APPLICATION OF THE QUADRATIC MINIMIZATION METHOD TO THE PROBLEM OF SIMULATED SYSTEM CHARACTERISTICS REPRESENTATION

V.K. Isaev, V.V. Sonin
Moscow Physical-Technical Institute,
Moscow, USSR

The problem of approximation of the experimental function $g(x)$ determined at discrete points $x = x_k$, $k = 1,2,\ldots, n$, can be represented as consisting of two steps:

1. The choice of an empirical formula

$$f(x, C_1, C_2, \ldots, C_m) \simeq g(x), \quad m < n. \tag{1}$$

2. The search of the constants $C_1, C_2, \ldots, C_m$ subject to minimum of discrepancies square sum

$$\Phi(C_1, C_2, \ldots, C_m) = \frac{1}{2}\sum_{k=1}^{n}\left[f(x_k, C_1, C_2, \ldots, C_m) - g(x_k)\right]^2. \tag{2}$$

The following technique is offered to determine the expression (1). The functions, approximating the experimental data at each of nonintersecting intervals, are constructed. On the basis of these expressions a broken line, whose behaviour qualitatively corres - ponds to the dependence sought for, is generated. Now it is possible to construct a smooth curve arbitrarily closely to the obtained broken line.

Let $H(x)$ be a broken line on $[a, b]$ formed by two smooth arcs, and

$$H(x) = \begin{cases} \varphi_1(x), & \varphi_1(x) \geqslant \varphi_2(x), \quad a \leqslant x \leqslant \xi, \\ \varphi_2(x), & \varphi_2(x) \geqslant \varphi_1(x), \quad \xi \leqslant x \leqslant b. \end{cases} \tag{3}$$

Then for a smooth function

$$\Phi(x,\varepsilon) = \frac{\varphi_1(x) + \varphi_2(x)}{2} + \sqrt{\left[\frac{\varphi_1(x) - \varphi_2(x)}{2}\right]^2 + \varepsilon^2}, \quad \varepsilon > 0, \tag{4}$$

the relation

$$0 < \Phi(x,\varepsilon) - H(x) \leqslant \varepsilon, \quad \varepsilon > 0, \tag{5}$$

is valid.

Similarly, it is possible to prove that for the functions

$$h(x)=\begin{cases}\varphi_2(x), \ \varphi_1(x)\geqslant\varphi_2(x), & a\leqslant x\leqslant\xi,\\ \varphi_1(x), \ \varphi_2(x)\geqslant\varphi_1(x), & \xi\leqslant x\leqslant b,\end{cases} \tag{6}$$

and

$$\varphi(x,\varepsilon)=\frac{\varphi_1(x)+\varphi_2(x)}{2}-\sqrt{\left[\frac{\varphi_1(x)-\varphi_2(x)}{2}\right]^2+\varepsilon^2}, \quad \varepsilon>0, \tag{7}$$

the following expression takes place:

$$0<h(x)-\varphi(x,\varepsilon)\leqslant\varepsilon, \quad \varepsilon>0. \tag{8}$$

Below the function $\Phi(x,\varepsilon)$ is referred to as the upper, and $\varphi(x,\varepsilon)$ - lower envelopes. Suppose that $\varphi_1(x)$ and $\varphi_2(x)$ exist everywhere on $[a,b]$. Despite this fact in general case the construction of the smooth envelope of a multiarc broken line by successive using the expressions (4) and (7) may prove impossible.

This difficulty can be overcome using slight modifications of the described technique and we don't discuss the additional modifications due to limited scope of the paper presented.

