{F}\phi$ can be derived, for example, in the calculus TJ adapted from [23], with the sole difference that we use the tableau notation whereas [23] uses the sequential one.

3 Calculus TJ^*

The results of this paper are based on our research published in [10]. In order to make this paper self-contained, we repeat the necessary notions and definitions in this section.

Let $\mu(\phi)$ be the quantifier-free result of removing all quantifiers from ϕ . Let for a formula ϕ fix the one-to-one function ω mapping *strong* in ϕ variable $x \in \text{mVr}$ into the parameter $^1 \bar{x} \in \text{iVr}$ and a weak in ϕ variable $x \in \text{mVr}$ into the free variable $^1 \underline{x} \in \text{iVr}$

(mind the left upper indices!). We also assign left upper indices to the occurrences of logical connectives in $\omega(\phi)$ —originally, the left upper index of all logical connectives is 1; in the process of derivation other indices are also assigned, see **Convention** below.

For example, if ϕ is $\forall x(\exists y P(x, y) \supset P(x, x))$, then $\mu(\phi) = P(x, y) \supset P(x, x)$, $\omega(\phi) = {}^1\forall {}^1x({}^1\exists {}^1y P({}^1x, {}^1y) \supset P({}^1x, {}^1x))$, and $\mu(\omega(\phi)) = P({}^1x, {}^1y) \supset P({}^1x, {}^1x)$.

We extend the definition of μ and ω to sequents and arbitrary sets of formulae in the obvious way. (There is no ambiguity in the definition of ω since all the formulae of any sequent have no common variables in pairs.)

In any tableaux-style calculus one has to deal with the necessity to apply quantifier rules. A distinctive feature of our approach is that we remove quantifiers from given formulae; and multiple quantifier rule applications can be modelled by means of the (TCopying) rule defined below.

If ψ is a formula, Q is one of \forall or \exists , and $Qx\phi$ is its subformula, we call $Qx\phi$ a *maximal Q -subformula* of ψ if $Qx\phi$ is not an immediate subformula of another Q -subformula of ψ and we call x a *principal variable* of $Qx\phi$. In addition, all variables bounded by quantifiers within $Qx\phi$ are called *latent* in $Qx\phi$.

For example, both $\phi_1 = \forall x \neg \forall y \forall z P(x, y, z)$ and $\phi_2 = \forall y \forall z P(x, y, z)$ are maximal \forall -subformulae of $\psi = \forall x \neg \forall y \forall z P(x, y, z)$. The variables x , y , and z are all latent in ϕ_1 , but only y and z are latent variables in ϕ_2 . Note that $\forall z P(x, y, z)$ is not a maximal subformula of ψ .

Convention. If ϕ is a maximal Q -subformula containing indexed variables, and j is an index, then ${}^j\phi$ denotes the result of replacing the indices of *all* logical connectives and the indexes of all *latent* variables in ϕ with j . For example, if $\phi = P({}^3x, {}^1y) \wedge Q({}^3x)$, $j = 5$, and both x and y are latent, then ${}^5\phi = P({}^5x, {}^5y) \wedge Q({}^5x)$. If, however, only y is latent in ϕ , then ${}^5\phi = P({}^3x, {}^5y) \wedge Q({}^3x)$.

The notion of a maximal Q -subformula is extended to the case of $\mu\omega$ -images of sequents in the following way: If a formula $Qx\phi$ is a maximal Q -subformula of a (usual) sequent S , then for every $j \in \mathbb{N}$, the formula ${}^j\mu(\omega(Qx\phi))$ is a maximal Q -subformula of $S_{\mu\omega}$. Besides, jx is called a *principal variable* of ${}^j\mu(\omega(Qx\phi))$. Moreover, if a variable y is latent in $Qx\phi$, then for every $j \in \mathbb{N}$, the variable jy is latent in ${}^j\mu(\omega(Qx\phi))$.

Assume we are interested in the validity of a closed formula ϕ . In our calculus TJ^* , proof search begins with the starting sequent $S_{\mu\omega} = \mathbf{F}\mu(\omega(\phi))$. The rules of TJ^* are given in Fig. 2. Note that the quantifier rules became redundant and are absent from the calculus.

