MULTILEVEL OPTIMAL CONTROL OF INTERCONNECTED DISTRIBUTED PARAMETER SYSTEMS

M. Amouroux - J.P. Babary - B. Pradin - A. Titli
Laboratoire d'Automatique et d'Analyse des Systèmes du C.N.R.S.
7, Avenue du Colonel Roche
B.P. 4036
31055 TOULOUSE CEDEX
France

I - INTRODUCTION

The control of distributed parameter systems presents a great many theoretical problems and has already given rise, over the last few years, to some fundamental research. However, the results obtained so far do not, in general, enable one to get round the difficulties which arise when the proposed control laws are applied. For this reason, a number of research teams have turned their attention to the use of new concepts such as that of classical decomposition-coordination in hierarchical control. This idea has been introduced either from a somewhat mathematical point of view [1, 2] or from more of a "control" angle [3 to 5] . In the latter case, techniques of dynamic and static hierarchical control (two level) are used to control a collection of interconnected subsystems. This collection is obtained by discretising the initial partial differential problem in space, or in time and space. It would seem at present that research in this direction should be limited to the field of applications. It is concerned at any rate with the resolution of an overall problem, whilst at the present time there is good reason for considering a more important problem, numerous cases of which are to be found in the economic sector, that of the optimal control of a group of interconnected sub-systems the behaviour of each of which is defined by partial differential equations.

The use of hierarchical control techniques to solve such a problem is being examined at the present time at the Laboratoire d'Automatique et d'Analyse des Systèmes in Toulouse as a joint project, undertaken by two groups : "Hierarchical Control" and "Distributed Parameter Processes".

The aim of this paper is to show how the problem has been tackled, and to present some of the results which have already been obtained. In the first part, we show how, for a certain class of distributed parameter systems, it is possible to decompose the overall problem into sub-problems, whilst at the same time retaining their "distributed parameter" nature ; in this part, the coordination task necessary to reach the overall solution is defined. The second part is devoted to the study of the sub-system and its controllability particularly as regards the coordination problems ; the implementation of the actuators is in fact the second essential aspect of this study.

II - OPTIMISATION OF INTERCONNECTED SYSTEMS

II-1 - Brief survey of the principles of decomposition-coordination [4]

As a result of the problems presented by the control of complex systems composed of interconnected sub-systems, (when the overall approach is too costly) or because of theoretical considerations (difficulties of convergence of the algorithms for large scale problems) to introduce new methods, particularly the decomposition-coordination methods which are found in hierarchical control. These techniques use multi-level and multi-objective pyramid shaped control structures, and are able to employ different types of decomposition (or division) of work :

- horizontal division based on the complexity of the process and its "interconnected sub-systems" aspect ;
- vertical division based on the complexity of the control with the levels : regulation, optimisation, self-adaptation, self-organisation ;
- functional division according to the situations which the process will come up against.

The principle of hierarchical control therefore consists of decomposing an overall problem into a certain number of sub-problems $P_i(\mu)$ with parameters related to μ ("intervention vector" or "coordination parameter") in such a way as to satisfy :

$$\text{Sol } P_1(\mu) \ldots P_i(\mu) \ldots P_N(\mu) \quad \Longrightarrow \quad \text{Sol } P \text{ (overall problem)}$$

The entire coordination problem consists, on the higher level, of forcing μ to approach a value μ^* which will lead to the solution of the overall problem. For the optimisation problem with which we are concerned, only the horizontal division can be used in, for example, a two level control structure. Each sub-problem is then defined by a group of two functions, essentially : the mathematical model of the process and the criterion which is associated with it. There are three possible types of coordination :

- manipulation of the criterion function ($\mu = \beta$, Lagrangian parameters associated with the interconnexion constraint ; modification of the criterion function by the coupling terms) ;
- action on the model (μ = X, coupling variables between sub-systems ;
- simultaneous action on the two functions ($\mu = \begin{bmatrix} \beta \\ X \end{bmatrix}$)

It can be shown that the choice of the coordination variables leads to a separable form, the additive $H = \sum_i H_i$ for the Hamiltonian associated with the optimisation problem (H_i = Hamiltonian i, bringing in only index i variables, except for the coordination variable). The examination of H_i enables the sub-problems to be formulated in optimisation terms.

