

## STRUCTURAL OPTIMIZATION

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### PART I : GENERAL CONCEPTS

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#### I. INTRODUCTION

Numerical methods of structural analysis have reached a high standard of efficiency. As a consequence they tend to overgrow their usefulness as numerical checks of stress distribution, amplitudes of displacement, natural frequencies and elastic stability to become adjuvants to design procedures.

The objectives of design vary according to the purpose of the structure. In aerospace engineering the weight is the prominent factor and is often the only goal of numerical optimization studies. In other cases the functional that is subject to minimization is more complex, economical factors of various kind being incorporated with their relative weights into the cost function.

Until recently the minimization was carried out by trial and error, the preliminary design and the modifications introduced after evaluation of a numerical structural analysis being largely based on engineering judgment. Presently there is a tendency to a more scientific approach in which the changes in design parameters are evaluated on the basis of algorithms. Efficient algorithms are those that tend to bring the functional to its minimum with the smallest number of iterations requiring a subsequent structural reanalysis. Moreover they have to satisfy many kinds of side constraints such as :

- remain within the elastic limits of the material in each structural member under a given set of load distributions;
- keep displacement-type limitations;
- avoid elastic instability;
- keep natural frequencies within prescribed limits;
- keep member sizes above minimum values.

The starting point of such automated Structural Optimization Programs is a given preliminary design. It is therefore difficult to evaluate the cost of optimization procedures, since the computer time devoted to reach a near-optimal stage will heavily depend on the quality of the preliminary design.

In case where the unicity of the optimal solution is not guaranteed a poor preliminary design can even lead to a local and not to the global minimum.

For this reason, while optimization programs will probably remain essential tools, the objectives of optimality will also tend to incorporate the computer in the preliminary design stage. This more direct approach towards an optimal structure is the aim of "Computer aided Design".

It is also a much more ambitious goal and, fortunately perhaps, will never obliterate the exercise of engineering art. It is indeed difficult to conceive a selection by the computer of the best "topology" of structural members to carry the loads according to the purpose of the structure, taking immediately the effect of side constraints into account. On the other hand, once the topology has been fixed by engineering judgement and experience, we will probably reach the stage where the computer will carry out from there the sizing of the members and even such other alterations in their geometry, permissible under the given topology and external constraints. Whether such ambitious programs will ever become operational within economical limits is a question that only experience will answer.

## 2. DESIGN VARIABLES

One can divide design variables in groups according to their relative importance. For aerospace structures, with a finite element method idealization, the following groups are proposed :

### 2.1. Element sizes

They comprize cross-sectional areas of beam, membrane and plate thicknesses ... The optimization of those variables alone leaves the topology (system of element interconnexions) and other geometrical characteristics (height, length, taper of beams, planforms of membranes and plates ...) unchanged.

### 2.2. Geometric variables

The choice of geometrical variables may alter the configuration of the structure but not its topology. In the finite element method they correspond to modifications in the nodal coordinates.

### 2.3. Material properties

The efficiency of the structure can be improved by a change of nature of the material selected for some of its members. For example Young's modulus and material density may be varied but this introduces discrete parameter modifications as opposed to the continuous variations possible in the previous design variables.

### 2.4. Topology

A change in topology is also, and more fundamentally so, a discrete modification to the structure. For example, a set of members may be replaced by a new one with different elements, differently connected.

The order in which the groups of design variables have been listed is roughly that of increasing complexity in an optimization program and attendant increasing cost.

This consideration has led numerous research teams to limit themselves to the first category. There is also some justification for it in the fact that the general layout of a structure is often dictated by other considerations than a certain definition of optimality. Aerodynamic shape, headroom, access facilities, failsafe design are characteristic examples.

The relative simplicity of dealing with element sizes only is enhanced by the choice of a finite element method for the discretization of the structure. As nodes are kept in place and element interconnexions are invariant, the statics and kinematics of the structure are not modified by alterations in element sizes. This can make a large part of the optimization program a fixed subroutine.

In the sequel we shall deal only with this restricted aspect of optimization.

### 3. NUMERICAL METHODS OF STRUCTURAL OPTIMIZATION

This section describes briefly two main approaches encountered in structural optimization and discusses their relative capabilities.

