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Increase of Accuracy of Projective-Difference Schemes

1. Abstract.

In this paper with simple examples there is examined one of the improvement methods of approximate solution, which is derived with integral equalities for elliptic differential problems. The improvement method is to use some approximate systems having low order of accuracy and depending on the mesh size as the parameter. A linear combination of solutions of these problems is made, which has a given order of accuracy limited by only a degree of smoothness and the data of the differential problem.

An idea of this method is due to L.F.Richardson, but E.A.Volkov and some other mathematicians obtained a constructive proof for some problems in the 1950's.

We research the realization of this method for an ordinary differential equation (in detail, as as illustration), an elliptic differential equation in a rectangle and in the domain with a smooth boundary, and an evolutional equation with a bounded operator.

2. An ordinary differential equation.

For a function $\varphi(x)$, which is defined in the segment $\bar{G} = [0,1]$, the notation $\varphi \in C_k(\bar{G})$ means existence of the continuous derivatives of $\varphi(x)$ on G up to order k .

Let the function $u \in C_2(G)$ be found from the equation

$$Lu \equiv -(au')' + bu = f \qquad \text{in} \qquad G = (0,1) \tag{1}$$

with boundary conditions

$$u(0)=0, \tag{2}$$

$$u'(1)+\gamma u(1)=g, \qquad \gamma \geqslant 0. \tag{3}$$

The coefficients of the equation (1) $\forall x \in \bar{G}$ are nonnegative:

$$a(x) \geqslant \alpha > 0, \qquad b(x) \geqslant 0 \tag{4}$$

Let :

$$(u,v)=\int_G uv\,dx, \quad [u,v]=\int_G (au'v'+buv)\,dx$$

be scalar products.

For to construct algebraic system approximating problem (1) - (3) we fix integer $n>0$ and denote

$$h=1/n, \quad G_h=\{y: y=ih, \quad i=1,\dots,n-1\}, \quad \bar{G}_h=G_h \cup \{1\}. \tag{5}$$

Let us introduce a set of n functions, setting $\forall y \in G_h$,

$$\omega_h(x,y)=\begin{cases} \left(\int_{-h}^{x-y} 1/a(y+t)\,dt\right)\Big/\int_{-h}^{0} 1/a(y+t)\,dt, & \text{if } x-y \in (-h,0), \\ \left(\int_{x-y}^{h} 1/a(y+t)\,dt\right)\Big/\int_{0}^{h} 1/a(y+t)\,dt, & \text{if } x-y \in [0,h), \\ 0 & \text{or else} \end{cases} \tag{6}$$

and for $y=1$

$$\omega_h(x,1)=\begin{cases} \left(\int_{-h}^{x-y} 1/a(y+t)\,dt\right)\Big/\int_{-h}^{0} 1/a(y+t)\,dt, & \text{if } x-y \in (-h,0). \\ 0 & \text{or else} \end{cases} \tag{7}$$

The equation (1) is multiplyed by every function (6) and is integrated over x:

$$(f(x),\omega_h(x,y))=(Lu(x),\omega_h(x,y))=[u(x),\omega_h(x,y)].$$

The methodic value of the test functions (6) - (7) is in fact that the term $\int a u' \omega_h' dx$ is approximated exactly. We approximate the other term so as to obtain Ritz's method system:

$$[u(x), \omega_h(x,y)] = \sum_{z \in \bar{G}_h} [\omega_h(x,y), \omega_h(x,z)]\, u^h(z). \tag{8.a}$$

It is notprincipal, but proof is simpler.

The boundary condition (2) yield

$$u^h(0) = 0 \tag{8.b}$$

and the condition (3) with the equation (1) permitsus to obtain approximate equation

$$\begin{aligned} (f(x), \omega_h(x,1)) &= [u(x), \omega_h(x,1)] - g a(1) + \gamma a(1) u(1) = \\ &\sum_{z \in \bar{G}_h} [\omega_h(x,z), \omega_h(x,1)]\, u^h(z) - g a(1) + \gamma a(1)\, u^h(1) . \end{aligned} \tag{8.c}$$

If we unite the equations (8.a) - (8.c) and denote $\theta_{z,y} = [\omega_h(x,z), \omega_h(x,y)]$, we have system

$$\begin{cases} u^h(0) = 0 \\ \sum\limits_{z \in \bar{G}_h} \theta_{z,y}\, u^h(z) = (f, \omega_h(x,y)), \qquad \forall y \in G_h, \\ \sum\limits_{z \in \bar{G}_h} \theta_{z,1}\, u^h(z) \neq \gamma a(1) u^h(1) = (f(x), \omega_h(x,1)) + g a(1) \end{cases} \tag{9}$$

which is equivalent to Ritz's method system, when one takes linear space of functions (6) - (7) as approximate subspace. Therefore the system (9) has a unique solution. For further account let us estimate solution of system

$$\begin{cases} v(0) = 0 \\ \sum\limits_{z \in \bar{G}_h} \theta_{z,y}\, v(z) = \sigma(y) \qquad \forall y \in G_h \\ \sum\limits_{z \in \bar{G}_h} \theta_{z,1}\, v(z) + \gamma a(1)\, u(1) = \sigma(1) \end{cases} \tag{10}$$

Lemma 1. For the solution of system (10) estimation

$$\max_{x\in\bar{G}_h} |v(x)| \leq \frac{1}{4\alpha} \sum_{y\in\bar{G}_h} |\sigma(y)|. \tag{11}$$

is valid.