Let us consider an example. In fig.1 the dependence $C_{x_0}(M)$ for

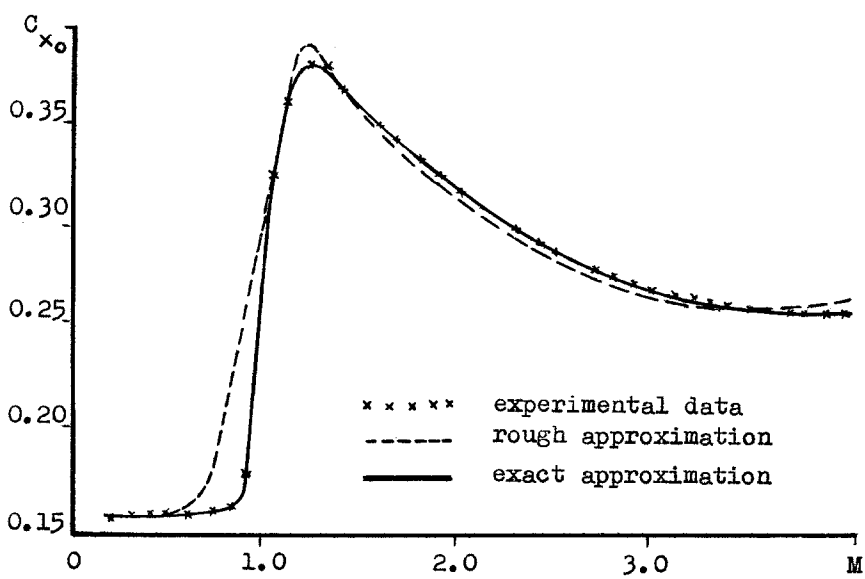

Fig.1

a missile - "the law of 1943" [1] is shown by crosses. The function $C_{x_0}(M)$ acquires practically constant value of 0.157 in the interval (0.1; 0.7). Put in this range of Mach numbers

$$C_{x_0}(M) \simeq \varphi_1(M) = C_1 M + C_2, \quad C_1 = 0, \quad C_2 = 0.157.$$

In the transonic region $C_{x_0}(M)$ increases nearly linear, so for the region $0.7 \leqslant M \leqslant 1.2$ put

$$C_{x_0}(M) \simeq \varphi_2(M) = C_4 M + C_5,$$

where $C_4=0.153$; $C_5=-0.205$. The supersonic part can be represented by the cubic

$$C_{x_0}(M) \simeq \varphi_3(M) = C_7 + C_8 M + C_9 M^2 + C_{10} M^3,$$

$$C_7 = 0.566445, \quad C_8=-0.1758, \quad C_9 = 0.02505, \quad C_{10} = 0.$$

In accordance with the technique described the upper envelope

$$\Phi_{12}(M) = \frac{\varphi_1(M)+\varphi_2(M)}{2} + \sqrt{\left[\frac{\varphi_1(M)-\varphi_2(M)}{2}\right]^2 + C_3^2} \tag{9}$$

($C_3=0.01$), representing $C_{x_0}(M)$ in the range of subsonic and transonic speeds, is constructed, and, finally, the lower envelope of the functions $\Phi_{12}(M)$ and $\varphi_3(M)$, representing $C_{x_0}(M)$ all over the range of $0.1 \leqslant M \leqslant 4$.

$$C_{x_0}(M) = f(M) = \frac{\Phi_{12}(M)+\varphi_3(M)}{2} - \sqrt{\left[\frac{\Phi_{12}(M)-\varphi_3(M)}{2}\right]^2 + C_6^2} \tag{10}$$

($C_6 = 0.01$). In fig.1 the function $f(M)$, corresponding to the given values of $C_1, C_2, \ldots, C_{10}$ is plotted by a dash line. It is shown that generally it correctly reflects the variation of the missile drag coefficient.

Now let us try to find exact values of C_i ($i = 1.2, \ldots, 10$) subject to minimum of square semisum of discrepancies at the points

$$\Phi(C_1, C_2, \ldots, C_{10}) = \frac{1}{2}\sum_{i=1}^{40}\left[f(0.1i;\ C_1, C_2, \ldots, C_{10} - C_{x_0}(0.1i)\right]^2. \tag{11}$$

It is evident that the expression (11) is not a quadratic function of the parameters $C_1, C_2, \ldots C_{10}$ to be sought. In addition, according to (9) and (10) $\Phi(C_1, C_2, \ldots C_{10})$ is an even function of the parameters C_3 and C_6, so an iterative procedure of the second order is offered for searching the minimum of Φ. A brief description of the algorithm is given below.

Let a certain initial value of C^o be known. Then any vector C may be presented in the form of

$$C = C^o + \varepsilon \bar{C}. \tag{12}$$

Here and below usual vectorial-matrix notation is used. When $\varepsilon \bar{C}$ is a sufficiently small vector, $\Phi(C)$ is properly described by its tangential quadratic form

$$\Phi(C) = \Phi(C^o) + \varepsilon \Phi_c^T \bar{C} + \frac{\varepsilon^2}{2} \bar{C}^T \Phi_{cc} \bar{C}. \tag{13}$$

During each iteration the vector $\bar{C}$ is chosen subject to decreasing right-hand part of the expression (13) along it at a sufficiently small ε :

1. Calculate $\Phi(C^o)$, $\Phi_c(C^o)$, $\Phi_{cc}(C^o)$; if there exists the solution of the equation

$$\Phi_{cc} \bar{C} = -\Phi_c, \tag{14}$$

put

$$\bar{C} = -\Phi_{cc}^{-1} \Phi_c \tag{15}$$

and go to 2, otherwise put $C = -\Phi_c$ and go to 4.