#### 3.1. Mathematical programming

In this relatively recent approach, minimum weight design is treated as the mathematical problem of extremizing a cost function in design space. Each dimension of this space is related to one design variable, so that each point corresponds to a possible design. The side-constraints consist of limits to the design variables (element sizes) themselves and to stresses or displacements, the latter constraints being generally functions of the design variables. Symbolically, denoting by  $A_i$  ( $i=1...n$ ) the design variables

$$W = W(A_1...A_n) \text{ min.}$$

$$\underline{A}_i \leq A_i \leq \bar{A}_i \quad i=1...n$$

$$g_j(A_1...A_n) \leq \bar{g}_j \quad j=1...p$$

The cost function and the nature of the second type of constraints determine whether the problem can be treated by linear or non-linear programming. The second case usually prevails for structural optimization.

Drawbacks inherent to the mathematical programming approach appear with large numbers of design variables as the number of cycles required to get close to the optimum rapidly rises. Each cycle involves a costly stress reanalysis and the computational expenditure rapidly becomes prohibitive. On the other hand the method is very general and reliable. If a solution converges to a local minimum instead of the global minimum

required, this can always be checked and the necessary steps be taken to reinitiate the procedure.

### 3.2. Optimality criteria

Intuitive considerations as to the nature of the optimal design may lead to adopt optimality criteria that are not directly related to the minimization of the given cost function but sometimes constitute a satisfactory approximation to it. They can then provide a basis for the search techniques and lead to simple recursions formulas for redesign. The best known and widely used example of such a procedure is the "fully stressed design" concept. According to it, each component of an optimal structure is stressed to its limit in at least one of the loading conditions.

Convergence to the optimal solution, according to the fully stressed design criterion, is obtained in one iteration for statically determinate structures. In statically determinate cases the internal loads are indeed independent of the design variables and optimality based on fully stressed design coincides with the exact minimal weight criterion if no limitations are put on displacements. In the statically indeterminate case each redesign modifies the internal loading distribution and fully stressed design does not yield the minimum weight but may be considered to approach it satisfactorily.

An attractive feature of fully stressed design that explains its relative success is its tendency to converge in a number of cycles independent of the number of design variables, in contrast to the more rigorous mathematical programming method. Moreover each redesign cycle is fairly simple.

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## PART II : AN ALGORITHM FOR MINIMUM WEIGHT DESIGN UNDER A =====

SET OF LOADING MODES WITH CONSTRAINTS ON STRESSES  
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AND DISPLACEMENTS.  
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### 1. PROBLEM DEFINITION

The structure is in the linear elastic regime and idealized by finite elements. Under all the specified loading distributions certain constraints on stresses and nodal displacements must be satisfied. The geometry and the material properties are predetermined.

The functional to be minimized

$$W = \sum_i \rho_i L_i A_i \quad i = 1 \dots n_e \quad (1)$$

is the structural weight, proportional in each element to the material density  $\rho_i$ , to the design variable  $A_i$  (cross-section of bar, thickness of membrane,...) and a geometrical parameter  $L_i$  (length of bar, area of membrane...).

### 2. DESCRIPTION OF THE CONSTRAINTS

#### 2.1. Production constraints

They place a lower and sometimes an upper limit to the design variables:

$$\underline{A}_i \leq A_i \leq \bar{A}_i \quad (2)$$

#### 2.2. Stress limitations

In bar-type elements the tensile stress limit is determined by elastic<sup>the</sup> properties of the material; the compressive limit may be reduced to take into consideration, in a simple manner, a safeguard against buckling.

If  $\sigma_i$  is the actual stress in the bar:

$$\underline{\sigma_i} \leq \sigma_i \leq \overline{\sigma_i} \quad (3)$$

In shear panels one assumes a maximum allowable shear stress, usually governed by buckling considerations:

$$\tau_i \leq \overline{\tau_i} \quad (4)$$

In more general membrane elements, where the three stress components  $\sigma_x$ ,  $\sigma_y$  and  $\tau_{xy}$  play equally important roles, a reference stress related to an elastic limit criterion may be introduced.

### 2.3. Displacement constraints

They assign upper bounds to generalized displacements. To determine analytical expressions for them in terms of the design variables, the virtual work theorem is used.

