Stability of Cohen-Grossberg Neural Networks with Impulsive and Mixed Time Delays

***Zhenjiang Zhao*†*, Qiankun Song*††**

† Department of Mathematics, Huzhou Teachers College, Huzhou, Zhejiang, 313000 China
††Department of Mathematics, Chongqing Jiaotong University, Chongqing, 400074 China

Abstract
In this paper, the problem of stability analysis for a class of impulsive Cohen-Grossberg neural networks with mixed time delays is considered. The mixed time delays comprise both the time-varying and distributed delays. By employing a combination of the M -matrix theory and analytic methods, several sufficient conditions are obtained to ensure the global exponential stability of equilibrium point for the addressed impulsive Cohen-Grossberg neural network with mixed delays. The proposed method, which does not make use of the Lyapunov functional, is shown to be simple yet effective for analyzing the stability of impulsive neural networks with variable and/or distributed delays. Moreover, the exponential convergence rate is estimated, which depends on the system parameters. The results obtained generalize a few previously known results by removing some restrictions or assumptions. An example with simulation is given to show the effectiveness of the obtained results.

Key words:
Cohen-Grossberg neural network, global exponential stability, time-varying delays, distributed delays, impulsive effect

1. Introduction

The Cohen-Grossberg neural network model, first proposed and studied by Cohen and Grossberg in 1983 [1], has attracted considerable attention due to its potential applications in classification, parallel computing, associative memory, signal and image processing, especially in solving some difficult optimization problems. In such applications, it is of prime importance to ensure that the designed neural networks be stable[2,3]. In practice, due to the finite speeds of the switching and transmission of signals, time delays do exist in a working network and thus should be incorporated into the model equation [2]. In recent years, the dynamical behaviors of Cohen-Grossberg neural networks with constant delays or time-varying delays or distributed delays have been studied, for example, see [2-26] and references therein.

On the other hand, impulsive effect likewise exists in a wide variety of evolutionary processes in which states are changed abruptly at certain moments of time in the fields such as medicine and biology, economics, electronics and telecommunications. Neural networks, which include Hopfield neural networks, cellular neural networks and Cohen-Grossberg neural networks, are often subject to impulsive perturbations that in turn affect dynamical behaviors of the systems. Therefore, it is necessary to consider both the impulsive effect and delay effect when investigating the stability of neural networks [27]. So far, several interesting results have been reported that focusing on the impulsive effect on delayed neural networks, see [27-39] and references therein. To the best of our knowledge, few authors have considered the dynamical behaviors of the impulsive Cohen-Grossberg neural network model with both time-varying and distributed delays.

Motivated by the above discussions, the objective of this paper is to study the global exponential stability of impulsive Cohen-Grossberg neural network with both time-varying and distributed delays, and estimate the exponential convergence rate index. By employing a combination of the M-matrix theory and analytic methods, we obtain several sufficient conditions for ensuring the global exponential stability. Our proposed method does not make use of the Lyapunov functional and is shown to be simple yet effective for analyzing the stability of impulsive neural networks with variable and/or distributed delays. Our main results generalize a few previously known results by removing some restrictions or assumptions. An example with simulation is given to show the effectiveness of the obtained results.

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2. Model description and preliminaries

In this paper, we consider the following model.

$$\begin{cases} \dfrac{du_i(t)}{dt} = -a_i(u_i(t))\Big[b_i(u_i(t)) - \sum_{j=1}^{n} c_{ij} g_j(u_j(t)) \\ \qquad - \sum_{j=1}^{n} d_{ij} f_j(u_j(t-\tau_{ij}(t))) \\ \qquad - \sum_{j=1}^{n} v_{ij} \int_{-\infty}^{t} K_{ij}(t-s) h_j(u_j(s)) ds + I_i \Big], \quad t \neq t_k, \\ u_i(t) = p_{ik}(u_1(t^-), \cdots, u_n(t^-)) + q_{ik}(u_1((t-\tau_{i1}(t))^-), \\ \qquad \cdots, u_n((t-\tau_{in}(t))^-)) + J_{ik}, \quad t = t_k, \end{cases} \tag{1}$$

for $i = 1, 2, \cdots, n$ and $k = 1, 2, \cdots$, where n corresponds to the number of units in the neural network; $u_i(t)$ corresponds to the state of the i th unit at time t. The first part is the continuous part of model (1), which describes the continuous evolution processes of the neural network, where g_j, f_j and h_j denote the activation functions; $\tau_{ij}(t)$ corresponds to the transmission delay along the axon of the j th unit from the i th unit and satisfies $0 \le \tau_{ij}(t) \le \tau_{ij}$ (τ_{ij} is a constant); $a_i(u_i(t))$ represents an amplification function at time t; $b_i(u_i(t))$ is an appropriately behaved function at time t such that the solutions of model (1) remain bounded; $C = (c_{ij})_{n\times n}$, $D = (d_{ij})_{n\times n}$ and $V = (v_{ij})_{n\times n}$ are connection matrices; K_{ij} is the delay kernel function; I_i is the constant input from outside of the network. The second part is the discrete part of model (1), which describes that the evolution processes experience abrupt change of state at the moments of time t_k (called impulsive moments), where $p_{ik}(u_1(t^-), \cdots, u_n(t^-))$ represents impulsive perturbations of the i th unit at time t_k and $u_j(t^-)$ denotes the left limit of $u_j(t)$; $q_{ik}(u_1((t-\tau_{i1}(t))^-), \cdots, u_n((t-\tau_{in}(t))^-))$ represents impulsive perturbations of the i th unit at time t_k which caused by transmission delays; J_{ik} represents external impulsive input at time t_k, the fixed moments of time t_k satisfy $t_1 < t_2 < \cdots$, $\lim_{k\to+\infty} t_k = +\infty$ and $\min_{2\le k\le\infty}\{t_k - t_{k-1}\} > \max_{1\le i,j\le n}\{\tau_{ij}\}$.