2. Calculate $\alpha = [(\Phi_c^T \Phi_c)(\bar{C}^T \bar{C})]^{-1/2} \Phi_c^T \bar{C}$. If $\alpha > 0$, change the sign $\bar{C}$:

$$\bar{C} = \Phi_{cc}^{-1} \Phi_c, \tag{16}$$

as the first order term of (13) is increasing when $\bar{C}$ is calculated according to (15), and go to 5. At $\alpha = 0$ put $\bar{C} = -\Phi_c$ and go to 4. Finally, when $\alpha < 0$ check up the condition

$$\|\bar{C}\| = (\bar{C}^T \bar{C})^{1/2} \leq \alpha, \tag{17}$$

where α is a sufficiently small number (in the example $\alpha = 10^{-6}$). If the condition (17) is fulfilled, go to 3, otherwise - to 5. The fulfilment of (17) means, that the extremum of the quadratic form (13) is at the distance of the order of α from the point C^o at $\varepsilon = 1$.

3. Print the values of components of the vector C^o, discrepancies

$$\delta_j = f(0.1j, C_1, C_2, \ldots, C_{10}) - C_{x_o}(0.1j), \; j = 1, 2, \ldots, 40, \tag{18}$$

and Silvester's determinants of the matrix Φ_{cc}: $(\Delta_1, \Delta_2, \ldots, \Delta_{10})$, end the calculation. If all the values $\Delta_i > 0$ $(i = 1, 2, \ldots, 10)$, then C^o is a local minimum of the function Φ . Otherwise it is necessary to recalculate the problem from another initial point or to choose another $\bar{C}$ on the basis of more detailed analysis and proceed

with the calculation.

4. Define ε subject to $\varepsilon = (\Phi_c^T \Phi_c)^{-1} D\Phi(C^o)$, where D is a specified number (in the example D = 0.1); then go to 6.

5. Put $\varepsilon = 1$ and go to 6.

6. Calculate for the chosen $\bar{C}$

$$\Phi_\varepsilon = \Phi_c^T \bar{C}, \tag{19}$$

$$\Phi_{\varepsilon\varepsilon} = \bar{C}^T \Phi_{cc} \bar{C}, \tag{20}$$

and the value of $\Phi(C^o + \varepsilon \bar{C})$ by the expression (13).

7. Define $\Phi(C^o + \varepsilon \bar{C})$ by the expression (11).

8. If

$$\Phi(C^o) > \Phi(C^o + \varepsilon \bar{C}), \tag{21}$$

put $C^1 = C^o + \varepsilon \bar{C}$ and repead 1-6, substituting C^1 for C^o. If (21) is not fulfilled, define $\varepsilon_{new} = \frac{1}{2[\Phi(C^o+\varepsilon\bar{C}) - \Phi(C^o) - \varepsilon \Phi_\varepsilon]} \cdot (-\Phi_\varepsilon \varepsilon^2)$, recalculate $\Phi(C^o + \varepsilon_{new}\bar{C})$ according to (11), check up (21) at $\varepsilon = \varepsilon_{new}$ etc until (21) is fulfilled.

The algorithm made it possible to obtain the solution of the set problem for 24 iterations. In fig.1 the final function, corresponding to the parameters values, that minimize Φ, is shown by a solid line. In fig.2 the function $-\lg\Phi$ is plotted versus iteration step number. It should be mentioned, that during iterations 4, 10-16 the tangential quadratic form (13) is nonconvex and $\bar{C}$ is defined by the expression (16).

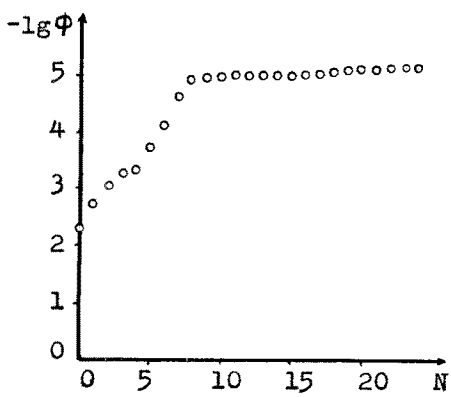

Fig.2

REFERENCE

I. Ф.Р. Гантмахер, Л.М. Левин "Теория полета неуправляемых ракет". Москва, Физматгиз, 1959 г.