ON SUFFICIENCY OF THE NECESSARY OPTIMALITY OF L.S.PONTRYAGIN'S MAXIMUM PRINCIPLE ANALOGUES TYPE

V.V.Leonov

Institute of Mathematics
Siberian Branch
U.S.S.R. Academy of Sciences
Novosibirsk

The creation of Pontryagin's maximum principle and R.Bellmann's dynamic programming method has promoted the intensive development of the optimal process theory in the recent time.The use of maximum principle analogues for the processes of quite different nature has proved to be most successful.But it has been established that the method has a limitation,for the maximum principle analogues give only necessary conditions of the control optimality.

We have found that for processes of a special form but of a different nature (such as descrete processes,processes with delay, distributed - parameter processes and those which obey ordinary differential equations) conditions of the corresponding maximum principle analogues are not only necessary but sufficient conditions of optimality.It turns out, that if the process describes a conflict situation the optimal behaviour of both parts is realized in pure strategies.

Because of restriction of the paper size we present two particular results only.

Let us use the notation

$$\operatorname{extr}^{\{z\}} = \begin{cases} \max \ldots, & \text{if } z > 0, \\ \min \ldots, & \text{if } z < 0 . \end{cases}$$

We shall be concerned with the process

$$\frac{dx}{dt}=Q(t)x+\sum_{j=1}^{m_1-1} a_j(u_j,t)g_j(x,t)+$$

$$+\sum_{\ell=1}^{m_2-1} b_\ell(v_\ell,t)f_\ell(x,t)+R(t)u_{m_1}+S(t)v_{m_2}+w(t), \qquad (1)$$

where the control parameters u_j ($j=1,\dots,m_1$) and v_ℓ ($\ell=1,\dots,m_2$) belong accordingly to the limited closed sets U and V_ℓ of Euclidean spaces E^{p_j} and E^{q_ℓ}, the x are elements of the n-space, $Q(t)$ are continuous for $t\in[0,T]$ $(n\times n)$- matrices, $a_j(u_j, t)$ and $b_\ell(v_\ell, t)$ are n-space vectors continuously dependent on their arguments on sets $[0,T]\times U_j$ and $[0,T]\times V_\ell$, $R(t)$ and $S(t)$ are continous for $t\in[0,T]$ $n\times p_{m_1}$ and $n\times q_{m_2}$ - dimension matrices respectively, while $w(t)$ is apriori known n - space vector function, continuous for $t\in[0,T]$. The sclar functions $g_j(x,t)$ and $f_\ell(x,t)$ will be considered to be linear in regard to x :

$$g_j(x,t)=\sum_{\nu=1}^{n} g_{j\nu}(t)x_\nu+g_{j0}(t) \qquad (j=1,\dots,m_1-1), \quad (2)$$

$$f_\ell(x,t)=\sum_{\mu=1}^{n} f_{\ell\mu}(t)x_\mu+f_{\ell0}(t) \qquad (\ell=1,\dots,m_2-1), \quad (3)$$

where $g_{j\nu}(t)$ and $f_{\ell\mu}(t)$ are continuous functions of $t\in[0,T]$.

Let the initial state of system (1) be set:

$$x(0)=x_0 . \qquad (4)$$

Let us assume that the first part can choose $u_j\in U_j$ $(j=1,\dots,m_1)$ and the second - $v_\ell\in V_\ell$ ($\ell=1,\dots,m_2$) arbitrarily, if only controls are measurable and the 1-st part means to maximize the criterion

$$R(x(T)) = c^* x(T), \qquad (5)$$

the 2-nd part, on the contrary, - to minimize (5).

It appears that if the quantities (2) and (3) maintain their fixed signs along the trajectory for $t<T$ at any permissible system (1) control with the set initial state (4), then any measurable permissible control, which for almost every $t<T$ obeys the equalities

$$\operatorname*{extr}_{u_j\in U_j}^{\{g_j\, x(t),t\}} \sum_{\nu=1}^{n} \psi_\nu(t)a_{j\nu}(u_j,t) =$$

$$=\sum_{\nu=1}^{n} \psi_\nu(t)a_{j\nu}(u_j(t),t) \qquad (j=1,\dots,m_1-1), \qquad (6)$$

$$\max_{u_{m_1}\in U_{m_1}} \psi^*(t)R(t)u_{m_1} = \psi^*(t)R(t)u_{m_1}(t), \tag{7}$$

$$\operatorname{extr}^{\{-f_\ell(x(t),t)\}} \sum_{\mu=1}^{n} \psi_\mu(t) b_{\ell\mu}(v_\ell, t) =$$

$$= \sum_{\mu=1}^{n} \psi_\mu(t) b_{\ell\mu}(v_\ell(t), t) \qquad (\ell = 1, \ldots, m_2 - 1), \tag{8}$$

$$\min_{v_{m_2}\in V_{m_2}} \psi^*(t)s(t)v_{m_2} = \psi^*(t)s(t)v_{m_2}(t), \tag{9}$$

where $\psi^*(t)=(\psi_1(t),\ldots,\psi_n(t))$, is optimal for the one and the other parts, i.e. it realizes the saddle strategy of both parts.In a general case the similar controls are not defined uniquely, but all of them obey the "saddle" condition. Then either player may choose his strategy in advance in virtue of equalities (6)-(9), having no information about the other player's behaviour. Quantities $\psi_\nu(t)$ (ν = 1,...,n) are defined uniquely by the system