Remark 1. When $p_{ik}(u_1(t^-), \cdots, u_n(t^-)) = u_i(t)$, $q_{ik}(u_1((t-\tau_{i1}(t))^-), \cdots, u_n((t-\tau_{in}(t))^-)) = 0$ and $J_{ik} = 0$ ($i = 1, 2, \cdots, n$; $k = 1, 2, \cdots$), model (1) turns to the following Cohen-Grossberg neural network model without impulses

$$\begin{aligned} \frac{du_i(t)}{dt} = -a_i(u_i(t))\Big[& b_i(u_i(t)) - \sum_{j=1}^{n} c_{ij} g_j(u_j(t)) \\ & - \sum_{j=1}^{n} d_{ij} f_j(u_j(t-\tau_{ij}(t))) \\ & - \sum_{j=1}^{n} \int_{-\infty}^{t} K_{ij}(t-s) h_j(u_j(s)) ds + I_i \Big] \end{aligned} \tag{2}$$

for $t > 0$, $i = 1, 2, \cdots, n$. Note that model (2) is a general neural network that covers some popular models such as delayed Hopfield neural networks, delayed cellular neural networks, delayed BAM neural networks.

For convenience, we introduce several notations. $u = (u_1, u_2, \cdots, u_n)^T \in R^n$ denotes a column vector; $|u|$ denotes the absolute-value vector given by $|u| = (|u_1|, |u_2|, \cdots, |u_n|)^T \in R^n$. For matrix $A = (a_{ij})_{n\times n} \in R^{n\times n}$, $|A|$ denotes the absolute-value matrix given by $|A| = (|a_{ij}|)_{n\times n}$; $\rho(A)$ denotes the spectral radius of A; $\|u\|$ denotes a vector norm defined by $\|u\| = \max_{1\le i\le n}|u_i|$. $C[X,Y]$ denotes the space of continuous mappings from the topological space X to the topological space Y. $PC[I, R^n] = \{\varphi : I \to R^n \mid \varphi(t^+) = \varphi(t)$ for $t \in I$, $\varphi(t^-)$ exists for $t \in (t_0, +\infty)$, $\varphi(t^-) = \varphi(t)$ for all but points $t_k \in (t_0, +\infty)\}$, where $I \subset R$ is an interval, $\varphi(t^+)$ and $\varphi(t^-)$ denote the left-hand limit and right-hand limit of the scalar function $\varphi(t)$, respectively.

Throughout this paper, we make the following assumptions:

(**H1**) Model (1) has at least one equilibrium point.

(**H2**) $a_i(u)$ is a continuous function and $0 < \underline{a}_i \le a_i(u)$ ($\underline{a}_i$ is a constant) for all $u \in R$, $i = 1, 2, \cdots, n$.

(**H3**) There exists a positive diagonal matrix $B = diag(b_1, b_2, \cdots, b_n)$ such that $\dfrac{b_i(u) - b_i(v)}{u - v} \ge b_i$ for all u, $v \in R$ ($u \neq v$), $i = 1, 2, \cdots, n$.

(**H4**) There exist three positive diagonal matrices $G = diag(G_1, G_2, \cdots, G_n)$, $F = diag(F_1, F_2, \cdots, F_n)$ and $H = diag(H_1, H_2, \cdots, H_n)$ such that $G_j = \sup_{u_1 \neq u_2} \left| \frac{g_j(u_1) - g_j(u_2)}{u_1 - u_2} \right|$, $F_j = \sup_{u_1 \neq u_2} \left| \frac{f_j(u_1) - f_j(u_2)}{u_1 - u_2} \right|$ and $H_j = \sup_{u_1 \neq u_2} \left| \frac{h_j(u_1) - h_j(u_2)}{u_1 - u_2} \right|$ for all $u_1 \neq u_2$, $j = 1, 2, \cdots, n$.

(**H5**) The delay kernel $K_{ij} : [0, +\infty) \to [0, +\infty)$ is real valued nonnegative continuous function and satisfies $\int_0^{+\infty} e^{\beta s} K_{ij}(s) ds = r_{ij}(\beta)$, where $r_{ij}(\beta)$ is continuous function in $[0, \delta)$, $\delta > 0$, and $r_{ij}(0) = 1$, $i, j = 1, 2, \cdots n$.