(Quasi)-proof. A sequent is said to be *closed* if it contains occurrences of both **TA** and **FA**, where A is an atomic formula. A sequent is *quasi-closed* if it contains occurrences of both **TA** and **FB**, where A and B are atomic formulae and $A \approx B$ modulo $\Sigma(A, B)$. Applying the above-mentioned rules ‘from top to bottom’ to a starting sequent and afterwards to its ‘consequences’, and so on, we construct a so-called *inference tree* for the starting sequent. An inference tree is called a *quasi-proof (proof) tree for a starting sequent* if all its leaves are quasi-closed (closed).

$$\begin{array}{c}
\frac{\Gamma, \mathbf{TA}^k \wedge B}{\Gamma, \mathbf{TA}, \mathbf{TB}} (\mathbf{T}\wedge) \quad \frac{\Gamma, \mathbf{FA}^k \wedge B}{\Gamma, \mathbf{FA} \quad \Gamma, \mathbf{FB}} (\mathbf{F}\wedge) \\
\frac{\Gamma, \mathbf{TA}^k \vee B}{\Gamma, \mathbf{TA} \quad \Gamma, \mathbf{TB}} (\mathbf{T}\vee) \quad \frac{\Gamma, \mathbf{FA}^k \vee B}{\Gamma, \mathbf{FA}} (\mathbf{F}\vee_1) \quad \frac{\Gamma, \mathbf{FA}^k \vee B}{\Gamma, \mathbf{FB}} (\mathbf{F}\vee_2) \\
\frac{\Gamma, \mathbf{TA}^k \supset B, \mathbf{F}\phi}{\Gamma, \mathbf{TA}^k \supset B, \mathbf{FA} \quad \Gamma, \mathbf{TB}, \mathbf{F}\phi} (\mathbf{T}\supset) \quad \frac{\Gamma, \mathbf{FA}^k \supset B}{\Gamma, \mathbf{TA}, \mathbf{FB}} (\mathbf{F}\supset) \\
\frac{\Gamma, \mathbf{T}^{k\neg A}, \mathbf{F}\phi}{\Gamma, \mathbf{T}^{k\neg A}, \mathbf{FA}} (\mathbf{T}\neg) \quad \frac{\Gamma, \mathbf{F}^{k\neg A}}{\Gamma, \mathbf{TA}} (\mathbf{F}\neg) \\
\frac{\Gamma, \mathbf{T}\phi}{\Gamma, \mathbf{T}\phi, \mathbf{T}^l\phi} (\mathbf{TCopying})
\end{array}$$

No sequent contains more than one formula of the form $\mathbf{F}\xi$. In the $(\mathbf{T}\supset)$ and $(\mathbf{T}\neg)$ rules, the expression $\mathbf{F}\phi$ might be empty (that is, the sequent contains no formula of the form $\mathbf{F}\psi$).

In the rule $(\mathbf{TCopying})$:

- ϕ is a maximal \forall -subformula of $S_{\mu\omega}$;
- l is a new index, that is, ${}^l\phi$ does not have common latent variables with other formulae of the sequent.

Fig. 2. Calculus TJ^*

3.1 Main Result for TJ^*

Let ϕ be a formula. By (i, ϕ) we denote the i -th occurrence of a logical connective (which could be a propositional connective or a quantifier) in ϕ when the formula ϕ is read from left to right. If (i, ϕ) is the occurrence of a logical connective \odot , we also refer to this occurrence as $i\odot$. Moreover, in what follows, any occurrence $i\odot$ of a symbol \odot in a formula F is treated as a new symbol. Therefore, $i\odot$ and $j\odot$ are different symbols which denote the same logical “operation” \odot . We also refer to a bound variable x bound by the quantifier iQx as $i x$.

For a formula ϕ , we write $i\odot \prec_\phi j\odot'$ if, and only if, in ϕ , the selected occurrence $j\odot'$ of the logical connective \odot' is in the scope of the selected occurrence $i\odot$ of \odot . For example, if ϕ is $(\neg \psi \wedge (\xi \vee \chi))$, then $2\wedge \prec_\phi 1\neg$ and $2\wedge \prec_\phi 3\vee$. We extend the relation \prec_ϕ to bound variables: $x \prec_\phi y$ if the quantifier Qy is in the scope of the quantifier $Q'x$ (recall that no two quantifiers in any formula have a common variable, which can be achieved by renaming bound variables). For example, for the formula $\phi = \neg \forall x \exists y P(x, y)$, we have: $x \prec_\phi y$.

We also extend the (transitive and irreflexive) relation \prec_ϕ to the case of indexed logical connectives in the following way: for any i and j and for any formula ϕ , we have ${}^k_i\odot \prec_\phi {}^l_j\odot'$ if, and only if, $i\odot \prec_\phi j\odot'$. Similarly for indexed variables, ${}^i_x \prec_\phi {}^j_y$ if, and only if, $x \prec_\phi y$.

Let Tr be an inference tree in the calculus TJ^* then the union of the relations \prec_ϕ , defined for all formulae ϕ from Tr , is a transitive and irreflexive relation denoted by \prec_{Tr} .

Any substitution σ induces a (possibly empty) relation \ll_σ as follows: $y \ll_\sigma x$ if, and only if, there exists $x \mapsto t \in \sigma$ such that x is a free variable, the term t contains y ,

and y is a parameter. For example, consider the substitution $\sigma = \{^1x \mapsto f(^2\bar{y}, ^1\bar{v}, ^1\bar{z})\}$. Then, $^2\bar{y} \ll_{\sigma} ^1x$ and $^1\bar{z} \ll_{\sigma} ^1x$ (note that 1x and $^1\bar{v}$ are not in the relation \ll_{σ}).

