ON APPROXIMATE SOLUTION OF THE PROBLEM WITH POINT AND BOUNDARY CONTROL

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Of obvious interest for practical applications is the investigation of the problems, where the process control is realized by means of the parameters effecting the plant both at the boundary and at its given points which do not belong to the boundary [I].In this connection we consider below an optimal control problem of heat transmission process in a uniform thin bar of a final length.Control is realized by means of a boundary and point parameters.We have found necessary and sufficient optimality condition,which makes possible to write down Fredholm's linear integral equations system of the second order relative to the desired controls.It is proved that this system has a unique solution whose determination is brought to the investigation of the infinite system of linear algebraic equations.We study the question of finding an approximate solution for the problem under consideration and we give the estimates of the errors admissible in this case.

Setting of the problem. Optimality condition

Let function $u(t,x)$,characterizing the controllable plant condition in the domain $Q=\{0<x<1,\ 0<t\leq T\}$ satisfy the equation

$$u_t - u_{xx} = p_1(t)\delta(x-x'), \tag{I}$$

while at the boundary Q let it satisfy additional conditions

$$u(0,x) = u_x(t,0) = 0\ , \quad u_x(t,1) = \alpha[p_2(t) - u(t,1)]\ , \quad \alpha = const>0 \tag{2}$$

where $p(t)=\{p_1(t), p_2(t)\}$ is controlling vector-function from the space $L_2^2(0,T)$ with the norm

$$\left(\int_0^T[p_1^2(t)+p_2^2(t)]dt\right)^{1/2}<\infty,$$

and $x'\in(0,1)$ is the control action $p_1(t)$ point of application.

In the following each function $p(t)\in L_2^2(0,T)$ will be called an admissible control.

With the fixed admissible control $p=p(t)$ there exists the unique generalized solution $u(t,x)\in W_2^{0,1}(\mathcal{Y})$ of the boundary problem (I)-(2) which satisfies the identity

$$\int_0^1 u(t_1,x)\Phi(t_1,x)dx-\iint_{\mathcal{Y}}[u\Phi_t-u_x\Phi_x]dx\,dt+\alpha\int_0^{t_1}u(t,1)\Phi(t,1)dt=$$

$$=\int_0^1[p_1(t)\Phi(t,x')+\alpha p_2(t)\Phi(t,1)]dt\quad\forall\Phi(t,x)\in W_2^{1,1}(\mathcal{Y}),\quad \mathcal{Y}=\{0\le x\le 1, 0\le t\le t_1\}$$

Here t_1 is an arbitrarily fixed time moment from $(0,T)$. This solution is determined by formula [2]

$$u(t,x)=\sum_{n=0}^{\infty}\frac{\cos\lambda_n x}{\omega_n}\int_0^t[p_1(\tau)\cos\lambda_n x'+\alpha p_2(\tau)\cos\lambda_n]e^{\lambda_n^2(\tau-t)}d\tau,\quad (3)$$

where $\{\cos\lambda_n x\}$, λ_n, ω_n are proper functions, values and normalizing multiplier of the boundary problem associated with (I)-(2).

We have to find such admissible control $p^0(t)$, which together with the corresponding generalized solution $u^0(t,x)$ of problem (I)-(2) gives the least possible value to the functional

$$S[p]=\int_0^1[u(T,x)-u_0(x)]^2dx+\beta\int_0^T[p_1^2(t)+p_2^2(t)]dt,\ \beta=const>0,\quad (4)$$

where $u_0(x)$ is the given function from $L_2(0,1)$. The control minimizing S we shall call optimal.

The existence of the unique control optimal on S follows from J.L.Lions' [I] results. Therefore we limit ourselves by its direct estimation. For this purpose we write out optimality conditions.

Theorem 1. In order to make admissible control $p^0(t)$ optimal by S it is necessary and sufficient to fulfil the condition

$$H(v(t,x'),v(t,1),p^0(t))\,(=)\,\sup_p H(v(t,x'),v(t,1),p),\quad (5)$$

where

$$H(v(t,x'),v(t,1),p)=v(t,x')p_1+\alpha v(t,1)p_2-\beta(p_1^2+p_2^2)$$

Symbol (=) denotes equality which holds good almost everywhere while the function $v(t,x)$ belongs to the space $W_2^{1,1}(Q)$ and satisfies the correlations

$$\iint_Q (\varphi v_t - v_x \varphi_x)\,dx\,dt = \alpha\int_0^T v(t,1)\,\varphi(t,1)\,dt \quad \forall \varphi \in W_2^{0,1}(Q),$$

$$\lim_{t\to T}\int_0^1 \{v(t,x) + 2[u(t,x) - u_0(x)]\}^2 dx = 0.$$

Such function as well as in the case of boundary problem (I)-(2), exists and in a unique way is determined by the formula

$$v(t,x) = 2\sum_{n=0}^{\infty}\frac{\cos\lambda_n x}{\omega_n}\left\{u_n^0\omega_n - \int_0^T [p_1(\tau)\cos\lambda_n x' + \alpha p_2(\tau)\cos\lambda_n]e^{\lambda_n^2(\tau-T)}d\tau\right\}e^{\lambda_n^2(t-T)}, \quad (6)$$

where u_n^0 are Fourier's coefficients of function $u_0(x)$ by the system $\{\cos\lambda_n x\}$.