II-2 - Formulation of the overall problem

Consider an overall system composed of N linear invariant, interconnected sub-systems defined by the following equations :

$$\left\{ \begin{array}{ll} \dfrac{\partial Y_i(x,t)}{\partial t} = M_i \left[Y_i(x,t) \right] + X_i(x,t) + B_i(x)\, U_i(x,t) & \\ \text{initial conditions : } Y_i(x,0) = Y_{i0}(x) & i = 1, \ldots, N \\ \text{boundary conditions : } L_i \left[Y_i(x',t) \right] = 0 & \end{array} \right. \quad (1)$$

where $Y_i(x, t)$ is the state variable of the ith sub-system

$x \in \Omega \subset E^m$; $x' \in \partial\Omega$; $t \in [0, T]$

M_i, L_i are matrix differential operators bringing in only derivatives related to the space variables x

$B_i(x)$ are matrices which are functions of x only.

The interconnexion between these N sub-systems is represented by a linear coupling :

$$X_i = \sum_{j=1}^{N} C_{ij}\, Y_j \qquad i = 1 \ldots N \qquad (2)$$

where C_{ij} are interconnexion matrices.

Several types of control are possible ; this study will be limited to the following types :

- distributed controls on the domain Ω : $U_i(x, t)$, $x \in \Omega$

- pointwise control defined on a finite number of points of the space domain :

$$U_i\,(x_k, t),\ k = 1,\ 2,\ \dots p \text{ or } U_i\,(x,\ t) = \sum_{k=1}^{p} U_k\,(t)\ .\ \delta\,(x - x_k)$$

The objective function of the overall system is assumed to be given in a separable additive quadratic form :

$$J = \sum_{i=1}^{N} J_i = \sum_{i=1}^{N} \int_0^T \int_{\Omega} F_i \left[Y_i\,(x,\ t),\ U_i\,(x,\ t)\right] dx\ dt \tag{3}$$

The overall problem is to minimise J subject to the constraints (1) and (2).

II-3 - Decomposition of the problem

Define the Hamiltonian H of the optimisation problem as

$$H = \sum_{i=1}^{N} F_i\left[Y_i\,(x,\ t),\ U_i\,(x,\ t)\right] + \sum_{i=1}^{N} \psi_i^T(x,\ t)\left\{ M_i\left[Y_i\,(x,t)\right] + X_i\,(x,t) + B_i\,(x)\,U_i\,(x,\ t)\right\} + \sum_{i=1}^{N} \beta_i\,(x,t)\left[X_i\,(x,\ t) - \sum_{j=1}^{N} C_{ij}\,Y_j\,(x,\ t)\right] \tag{4}$$

The conditions of optimality are obtained using the Maximum Principle applied to systems governed by partial differential equations.

Since the coupling between the sub-systems is a state variable one, the decomposition method chosen -in fact, the only one possible- is the infeasible method of coordination using the criterion function [4] . In this method, the coupling equations are treated on the coordination level which fixes the β_i (x, t) for the first level sub-problems. In order to avoid the singular problems which could arise (X_i appearing linearly in H_i) Y_i (x, t) is replaced in F_i by its expression in terms of the X_i (x, t). The sub-problems which must be solved can be written as follows :

minimise the criterion
$$J'_i = \int_0^T \int_{\Omega} \left\{ F_i\left[X_i\,(x,\ t),\ U_i\,(x,t)\right] + \beta_i^T\,(x,t)\,X_i\,(x,\ t) - \sum_{j=1}^{N} \beta_j^T\,(x,\ t)\,C_{ji}\,Y_i\,(x,\ t)\right\} dx\ dt \tag{5}$$

subject to the constraints
and the initial and boundary conditions
$$\frac{\partial Y_i}{\partial t} = M_i\left[Y_i\right] + X_i + B_i\,U_i$$