A substitution σ is *admissible* for a formula ϕ (for an inference tree Tr) if, and only if, for every $x \mapsto t \in \sigma$, x is a free variable, and the transitive closure $\triangleleft_{\phi, \sigma}$ of $\prec_{\phi} \cup \ll_{\sigma}$ ($\triangleleft_{Tr, \sigma}$ of $\prec_{Tr} \cup \ll_{\sigma}$) is an irreflexive relation.

Let Tr be an inference tree for a starting sequent $S_{\mu\omega}$ in TJ^* . Suppose $^{l_1}_{j_1} \odot_1, \dots, ^{l_r}_{j_r} \odot_r$ is the sequence of propositional connectives occurrences in formulae from $S_{\mu\omega}$, which are eliminated in Tr by applying inference rules, written in the rules applications order leading to the construction of Tr . Then the sequence of such rules applications is called *proper* for Tr . We denote such sequence by $\alpha_{Tr} (^{l_1}_{j_1} \odot_1), \dots, \alpha_{Tr} (^{l_r}_{j_r} \odot_r)$. (It must be clear that there can exist more than one proper sequence for an inference tree Tr .)

Further, an inference tree Tr for $S_{\mu\omega}$ is called *compatible* with the substitution σ if, and only if, there exists a proper sequence $\alpha_{Tr} (^{l_1}_{j_1} \odot_1), \dots, \alpha_{Tr} (^{l_r}_{j_r} \odot_r)$ for Tr such that for any natural numbers m and n , the property $m < n$ implies that the ordered pair $\langle ^{l_m}_{j_m} \odot_m, ^{l_n}_{j_n} \odot_n \rangle$ does not belong to $\triangleleft_{Tr, \sigma}$.

The following result can easily be extracted from [10].

Proposition 1. *A sequent $\mathbf{F}\phi$ is deducible in the calculus TJ if, and only if, a quasi-proof tree Tr for $\mathbf{F}(\mu(\omega(\phi)))$ can be constructed in the calculus TJ^* , and there exists a substitution σ such that (i) $Tr \cdot \sigma$ is a proof tree, (ii) σ is an admissible substitution for Tr , and (iii) the tree Tr is compatible with σ .*

4 Herbrand's Theorem

This section develops the ideas suggested in [12]: Given a first-order intuitionistic formula, we generate ground instances of this formula and then check whether the instances are deducible in a propositional intuitionistic tableaux calculus, provided that the propositional proof is compatible with the way how the instances were generated.

First, we introduce a specialised *convolution calculus*, which allows one to “gather” the required multiple occurrences of subformulae. Then, we introduce the notion of a *Herbrand quasi-universe* and formulate the main result of this paper.

4.1 The Convolution Calculus

We reduce the deducibility of first-order sequents to the deducibility of quantifier-free sequents. Let Tr be an inference tree for a starting sequent S (of the form $\mathbf{F}\phi$) in the calculus TJ^* . To every sequent Sq in Tr , we assign the sequent $\mathfrak{t}(Sq)$, termed the *spur* of Sq , as follows.

- If Sq is a leaf of Tr , having the form $\mathbf{TF}_1, \dots, \mathbf{TF}_n, \mathbf{FG}$, then $\mathfrak{t}(Sq)$ is $\mathbf{TF}_1, \dots, \mathbf{TF}_n, \mathbf{FG}$.
- If Sq is not a leaf node and spurs are assigned to all its successors, we assign $\mathfrak{t}(Sq)$ to Sq in accordance with the rules of the convolution calculus given in Fig. 3: If a rule R of the calculus TJ^* is applied to the sequent Sq in Tr , the spur is assigned to Sq as prescribed by the rule $\uparrow R$ of the convolution calculus applied “bottom up”.