Proof. Let us denote $\bar{v}(x) = \sum_{z\in\bar{G}_h} v(z)\,\omega_h(x,z)$, multiply every equation of system (10) by $v(y)$ and sum up over $y\in\bar{G}_h$:

$$[\bar{v}(x),\bar{v}(x)] + \gamma a(1)\, v^2(1) = \sum_{y\in\bar{G}_h} \sigma(y)\, v(y).$$

With the condition $v(0)=0$ the left part may be decreased with help of Sobolev's theorem [3] :

$$[v(x),\bar{v}(x)] \geq \alpha\,(\bar{v}'(x),\bar{v}'(x)) \geq 4\alpha \max_{x\in\bar{G}} |\bar{v}(x)| \geq 4\alpha \max_{x\in\bar{G}_h} |v(x)|.$$

There is the inequality

$$\sum_{y\in\bar{G}_h} \sigma(y)\, v(y) \leq \max_{x\in\bar{G}_h} |v(x)| \sum_{y\in\bar{G}_h} |\sigma(y)| \text{ , that}$$

is valid for the right part. If we devide these inequalities by $\max|v(x)|$, we obtain the lemma statement.

Let us consider a connection between solution of system (9) and problem (1) - (3).

Lemma 2. Let us suppose that $a \in C_{2k+1}(\bar{G})$, b , $f \in C_{2k}(G)$ and integer $k \geq 0$ in equation (1). Then there are k functions $v_\ell \in C_{2(k-\ell+1)}(\bar{G})$ which do not depend on h and $\forall h\in(0,1)$

$$u^h(x) = u(x) + \sum_{\ell=1}^{k} h^{2\ell}\, v_\ell(x) + h^{2k+2}\,\xi^h(x) \qquad \forall\, x\in\bar{G}_h \tag{12}$$

where descrete function is bounded:

$$\max_{x\in\bar{G}_h} |\xi^h(x)| \leq c \qquad \forall\, h\in(0,1). \tag{13}$$

Proof. In analogy with [1] let us suppose decomposition (12) being. We shall find necessary conditions and make them sufficient.

So, let us substitute phrase (12) in system (9) and change its right part:

$$\sum_{\ell=1}^{k} h^{2\ell} v_\ell(0) + h^{2k+2}\xi^h(0) = 0 \tag{14.a}$$

$$\sum_{z\in\bar G_h} \theta_{z,y}\left\{u(z)+\sum_{\ell=1}^{k} h^{2\ell} v_\ell(z) + h^{2k+2}\xi^h(z)\right\} = \tag{14.b}$$

$$[u(x),\, \omega_h(x,y)] \qquad \forall\, y\in G_h \ ,$$

$$\sum_{z\in\bar G_h} \theta_{z,1}\left\{u(z)+\sum_{\ell=1}^{k} h^{2\ell} v_\ell(z) + h^{2k+2}\xi^h(z)\right\} + \tag{14.c}$$

$$\gamma\alpha(1)\left\{\sum_{\ell=1}^{k} h^{2\ell} v_\ell(z) + h^{2k+2}\xi^h(z)\right\} = [u(x),\omega_h(x,1)] .$$

Let us note that $\forall\, y\in\bar G_h$ $\theta_{z,y}\neq 0$ only for three values $z=y\pm h, y$. Let $y\in(0,1)$ and examine quantity

$$A(h)=\sum_{z=y\pm h,y}\theta_{z,y}\, v(z) - [v(x),\omega_h(x,y)] =$$

$$\sum_{z=y\pm h,y}\big(b(x)\omega_h(x,y),\omega_h(z,y)\big)v(z) - \big(b(x)v(x)\omega_h(x,y)\big).$$

Let function $v\in C_{2\ell}(\bar G)$ and $h\le \min\{y,1-y\}$, then

$$A(h)=\sum_{j=1}^{\ell-1} h^{2j+1} d_j(y) + h^{2\ell+1}\psi^h(y) \tag{15}$$

with functions $d_j\in C_{2(\ell-j)}(\bar G)$, which do not depend on h, and with a discrete function ψ^h, which is bounded by costant, consisting of moduli of derivatives of the functions a, b, v. It follows from derivation:

$$\frac{\partial^{2j}A}{\partial h^{2j}}(+0)=0, \qquad d_j=\frac{\partial^{2j+1}A}{\partial h^{2j+1}}(+0),$$

$$\Psi_h = \frac{\partial^{2\ell+1} A}{\partial h^{2\ell+1}}(\nu), \qquad \nu \in (0,h).$$