$$\frac{d\psi_\nu}{dt} = -\sum_{s=1}^{n} q_{s\nu}(t)\psi_\nu - \sum_{j=1}^{m_1-1} q_{j\nu}(t) \operatorname{extr}_{u_j\in U_j}^{\{g_j\}} \sum_{s=1}^{n} \psi_s a_{js}(u_j, t) - \tag{10}$$
$$- \sum_{\ell=1}^{m_1-1} f_{\ell\nu}(t) \operatorname{extr}^{\{-f_\ell\}} \sum_{s=1}^{n} \psi_s b_{\ell s}(v_\ell, t) \qquad (\nu = 1, \ldots, n),$$

under the right-end conditions

$$\psi(t) = c. \tag{11}$$

An algorithm for determining the sign definiteness of quantities (2), (3) has been worked out.

The establishment of optimality conditions in the multi-stage processes is analogous.

Let us consider the multi-stage process

$$x(n) = Q_n x(n-1) - \sum_{j=1}^{\ell_n - 1} a_j^n(u_j(n)) g_j^n(x(n-1)) +$$
$$+ \sum_{j=1}^{t_n - 1} b_j^n(v_j(n)) f_j^n(x(n-1)) + R_n u_{\ell_m}(n) + s_n v_{t_n}(n) + w_n , \tag{1'}$$

where $x(n) \in E^{m_n}$, $Q_n - m_n \times m_{n-1}$, is matrix, $a^n(u_j(n))$ and $b_j^n(v_j(n))$ depend continuously on their arguments on $u_j(n) \in U_j^n \subset E^{q_j^n}$, $v_j(n) \in V_j^n \subset E^{q_j^n}$, R_n and S_n are $m_n \times q_{e_n}^n -$, $m_n \times p_{t_n}^n$ -dimension matrices respectively w_n is m_n - dimension vector $g_j^n(x(n-1))$ and $f_j^n(x(n-1))$ are assumed to be of the form (2) and (3) , hence

$$g_j^n(x(n-1)) = \sum_{\nu=1}^{m_n} g_{j\nu}^n x_\nu(n-1) + g_{j0}^n \quad (j=1,\dots,\ell_n - 1), \tag{2'}$$

$$f_j(x(n-1)) = \sum_{\mu=1}^{m_n} f_{j\mu} x_\mu(n-1) + f_{j0} \quad (j=1,\dots,t_n - 1). \tag{3'}$$

The first part means to maximize the criterion

$$R(x(N)) = c^* x(N), \tag{5'}$$

by a permissible control u , the second - to minimize (5').

If the quantities (2') and (3') maintain the fixed sign on the process (1') at every n - stage, i.e. $sgn\, g_j^n(x(n-1)) \equiv (-1)^{\pi_j^n}$, $sgn\, f_j^n(x(n-1)) \equiv (-1)^{\delta_j^n}$ for any permissible control, then every optimum (saddle) control of both parts $\{\bar{u}_j(n)\}$, $\{\bar{v}_j(n)\}$ obeys the equalities

$$\underset{u_j(n)\in U_j^n}{extr}\{(-1)^{\pi_j^n}\} \sum_{\nu=1}^{m_n} \psi_\nu(n)\, a_{jn}^n(u_j(n)) = \sum_{\nu=1}^{m_n} \psi_j(n)\, a_{j\nu}^n(\bar{u}_j(n)), \tag{6'}$$

$$\max_{u_{\ell_n}(n)\in U_{\ell_n}^n} \psi^*(n) R_n u_{m_n}(n) = \psi^*(n) R_n \bar{u}_{m_n}(n), \tag{7'}$$

$$\underset{v_j(n)\in V_j^n}{extr}\{(-1)^{\delta_j^n}+1\} \sum_{\nu=1}^{m_n} \psi_\nu(n)\, b_{j\nu}^n(b_j(n)) = \sum_{\nu=1}^{m_n} \psi_\nu(n)\, b_{j\nu}^n(\bar{v}_j(n)) \tag{8'}$$

$$(j=1,\dots,t_n - 1),$$

$$\min_{v_{t_n}(n)\in V_{t_n}^n} \psi^*(n) S_n v_{t_n}(n) = \psi^*(n) S_n \bar{v}_{t_n}(n), \tag{9'}$$

where $\psi^*(n) = (\psi_1(n),\dots,\psi_{m_K}(n))$ and quantities $\psi_\nu(n)$ $(\nu=1,\dots,m_n)$ are defined by the system

$$\psi_0(n-1) = \sum_{s=1}^{m_n} q_{s0}^n \psi_s(n) + \sum_{j=1}^{\ell_n - 1} q_{j\nu}^n\, extr\{(-1)^{\pi_j^n}\} \sum_{s=1}^{\ell_n - 1} a_{js}^n(u_j(n))\psi_s(n) +$$

$$+\sum_{j=1}^{t_n-1} b_{j\nu}\, extr\left\{(-1)^{\delta_j^n}+1\right\}\sum_{s=1}^{m_n}\psi_s(n)\,b_{js}(v_j(n)) \quad (\nu=1,\dots,m_{n-1}), \qquad (10')$$

to be solved under the right-end conditions

$$\psi(N) = c \qquad (11')$$

It should be noticed in conclusion, that if (1), (1') have no members with variables $v-1$ the problem becomes an ordinary optimal control one, and the above cited results are usually true.