$$\begin{array}{c}
\frac{\Gamma, \mathbf{T}\mathbf{t}(A) \wedge \mathbf{t}(B)}{\Gamma, \mathbf{T}\mathbf{t}(A), \mathbf{T}\mathbf{t}(B)} (\uparrow \mathbf{T}\wedge) \quad \frac{\Gamma, \mathbf{F}\mathbf{t}(A) \wedge \mathbf{t}(B)}{\Gamma, \mathbf{F}\mathbf{t}(A) \quad \Gamma, \mathbf{F}\mathbf{t}(B)} (\uparrow \mathbf{F}\wedge) \\
\frac{\Gamma, \mathbf{T}\mathbf{t}(A) \vee \mathbf{t}(B)}{\Gamma, \mathbf{T}\mathbf{t}(A) \quad \Gamma, \mathbf{T}\mathbf{t}(B)} (\uparrow \mathbf{T}\vee) \quad \frac{\Gamma, \mathbf{F}\mathbf{t}(A) \vee \mathbf{t}(B)}{\Gamma, \mathbf{F}\mathbf{t}(A)} (\uparrow \mathbf{F}\vee_1) \quad \frac{\Gamma, \mathbf{F}\mathbf{t}(A) \vee B}{\Gamma, \mathbf{F}\mathbf{t}(B)} (\uparrow \mathbf{F}\vee_2) \\
\frac{\Gamma, \mathbf{T}(\mathbf{t}(A) \supset \mathbf{t}(B)) \wedge (\mathbf{t}(A) \supset \mathbf{t}(B)), \mathbf{F}\mathbf{t}(\phi)}{\Gamma, \mathbf{T}\mathbf{t}(A) \supset \mathbf{t}(B), \mathbf{F}\mathbf{t}(A) \quad \Gamma, \mathbf{T}\mathbf{t}(B), \mathbf{F}\mathbf{t}(\phi)} (\uparrow \mathbf{T}\supset) \quad \frac{\Gamma, \mathbf{F}\mathbf{t}(A) \supset \mathbf{t}(B)}{\Gamma, \mathbf{T}\mathbf{t}(A), \mathbf{F}\mathbf{t}(B)} (\uparrow \mathbf{F}\supset) \\
\frac{\Gamma, \mathbf{T} \neg \mathbf{t}(A) \wedge \neg \mathbf{t}(A), \mathbf{F}\phi}{\Gamma, \mathbf{T} \neg \mathbf{t}(A), \mathbf{F}\mathbf{t}(A)} (\uparrow \mathbf{T}\neg) \quad \frac{\Gamma, \mathbf{F} \neg \mathbf{t}(A)}{\Gamma, \mathbf{T}\mathbf{t}(A)} (\uparrow \mathbf{F}\neg) \\
\frac{\Gamma, \mathbf{T}\mathbf{t}(\phi) \wedge \mathbf{t}(\phi)}{\Gamma, \mathbf{T}\mathbf{t}(\phi), \mathbf{T}\mathbf{t}(\phi)} (\uparrow \mathbf{TCopying})
\end{array}$$

The rule $(\uparrow \mathbf{T}\neg)$ assigns the spur $\Gamma, \mathbf{T} \neg \mathbf{t}(A) \wedge \neg \mathbf{t}(A), \mathbf{F}\phi$ to the sequent $\Gamma, \mathbf{T}^k \neg A, \mathbf{F}\phi$.

Fig. 3. Convolution Calculus

The result of the assignment of spurs to all sequents in Tr is called the *spurred image* of Tr and is denoted by $\mathbf{t}(Tr)$. The top node of $\mathbf{t}(Tr)$ is denoted by $\mathbf{t}(S)$, where S is a starting sequent. It should be clear that $\mathbf{t}(Tr)$ consists of quantifier-free formulae only. Moreover, any formula in $\mathbf{t}(Tr)$ consists of the symbols of the original signature only.

Next, we are going to reduce the deducibility of sequences to the deducibility of spurs in a tableau propositional calculus. Note that, since all the necessary multiple occurrences of subformulae are introduced to the spur, the propositional calculus is contraction free.

Let pTJ be the calculus obtained from TJ by deleting all its quantifier rules and replacing the rules $(\mathbf{T}\supset)$ and $(\mathbf{T}\neg)$ with

$$\frac{\Gamma, \mathbf{T}A \supset B, \mathbf{F}\phi}{\Gamma, \mathbf{F}A \quad \Gamma, \mathbf{T}B, \mathbf{F}\phi} (p\mathbf{T}\supset) \quad \frac{\Gamma, \mathbf{T}\neg A, \mathbf{F}\phi}{\Gamma, \mathbf{F}A} (p\mathbf{T}\neg),$$

respectively. We extend the definitions of admissible substitutions and compatibility of inference trees and substitutions to the case of pTJ .

The following properties of proof trees can be easily proved by induction on the number of rules applications.

Proposition 2. *Let Tr be an inference tree for a starting sequent S in the calculus TJ^* and σ a substitution. Then the following properties hold w.r.t. $\mathbf{t}(Tr)$, $\mathbf{t}(S)$, and σ :*

- 1) $\mathbf{t}(Tr)$ and Tr contain the same variables;
- 2) $\mathbf{t}(Tr)$ is an inference tree in TJ^* for $\mathbf{t}(S)$ (up to multiple applications of the $\mathbf{F}\wedge$);
- 3) $\mathbf{t}(Tr) \cdot \sigma$ is an inference tree in the calculus pTJ for the initial sequent $\mathbf{t}(S) \cdot \sigma$;
- 4) $\mathbf{t}(Tr) \cdot \sigma$ is a proof tree in pTJ if, and only if, $Tr \cdot \sigma$ is a proof tree in TJ^* ;
- 5) σ is admissible for $\mathbf{t}(Tr)$ if, and only if, σ is admissible for Tr ;
- 6) $\mathbf{t}(Tr)$ is compatible with σ if, and only if, Tr is compatible with σ .