Provided that there exists a generalized solution of the boundary problem (I)-(2) and the function $v(t,x)\in W_2^{1,1}(Q)$ theorem 1 may be proved by calculating the augmentation of functional S [2].

Since different vector-functions from $L_2^2(0,T)$ are admissible controls, then condition (5) enables us to determine the optimal control by correlations

$$p_1^0(t) = \frac{1}{2\beta}v(t,x'), \qquad p_2^0(t) = \frac{\alpha}{2\beta}v(t,1). \quad (7)$$

Note 1.1. In a similar way may be studied the cases of problems with linear boundary conditions which differ from (2).

Note 1.2. No principal difficulty is caused by the investigation of the case when any finite number of point parameters participate in the process control. In this case correlations (7) are supplemented with ones analogous to the first one from (7).

Optimal control estimation

Here holds good the following

Theorem 2. Optimal control $p^0(t) = \{p_1^0(t), p_2^0(t)\}$ is the unique solution of the integral equations system

$$p^0(t) + \int_0^T K(t,\tau)\,p^0(\tau)\,d\tau = F(t), \quad (8)$$

where $K(t,\tau)$ is matrix nucleus with elements $K_{ij}(t,\tau)$:

$$K_{11} = \sum_{n=0}^{\infty}\mu_n\cos\lambda_n x' e^{\lambda_n^2(t+\tau-2T)}, \quad K_{12} = K_{21} = \alpha\sum_{n=0}\mu_n\cos\lambda_n\cos\lambda_n x' e^{\lambda_n^2(t+\tau-2T)},$$

$$K_{22} = \alpha^2\sum_{n=0}^{\infty}\mu_n\cos^2\lambda_n e^{\lambda_n^2(t+\tau-2T)}, \quad F(t) = \{F_1(t), F_2(t)\}, \quad \mu_n = \frac{1}{\beta\omega_n},$$

$$F_1(t)=\frac{1}{\beta}\sum_{n=0}^{\infty}u_n^0\cos\lambda_n x^1 e^{\lambda_n^2(t-T)},\qquad F_2(t)=\frac{\alpha}{\beta}\sum_{n=0}^{\infty}u_n^0\cos\lambda_n e^{\lambda_n^2(t-T)}.$$

Demonstration. By considering (6) (at $p=p^0(t)$) and (7), we make sure that the optimal control satisfies system (8). Since $\|K\|<\infty$, $\|F\|<\infty$ and $(K\varphi,\varphi)\geqslant 0$*), then according to the integral equations theory, there exists the unique solution for system (8).

In order to find the optimal control $p^0(t)$ we write down system (8) in the form

$$p_1^0(t)=\sum_{n=0}^{\infty}R_n\cos\lambda_n x^1 e^{\lambda_n^2(t-T)},\qquad p_2^0(t)=\alpha\sum_{n=0}^{\infty}R_n\cos\lambda_n e^{\lambda_n^2(t-T)},\tag{9}$$

where

$$R_n=\mu_n\Big[u_n^0\omega_n-\cos\lambda_n x^1\int_0^T p_1^0(t)e^{\lambda_n^2(t-T)}dt-\alpha\cos\lambda_n\int_0^T p_2^0(t)e^{\lambda_n^2(t-T)}dt\Big].$$

By substituting (9) into (8) and considering the linear independence of the aggregate $\exp\lambda_n^2(t-T)$ we make sure that R_n satisfies the system of linear algebraic equations

$$R_n+\sum_{k=0}^{\infty}K_{nk}R_k=b_n,\quad n=0,1,\dots,\tag{10}$$

where

$$K_{nk}=\frac{a_{nk}}{\lambda_n^2+\lambda_k^2}\left(1-e^{-(\lambda_n^2+\lambda_k^2)T}\right),\quad a_{nk}=\mu_n(\cos\lambda_n x^1\cos\lambda_k x^1+\alpha^2\cos\lambda_n\cos\lambda_k),$$
$$b_n=\mu_n\omega_n u_n^0.$$

Equations system (10) has the unique solution with a boundary property in l_2.

Thus, optimal control has the form (9), where R_n is determined in the unique way from system (10).

Assuming in (3) that $p=p^0(t)$, we have

$$u(T,x)=\sum_{n=0}^{\infty}u_n(T)\cos\lambda_n x,\qquad u_n(T)=\beta\sum_{k=0}^{\infty}K_{nk}R_k.\tag{11}$$

Considering (10) and (11) for finding S we obtain the formula

$$S[p^0]=\sum_{n=0}^{\infty}\omega_n\Big\{[u_n(T)-u_n^0]^2+[u_n^0-u_n(T)]u_n(T)\Big\}=\|u_0(x)\|^2-(u(T,x),u_0(x)).$$

Note 2.1. Analogous results can be obtained when investigating the case when a finite number of pointed parameters participate in the control (see note 1.2.). Here the order of system (8) augments but the properties of nucleous $K(t,x)$ and function $F(t)$ remain unchanged.