On the coordination level there is the possibility of using different algorithms to determine the β_i (x, t). The gradient algorithm has been chosen here because of its ease of application

$$\beta_i^{n+1}(x,\ t) = \beta_i^{n}(x,\ t) - K_c\left[\frac{\partial H}{\partial \beta_i(x,t)}\right]^n \tag{6}$$

where n is the iteration index on the coordination level and K_c is the iteration constant.

II-4 - Resolution of the sub-problems

Discretisation of the sub-system equations with respect to the space variables (or with respect to the space and time variables) enables one to arrive at dynamic (or static) interconnected sub-systems, and to obtain optimum control of these ; or rather, it enables one to define interconnected dynamic (or static) sub-problems which are solved by classical hierarchical control techniques [3, 5, 6].

Here, the sub-problem will be solved by the application of the Maximum Principle [7] . The conditions of optimality are written as [8] :

$$\left\{\begin{aligned} &\frac{\partial Y_i}{\partial t} = \frac{\partial H}{\partial \Psi_i} = M_i\left[Y_i\right] + X_i + B_i\ U_i \\ &\frac{\partial \Psi_i}{\partial t} = -\frac{\partial H}{\partial Y_i} - (-1)^{\ell}\ \frac{\partial^{\ell}}{\partial x^{\ell}}\left(\frac{\partial H}{\partial\left[\frac{\partial^{\ell} Y_i}{\partial x^{\ell}}\right]}\right) \\ &\frac{\partial H}{\partial U_i} = 0 \\ &\frac{\partial H}{\partial X_i} = 0 \end{aligned}\right. \tag{7}$$

with the initial and boundary conditions on Y_i as defined earlier those relative to $\Psi_i(x,\ t)$:

$$\Psi_i(x,\ T) = 0 \quad ; \quad \frac{\partial^{\ell-1}}{\partial x^{\ell-1}}\left(\frac{\partial H}{\frac{\partial^{\ell} Y_i}{\partial x^{\ell}}}\right)\Bigg|_{\partial\Omega} = 0 \tag{8}$$

These distributed parameter are solved by the eigenfunctions method by seeking a solution in the form :

$$\begin{aligned} U_i(x,\ t) &= \sum_n u_{in}(t)\ \varphi_{in}(x) \\ Y_i(x,\ t) &= \sum_n y_{in}(t)\ \varphi_{in}(x) \end{aligned} \tag{9}$$

It will be supposed that such Eigenfunctions exist. Thus the problem amounts to the resolution of ordinary differential systems in $u_{in}(t)$ and $y_{in}(t)$. Several types of coordination could then be considered :

- the decomposition of the $\beta_i(x, t)$ in terms of the $\varphi_{in}(x)$:

$$\beta_i(x, t) = \sum_n \beta_{in}(t)\,\varphi_{in}(x) \tag{10}$$

In this case, the coordination is carried out by the $\beta_{in}(t)$ which considerably reduces the transfer of information between levels.

- coordination by the functions $\tilde{\beta}_i(x_k, t)$, defined as a certain number of points in the space domain. In this case, the $\tilde{\beta}_i$ can be considered as "pseudo-controls" for sub-system i. This type of coordination presents the problem of finding optimal actuating points x_k and this problem is tackled in the second part of the paper.