If  $F$  denotes a vector (column matrix) of externally applied loads,

$u$  the conjugate vector of generalized displacements,

$\sigma$  the stress vector,

$\epsilon$  the conjugate strain vector,

the virtual work is given by :

$$\Delta = F_{(v)}^T u_{(r)} = \int_V \sigma_{(r)}^T \epsilon_{(v)} dV = \int_V \sigma_{(v)}^T \epsilon_{(r)} dV \quad (6)$$

The subscripts between brackets refer to either a virtual or a real vector, the superscript T denotes transposition. Splitting the integral into the sum of contributions of each finite element:

$$\Delta = \sum_i \int_{V_i} (\sigma_{(r)}^T \epsilon_{(v)})_i dV_i \quad (7)$$

According to the finite element theory we have:

$$\int_{V_i} (\sigma_{(r)}^T \epsilon_{(v)})_i dV_i = q_{(r)i}^T K_i q_{(v)i} = q_{(r)i}^T g_{(v)i} \quad (8)$$

where  $q_i$  is the vector of generalized displacements of element  $i$  and  $g_i$  its conjugate of generalized loads.  $K_i$  is the stiffness matrix of the element.

Let now  $u_j$  denote a displacement component of the nodal displacement vector of the structure. Applying a corresponding virtual unit load to the structure, (7) and (8) give:

$$u_j = \sum_i q_{(r)i}^T g_{i(j)} \quad (9)$$

where the  $g_{i(j)}$  are the corresponding virtual loads generated at each element level. In statically determinate structures the loads  $g_{i(j)}$  are uniquely determined by the unit load and (9) turns out to be given in terms of the design variables by;

$$u_j = \sum_i \frac{c_{ij}}{A_i} \quad (10)$$

where the  $c_{ij}$  are constants. In redundant structures those coefficients are themselves implicit functions of the design variables.

### 3. FORMULATION

Let us begin with the statically determinate case

#### 3.1. Analysis stage

The structure is analyzed under

- the  $n_r$  real loading systems of the design specification
- the  $n_v$  virtual loading cases connected with each displacement constraint.

#### 3.2. Redesign stage

If the stress constraints are;

$$\begin{aligned} \sigma_{il} &\leq \bar{\sigma}_{il} & i &= 1 \dots n_e \\ & & l &= 1 \dots n_r \end{aligned} \quad (11)$$

where  $\sigma_{il}$  is the actual stress in element  $i$  under the loading case  $l$ , the redesign is effected in a single step by:

$$A_i^* = A_i \max_l \left\{ \frac{\sigma_{il}}{\bar{\sigma}_{il}} \right\} \quad (12)$$

with, in addition, the minimum size requirement:

$$A_i^* \geq \underline{A_i} \quad (13)$$

This method provides a "fully stressed" design, each element reaching its limiting stress (or having its minimal size) under at least one of the loading cases.

If we have displacement constraints, the analysis stage will provide the matrix of  $c_{ij}$  coefficients appearing in (10). Taking the  $A_i^*$  values appearing in (12) as minimal, the problem with the addition of displacement constraints can be stated as follows:

$$W = \sum_i \rho_i L_i A_i \quad \min$$

under

$$\sum_i \frac{c_{ij}}{A_i} \leq \bar{u}_j \quad j = 1 \dots n_t \quad (14)$$

$$\bar{A}_i \geq A_i \geq A_i^* \quad i = 1 \dots n_e \quad (15)$$

where  $n_t = n_r \times n_v$ .

Because of the assumption of statical determinacy the formulation is rigorous, the solution unique and only one stress analysis is required.

Real structures, however, are rarely statically determinate. If subjected to different load distributions they are in fact both stiffer and even lighter if proper use is made of the stress cooperation provided by redundancy. But in this case both the  $c_{ij}$  and the  $A_i^*$  become implicit functions of the design variables. Each change in those will produce new  $c_{ij}$  and  $A_i^*$  that can only be known exactly through a costly stress reanalysis. The following approach is suggested. The problem as defined by equations (14) and (15) is solved by considering the  $c_{ij}$  and  $A_i^*$  as constants. The evolved solution for the design variables is inserted in a new stress analysis to provide new  $c_{ij}$  and  $A_i^*$  values with which to reinitiate problem (14), (15) until close to convergence.

#### 4. SOLUTION OF THE LINEARIZED PROBLEM (14), (15)

In order to solve this problem, in which the  $c_{ij}$  and  $A_i^*$  are assumed to be given, it is beneficial to take the reciprocals of sizing variables as new design variables. The recast problem may then be solved by means of the gradient projection method for linear constraints (ref. [6]) adapted to the problem under consideration.