The estimation of ψ^h (similar (13)) is obtaind by change of $\vartheta(x)$ for Taylor's line

$$\mu \in (x,y), \qquad \mu \in (y,x),$$

$$\vartheta(x) = \vartheta(y) + \sum_{i=1}^{2\ell-1} \frac{(x-y)^i}{i!} \vartheta^{(i)}(y) + h^{2\ell}\left(\frac{x-y}{h}\right)^{2\ell} \frac{\vartheta^{(2\ell)}(\mu)}{(2\ell)!}.$$

Further, similar way gives us

$$(\vartheta(x), \omega_h(x,y)) = h\vartheta(y) + \sum_{j=1}^{\ell-1} h^{2j+1}\pi_j(y) + h^{2\ell+1}\rho_h(y) \tag{16}$$

where functions π_j , ρ_h have such properties as d_j , Ψ_h accordingly. Using now statement (16) for $(\ell-1)$ of different functions from (15) we have by the induction

$$\sum_{z\in\bar{G}_h} \Theta_{z,y}\,\vartheta(z) = [\vartheta(x), \omega_h(x,y)] + \sum_{j=1}^{\ell-1} h^{2j}(\varphi_j(x), \omega_h(x,y)) + h^{2\ell+1}\delta_h(y) \qquad \forall\, y \in G_h. \tag{17}$$

The functions φ_j , δ_h have such properties as d_j , Ψ_h .

Similar way gives us formula

$$\sum_{z\in\bar{G}_h} \Theta_{z,1}\,\vartheta(z) = [\vartheta(x), \omega_h(x,1)] + \sum_{j=1}^{\ell-1} h^{2j}(\varphi_j(x), \omega_h(x,1)) + \sum_{j=1}^{\ell-1} h^{2j}\zeta_j + h^{2\ell}\delta_h(1), \tag{18}$$

and φ_j are the same functions as in (17), ζ_j are constants which do not depend on h .

Now we use these formulae by changing u , k for ϑ , ℓ and substracting (17), (18) from (14.b), (14.c)

$$\sum_{z\in\bar{G}_h} \Theta_{z,y}\left\{\sum_{\ell=1}^{k} h^{2\ell} u_\ell(z) + h^{2k+2}\xi_h(z)\right\} = \sum_{j=1}^{k} h^{2j}(\bar{\varphi}_j(x), \omega_h(x,y)) + h^{2k+3}\bar{\delta}_h(y) \qquad \forall\, y \in G_h \tag{19}$$

$$\sum_{z\in\overline{G}_h}\theta_{z,1}\Big\{\sum_{\ell=1}^{k}h^{2\ell}\,u_\ell(z)+h^{2k+2}\,\xi_h(z)\Big\}=$$
$$=\sum_{j=1}^{k}h^{2j}\left(\overline{\varphi}_j(x),\,\omega_h(x,1)\right)+\sum_{j=1}^{k}h^{2j}\,\overline{\zeta}_j+h^{2\ell}\overline{\delta}_h(1). \tag{20}$$

Here $\overline{\varphi}_j \in C_{2(k-j)}(\overline{G})$ and do not depend on h ; $\overline{\delta}_h$ is bounded: $|\overline{\delta}_h(y)|\leq C \quad \forall y\in\overline{G}_h \quad \forall h\in(0,1)$; $\overline{\zeta}_j$ are constants which do not depend on h .

Then we take u_1 as solution of problem

$$\begin{aligned} &\mathcal{L}u_1=\overline{\varphi}_1 \qquad \text{в} \qquad G\\ &u_1(0)=0, \qquad u_1'(1)+\gamma u_1(1)=\overline{\zeta}_1/a(1). \end{aligned}\tag{21}$$

It necessarily follows from this that $u_1\in C_{2k}(\overline{G})$ and does not depend on h . Let us substitute u_1 in identities (17) - (18), multiply them by h^2 and substract from (19) - (20)

$$\begin{aligned} &\sum_{z\in\overline{G}_h}\theta_{z,y}\Big\{\sum_{\ell=2}^{k}h^{2\ell}u_\ell(z)+h^{2k+2}\,\xi_h(z)\Big\}=\sum_{j=2}^{k}h^{2j}(\overline{\overline{\varphi}}_j(x),\omega_h(x,y))+\\ &\qquad +h^{2k+3}\overline{\overline{\delta}}_h(y) \qquad \forall y\in G_h,\\ &\sum_{z\in\overline{G}_h}\theta_{z,1}\Big\{\sum_{\ell=2}^{k}h^{2\ell}u_\ell(z)+h^{2k+2}\,\xi_h(z)\Big\}=\sum_{j=2}^{k}h^{2j}(\overline{\overline{\varphi}}_j(x),\omega_h(x,y))+\\ &\qquad +\sum_{j=2}^{k}h^{2j}\,\overline{\overline{\zeta}}_j+h^{2k+2}\,\overline{\overline{\delta}}_h(1). \end{aligned}\tag{22}$$