II-5 - Example

In this section, the results obtained on an example made up of two sub-systems defined on the domain $\Omega =]0, 1[$ with :

$$M_i = \left[k_i \frac{\partial^2}{\partial x^2} + 1 \right] \quad ; \quad B_i = 1 \quad ; \quad C_{ij} = \begin{cases} 0 & \text{if } i = j \\ -1 & \text{if } i \neq j \end{cases}$$

$$L_i = \frac{\partial}{\partial x} \quad ; \quad J_i = \int_0^T \int_\Omega \sum_{i=1}^{2} \left[Y_i^2 + k_3 U_i^2 \right] dx\, dt$$

The conditions of optimality are written as :

$$\begin{cases} \dfrac{\partial Y_i}{\partial t} = k_i \dfrac{\partial^2 Y_i}{\partial x^2} + Y_i - X_i + U_i \\ \dfrac{\partial \psi_i}{\partial t} = -k_i \dfrac{\partial^2 \psi_i}{\partial x^2} - \psi_i - \sum_{j=1}^{2} C_{ji}\,\beta_j \end{cases} \qquad \begin{cases} 2 k_3 U_i + \psi_i = 0 \\ -\psi_i + \beta_i + 2X_i = 0 \end{cases} \qquad i = 1, 2$$

with the boundary conditions :

$$L_i\left[Y_i\right] = 0 \quad ; \quad L_i\left[\psi_i\right] = 0$$

$$Y_i(x, 0) = Y_{i0}(x) \quad ; \quad \psi_i(x, T) = 0$$

The following criterion is to be minimised :

$$J'_i = \int_0^T \int_\Omega \left[Y_i^2 + k_3 U_i^2 + \beta_i X_i - \sum_{j=1}^{2} \beta_j^T C_{ji} Y_i \right] dx\, dt$$

The resolution of the sub-problems by decomposition into Eingenfunctions and the use of the coordinator [10] gave the results shown in figure 1.

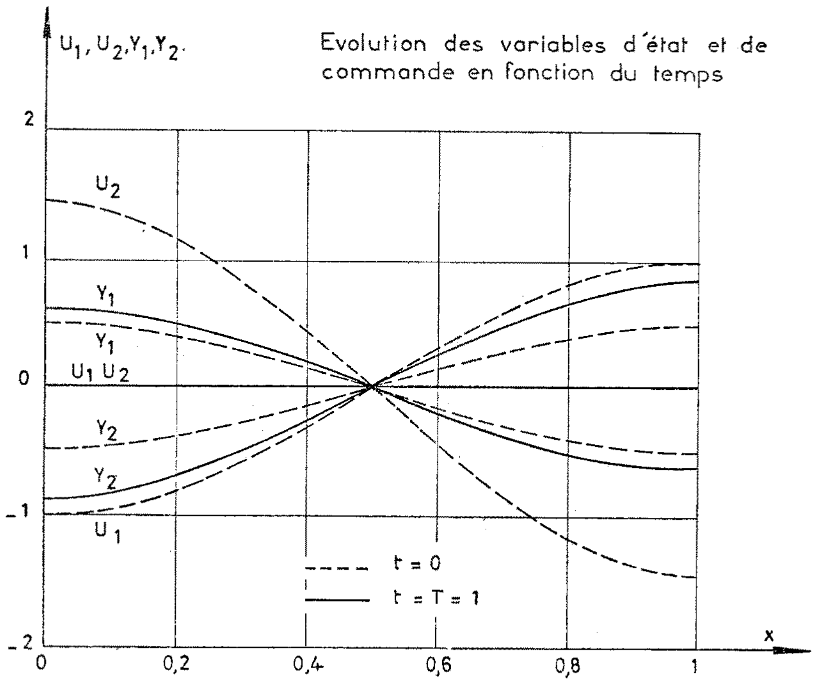

Figure 1

III - STUDY OF THE SUB-SYSTEM

In order to apply control to a distributed parameter system it is necessary to study a certain number of problems such as the observability of the system in relation to the choice of the type and number of sensors, and its controllability, depending on the control action chosen.