Taking this proposition into account, we can reformulate Proposition 1 as follows.

Proposition 3. *A sequent $\mathbf{F}\phi$ is deducible in the calculus TJ if, and only if, a quasi-proof tree Tr for $\mathbf{F}(\mu(\omega(\phi)))$ can be constructed in the calculus TJ^* , and there exists a substitution σ such that (i) $\imath(Tr) \cdot \sigma$ is deducible in pTJ , (ii) σ is an admissible substitution for $\imath(Tr)$, and (iii) the tree $\imath(Tr)$ is compatible with σ .*

4.2 Intuitionistic Herbrand Theorem

Now we introduce our modification of the notions of the multiplicity [2, 25] and the Herbrand quasi-universe [12].

Let ϕ be a formula and ϕ_1, \dots, ϕ_n its variants. If $\phi, \phi_1, \dots, \phi_n$ does not have any bound and latent variables in pairs, then $\phi_1 \wedge \dots \wedge \phi_n$ ($n > 0$) is called a *variant n -fold \wedge -duplication*. The formula $\phi \wedge \dots \wedge \phi$ is called an *identical n -fold \wedge -duplication* of ϕ .

Herbrand extension. Let ψ be a formula and ξ a \wedge -duplication of ϕ . The result of the replacement of ϕ in ψ with ξ is called a *one-step Herbrand extension of ψ* if one of the following condition is satisfied:

- (i) ϕ is a maximal negative \forall -subformula of ψ and ξ is a variant duplication of ϕ , which has no common bound and latent variables with ψ ;
- (ii) ϕ is a negative \supset -subformula (negative \neg -subformula) of ψ , and ξ is a identical duplication of ϕ .

Finally, the result of a finite sequence of one-step extensions consequently applied to a given formula ψ is called a *Herbrand extension* of ψ .

The notion of a Herbrand quasi-universe, introduced in [12] and modified here for the intuitionistic case, plays the same role in our research as the usual Herbrand universe in the case of classical logic. Unlike the “usual” Herbrand universe, the quasi-universe also contains parameters in the case where strong variables occur in an initial sequent.

Herbrand quasi-universe. Let S be a sequent. Then $HQ(S)$ denotes the following minimal set of terms called a *Herbrand quasi-universe*: (i) every constant and every parameter, occurring in S , belong to $HQ(S)$ (if there is no constant in S then the special constant $c_0 \in HQ(S)$); (ii) if f is a k -ary functional symbol and terms $t_1, \dots, t_k \in HQ(S)$ then $f(t_1, \dots, t_k) \in HQ(S)$.

Theorem 1. *A sequent $\mathbf{F}\phi$ is deducible in the calculus TJ if, and only if, there exist a Herbrand extension $HE(\phi)$ of ϕ and a substitution σ of terms from the Herbrand quasi-universe $HQ(\mathbf{F}\mu(HE(\phi)))$ for all free variables in $\mu(HE(\phi))$ such that (i) σ is an admissible substitution for $\mu(HE(\phi))$ and (ii) there exists a proof tree for $\mathbf{F}\mu(HE(\phi)) \cdot \sigma$ in pTJ , compatible with σ .*

Proof. Note that we can assume that σ substitutes terms from a Herbrand quasi-universe only, which is followed from the subformula property of the calculi given above and from the fact that we can restrict ourselves with the consideration of substitutions being simultaneous most general unifiers of certain sets of terms.

Necessity. Let $\mathbf{F}\phi$ be deducible in the calculus TJ . By Proposition 3, there exist a quasi-proof tree Tr for the sequent $\mathbf{F}\mu(\omega(\phi))$ in TJ^* and a substitution σ such that (i) $\mathfrak{t}(Tr) \cdot \sigma$ is a proof tree in pTJ , (ii) σ is admissible for Tr , and (iii) $\mathfrak{t}(Tr)$ is compatible with σ .

Consider the top sequent $\mathfrak{t}(\mathbf{F}\phi)$ of $\mathfrak{t}(Tr)$. By definitions of the convolution calculus and a Herbrand extension, there exists a sequence of one-step Herbrand extensions such that the sequents $\mathfrak{t}(\mathbf{F}\phi)$ and $\mathbf{F}\mu(HE(\phi))$ coincide. This proves both items (i) and (ii).

Sufficiency. As in the case on the necessity, it follows from Proposition 3 and the properties of the convolution calculus and $HE(\phi)$.