As required by this method, the initial point must be a feasible point, a point lying in the convex region formed by the prescribed constraints.

In any given case, such a point can readily be found by linear scaling of all member sizes, so that a feasible bounded design is generated (one constraint at its critical value, others subcritical). This scaling of all the design variables does not introduce stress redistribution : each stress and each displacement are simply divided by the same scaling factor.

#### 5. APPLICATION OF THE METHOD

Amongst known optimization programs we can mention :

- GELLATLY and BERKE (ref. [2])
- TAIG and KERR (ref. [3]).

As mentioned previously a structural optimization program performs iterative cycling between a structural analysis stage and a redesign stage. The programs under study at the Aerospace Laboratory of the University of Liège are coupled to the extensively developed ASEF code as the analysis module. The structural idealization consists up to now of axial force members and triangular or quadrilateral membrane elements. The degree of displacement polynomials within the elements is allowed to vary from 1 to 3. A symmetry option has been introduced, that constrains members of any specified group to be identical; in this case, the number of design variables is reduced to the number of element groups.

## 6. EXAMPLES

The method proposed in section 3 has been tested against solutions to classical problems found in the literature.

The two first examples show clearly that the rate of convergence of the redesign procedure is not directly related to the size of the problem under consideration.

### 6.1. Four-Bar Pyramid (fig. 1 )

This very simple structure is subjected to the single loading case given fig.1. Constraints are placed on maximum stress (25000 psi), minimum area ( $0.1 \text{ in}^2$ ) and node displacement in z-direction (0.3 in.). The present results (table I ) duplicate those of ref. [2] and [3] . Fig. 4.a shows the strange pattern followed by the iteration procedure : the design seems to converge after 2 cycles and only after several more cycles does the rate of weight reduction accelerate till the final design is generated (after 20 iterations).

### 6.2. 72- Bar Four Level Tower (fig.3)

This doubly symmetric tower is subjected to two loading cases (table II-a). Symmetry is achieved by use of the input option, which reduces the number of design variables from 72 to 16. The stress limits are again 25000 psi. with  $0.1 \text{ in}^2$  minimum area. The displacements of the four uppermost nodes are limited to 0.25 in. in the x and y-directions. In spite of the larger size of the problem, convergence is very rapid and optimal design is reached in only five iterations (see fig. 4.b). The results (table II-b) are the same as those of ref. [3] .

### 6.3. Cantilever Frame (fig.2)

The 10 bar-truss is subjected to the single loading case indicated on fig. 2. The stress limit in all members is 25000 psi. with  $0.1 \text{ in}^2$  minimum area. The node

displacements in y-direction are prescribed to be less than 0.2 in. Table III shows the results obtained from the present method and, for comparison, from other methods (ref. [1], [2], [3]).

In addition, stress constraints have been formulated in a similar way than for displacement constraints: each member stress is linearly expressed in terms of the inverses of the design variables. Corresponding results are given in table III under the title "Experimental Method". This method generates a design weighting 5060.8 lb. which exhibits the following particular characteristics. Member 6 is fully stressed, while being at its minimum area. Furthermore only one displacement constraint is exactly satisfied (node 1) while another prescribed displacement is close to its limiting value (node 3).

For the other designs, shown on table III, these two displacements reach simultaneously their limiting values.

## 7. CONCLUSION

While using a mathematical programming algorithm, the method that was presented has the convergence characteristics of an optimality criterion approach. Except for the last example (cantilever frame), the same results as those of Taig and Kerr (ref. [3]) have been obtained for each analysis and redesign step. In addition, when there is only one active displacement constraint, the results of Gellatly and Berke (ref. [2]) are also identical to ours. In fact, all these methods are based on the same technique : by means of the virtual work theorem each limited displacement (or linear combination of displacements) is expressed in terms of the design variables. The resulting relations remain exact only in the case of a statically determinate structure; for a redundant structure, they become approximated. The redesign procedures are characterized by the algorithm used in order to resolve the ensuing linearized problem (14) (15).

The present method has the advantage of using a particularly suggestive algorithm : each path up to an "approached" optimum readily shows whether a constraint becomes active or not.

Furthermore each point of this path is a feasible bounded point. That important feature allows the algorithm to be eventually stopped before reaching the optimum, in order to avoid a final divergence due to a too strong internal redundancy.

