ON FINAL STOPPING TIME PROBLEMS

(Summary) [1]

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§1. Let us consider the state equation of a dynamical system

$$(1)\qquad \begin{cases} \dot{y}_{xt}(s) = f(s, y_{xt}(s)) & 0 \leqq t \leqq s \leqq T < +\infty \\ y_{xt}(t) = x & x \in \mathbb{R}^n \end{cases}$$

with the cost function

$$(2)\qquad J_{xt}(\tau) = g(\tau, y_{xt}(\tau)) + \int_t^\tau l(s, y_{xt}(s))ds \qquad 0 \leqq t \leqq \tau \leqq T\ ,\quad x \in \mathbb{R}^n\ .$$

Our purpose is to find the function

$$(3)\qquad \rho(t,x) = \min\left\{J_{xt}(\tau)\ /\ t \leqq \tau \leqq T\right\}\quad ,\ \forall\ (t,x) \in [0,T] \times \mathbb{R}^n$$

and the smallest time $\tau_{xt} \in [t,T]$ for which we get

$$(4)\qquad \rho(t,x) = J_{xt}\left(\tau_{xt}\right)\quad ,\qquad \forall\ (t,x) \in [0,T] \times \mathbb{R}^n\ .$$

(1) — (4) will be called Pb I .

In §2 we show (Theorem 1) properties of regularity of the function $\rho(t,x)$ and we get some necessary conditions (Theorem 2) that $\rho(t,x)$ must satisfy if (3) holds. As we intend to study Pb I with the tools of the theory of variational inequalities we introduce:

.) the new unknown

$$u(t,x) = [\exp(-b(t))]\,[\rho(T-t, x) - g(T-t, x)]$$

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with $b(t) \in C^{\circ}[0,T]$ and $\frac{db}{dt} \in L^2(0,T)$;

.) the functions

$$f^*(t,x) = - f(T-t,\ x)$$

$$l^*(t,x) = [\exp(-\ b(t))]\left[\frac{\partial g}{\partial s}(T-t,\ x) + \frac{\partial g}{\partial y}(T-t,\ x)\ .\ f(T-t,x) + l(T-t,\ x)\right] \quad ;$$

.) the weight functions

$$w_i(x) = (1 + |x|^2)^{-\frac{s_i}{2}} \quad , \quad i = 1,2 \quad ,$$

$$s_1 - \frac{n}{2} > \gamma + 1 \quad , \quad s_2 + 1 \leqq s_1 \quad , \quad \gamma \text{ given in Theorem 1} \quad ;$$

.) the spaces

$$H = \left\{v\,/\,v.w_1 \in L^2(\mathbb{R}^n)\right\} \quad ; \quad V = \left\{v \in H\,/\,\frac{\partial v}{\partial x_i}\ .\ w_2 \in L^2(\mathbb{R}^n)\ ;\ i = 1,2,\dots,n\right\} \quad ;$$

.) the operator

$$A(t) : V \longrightarrow H \quad , \quad (A(t)v)_{(x)} = \frac{db}{dt}(t).v(x) + \sum_{i=1}^{n} f_i^*(t,x)\,\frac{\partial v}{\partial x_i}(x) \quad ;$$

.) the functional

$$\varphi(t) : H \longrightarrow \mathbb{R} \quad , \quad (\varphi(t), v)_H = \int_{\mathbb{R}^n} l^*(t,x)\, v(x)\, w_1^2(x)\, dx \quad ;$$

.) and the cone

$$K_0 = \{v \in H\,/\,v(x) \leqq 0 \quad \forall\, x \in \mathbb{R}^n \quad , \quad (a.e)\} \quad .$$

Finally, we put the

Pb II
$$\left\{\begin{array}{l} \text{to find } u(.) \in H^1(0,T\,;H) \cap L^2(0,T;V)\ ;\ u(0) = 0\ ;\ u(t) \in K_0\ \forall t \in [0,T] \\ \text{such that a.e. in }]0,T[\text{ the following inequality holds:} \\ \left(\frac{du}{dt}(t)\ ,\ v-u(t)\right)_H + \left(A(t)\,u(t)\ ,\ v-u(t)\right)_H \geqq \left(\varphi(t)\ ,\ v-u(t)\right)_H \quad , \quad \forall\, v \in K_0 \quad . \end{array}\right.$$

In Theorem 3 we show the equivalence between Pb I and Pb II , and the existence and uniqueness of the solution of these problems.

In §3 we study the analogous of Pb I for a bounded set $\Omega \subset \mathbb{R}^n$.

In this case we have to do with boundary conditions. Following the same ideas of §2 we obtain the existence and uniqueness of the solution of this new problem.

In §4 we present a numerical approach to solve the problems which have just been introduced in §2 and §3 (the proof of the convergence is included). It consist in a combination of the penalty method with an internal approximation technique.

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