(**H6**) There exist nonnegative matrices $P_k = (p_{ij}^{(k)})_{n \times n}$, $Q_k = (q_{ij}^{(k)})_{n \times n}$ such that

$$\left| p_{ik}(u_1, \cdots, u_n) - p_{ik}(v_1, \cdots, v_n) \right| \le \sum_{j=1}^{n} p_{ij}^{(k)} \left| u_j - v_j \right|,$$

$$\left| q_{ik}(u_1, \cdots, u_n) - q_{ik}(v_1, \cdots, v_n) \right| \le \sum_{j=1}^{n} q_{ij}^{(k)} \left| u_j - v_j \right|$$

for all $(u_1, \cdots, u_n)^T \in R^n$, $(v_1, \cdots, v_n)^T \in R^n$, $i = 1, 2, \cdots, n$, $k = 1, 2, \cdots$.

Definition 1: The equilibrium point $u^* = (u_1^*, u_2^*, \cdots, u_n^*)^T$ of model (1) is said to be globally exponentially stable if there exist constants $\varepsilon > 0$ and $M > 0$ such that

$$\left\| u(t) - u^* \right\| \le M \left\| \phi - u^* \right\| e^{-\varepsilon(t - t_0)} \tag{3}$$

for all $t > 0$, where $u(t) = (u_1(t), u_2(t), \cdots, u_n(t))^T$ is any solution of model (1) with initial value $u_i(t_0 + s) = \phi_i(s) \in PC((-\infty, 0], R)$, $i = 1, 2, \cdots, n$, and

$$\left\| \phi - u^* \right\| = \max_{1 \le i \le n} \sup_{s \in (-\infty, 0]} \left| \phi_i(s) - u_i^* \right|.$$

Definition 2[18]: A real matrix $A = (a_{ij})_{n \times n}$ is said to be an M-matrix if $a_{ij} \le 0$ ($i, j = 1, 2, \cdots, n$, $i \neq j$) and successive principle minors of A are positive.

To prove our results, the following lemmas that can be found in [18, 33] are necessary.

Lemma 1[18]: Let Q be $n \times n$ matrix with non-positive off-diagonal elements, then Q is an M-matrix if and only if one of the following conditions holds.

(i) The real parts of all eigenvalues of Q are positive.

(ii) There exists a vector $\xi > 0$ such that $\xi^T Q > 0$.

When A is an M-matrix, denoting

$$\Omega(A) = \{\xi \in R^n \mid A\xi > 0, \xi > 0\},$$

we know from Lemma 1 that $\Omega(A)$ is nonempty.

Lemma 2[33]: Let A be a nonnegative matrix, then A has an nonnegative eigenvalue that is $\rho(A)$ and its eigenvectors are nonnegative.

Let

$$\Gamma(A) = \{\xi \in R^n \mid A\xi = \rho(A)\xi\}.$$

When A is an nonnegative matrix, it follows from Lemma 2 that $\Gamma(A)$ is nonempty.

3. Main Results

Theorem 1: Under assumptions (H1)-(H6), the equilibrium point of model (1) is globally exponentially stable and the exponential convergence rate equals $\varepsilon - \alpha$ if the following conditions are satisfied

(i) $W = B - |C|G - |D|F - |V|H$ is an M-matrix.

(ii) $\Delta = \bigcap_{k=1}^{\infty} \left[\Gamma(P_k) \cap \Gamma(Q_k) \right] \cap \Omega(W)$ is nonempty.

(iii) There exists a constant α such that

$$\frac{\ln \alpha_k}{t_k - t_{k-1}} \le \alpha < \varepsilon, k = 1, 2, \cdots \tag{4}$$

where the sequence α_k satisfies

$$\alpha_k \ge \max\{1, \rho(P_k) + e^{\varepsilon\tau} \rho(Q_k)\} \tag{5}$$

and the scalar $\varepsilon > 0$ is determined by the inequality

$$\xi_i \left(\frac{\varepsilon}{a_i} - b_i \right) + \sum_{j=1}^{n} \xi_j \left(|c_{ij}| G_j + e^{\varepsilon\tau} |d_{ij}| F_j + |v_{ij}| r_{ij}(\varepsilon) H_j \right) < 0 \tag{6}$$

for a given $\xi = (\xi_1, \xi_2, \cdots, \xi_n)^T \in \Delta$, $\tau = \max_{1 \le i \le n, 1 \le j \le n} \{\tau_{ij}\}$.

Proof. From assumption (**H1**), we let $u^* = (u_1^*, u_2^*, \cdots, u_n^*)^T$ be an equilibrium point of model (1). By denoting

$$y_i(t) = u_i(t) - u_i^*, \tilde{a}_i(y_i(t)) = a_i(y_i(t) + u_i^*),$$

$$\tilde{b}_i(y_i(t)) = b_i(y_i(t) + u_i^*) - b_i(u_i^*),$$

$$\tilde{g}_j(y_j(t)) = g_j(y_j(t) + u_j^*) - g_j(u_j^*),$$

$$\tilde{f}_j(y_j(t)) = f_j(y_j(t) + u_j^*) - f_j(u_j^*),$$

$$\tilde{h}_j(y_j(t)) = h_j(y_j(t) + u_j^*) - h_j(u_j^*),$$

$$\tilde{p}_{ik}(y_1(t), \cdots, y_n(t)) = p_{ik}(y_1(t) + u_1^*, \cdots, y_n(t) + u_n^*) - p_{ik}(u_1^*, \cdots, u_n^*),$$