Here, the case in which the sub-systems (1) are controlled at a certain number of points situated in the interior of the space domain is considered :

$$\frac{\partial Y}{\partial t}(x,t) = M\left[Y(x,\ t)\right] + X(x,\ t) + \sum_{j=1}^{p} U_j(t)\ .\ \delta(x - x_j) \qquad (11)$$

$$L\left[Y(x',\ t)\right] = 0 \quad ; \quad Y(x,\ 0) = Y_0(x)$$

Using the hypothesis of the separability of the criterion formulated in the previous paragraph, the criterion :

$$J(U) = \int_0^T \int_\Omega F(Y(x,\ t),\ U(t))\, dx\, dt \qquad (12)$$

with $\quad U = \left[U_1,\ \ldots,\ U_p\right]^T$

is associated with equations (11).

The problem is one of minimising J (u) subject to the constraint (11), with respect to U and to x_j (j = 1 . . . p). In what follows, the problem of obtaining the control law is decoupled from that of determining the optimal positions of the actuating points. The former can be tackled in different ways [9], for example :

(i) by minimising the criterion J (u_{opt}) with respect to x_j (j=1. . . p) ; in this case, in general, the solution depends on the state of the system (cf. III-1).

(ii) by extremalising a characteristic function which depends on the system only and sometimes on the type of criterion considered (cf. III-2).

The state mode is obtained from the transformation defined in the preceding paragraph ; it is in the form :

$$\dot{y}(t) = A\, y(t) + B\, u(t) + C\, \xi(t)$$

$$A = \begin{bmatrix} \lambda_1 & & \mathbb{O} \\ & \ddots & \\ \mathbb{O} & & \lambda_n \\ & & & \ddots \end{bmatrix}_{\infty \times \infty} \qquad B = \begin{bmatrix} \varphi_1^*(x_1) & \cdots & \varphi_1^*(x_p) \\ \vdots & & \vdots \\ \varphi_n^*(x_1) & \cdots & \varphi_n^*(x_p) \\ \vdots & & \vdots \end{bmatrix}_{\infty \times p} \tag{13}$$

where φ^* are the Eigenfunctions of the adjoint operator M^*.

Certain intrinsic properties of the system can be studied from this model. Indeed, whatever the approach adopted, it is necessary to study the controllability of the system so as to define the collection of admissible positions for the actuating points ; more precisely, the hypothesis of the controllability of the system is linked to the property [10] :

$$\text{row}\,\left[\varphi_i^*(x_1)\ \ldots\ \varphi_i^*(x_p)\right] = 1 \qquad i = 1, 2 \ldots \tag{14}$$

In practice, the application of such a property gives results which are unusable. Because of this, a modal reduction of the system is considered and in particular the influence of this reduction on the optimal position of the actuating points is studied.

III-1 - Minimisation of the criterion J (u_{opt})

Two types of criteria are considered :

III-1-a - Quadratic criterion

$$J(U) = \int_0^{\infty} \left[\int_{\Omega} Y^T(x, t)\, Q\, Y(x, t)\, dx + U^T R\, U(t) \right] dt \qquad (15)$$

The use of the transformation (9) leads to the equivalent criterion :

$$J(u) = \int_0^{\infty} \left[y^T(t)\, Q'\, y(t) + u^T(t)\, R\, u(t) \right] dt \qquad (16)$$

with $Q' = \int_{\Omega} \varphi^T(x)\, Q\, \varphi(x)\, dx$

The optimal control, minimising the criterion J (U) with respect to U is given by the formula [11] :

$$U_{opt} = - R^{-1} B^T(x_1, x_2, \ldots, x_p)\, K(x_1, x_2, \ldots, x_p)\, y(t) \qquad (17)$$

where K is the matrix solution of the algebraic Riccati equation.