We demonstrate our approach on a series of examples.

Example 1 (The rôle of admissibility).

Let ϕ be the following formula $\forall x \exists y P(x, y) \supset \exists y' \forall x' P(x', y')$. All Herbrand extensions $\xi_k (= HE(\phi))$ of ϕ are of the form:

$$(\forall {}^1x_1 \exists {}^1y_1 P({}^1x_1, {}^1y_1) \wedge \dots \wedge \forall {}^1x_k \exists {}^1y_k P({}^1x_k, {}^1y_k)) \supset \exists {}^1y' \forall {}^1x' P({}^1x', {}^1y').$$

The Herbrand quasi-universe for this case is $QH(HE(\mu(\phi))) = \{c_0, {}^1y_1, \dots, {}^1y_k, {}^1x'\}$. It is easy to see that the substitution σ_i of the form $\{{}^1x_i \mapsto {}^1x', {}^1y' \mapsto {}^1y_i, {}^jx \mapsto c_0 : j \neq i\}$ transforms $\mathbf{F}\mu(\xi_k)$ into the sequent

$$\begin{aligned} \mathbf{F}\mu(\xi_k) \cdot \sigma_i &= \mathbf{F}(P(c_0, {}^1y_1) \wedge \dots \wedge P(c_0, {}^1y_{i-1}) \wedge P({}^1x', {}^1y_i)) \wedge P(c_0, {}^1y_{i+1}) \wedge \dots \\ &\quad \wedge P(c_0, {}^1y_k) \supset P({}^1x', {}^1y_i), \end{aligned}$$

which is deducible in pTJ .

However, σ_i is not admissible for ξ_k since we have ${}^1x_i \prec_{\xi_k} {}^1y_i, {}^1y' \prec_{\xi_k} {}^1x', {}^1x' \ll_{\sigma_i} {}^1x_i$, and ${}^1y_i \ll_{\sigma_i} {}^1y'$, and therefore, ${}^1x_i \triangleleft_{\xi_k, \sigma_i} {}^1x_i$. As a consequence of Theorem 1, $\mathbf{F}\phi$ is not deducible in TJ . Note $\mathbf{F}\phi$ is also not classically deducible as shown in [12].

Next, consider the formula $\phi = \exists y \forall x P(x, y) \supset \forall x' \exists y' P(x', y')$. Then similarly to the case considered above, $HE(\phi) = \exists {}^1y_1 \forall {}^1x_1 P({}^1x_1, {}^1y_1) \supset \forall {}^1x' \exists {}^1y' P({}^1x', {}^1y')$ and the substitution $\sigma = \{{}^1x_1 \mapsto {}^1x', {}^1y' \mapsto {}^1y_1\}$, is admissible for $HE(\phi)$. Moreover, the sequent $\mathbf{F}\mu(HE(\phi)) \cdot \sigma$ is deducible in pTJ and any its proof tree is compatible with σ . Thus, the sequent $\mathbf{F}\phi$ is deducible in TJ , i.e. it is intuitionistically valid.

Example 2 (The rôle of multiplicities). Let ϕ be the formula $\exists y \forall x B(x, y) \supset \exists z (B(a, z) \wedge B(b, z))$ and S be the sequent $\mathbf{F}\phi$, where x, y , and z are variables; a and b are constants; and B is a predicate symbol. We show that ϕ is an intuitionistically valid formula.

If we do not introduce copies of subformulae, there is no substitution σ of a term from $HQ(\mu(S)) = \{a, b, y\}$ for the free variable x such that $HQ(\mu(S)) \cdot \sigma$ is deducible in pTJ . Therefore, Theorem 1 is not applicable.

If, however, we consider a Herbrand extension of ϕ ,

$$HE(\phi) = \exists y (\forall {}^1xB({}^1x, y) \wedge \forall {}^2xB({}^2x, y)) \supset \exists z (B(a, z) \wedge B(b, z)),$$

and substitution $\sigma = \{{}^1x \mapsto a, {}^2x \mapsto b, z \mapsto y\}$, then the sequent $\mathbf{F}HQ(\mu(S)) \cdot \sigma$ is deducible in pTJ . Obviously, σ substitutes terms from $HQ(\mu(HE(\phi)))$ and it is admissible for $\mu(HE(\phi))$. Moreover, any proof tree for $\mathbf{F}\mu(HE(\phi)) \cdot \sigma$ in pTJ is compatible with σ . By Theorem 1, we come to the conclusion that the sequent $\mathbf{F}\phi$ is deducible in TJ and, therefore, ϕ is an intuitionistically valid formula.