It is obvious from (22), that it is necessary to take u_2 as a solution of a problem

$$\mathcal{L}u_2=\overline{\overline{\varphi}}_2 \qquad \text{в} \qquad G$$

$$u_2(0)=0, \qquad u_2'(1)+\gamma u_2(1)=\overline{\overline{\zeta}}_2/a(1).$$

Such choice guarantees that the requirement to u_2 is valid and that we have possibility to take away elements in (22) with multipliers h_4 , h_5 .

Continuing in this manner, over k steps we come to system

$$\xi_h(0)=0$$

$$\sum_{z \in G_h} \Theta_{z,y}\, \xi_h(z) = \hbar\, \hat{\delta}_h(y) \qquad \forall\ y \in G_h$$

$$\sum_{z \in G_h} \Theta_{z,1}\, \xi_h(z) = \hat{\delta}_h(1) \tag{23}$$

with bounded discrete function $\hat{\delta}_h$. This system has a unique solution, i.e. ξ_h is found by a unique way so that (12) is valid when the functions u_ℓ are chosen. The estimation (13) follows from lemma 1 and from a fact that the function $\hat{\delta}_h$ is bounded.

Lemma is proved.

The decomposition permits us to basis improvement method.

Theorem 1. Given the conditions of lemma 2, one may find solution of problem (1) - (4) which has accuracy of order $\hbar^{2k+2}$, where $\hbar = \max\limits_{\ell=1}^{k+1} h_\ell$.

Proof. When a point x (where we find value of $u(x)$) is a common one for all regular meshes, a higher accuracy solution is made up as follows

$$\bar{u}(x) = \sum_{\ell=1}^{k+1} \gamma_\ell u^{h_\ell}(x).$$

Here u^{h_ℓ} is a solution of problem (9), when the mesh size G_h is equal to h_ℓ , and the γ_ℓ are chosen so that the coefficients of all u_ℓ vanish in the linear combination. That is achieved by a choice of γ_ℓ from system

$$\sum_{\ell=1}^{k+1} \gamma_\ell = 1$$

$$\sum_{\ell=1}^{k+1} \gamma_\ell h_\ell^{2s} = 0\,, \qquad s = 1, \ldots, k\,. \tag{24}$$

In this case on basis (13) it follows that

$$|u(x) - \bar{u}(x)| \leq C \sum_{\ell=1}^{k+1} |\gamma_\ell| h_\ell^{2k+1}.$$

For to estimate γ_k one may solve the system (24) by Kramer's method using results of [2] on Vandermond's determinants:

$$\gamma_i = \prod_{\substack{1 \le \ell \le k+1 \\ \ell \ne i}} \frac{h_\ell^2}{h_\ell^2 - h_i^2}$$

From these formulae with a condition $h_\ell / h_{\ell+1} \geqslant c_1 > 1$, $\forall\ \ell = 1, \dots, k$ it follows that

$$|\gamma_i| \le \left(\frac{c_1^2}{c_1^2 - 1}\right)^{k+1} \qquad \forall\ i = 1, \dots, k+1 .$$

When point x is not common one for all meshes it is necessary to use an interpolation. The smoothness of functions u and u_ℓ permits us to conclude that using Lagrange interpolation to point x from $(k+1)$ neighbouring points of the discrete mesh G_{h_ℓ} we may obtain the decomposition (12) with the same functions u_ℓ. There is changed only the constant c in the estimation of functions ξ_{h_ℓ} where derivation estimations of u, u_ℓ and interpolation weights appear in addition. If $(k+1)$ of decompositions (12) is made in the point x by interpolation, the higher accuracy method is like above.

Remark. The proof may be used without any changes in a case when the coefficients of the right part (and the solution) are piecewise smooth and there are conditions in every point of discontinuities of function a

$$u(\varepsilon + 0) = u(\varepsilon - 0)$$

$$a(\varepsilon + 0)\, u'(\varepsilon + 0) = a(\varepsilon - 0)\, u'(\varepsilon - 0) + w_\varepsilon ,$$

where w_ε is a certain constant. To this end the discrete meshes must be regular in each piece of smoothness and the points of discontinuities must be points of meshes.

3. The Laplace equation in a rectangle.

Considering the Laplace equation in a rectangle we try to show one of way of work with angular points. The main difficulty is bad solution smoothness near angular points notwithstanding good smoothness of all problem data except boundary.

In this section G is an open square: $G = \{x : x = (x_1, x_2)$; $0 < x_1, x_2 < 1\}$ in R^2 with boundary Γ , $\bar{G}$ is $G \cup \Gamma$ and for two points $x, x' \in R^2$ the distance is: $|x - x'| = ((x_1 - x'_1)^2 - - (x_2 - x'_2)^2)^{1/2}$.