The optimal distribution, if it exists, is the solution of

$$\min_{(x_1, \ldots, x_p)} J_{opt}(x_1, \ldots, x_p) = \min_{x_1, \ldots, x_p} \left[\frac{1}{2} y^T(0)\, K(x_1, \ldots, x_p)\, y(0) \right] \qquad (18)$$

III-1-b - Energy criterion

$$J(U) = \int_0^T U^T(t)\, U(t)\, dt \qquad (19)$$

The minimisation of the criterion with respect to U leads to the formulation of the control law [12] :

$$U_{opt}(x_1, \ldots, x_p, t) = B^*(x_1, \ldots, x_p)\, \Phi^*(T-t) \left[\int_0^T \Phi(T-\tau)\, BB^*\, \Phi^*(T-\tau)\, d\tau \right]^{-1} y\,d \qquad (20)$$

where B^* represents the adjoint matrix of B, Φ^*, the adjoint matrix of the state transition matrix Φ and yd the desired state.

Assume that :

$$U_{opt}(x_1, \ldots, x_p, t) \triangleq M(x_1, \ldots, x_p, t)\, y\,d \qquad (21)$$

The optimal distribution is the solution of :

$$\min_{(x_1, \ldots, x_p)} \left\| U_{opt}(x_1, \ldots, x_p, t) \right\| \qquad (22)$$

III-2 - Extremalisation of a characteristic function

To each type of criterion defined above it is possible to associate a characteristic function, and this makes it possible to obtain an optimal distribution of the actuating points independently of the state of the system.

III-2-a - Quadratic criterion

Since the matrix K is positive definite and symmetric :

$$0 < \frac{J\,(U_{opt})}{y^T\,(0)\,y\,(0)} \leqslant \frac{1}{2}\,\|K\| \tag{23}$$

where $\|K\| = \left(\sum_i \sum_j k_{ij}^{\;2}\right)^{1/2}$

The problem of the minimisation of the criterion $J\,(u_{opt})$ with respect to $(x_1, \ldots, x_p)$ is therefore transformed into that of minimising its upper limit :

$$\min_{x_1, \ldots, x_p} \|K\,(x_1, \ldots, x_p\,\| \tag{24}$$

III-2-b - Energy criterion

In the same way the upper limit on the norm of the control U on (O, T) is minimised :

$$\min_{x_1, \ldots, x_p} \|M\,(x_1, \ldots, x_p, t)\,\| \tag{25}$$

III-2-c - Criterion of controllability [10]

If W represents the upper limit of $\|U\,(t)\|$, the collection of states obtainable from this control is defined by :

$$y^T\,(t)\,P^{-1}\,(x_1, \ldots, x_p)\,y\,(t) \leqslant W \tag{26}$$

where P is a square matrix the general term of which is given by :

$$P_{ij} = \sum_{k=1}^{P} \varphi_i^*\,(x_k)\,\varphi_j^*\,(x_k) \int_0^T e^{(\lambda_i + \lambda_j)(T-\tau)}\,d\tau \tag{27}$$

This collection forms a hyper-ellipsoid the square of whose volume is proportional to the determinant of the matrix P. The optimal distribution of the actuating points can be linked to the maximisation with respect to $(x_1, \ldots, x_p)$ of the volume of the attainable domain, therefore to that of the determinant P :

$$\max_{x_1, \ldots, x_p} \left[\det P (x_1, \ldots, x_p) \right] \tag{28}$$

III-3 - Example

From a practical point of view, these methods can be applied to distributed systems whose model is reduced to the order ν . In particular, the influence of this order on the distribution of the actuating points can be studied. Consider the example (with only one actuating point) :

$$\begin{cases} \dfrac{\partial Y}{\partial t} = \beta \dfrac{\partial^2 Y}{\partial x^2} + \gamma Y + U(t)\, \delta (x - x_1) \\ Y(0, t) = Y(1, t) = 0 \quad ; \quad x, x_1 \in]0, 1[\quad ; \quad t \in [0\ T] \end{cases}$$

The optimal distribution of the actuating points in relation to the different criteria considered above is shown in figures 2 and 3.