$$\tilde{q}_{ik}(y_1(t), \cdots, y_n(t)) = q_{ik}(y_1(t) + u_1^*, \cdots, y_n(t) + u_n^*) - q_{ik}(u_1^*, \cdots, u_n^*),$$

we can rewrite model (1) as follows:

$$\begin{cases} \dfrac{dy_i(t)}{dt} = -\tilde{a}_i(y_i(t))\Big[\tilde{b}_i(y_i(t)) - \sum\limits_{j=1}^{n} c_{ij}\tilde{g}_j(y_j(t)) \\ \qquad -\sum\limits_{j=1}^{n} d_{ij}\tilde{f}_j(y_j(t-\tau_{ij}(t))) \\ \qquad -\sum\limits_{j=1}^{n} v_{ij}\int_{-\infty}^{t} K_{ij}(t-s)\tilde{h}_j(u_j(s))ds\Big], \quad t \neq t_k, \\ y_i(t) = \tilde{p}_{ik}(y_1(t^-),\cdots,y_n(t^-)) + \tilde{q}_{ik}(y_1((t-\tau_{i1}(t))^-), \\ \qquad \cdots, y_n((t-\tau_{in}(t))^-)), \quad t = t_k \end{cases} \tag{7}$$

for $i = 1,2,\cdots,n$, $k = 1,2,\cdots$.

Since W is an M-matrix and the set Δ is nonempty, from lemma 1, there exists $\xi = (\xi_1,\xi_2,\cdots,\xi_n)^T \in \Delta \subseteq \Omega(W)$ such that

$$-\xi_i b_i + \sum_{j=1}^{n}\xi_j\left(|c_{ij}|G_j + |d_{ij}|F_j + |v_{ij}|H_j\right) < 0, i = 1,2,\cdots,n. \tag{8}$$

We can choose a sufficiently small positive constant $\varepsilon > 0$ such that

$$-\xi_i\left(b_i - \frac{\varepsilon}{a_i}\right) + \sum_{j=1}^{n}\xi_j\left(|c_{ij}|G_j + e^{\tau\varepsilon}|d_{ij}|F_j + |v_{ij}|r_{ij}(\varepsilon)H_j\right) < 0, \quad i = 1,2,\cdots,n. \tag{9}$$

Let

$$x_i(t) = e^{\varepsilon(t-t_0)}|y_i(t)|, i = 1,2,\cdots,n.$$

Calculating the upper right derivative $D^+x_i(t)$ of $x_i(t)$ along the solutions of (7), from the assumption (**H2**), (**H3**), (**H4**) and (**H5**), we can get

$$\begin{aligned} D^+x_i(t) &= \varepsilon e^{\varepsilon(t-t_0)}|y_i(t)| + \varepsilon e^{\varepsilon(t-t_0)}\operatorname{sgn}(y_i(t))\Big\{ \\ &\quad -\tilde{a}_i(y_i(t))\Big[\tilde{b}_i(y_i(t)) - \sum_{j=1}^{n} c_{ij}\tilde{g}_j(y_j(t)) \\ &\quad -\sum_{j=1}^{n} d_{ij}\tilde{f}_j(y_j(t-\tau_{ij}(t))) \\ &\quad -\sum_{j=1}^{n} v_{ij}\int_{-\infty}^{t} K_{ij}(t-s)\tilde{h}_j(y_j(s))ds\Big]\Big\} \\ &\le \varepsilon e^{\varepsilon(t-t_0)}\frac{\tilde{a}_i(y_i(t))}{a_i}|y_i(t)| + \varepsilon e^{\varepsilon(t-t_0)}\Big\{ \\ &\quad -\tilde{a}_i(y_i(t))\Big[b_i|y_i(t)| - \sum_{j=1}^{n}|c_{ij}|G_j|y_j(t)| \\ &\quad -\sum_{j=1}^{n}|d_{ij}|F_j|y_j(t-\tau_{ij}(t))| \\ &\quad -\sum_{j=1}^{n}|v_{ij}|\int_{-\infty}^{t} K_{ij}(t-s)|y_j(t)|H_j ds\Big]\Big\} \\ &= \varepsilon\frac{\tilde{a}_i(y_i(t))}{a_i}x_i(t) - \tilde{a}_i(y_i(t))\Big[\ b_i x_i(t) \\ &\quad -\sum_{j=1}^{n}|c_{ij}|G_j x_j(t) - \sum_{j=1}^{n}|d_{ij}|F_j e^{\varepsilon\tau_{ij}(t)}x_j(t-\tau_{ij}(t)) \\ &\quad -\sum_{j=1}^{n}|v_{ij}|\int_{-\infty}^{t} K_{ij}(t-s)|x_j(t)|H_j ds\Big]\Big\} \\ &\le \tilde{a}_i(y_i(t))\Big[-\left(b_i - \frac{\varepsilon}{a_i}\right)x_i(t) + \sum_{j=1}^{n}|c_{ij}|G_j x_j(t) \\ &\quad + e^{\varepsilon\tau}\sum_{j=1}^{n}|d_{ij}|F_j x_j(t-\tau_{ij}(t)) \\ &\quad + \sum_{j=1}^{n}|v_{ij}|H_j\int_{-\infty}^{t} K_{ij}(t-s)|x_j(t)|ds\Big] \end{aligned}$$

for $i = 1,2,\cdots,n;$ $t_{k-1} < t < t_k$, $k = 1,2,\cdots$.