For to simplify our considerations let us examine an equation with constant coefficients, Laplace's equation

$$-\Delta u = f \quad \text{in} \quad G . \tag{1}$$

Our problem is to find function u which satisfies equation (1) and condition

$$u = 0 \quad \text{on} \quad \Gamma . \tag{2}$$

For to describe differential properties of the solution let us introduce norms

$$M_m^k [u] = \sum_{(m)} \max_{x \in \bar{G}} d^{m-k}(x) |\mathcal{D}^m u(x)| \tag{3}$$

and

$$M_{m+\alpha}^k [u] = \sum_{(m)} \max_{x, x' \in \bar{G}} d^{m+\alpha-k}(x, x') \frac{|\mathcal{D}^m u(x) - \mathcal{D}^m u(x')|}{|x - x'|^{\alpha}} , \tag{4}$$

where k , m are integer nonnegative numbers, $\alpha \in [0,1]$, $d(x)$ is distance from x to nearest square angle,

$$d(x, x') = \min\{d(x), d(x')\}, \quad \mathcal{D} \text{ is } \frac{\partial}{\partial x} \text{ or } \frac{\partial}{\partial y} .$$

We use usual classes of smoothness (s.f. [5]): $C_{\ell,\alpha}(\bar{G})$ is a class of functions which have in $\bar{G}$ ℓ continuous derivatives and the quantity

$$\|f\|_{\ell,\alpha} = \sum_{m=0}^{\ell} M_\ell^\circ[f] + M_{\ell+\alpha}^\circ[f] \; ; \qquad (5)$$

is limited.

$C_{\ell,\alpha}(G)$ is the set of functions from $C_{\ell,\alpha}(\bar{\Omega})$ for any closed set $\Omega \subset G$.

It follows from monograph [5] that notwithstanding good smoothness of the right part:

$$f \in C_{2\ell+1,\alpha}(G) \qquad (6)$$

the solution of (1)-(2) is valid for

$$u \in C_{2\ell+3,\alpha}(G), \qquad (7)$$

but not for

$$u \in C_{2\ell+3,\alpha}(\bar{G}).$$

I.e. quantities in the right part of (5) may be infinite. But investigation of [5] shows that asymptotic behaviour of discontinuities is less than some orders of $1/x$, namely, quantities

$$M_0^\circ[u], M_1^\circ[u], M_m^1[u] \quad \forall m=2,\dots,2\ell+3, \; M_{2\ell+3,\alpha}^1[u] \qquad (8)$$

are limited.

Moreover it is sufficient that one has more weak assumptions for it

$f \in C_{2\ell+1,\alpha}(G)$ and quantities

$$M_0^\circ[f], M_m^1[f] \quad \forall m = 1,\dots,2\ell+1, \; M_{2\ell+1,\alpha}^1[f] \qquad (9)$$

are limited.

Some difficulties with infinite derivatives are avoided with the help of a special choice of a discrete mesh which is condensed near the discontinuities points. Let

$$\varphi(t) = \left(\int_0^t z^\gamma (1-z)^\gamma dz\right) \Big/ \int_0^1 z^\gamma (1-z)^\gamma dz \quad \forall t \in [0,1]. \qquad (10)$$

A positive parameter γ will be chosen later. Let us fix integer $n > 0$ and let $h = 1/n$ and

$$\begin{aligned} \bar{G}_h = \{ x : x_1 = \varphi(ih),\ x_2 = \varphi(jh),\ \forall i = 0,\dots,n, \\ \forall j = 0,\dots,n\},\quad G_h = \bar{G}_h \cap G,\quad \Gamma_h = \bar{G}_h \cap \Gamma. \end{aligned} \qquad (11)$$

The basic functions are introduced as follows :

$$\omega_h(x,x') = \rho(x_1,x_1')\rho(x_2 x_2') \qquad \forall x \in G_h$$

$$\rho(t,t') = \begin{cases} \dfrac{t' - \varphi(t-h)}{\varphi(t) - \varphi(t-h)} & \text{if} \quad t' \in (\varphi(t-h), \varphi(t)] \\ \dfrac{\varphi(t+h) - t'}{\varphi(t+h) - \varphi(t)} & \text{if} \quad t' \in (\varphi(t), \varphi(t+h)) \\ 0 \text{ or else} & \end{cases} \qquad (12)$$