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*) Here and in the following statement $\|\cdot\|$ and $(\cdot,\cdot)$ there is a norm and scale product in the space $L_2^2(0,T)$.

Approximate optimal control estimation

It is natural that for algebraic equations system (10) it is impossible to find an accurate solution;therefore, we shall estimate it approximately.For this purpose we shall limit ourselves with a finite number of items in all the root determining $K(t,\tau)$ and $F(t)$ .As a result we obtain the integral equation

$$P_m^0(t) + \int_0^T K_m(t,\tau) P_m^0(\tau) d\tau = F_m(t). \tag{12}$$

Solution of equation (12) we call m-approximation of optimal control.It is of the form

$$P_{1m}^0(t) = \sum_{n=0}^{m} R_n^m \cos\lambda_n x' e^{\lambda_n^2(t-T)}, \quad P_{2m}^0(t) = \alpha \sum_{n=0}^{m} R_n^m \cos\lambda_n e^{\lambda_n^2(t-T)},$$

where R_n^m is determined in the unique way from the equations system

$$R_n^m + \sum_{k=0}^{m} K_{nk} R_k^m = \beta_n, \qquad n = 0, 1, \dots, m.$$

which corresponds to (10).

Approximate value of S is found by the formula

$$S_m[P_m^0] = \|u_0(x)\|^2 - (u_m(T,x), u_0(x)).$$

Now let us estimate the convergence rate of selected approximations to the accurate solution of the problem.From (8) and (12) we have $p^0 - p_m^0 = \psi_m - K_m(p^0 - p_m^0)$, $\psi_m = F - F_m - (K - K_m)p^0$, where K and K_m are Fredholm's operators in the equations (8) and (12).By multiplying the both parts of the found equality by $p^0 - p_m^0$ with the consideration of non-negativity of operator K_m , we have

$$\|p^0 - p_m^0\| \le \|\psi_m\| \tag{13}$$

Similarly we find $\|p^0\| \le \|F\|$ .Consequently,

$$\|\psi_m\| \le \|F - F_m\| + \|K - K_m\| \|F\|$$

For estimating the convergence rate of p^0 to p^0 with $m \to \infty$ we estimate the values $\|F - F_m\|$, $\|K - K_m\|$ and $\|F\|$ .We have [3]

$$\|F - F_m\| \le \frac{\sqrt{1+\alpha^2}}{\pi\beta} \|u_0(x)\| \Big(\sum_{n=m+1}^{\infty} \frac{1}{n^2}\Big)^{1/2} = \frac{\sqrt{1+\alpha^2}}{\pi\beta} \|u_0(x)\| \frac{\sqrt{m+2}}{m+1},$$

$$\|K - K_m\| \le \Big(\sum_{i,j=1}^{2} \int_0^T\int_0^T [K_{ij}(t,\tau) - K_{ij}^m(t,\tau)]^2 d\tau dt\Big)^{1/2} \le \frac{1+\alpha^2}{\pi\beta} \frac{m+2}{(m+1)^2},$$

$$\|F\| \le \frac{\sqrt{1+\alpha^2}}{\beta}\|u_0(x)\|\left(\frac{1}{\lambda^2}+\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{1/2} < \frac{\sqrt{1+\alpha^2}}{\beta}\|u_0(x)\|\left(\frac{1}{\lambda_0^2}+\frac{1}{6}\right)^{1/2}$$

Now, considering the obtained inequalities, it follows from (13) that

$$\|p^0-p_m^0\| \le \frac{\sqrt{1+\alpha^2}}{\pi\beta}\|u_0(x)\|\left[\frac{\sqrt{m+2}}{m+1}+\frac{1+\alpha^2}{\beta}\left(\frac{1}{\lambda_0^2}+\frac{1}{6}\right)^{1/2}\frac{m+2}{(m+1)^2}\right].$$

Thus, $\|p^0-p_m^0\|\to 0$ with $m\to\infty$ not sooner than $\frac{1}{m}$.

For estimating the convergence rate $S_m[p_m^0]\to S[p^0]$ with $m\to\infty$ we have inequality

$$|S[p^0]-S_m[p_m^0]| \le \mu\left[\|p^0-p_m^0\|\sum_{n=0}^{\infty}\frac{1}{\lambda^2}+\frac{\|F\|^2}{\pi^2}\frac{m+2}{(m+1)^2}\right], \quad \mu = const>0.$$

Hence we find that $S_m[p_m^0]\to S[p^0]$ with $m\to\infty$ not sooner than $\frac{1}{m}$.

Note 3.1. The results expounded above represent the generalization of paper [3].

It should be noted that generally speaking, it seems impossible to essentially improve the estimates obtained by minimization of the functionals of type (4) by the method expounded above (compare with [3]).

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