Leting

$$l_0 = \frac{\|\phi - u^*\|}{\min\limits_{1\le i\le n}\{\xi_i\}},$$

then we have

$$x_i(s) = e^{\varepsilon(s-t_0)}|y_i(s)| \le |y_i(s)| = |\phi_i(s-t_0) - u_i^*| \le \|\phi - u^*\| \le \xi_i l_0$$

, $-\infty < s \le t_0$, $i = 1,2,\cdots,n$.

Let us prove

$$x_i(t) \le \xi_i l_0, \quad t_0 \le t < t_1, \quad i = 1,2,\cdots,n. \tag{11}$$

If (11) is not true, then there exist some i and $t^* \in [t_0, t_1]$ such that

$$x_i(t^*) = \xi_i l_0, \quad D^+x_i(t^*) \ge 0 \quad \text{and} \quad x_j(t) \le \xi_j l_0$$

for $-\infty < t \le t^*$, $j = 1,2,\cdots,n$. However, from (9), (10) and (**H5**), we get

$$\begin{aligned} D^+x_i(t) &\le \tilde{a}_i(y_j(t^*))\Big[-\left(b_i - \frac{\varepsilon}{a_i}\right)\xi_i \\ &\quad + \sum_{j=1}^{n}|c_{ij}|G_j\xi_j + e^{\varepsilon\tau}\sum_{j=1}^{n}|d_{ij}|F_j\xi_j \\ &\quad \sum_{j=1}^{n} v_{ij}H_j r_{ij}(\varepsilon)\xi_j\Big]l_0 \\ &< 0, \end{aligned}$$

and this is a contradiction. So

$$x_i(t) \le \xi_i l_0, \quad t_0 \le t < t_1, \quad i = 1,2,\cdots,n,$$

which is

$$|y_i(t)| \le \xi_i l_0 e^{-\varepsilon(t-t_0)}, \quad t_0 \le t < t_1, \quad i = 1,2,\cdots,n. \tag{12}$$

In the following, we will use the mathematical induction to prove that

$$|y_i(t)| \le \alpha_0\alpha_1\cdots\alpha_{k-1}\xi_i l_0 e^{-\varepsilon(t-t_0)}, \quad t_{k-1} \le t < t_k, \quad i = 1,2,\cdots,n, \quad k = 1,2,\cdots, \tag{13}$$

holds for $\alpha_0 = 1$.

When $k=1$, from (12) we know that (13) holds. Suppose that the inequalities

$$\left|y_i(t)\right| \le \alpha_0\alpha_1\cdots\alpha_{k-1}\xi_i l_0 e^{-\varepsilon(t-t_0)}, \quad t_{k-1}\le t<t_k, \quad i=1,2,\cdots,n$$

holds for $k=1,2,\cdots,m$.

From assumption (**H6**) and (14), the discrete part of model (7) satisfies that

$$\begin{aligned}\left|y_i(t_m)\right| &\le \left|\tilde{p}_{im}(y_1(t_m^-),\cdots,y_n(t_m^-))\right| \\ &+\left|\tilde{q}_{im}(y_1((t_m-\tau_{i1}(t_m))^-),\cdots,y_n((t_m-\tau_{in}(t_m))^-))\right| \\ &\le \sum_{j=1}^{n} p_{ij}^{(m)}\left|y_j(t_m^-)\right| + \sum_{j=1}^{n} q_{ij}^{(m)}\left|y_j((t_m-\tau_{ij}(t_m))^-)\right| \\ &\le \sum_{j=1}^{n} p_{ij}^{(m)}\alpha_0\alpha_1\cdots\alpha_{m-1}\xi_j l_0 e^{-\varepsilon(t_m-t_0)} \\ &+\sum_{j=1}^{n} q_{ij}^{(m)}\alpha_0\alpha_1\cdots\alpha_{m-1}\xi_j l_0 e^{-\varepsilon(t_m-\tau_{ij}(t_m)-t_0)} \\ &\le \left(\sum_{j=1}^{n} p_{ij}^{(m)}\xi_j + e^{\varepsilon\tau}\sum_{j=1}^{n} q_{ij}^{(m)}\xi_j\right)\alpha_0\alpha_1\cdots\alpha_{m-1} l_0 e^{-\varepsilon(t_m-t_0)}\end{aligned} \tag{15}$$

for $i=1,2,\cdots,n$.

From $\xi=(\xi_1,\xi_2,\cdots,\xi_n)^T\in\Delta$ and **Lemma 2**, we know that $\xi\in\Gamma(P_m)$ and $\xi\in\Gamma(Q_m)$. Thus

$$P_m\xi=\rho(P_m)\xi, \quad Q_m\xi=\rho(Q_m)\xi,$$

i.e.,

From (15), (16) and (4), we get

$$\begin{aligned}\left|y_i(t_m)\right| &\le \left(\rho(P_m)+e^{\varepsilon\tau}\rho(Q_m)\right)\alpha_0\alpha_1\cdots\alpha_{m-1}\xi_i l_0 e^{-\varepsilon(t_m-t_0)} \\ &\le \alpha_0\alpha_1\cdots\alpha_{m-1}\alpha_m\xi_i l_0 e^{-\varepsilon(t_m-t_0)}\end{aligned} \tag{17}$$

for $i=1,2,\cdots,n$. This, together with both (14) and (12), lead to

$$\left|y_i(t_m)\right| \le \alpha_0\alpha_1\cdots\alpha_{m-1}\alpha_m\xi_i l_0 e^{-\varepsilon(t-t_0)}, \quad i=1,2,\cdots,n; \quad t\in(-\infty,t_m] \tag{18}$$

i.e.,

$$x_i(t)\le \alpha_0\alpha_1\cdots\alpha_{m-1}\alpha_m\xi_i l_0, \quad i=1,2,\cdots,n; \quad t\in(-\infty,t_m] \tag{19}$$