We multiply every term of the equation (1) by a basic function, integrate it over x'. Then we change some integrals (after integration by parts) for simplest quadrature formulae: $\forall x \in G_h$

$$\int_G \frac{\partial^2 u}{\partial x_1^2}(x')\omega_h(x,x')dx' \approx \big(\varphi(x_2+h) - \varphi(x_2-h)\big)/2 \times$$

$$\times \left\{ \frac{u^h(x_1,x_2) - u^h(x_1-h,x_2)}{\varphi(x_1) - \varphi(x_1-h)} - \frac{u^h(x_1+h,x_2) - u^h(x_1,x_2)}{\varphi(x_1+h) - \varphi(x_1)} \right\} \equiv A(x)u(x), \qquad (13.a)$$

$$\int_G \frac{\partial^2 u}{\partial x_2^2}(x')\omega_h(x,x')dx' \approx \big(\varphi(x_1+h) - \varphi(x_1-h)\big)/2 \times$$

$$\times \left\{ \frac{u^h(x_1,x_2) - u^h(x_1,x_2-h)}{\varphi(x_2) - \varphi(x_2-h)} - \frac{u^h(x_1,x_2+h) - u^h(x_1,x_2)}{\varphi(x_2+h) - \varphi(x_2)} \right\} \equiv B(x)u(x). \qquad (13.b)$$

Thus let us note that discrete operators A and B are defined. An approximate system may be written in a form

$$A(x)u^h(x) + B(x)u^h(x) = \int_G f(x)\,\omega_h(x,x')\,dx' \quad \forall x \in G_h$$

$$u^h(x) = 0 \qquad \forall x \in \Gamma_h. \tag{14}$$

The special form of an approximating error permits us to obtain (even in case $f \in L_2(G)$) the speed of convergence is equal to $h^{1+\alpha}$ $(\alpha \in (0,1))$ in the norm

$$\|v^h(x)\|_0 = \Big[\sum_{x \in G_h} \frac{1}{4} P(x)\big(v^h(x)\big)^2\Big]^{1/2}, \tag{15}$$

where $P(x)$ is the area of the support of the function $\omega_h(x,x')$ with respect to x'. Namely for any $\gamma > 0$ and even for $\varphi(t) = t$ (i.e. the mesh is regular) the solution of the system (14) converges to the solution of the problem (1) - (2) and there is an estimation

$$\|u(x) - u^h(x)\|_0 \leq c\,\|f\|_{L_2(G)} \times h^{1+\alpha} \qquad \alpha \in (0,1)$$

where constant c depends on α only.

To prove this statement and the more general one we shall heed some results about a steady characteristic and approximation.

<u>Lemma 1</u>. If functions φ and ψ are defined on $\bar{G}_h$ and satisfy systems

$$\big(A(x) + B(x)\big)\varphi(x) = A(x)\,\delta(x) \qquad \forall x \in G_h$$

$$\varphi(x) = 0 \qquad \forall x \in \Gamma_h$$

and

$$\big(A(x) + B(x)\big)\psi(x) = B(x)\,\delta(x) \qquad \forall x \in G_h$$

$$\psi(x) = 0 \qquad \forall x \in \Gamma_h$$

then the estimations are valid

$$\|\varphi\|_0 \leqslant c_1 \|\delta\|_0 \qquad \text{and} \qquad \|\psi\|_0 \leqslant c_2 \|\delta\|_0 ,$$

where c_1 , c_2 do not depend on h and δ .

Lemma 2. Let $u(x)$ be the solution of the problem (1) - (2) and the condition (8). Then

$$A(x)u(x) + B(x)u(x) = \int_G f(x)\omega_h(x,x')\,dx' + \sum_{i=1}^{\ell} h^{2i} \int_G g_i(x)\omega_h(x,x')\,dx' +$$

$$+ h^{2\ell+3+\alpha} B(x)\xi_h(x) + h^{2\ell+3+\alpha} A(x)\eta_h(x) \qquad \forall x \in G_h$$

where the functions g_i do not depend on h . And if in formula (10) γ ies condition

$$\gamma \geqslant 2\ell + 3 + \alpha , \tag{16}$$

then g_i satisfies(9), where ℓ -i is instead of ℓ ; ξ_h and η_h depend on h but they are regular bounded with respect to h :

$$\|\eta_h\|_0 \leqslant c_3 \qquad \text{and} \qquad \|\xi_h\|_0 \leqslant c_4 . \tag{17}$$

This result is sufficient for using the proof which is similar to the second section proof (see also the proof scheme in [1]). This gives us

Lemma 3. In conditions (9) the solution of the system (14) converges to the solution of the problem (1) - (2) and there are $(\ell+1)$ functions $v_k \in C_{2\ell-2k+3+\alpha}(G)$ which do not depend on h and for any $n > 1$ $(0 < h < 1)$

$$u^h(x) = u(x) + \sum_{k=1}^{\ell+1} h^{2k} v_k(x) + h^{2\ell+3+\alpha} \varepsilon^h(x) \qquad \forall x \in G_h. \tag{18}$$

If $\gamma \geqslant 2\ell + 3 + \alpha$ then for each v_k the condition (8) is valid ($\ell - k$ is instead of ℓ) and the function ε^h is bounded for all h in the sense:

$$\| \varepsilon^h \|_0 \leqslant c_5 . \tag{19}$$

Remark. The estimate (19) involves estimate modulo

$$| \varepsilon^h(x) | \leqslant c_6 h^{-1} d^{-\gamma}(x) \| \varepsilon^h \|_0 \qquad \forall \ x \in G_h, \tag{20}$$

where $d(x)$ is defined by (4).