Example 3 (The rôle of compatibility). Let us consider the formula $\phi = (\neg\forall xP(x) \supset \exists y\neg P(y))$. There exists no Herbrand extension of ϕ but ϕ itself. Therefore, $QH(HE(\phi)) = \{c_0, x\}$ and the only possible admissible substitution $\sigma = \{y \mapsto x\}$ transforms $\mathbf{F}\mu(\phi)$ into the sequent $S = \mathbf{F}\neg P(x) \supset \neg P(x)$, which is deducible in pTJ .

The only possible proof tree Tr for S in pTJ is as follows.

1. $\mathbf{F}\neg P(x) \supset \neg P(x)$ (starting sequent)
2. $\mathbf{T}\neg P(x), \mathbf{F}\neg P(x)$ (from (1), by $(\mathbf{F} \supset)$ -rule)
3. $\mathbf{T}\neg P(x), \mathbf{TP}(x)$ (from (2), by $(\mathbf{F}\neg)$ -rule)
4. $\mathbf{TP}(x), \mathbf{FP}(x)$ (from (3), by $(\mathbf{T}\neg)$ -rule: Axiom)

Notice that Tr is not compatible with σ : the $(\mathbf{F} \supset)$ -rule application (the 2nd step) precedes to the $(\mathbf{T} \supset)$ -rule application (the 3rd step), although the compatibility condition requires the inverse order of rule applications. Since there is no other way to construct a proof tree for S , we conclude that the sequent $\mathbf{F}\phi$ is not deducible in TJ , as implied by Theorem 1, and the formula ϕ is not intuitionistically valid.

5 Conclusions and Future Work

Herbrand's theorem in intuitionistic context is inherently complex. The ultimate goal is to be able to compute given a first-order formula ϕ a series of its ground Herbrand extensions $\widehat{\phi}_1, \widehat{\phi}_2, \dots$ in such a way that

ϕ is intuitionistically valid if, and only if, for some $n \geq 1$, $\widehat{\phi}_n$ is intuitionistically valid. (1)

To our best knowledge, nobody yet succeeded to do that for arbitrary intuitionistic formulae. We are aware of three approaches to this problem.

Classical Herbrand extension for fragments of intuitionistic logic. For certain fragments of intuitionistic logic, (1) still holds for the classical Skolemisation and Herbrand extensions. The simplest case of intuitionistic formulae in prenex form is considered in [15, 3] while [15, 16] gives a full characterisation of intuitionistic formulae for which the classical Skolemisation and Herbrand extensions do the job. As the example considered in the Introduction shows, the classical Skolemisation does not work for all intuitionistic formulae.

Reduction to a different language. Alternatively, one can reduce validity of first-order formulae to validity of formulae in logics “between” propositional and first-order. This approach was pursued by Fitting through predicate abstractions [5] and by Baaz&Iemhoff through existence predicates and the eSkolemisation [1]. The main disadvantage of such an approach for the automated reasoning community is the necessity to develop specialised provers for these “intermediate” logics.

Constraining proofs. Finally, one can extract an admissible substitution and a Herbrand extension from a sequent, tableau, or connection method proof [20, 13, 22, 23, 25, 18, 19, 11, 24]. Then a given formula is valid if, and only if, there exists a sequent, tableau, or connection method (resp.) *restricted* proof for the Herbrand extension. Note that the resulting Herbrand extension and the proof search process are tightly integrated with each other, and it is not possible, say, to use Herbrand extension from one approach and proof method from the other.

This paper extends further the third approach. Our main contribution is that we separate the generation of ground instances from checking the propositional deducibility and these two processes are only connected through compatibility check. We can formulate the notion of compatibility with a substitution *regardless* of the particular set of tableaux inference rules used. For example, one can introduce *pLJ*, a variant of Gentzen’s intuitionistic calculus *LJ*, which differs from *LJ*, in only that it lacks quantifier rules and the thinning rule. The definition of an inference tree compatible with a substitution can be easily transferred to *LJ*. Then, the following proposition can be proved. (We use tableau form of *LJ*.)

Proposition 4. *A sequent $\mathbf{F}\phi$ is deducible in the calculus *LJ* if, and only if, there exist a Herbrand extension $HE(\phi)$ of ϕ and a substitution σ of terms from the Herbrand quasi-universe $HQ(\mathbf{F}\mu(HE(\phi)))$ for all free variables in $\mu(HE(\phi))$ such that (i) σ is an admissible substitution for $\mu(HE(\phi))$ and (ii) there exists a proof tree for $\mathbf{F}\mu(HE(\phi)) \cdot \sigma$ in *pLJ*, compatible with σ .*

This result suggests the following conjecture.