In the following, we will prove that

$$x_i(t)\le \alpha_0\alpha_1\cdots\alpha_{m-1}\alpha_m\xi_i l_0, \quad i=1,2,\cdots,n; \quad t\in[t_m,t_{m+1}) \tag{20}$$

holds.

If (20) is not true, then there exist some i and $t^{**}\in[t_m,t_{m+1})$ such that

$$x_i(t^{**})=\alpha_0\alpha_1\cdots\alpha_{m-1}\alpha_m\xi_i l_0, \quad D^+x_i(t^{**})\ge 0 \quad \text{and}$$

$$x_j(t)\le \alpha_0\alpha_1\cdots\alpha_{m-1}\alpha_m\xi_i l_0$$

For $-\infty<t\le t^{**}$, $j=1,2,\cdots,n$. However, from (9), (10) and (**H5**), we get

$$\begin{aligned}D^+x_i(t^{**}) &\le \tilde{a}_i(y_i(t^{**}))\left[-\left(b_i-\frac{\varepsilon}{a_i}\right)\xi_i\right. \\ &+\sum_{j=1}^{n}\left|c_{ij}\right|G_j\xi_j + e^{\varepsilon\tau}\sum_{j=1}^{n}\left|d_{ij}\right|F_j\xi_j \\ &\left.+\sum_{j=1}^{n} v_{ij}H_j r_{ij}(\varepsilon)\xi_j\right]\alpha_0\alpha_1\cdots\alpha_{m-1}\alpha_m l_0 \\ &<0,\end{aligned}$$

which is a contradiction. This indicates that (20) holds. To this end, by the mathematical induction, we can conclude that (13) holds.

From (5), we have

$$\alpha_k\le e^{\alpha(t_k-t_{k-1})}, \quad k=1,2,\cdots.$$

It follows from (13) that

$$\begin{aligned}\left|y_i(t)\right| &\le e^{\alpha(t_1-t_0)}e^{\alpha(t_2-t_1)}\cdots e^{\alpha(t_{k-1}-t_{k-2})}\xi_i l_0 e^{-\varepsilon(t-t_0)} \\ &=\frac{\xi_i}{\min\limits_{1\le i\le n}\{\xi_i\}}\left\|\phi-u^*\right\|e^{\alpha(t_{k-1}-t_0)}e^{-\varepsilon(t-t_0)} \\ &\le\frac{\xi_i}{\min\limits_{1\le i\le n}\{\xi_i\}}\left\|\phi-u^*\right\|e^{\alpha(t-t_0)}e^{-\varepsilon(t-t_0)} \\ &=\frac{\xi_i}{\min\limits_{1\le i\le n}\{\xi_i\}}\left\|\phi-u^*\right\|e^{-(\varepsilon-\alpha)(t-t_0)}\end{aligned}$$

for any $t\in[t_{k-1},t_k)$, $k=1,2,\cdots$, that is

$$\left|u_i(t)-u^*\right|\le\frac{\xi_i}{\min\limits_{1\le i\le n}\{\xi_i\}}\left\|\phi-u^*\right\|e^{-(\varepsilon-\alpha)(t-t_0)}$$

for $t\ge t_0$, So

$$\left|u(t)-u^*\right|\le M\left\|\phi-u^*\right\|e^{-(\varepsilon-\alpha)(t-t_0)}$$

for $t\ge t_0$, where $M=\dfrac{\xi_i}{\min\limits_{1\le i\le n}\{\xi_i\}}\ge 1$. This means that the equilibrium point u^* of model (1) is globally exponentially stable, and the exponential convergence rate equals $\varepsilon-\alpha$. The proof is completed.

Remark 2. We may choose appropriate matrices P_k and Q_k in assumption (**H6**) to guarantee that the set Δ in **Theorem 1** is nonempty. In particular, when $P_k=p_kE$ and $Q_k=q_kE$ (p_k, q_k are nonnegative constants and E is a unit matrix), Δ is certainly nonempty. So, by using **Theorem 1**, we can obtain the following corollary easily.

Corollary 1: Under assumptions (**H1**)-(**H5**), the equilibrium point of model (1) is globally exponentially stable if the following conditions are satisfied

(i) There exist nonnegative constants p_k and q_k such that

$$\left|p_{ik}(u_1,\cdots u_n)-p_{ik}(v_1,\cdots v_n)\right|\le p_k\left|u_i-v_i\right|$$

$$|q_{ik}(u_1,\cdots u_n)-q_{ik}(v_1,\cdots v_n)|\le q_k|u_i-v_i|$$

for all $(u_1,\cdots,u_n)^T\in R^n$, $(v_1,\cdots,v_n)^T\in R^n$, $i=1,2,\cdots,n$, $k=1,2,\cdots$.