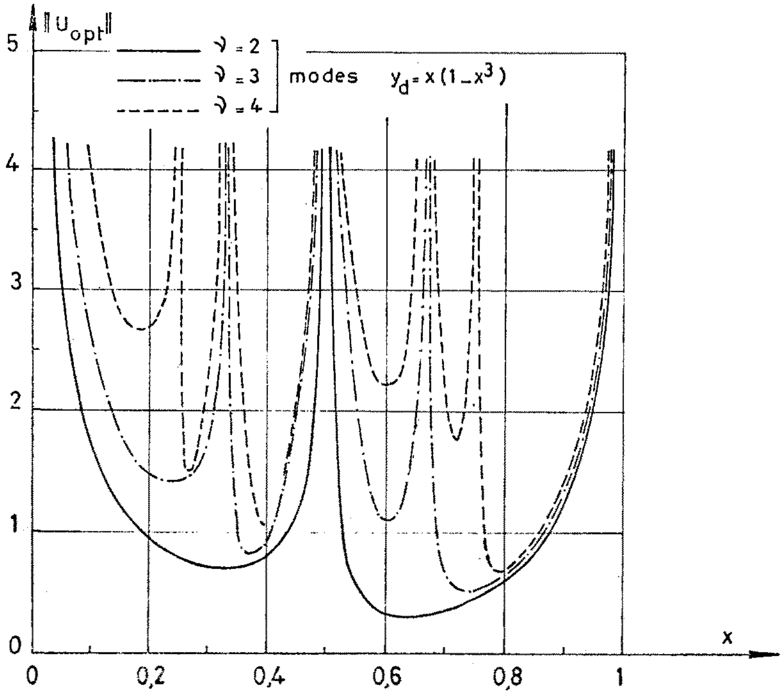

Figure 2 - Critères III-1-b

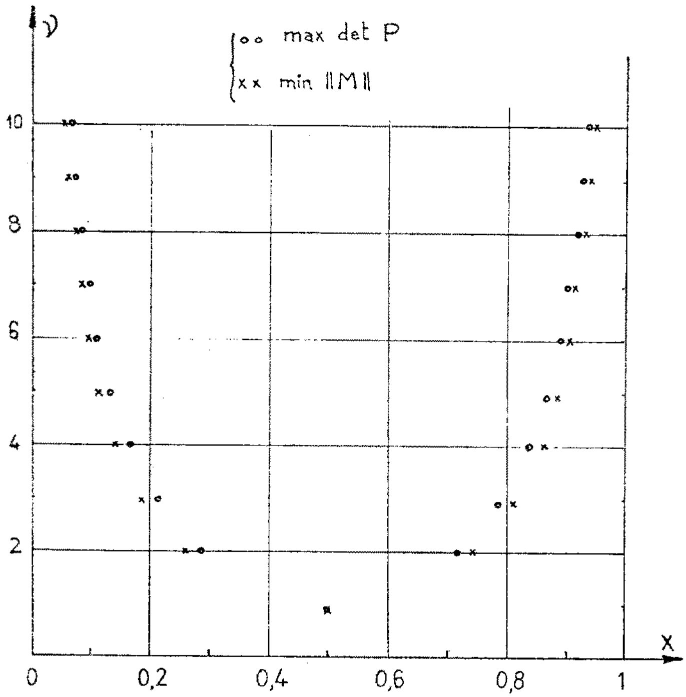

Figure 3 - Critères III-2-b et III-2-c

IV - CONCLUSION

In this paper, we have shown how the problem of the optimal control of interconnected distributed parameter systems using hierarchical control techniques can be tackled. The techniques given here enable one to obtain a collection of control sub-problems which retain their "distributed parameter" nature.

In this decomposition, it is necessary to choose a set of coordination variables which lead to an additive separable form for the Hamiltonian. Each of the sub-problems can be solved using the Maximum Principle. On the level of each sub-system, different types of criteria have been defined and these enable an optimal distribution of a collection of actuating points to be determined. In addition, it was considered necessary to include the study of a sub-system (control, controllability, application of actuators, and the dual problem of observation, observability and the implementation of sensors) taking into account the exchange of information between the different levels of the hierarchical structure .

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