That is why the final result is formulated as follows.

Theorem 1. Given the conditions of lemma 3, one may find (with the help of $(\ell+2)$ solutions of the system (14) with different mesh sizes h_k) the approximate solution of the problem (1) - (2) which has a precision of order $\bar{h}^{2\ell+2+\alpha}$:

$$| u(x) - \bar{u}(x) | \leqslant c_7 d^{-\gamma}(x) \bar{h}^{2\ell+2+\alpha}$$

where c_7 do not depend on x , h and $\bar{h} = \max\limits_{1 \leqslant k \leqslant \ell+2} h_k$.

Remark. The proof and the above technique are suitable for the problem (1) - (2) in the cases:

a). when the right part of (1) has first sort discontinuity lines which are parallel to coordinate axes;

b). when the solution has first sort discontinuity on such lines and there are conditions for the solution

$$u(x-0) = u(x+0)$$

and for its normal derivative

$$\frac{\partial u}{\partial n}(x-0) = \frac{\partial u}{\partial n}(x+0) + w$$

(n is a unit normal direction). An argument $x+0$ means that we take a limit on the right side of the discontinuity lines. It is similar for $x-0$.

Both in the first case and in the second case it is necessary that the discontinuity lines are the mesh lines. But in the second case we must condense the mesh lines near intersection of boundary and the discontinuity lines.

4. The elliptic equation in a domain with a smooth boundary.

When we solve the elliptic equation in a domain with a smooth boundary we have some difficulties of an approximation because the mesh is not regular near boundary.

A way to avoid this difficulty is to adjust mesh and the domain. We explain it by example with a domain with one boundary component. Let there be transformation which is smooth enough and which transforms the initial domain to the circle. Then we need introduce polar coordinates. Now the domain is a rectangle, where the equation is defined. If the domain has two boundary components we reduce it to ring and so on.

When the transformation may be found easily such a way has an algorithmical profit.

If the Dirichlet problem is examined then one may use a method, described in [4] . This method contains a multipoint interpolation formula which has a high precision. By choice of free parameters in formula one may take a diagonal dominance in the obtain algebraic equations. It permits us to put up stable system of algebraic equations.

Using the indicated methods one may solve Dirichlet's problem in a different domain.

In this section we examine the question about an improvement of a solution of the penalty method. It follows from [6] that the use of a boundary penalty in the Ritz method permits us to come from the Dirichlet problem to the third boundary problem. And if the solution is smooth enough then adding the penalty to the variational functional is equivalent to coming from a problem

$$\begin{aligned} \mathcal{L}u &= f \quad \text{in} \quad G \\ u &= g \quad \text{on} \quad \Gamma \end{aligned} \tag{1}$$

to the problem

$$\begin{aligned} \mathcal{L}u^\varepsilon &= f \quad \text{in} \quad G \\ u^\varepsilon + \frac{\partial u^\varepsilon}{\partial n} &= g \quad \text{on} \quad \Gamma. \end{aligned} \tag{2}$$

Here

$$\mathcal{L} \equiv -\sum_{i,j=1}^{p} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{p} b_i \frac{\partial u}{\partial x_i} + c \tag{3}$$

is the elliptic differential operator, G is a domain in p - space R^p with a boundary Γ , n is an unit external co-normal.

Similar as above let us suppose the existence of the decomposition

$$u^\varepsilon = u + \sum_{i=1}^{\ell} \varepsilon^i v_i + \varepsilon^{\ell+\mu} \xi^\varepsilon, \qquad \ell > 0, \ \mu \in [0,1] \tag{4}$$

where the functions v_i do not depend on ε and the function ξ^ε is bounded. Then we substitute (4) in the problem (2) and compare coefficients for every power ε . Thus we put up a sequence of problems:

1) $$\Delta u = f \quad \text{in} \quad G$$
$$u = g \quad \text{on} \quad \Gamma,$$

2) $$\Delta v_1 = 0 \quad \text{in} \quad G$$
$$v_1 = \frac{\partial u}{\partial n} \quad \text{on} \quad \Gamma,$$

i) $$\Delta v_i = 0 \quad \text{in} \quad G$$
$$v_i = -\frac{\partial v_{i-1}}{\partial n} \quad \text{on} \quad \Gamma \quad \forall i = 1, \dots, \ell$$

$\ell + \mu$) $$\Delta \xi^\varepsilon = 0 \quad \text{in} \quad G$$
$$\xi^\varepsilon + \varepsilon \frac{\partial \xi^\varepsilon}{\partial n} = -\frac{\partial v_\ell}{\partial n}.$$

If this problem has a unique solution then the functions are found recurrently . The quantity of the members in (4) is bounded by a power of smoothness of the problem data.