(ii) $W=B-|C|G-|D|F-|V|H$ is M-matrix.

(iii) Let $\alpha_k\ge\max\{1,p_k+q_ke^{\varepsilon\tau}\}$. Assume that there exists a constant α such that

$$\frac{\ln\alpha_k}{t_k-t_{k-1}}\le\alpha<\varepsilon \qquad k=1,2,\cdots,$$

where the scalar ε is determined by the inequality

$$\xi_i\left(\frac{\varepsilon}{a_i}-b_i\right)+\sum_{j=1}^{n}\xi_j\left(|c_{ij}|G_j+e^{\varepsilon\tau}|d_{ij}|F_j+|v_{ij}|r_{ij}(\varepsilon)H_j\right)<0$$

for a given $\xi=(\xi_1,\xi_2,\cdots,\xi_n)^T\in\Omega(W)$.

Proof. Noticing condition (i) is a special case of (**H6**) with $P_k=p_kE$ and $Q_k=q_kE$, we know that (**H6**) is satisfied. It can be easily computed that $\Delta=\bigcap_{k=1}^{\infty}[\Gamma(P_k)\cap\Gamma(Q_k)]\cap\Omega(W)=\Omega(W)$. From condition (ii), we know that $\Omega(W)$ is nonempty, and therefore the condition (ii) of **Theorem 1** is satisfied. By using **Theorem 1** we can deduce the conclusion, and the proof is complete.

From Theorem 1 of [18] and **Corollary 1** of this paper, we can prove the following result. For simplicity of the presentation, the proof is skipped.

Corollary 2: Under assumptions (**H2**)-(**H5**), model (2) has a unique equilibrium point, which is globally exponentially stable if $W=B-|C|G-|D|F-|V|H$ is an M-matrix.

Remark 3. In [2-16], [19, 20, 21, 23, 25, 26], the amplification functions were required to satisfy $0<\underline{a}_i\le a_i(u)\le\overline{a}_i<+\infty$ for all $u\in R$, $i=1,2,\cdots,n$. It is worth pointing out, in our paper, the upper bound constraint on the amplification functions is no longer needed. In addition, assumption (H3) on the behaved functions in our results is the same as that in [5, 8, 18], and the condition for differentiability imposed on behaved functions in [2-4], [6], [7], [9-11] is removed in our results.

Remark 4. Corollary 2 of this paper shows that there is a unique equilibrium point u^* of the continuous part of the system (1) under assumptions (**H2**)-(**H5**). In many cases, u^* may not be a solution of the discrete part of the system (1) without the external impulsive input. In other words, the entire system (1) may have no equilibrium point. In order to guarantee that the entire system (1) has an equilibrium point, as in [33], we can introduce the external impulsive input J_{ik} so that u^* is also an equilibrium point of the discrete part of the system (1).

4. Example

Example 1. Consider the following model

$$\begin{cases}\dfrac{du_1(t)}{dt}=-(2+\cos u_1(t))\big[11u_1(t)-f_1(u_1(t))+f_2(u_2(t))\\ \qquad -g_1(u_1(t-\tau(t)))-2\int_{-\infty}^{t}K_{11}(t-s)h_1(s)ds\\ \qquad +\int_{-\infty}^{t}K_{12}(t-s)h_2(s)ds-11\big],\quad t>0,t\ne t_k,\\ \dfrac{du_2(t)}{dt}=-(3-\sin u_2(t))\big[12u_2(t)+2f_2(u_2(t))\\ \qquad -2g_1(u_1(t-\tau(t)))-3g_2(u_2(t-\tau(t)))\\ \qquad -\int_{-\infty}^{t}K_{11}(t-s)h_1(s)ds\\ \qquad +2\int_{-\infty}^{t}K_{12}(t-s)h_2(s)ds-23\big],\ t>0,t\ne t_k,\\ u_1(t)=0.2e^{0.05k}u_1(t^-)-0.1e^{0.05k}u_2(t^-)\\ \qquad +0.4e^{0.05k}u_1((t-\tau(t))^-)+1-0.4e^{0.05k},\quad t=t_k,\\ u_2(t)=-0.4e^{0.05k}u_1(t^-)+0.2e^{0.05k}u_2(t^-)\\ \qquad -0.4e^{0.05k}u_2((t-\tau(t))^-)+2-0.8e^{0.05k},\quad t=t_k,\end{cases} \tag{30}$$

where $g_1(x)=g_2(x)=f_1(x)=f_2(x)=h_1(x)=h_2(x)=x$, $K_{11}(x)=K_{12}(x)=K_{21}(x)=K_{22}(x)=te^{-t}$, $\tau(t)=|\cos t|+0.1$, $t_0=0$, $t_k=t_{k-1}+0.5k$, $k=1,2,\cdots$.

One can verify that the point $(1,2)^T$ is an equilibrium point of model (30), and model (30) satisfies assumptions (**H2**)-(**H6**) with $a_1=1$, $a_2=2$, $B=\begin{pmatrix}11&0\\0&12\end{pmatrix}$, $C=\begin{pmatrix}1&-1\\0&-2\end{pmatrix}$, $D=\begin{pmatrix}1&0\\2&3\end{pmatrix}$, $V=\begin{pmatrix}2&-1\\1&-2\end{pmatrix}$, $F=G=H=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $P_k=0.1e^{0.05k}\begin{pmatrix}2&1\\4&2\end{pmatrix}$, $Q_k=0.4e^{0.05k}\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $\tau=1.1$.