Conjecture 1. For any propositional tableau intuitionistic calculus *pC* it is possible to formulate the notion of compatibility with a substitution such that any formula ϕ is intuitionistically valid if, and only if, there are an Herbrand extension $HE(\phi)$ of ϕ and a substitution σ of terms from the Herbrand quasi-universe $HQ(\mathbf{F}\mu(HE(\phi)))$ for all free variables in $\mu(HE(\phi))$ such that (i) $\mu(HE(\phi)) \cdot \sigma$ is intuitionistically valid as a propositional formula, (ii) σ is an admissible substitution for $\mu(HE(\phi))$, and (iii) for $\mathbf{F}\mu(HE(\phi)) \cdot \sigma$, there exists a proof tree in *pC* compatible with σ .

We are planning to further explore this conjecture especially for contraction-free [4], multi-succedent [24], and labelled [6] tableau proof systems.

References

1. M. Baaz and R. Iemhoff. On the skolemization of existential quantifiers in intuitionistic logic. *Annals of Pure and Applied Logic*, 2006. to appear.
2. W. Bibel. *Automated Theorem Proving*. Vieweg, Braunschweig, second edition, 1987.
3. K. A. Bowen. An Herbrand theorem for prenex formulas of *LJ*. *Notre Dame Journal of Formal Logic*, 17(2):263–266, 1976.
4. Roy Dyckhoff. Contraction-free sequent calculi for intuitionistic logic. *J. Symb. Log.*, 57(3):795–807, 1992.
5. Melvin Fitting. A modal herbrand theorem. *Fundam. Inform.*, 28(1-2):101–122, 1996.
6. D. Gabbay. *Labelled deductive systems*. Oxford university press, 1996.

7. G. Gentzen. Untersuchungen über das logische Schließen. *Mathematische Zeitschrift*, 39:176–210, 405–433, 1934.
8. R. Hähnle. Tableaux and related methods. In A. Robinson and A. Voronkov, editors, *Handbook of Automated Reasoning*, volume I, chapter 3, pages 101–178. Elsevier, 2001.
9. J. Herbrand. Recherches sur la théorie de la démonstration, (thesis). In W. Goldfarb, editor, *Logical writings*. Cambridge, 1977.
10. B. Konev and A. Lyaletski. Tableau method with free variables for intuitionistic logic. In *Proceedings of the International IIS: IIPWM'06 Conference*, Advances in Soft Computing, 2006. To appear.
11. C. Kreitz and J. Otten. Connection-based theorem proving in classical and non-classical logics. *J. UCS*, 5(3):88–112, 1999.
12. A. Lyaletski. Sequent forms of Herbrand theorem and their applications. *Annals of Mathematics and Artificial Intelligence*. In press.
13. A. V. Lyaletski. Gentzen calculi and admissible substitutions. In *Actes Préliminaires du Symposium Franco-Sovietique "Informatika-91"*, pages 99–111, Grenoble, France, 1991.
14. S. Y. Maslov. An inverse method for establishing deducibility of nonprenex formulas of the predicate calculus. In J. Siekmann and G. Wrightson, editors, *Automation of Reasoning 2: Classical Papers on Computational Logic 1967-1970*, pages 48–54. Springer, Berlin, Heidelberg, 1983.
15. G. Mints. Herbrand theorem. In *Mathematical Theory of Logical Inference*, pages 311–350. Nauka, Moscow, 1967.
16. G. Mints. The Skolem method in intuitionistic calculi. In *Proc. Steklov Inst. Math.*, volume 121, pages 73–109. 1972.
17. J. Otten. ileanTAP: An intuitionistic theorem prover. In *Proc. TABLEAUX'97*, volume 1227 of *LNCS*, pages 307–312, 1997.
18. J. Otten and C. Kreitz. A connection based proof method for intuitionistic logic. In *Proc. TABLEAUX'95*, volume 918 of *LNCS*, pages 122–137, 1995.
19. J. Otten and C. Kreitz. A uniform proof procedure for classical and non-classical logics. In *KI-96*, volume 1137 of *LNCS*, pages 307–319, 1996.
20. S. Reeves. Semantic tableaux as framework for automated theorem-proving. In C. S. Mellish and J. Hallam, editors, *Proc. AISB-87*, pages 125–139, 1987.
21. J. A. Robinson. A machine oriented logic based on the resolution principle. *J. Assoc. Comput. Mach.*, 12:23–41, 1965.
22. N. Shankar. Proof search in the intuitionistic sequent calculus. In *Proc. CADE'92*, volume 607 of *LNCS*, pages 522–536, 1992.
23. A. Voronkov. Proof search in intuitionistic logic based on constraint satisfaction. In *Proc. TABLEAUX'96*, volume 1071 of *LNCS*, pages 312–329, 1996.
24. A. Waaler and L. Wallen. Tableaux for intuitionistic logics. In M. D'Agostino, D. Gabbay, R. Hähnle, and J. Posegga, editors, *Handbook of Tableau Methods*, pages 255–296. Kluwer, Dordrecht, 1999.
25. L. Wallen. *Automated Deduction in Nonclassical Logics*. MIT Press: Cambridge, 1990.