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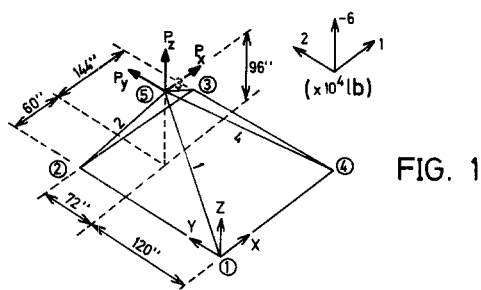


FIG. 1

FOUR BAR PYRAMID

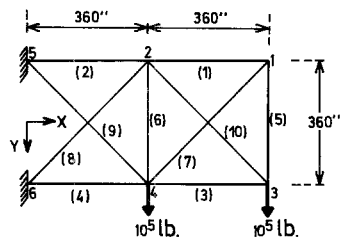


FIG. 2

CANTILEVER FRAME

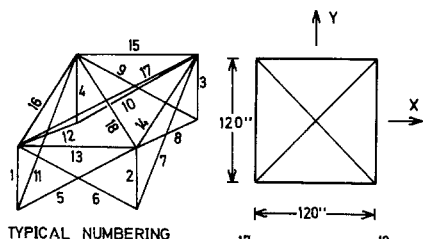
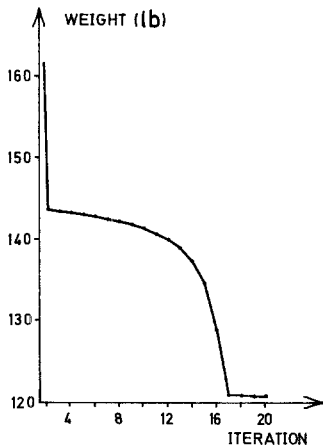
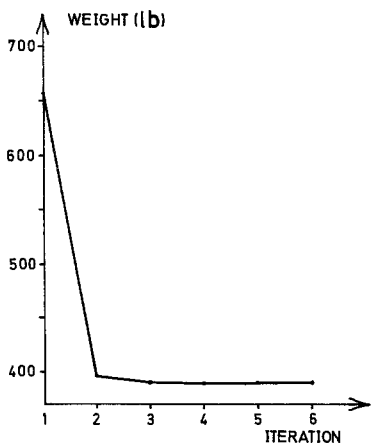


FIG. 3

72-BAR TOWER



(a) four-bar pyramid



(b) 72-bar tower

ITERATION HISTORY

TABLE I FOUR BAR PYRAMID

WEIGHT	N <sub>o</sub> OF ITERATION	MEMBER SIZES			
		1	2	3	4
120.73	22	0.100	3.893	0.747	2.510

TABLE II 72-BAR TOWER

(a) Loading systems (lb)

Load case	Node	X	Y	Z
1	17	5000	5000	-5000
2	17	0	0	-5000
	18	0	0	-5000
	19	0	0	-5000
	20	0	0	-5000

Final Weight : 379.66 lb  
No of iteration : 5

(b) Final Design

Member Size		Member Size	
1-4	1.897	37-40	0.507
5-12	0.516	41-48	0.520
13-16	0.100	49-52	0.100
17,18	0.100	53,54	0.100
22-12	1.280	55-58	0.157
23-30	0.515	59-66	0.536
31-34	0.100	67-70	0.410
35,36	0.100	71,72	0.654

TABLE III CANTILEVER FRAME

MEMBER N <sub>o</sub>	REF. [3]	REF. [2]	REF. [1]	PRESENT METHOD (*)	EXP. METHOD
1	0.10	0.10	0.13	0.10	0.10
2	31.98	31.35	30.42	30.73	30.52
3	15.43	15.60	14.90	14.73	15.22
4	22.57	20.03	23.41	23.94	23.20
5	0.57	0.24	0.10	0.10	0.55
6	0.58	0.14	0.10	0.10	0.10
7	0.10	0.10	0.19	0.10	0.10
8	22.76	22.21	21.08	20.95	21.03
9	21.82	22.06	21.08	20.84	21.53
10	6.44	8.35	8.70	8.54	7.46
TOTAL WEIGHT	5167.	5112.17	5084.90	5076.67	5060.85
No. OF ITERATIONS	32	19	25	14	19

(\*) Taig and Kerr have also obtained this result by fixing member 5 at its minimum area.