It can be easily checked that

$$\text{W=B-}|\text{C}|F-|D|G-|V|H=\begin{pmatrix}7&-2\\-3&5\end{pmatrix}$$

is an M-matrix, and $\rho(P_k)=\rho(Q_k)=0.4e^{0.05k}$. Therefore, $\Gamma(P_k)=\{(z_1,z_2)^T\mid z_2=2z_1\}$, $\Gamma(Q_k)=R^2$, $\Omega(W)=\{(z_1,z_2)^T\mid 0.6z_1<z_2<3.5z_1,z_1>0,z_2>0\}$, so $\Delta=\{(z_1,z_2)^T\mid z_2=2z_1,z_1>0,z_2>0\}$ is nonempty. Let $\xi=(1,2)^T\in\Delta$ and $\varepsilon=0.2164$ so that the following inequalities

$$\xi_i\left(\frac{\varepsilon}{a_i}-b_i\right)+\sum_{j=1}^{n}\xi_j\left(\left|c_{ij}\right|G_j+e^{\varepsilon\tau}\left|d_{ij}\right|F_j+\left|v_{ij}\right|r_{ij}(\varepsilon)H_j\right)<0$$

hold for $i, j=1,2$. Taking $\alpha_k=e^{0.05k}$, $\alpha=0.1$, we know that the inequalities

$$\alpha_k \geq \max\left\{1, 0.4e^{0.05k}+0.4e^{0.05k}e^{0.23804}\right\},\quad k=1,2,\cdots,$$

and

$$\frac{\ln\alpha_k}{t_k-t_{k-1}}=\frac{\ln e^{0.05k}}{0.5k}\leq\alpha<\varepsilon,\quad k=1,2,\cdots,$$

are satisfied. Clearly, all conditions of **Theorem 1** are satisfied. From **Theorem 1**, we know that the unique equilibrium point $(1,2)^T$ of model (30) is globally exponentially stable, and the exponential convergence rate equals 0.1164. The global exponential stability of equilibrium point $(1,2)^T$ of model (30) is further verified by the simulation given in **Figure 1**, where the initial state is taken as $u_1(s)=2-\sin(3s)$, $u_2(s)=-1+\cos(4s)$, $s\in(-\infty,0]$.

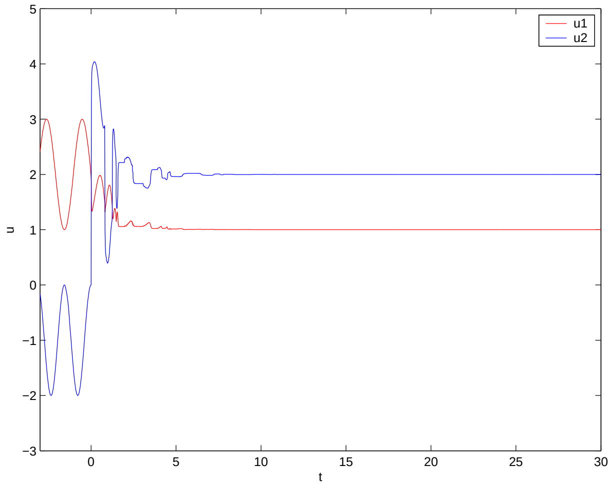

Fig. 1 Quantization procedure for measurement of noise level fluctuation.

5. Conclusion

In this paper, the problem on exponential stability has been investigated for a class of impulsive Cohen-Grossberg neural networks with both the time-varying and distributed delays. Several sufficient conditions for checking the global exponential stability of equilibrium point have been established by using the M-matrix theory and analytic methods. Moreover, the exponential convergence rate index has been estimated, which depends on the system parameters. The proposed results have generalized some recently known ones in the literature, and removed some restrictions on the neural networks. An example with simulation has been given to show the effectiveness of the obtained results.

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Zhenjiang Zhao received the M.S. degrees in Department of Mathematics, OKAYAMA University, of Japan in 1996 and 1998. During 1998-2002, he stayed in TSS Software Co. Ltd. in Japan to develop software of computer. Since 2002, He has worked in Department of Mathematics, Huzhou Teachers College, Zhejiang, China.

Qiankun Song was born in 1964. He received the B.S. degree in Mathematics in 1986 from Sichuan Normal University, Chengdu, China, and the M.S. degree in Applied Mathematics in 1996 from Northwestern Polytechnical University, Xi'an, China. He was a student at refresher class in the Department of Mathematics, Sichuan University, Chengdu, China, from September 1989 to July 1990. From July 1986 to December 2000, he was with Department of Mathematics, Sichuan University of Science and Engineering, Sichuan, China. From January 2001 to June 2006, he was with the Department of Mathematics, Huzhou University, Zhejiang, China. In July 2006, he moved to the Department of Mathematics, Chongqing Jiaotong University, Chongqing, China. He is currently a Professor at Chongqing Jiaotong University. He is the author or coauthor of more than 40 journal papers and one edited book. His current research interests include neural networks, chaos synchronization and stability theory.