If one has the decomposition (4) one may obtain the solution of (1) by linear combination with accuracy of order $\varepsilon^{\ell+\mu}$, using for it $\ell+1$ solutions with different ε_k .

5. The numerical solution of an evolutional problem with a bounded operator.

We need this section to show that this method is universal for any type of equations.

In Hilbert's space X with a norm $\|x\| = (x, x)^{1/2}$ we consider functions of one real variable $t \in [0,T]$. In this section we write $f \in C^k$ if f has a value in X and has k continuous derivations (s.f. [8]).

Let us examine a problem

$$\frac{\partial u}{\partial t} + Au = f, \qquad u(0) = u_0, \tag{1}$$

where $A: X \to X$ is a linear operator, which is regular bounded on $[0,T]$ and positive semi-definite in a sense: $(Ax, x) = 0$ $\forall x \in X$, $t \in [0,T]$. Let the operator A be decomposed in the sum $A(t) = \sum_{i=1}^{n} A_i(t)$, where $A_i: X \to X$ are linear operators, which are regular bounded and positive semi-definite on $[0,T]$.

For numerical solving we use a splitting-up scheme

$$\begin{aligned}
&(I + \tau A_1(t))\, u^\tau(t - \tau \tfrac{n-1}{n}) = u^\tau(t-\tau) + \tau f(t) \\
&(I + \tau A_2(t))\, u^\tau(t - \tau \tfrac{n-2}{n}) = u^\tau(t - \tau \tfrac{n-1}{n}) \\
&(I + \tau A_n(t))\, u^\tau(t) = u^\tau(t - \tau \tfrac{1}{n})
\end{aligned} \tag{2}$$

$$\forall t \in \omega_\tau$$

and

$$u^\tau(0) = u_0,$$

Here ω_τ is a regular net with the size $\tau = T/M$, I is a unit operator. It follows from [1] that the scheme is stable:

$$\max_{t \in \omega_\tau} \|u^\tau(t)\| \leqslant \|u_0\| + T \max_{t \in \omega_\tau} \|f(t)\|.$$

Let us note that the semi-definiteness is taken for the stableness. We may lay aside this supposition if we choose the size τ from the condition of the limitation of the operators A_i.

Theorem 1. Let $f \in C^s$, and $A(t)$ has a smoothness which is enough to have a unique solution $u \in C^{k+1}$ for any right part $f \in C^k$ and any initial value $u_0 \in X$ $(0 \leqslant k \leqslant s)$.

Then there are $(s-1)$ functions $v_j \in C^{s-j+1}$ which do not depend on τ and decomposition

$$u^{\tau}(t) = u(t) + \sum_{j=1}^{s-1} \tau^j v_j(t) + \tau^s \xi_{\tau}(t), \qquad t \in \omega_{\tau} \qquad (3)$$

is valid. Here discrete function ξ_s is bounded:

$$\xi_{\tau} : \max_{t \in \omega_{\tau}} \| \xi_{\tau} \| \leq c ,$$

where constant c depend on norms of derivatives of f and does not depend on τ.

<u>Proof</u> may be constructed such a way. At first let us suppose the decomposition (3) exist.

Let us fix any $t \in \omega_{\tau}$, expel $u^{\tau}(t)$ and $u^{\tau}(t-\tau)$ from (2) (using decomposition (3)) and substitute Taylor's formula for any functions so that we have functions only in the point t. Comparing coefficient of all powers of τ we put up discrete systems

$$\frac{\partial v_i}{\partial t} + A v_i = f_i , \qquad t \in \omega_{\tau} , \qquad v_i(0) = 0 . \qquad (4)$$

Besides, expressions taking part in f_i include only $u, v_1, \ldots, v_{i-1}$. If we change the domain ω_{τ} on the interval $(0, T)$ we obtain list of problems and v_i may be found recurrently The smoothness of v_i follows from supposition of the theorem and the independence of τ follows from a kind of the equations and their right part.

Assuming v_i to be given we define ξ_{τ} so that (3) is valid. Then from the way which we find v_i it follows that

$$\prod_{i=1}^{n} (I + \tau A_i(t)) \xi_{\tau}(t) = \xi_{\tau}(t-\tau) + \tau f_{\tau}(t), \quad t \in \omega_{\tau}, \quad \xi_{\tau}(0) = 0 . \qquad (5)$$

From the stability of this system and from a kind of f_{τ}

(which is a combination of derivatives of u, v_j and any multiplications by A_i) estimates follow for ξ_τ .

On the basis of this result the usual considerations give us

<u>Theorem 2</u>. Given the conditions of theorem 1 and solving s problem with different mesh size τ_z one may find approximate solution of the problem (1) which has precision of order $O(\tau_0^s)$ in the norm of space X in each point of $[0,T]$ (here $\tau_0 = \max\limits_{1 \le z \le s} \tau_z$).

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