THE STABILITY OF OPTIMAL VALUES IN PROBLEMS OF DISCRETE PROGRAMMING

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Let us consider function $f(X)$, that is given and limited to the set $\mathcal{D}$, consisting of a finite number of elements $X \in \mathcal{D}$.

For an arbitrary $X^c \in \mathcal{D}$ and any number $R \geq 0$ there exists a non-empty subset $\mathcal{D}(X^0, R) \subset \mathcal{D}$, possessing the following properties:

if $X \in \mathcal{D}(X^0, R)$,

then $|f(X) - f(X^0)| \leq R$;

if $X \notin \mathcal{D}(X^0, R) \quad (X \in \mathcal{D} \setminus \mathcal{D}(X^0, R))$,

then $|f(X) - f(X^0)| > R$.

Let us introduce function $\varphi(X) \equiv f(X) - f(X^0)$ for all $X \in \mathcal{D}$. It is clear that when $X = X^0$, $\varphi(X^0) = 0$. We shall provide several definitions.

Let us designate subset $\mathcal{D}(X^0, R)$ as the region of R-stability for the values $\varphi(X^c) = 0$, and let us say that the value $\varphi(X^0)$ at $\mathcal{D}(X^0, R)$ is R-stable.

The value $\varphi(X^0)$ is absolutely R-stable, if $\varphi(X) = 0$ for all $X \in \mathcal{D}(X^0, R)$.

The value $\varphi(X^0)$ is absolutely stable, if $\varphi(X) = 0$ for all $X \in \mathcal{D}(f(X) = const)$.

The following assertion is evident. If $f(X) \neq const$, then a number R^0 exists, which is the upper boundry of the absolute R-stability of the value $\varphi(X^0)$, i.e., if $0 \leq R < R^0$, then all regions $\mathcal{D}(X^0, R)$ are regions of absolute R-stability, if $R \geq R^0$, then no single region $\mathcal{D}(X^0, R)$ is a region of absolute R-stability.

It is not difficult to note that $\mathcal{D}(X^0, R) = \mathcal{D}(X^0, 0)$ for all $0 \leq R < R^0$. For this reason in order to determine the region of absolute R-stability for any $0 \le R < R^0$ it is necessary to find all of the solutions for equation $\varphi(X) = 0$ with the condition that $X \in \mathcal{D}$. The solution of such equations, as a rule, is often quite difficult (it is sufficient to recall the Diofan equations).

During the investigation of the stability of value $\varphi(X^0)$ the following problems emerge:

- to determine R^0 – the upper boundry of absolute R-stability;
- for any $R \geqslant 0$ to evaluate the quantity $|\mathcal{D}(X^0, R)|$ – the number of elements in the set $\mathcal{D}(X^0, R)$;
- for any $R \geqslant 0$ to construct an algorithm for obtaining all elements of the existing set $\mathcal{D}(X^0, R)$.

In many applications the solutions of these problems are of interest in cases in which the optimal value of the function $f(X)$ is attained at X^0. For the sake of definition let $f(X^0) = \min f(X)$ for $X \in \mathcal{D}$. Then $\varphi(X) \geqslant 0$ for all $X \in \mathcal{D}$ and $\min \varphi(X) = \varphi(X^0) = 0$.

Let us cite a few examples and classes of functions for which a solution proved to be possible for the problems mentioned above during the investigation of the stability of their optimal values.

1. $X \subset I = \{1, \ldots, m\}$, $\mathcal{D} = \{X\}$

$f(X)$ satisfies the conditions

$$f(X^1) + f(X^2) - f(X^1 \cup X^2) - f(X^1 \cap X^2) \le 0. \qquad (1)$$

For the determination of the minimal value of $f(X^0)$ of such functions let us employ the method of sequential calculation [1].

The function $\varphi(X)$ likewise satisfies conditions (1). For the determination of the region $\mathcal{D}(X^0, R)$ the following generalizations of the rejection rules are employed, the proof of which is given in [2,3].

<u>The first generalized rejection rule</u>. If for X^1 and X^2 the following conditions are fulfilled:

$X^1 \subset X^2$ and $\varphi(X^1) + R < \varphi(X^2)$, then no single $X \supset X^2$ enters into $\mathcal{D}(X^0, R)$.

<u>The second generalized rejection rule</u>. If for X^1 and X^2 the following conditions are fulfilled:

$X^1 \subset X^2$ and $\varphi(X^1) > \varphi(X^2) + R$, then no single

$X \subset X^1$ enters into $\mathcal{D}(X^0, R)$.

The third generalized rejection rule. If for X^1 and X^2 the following conditions are fulfilled: $X^1 \subset X^2$ and

$$\varphi_1(\underline{X}) > \varphi(\tilde{X}) + R \quad \text{, or} \quad \varphi_2(\underline{X}) > \varphi(\tilde{X}) + R \;,$$

then no single X of the type $X^1 \subset X \subset X^2$ enters into $\mathcal{D}(X^0, R)$. Here: $\varphi(\underline{X}) = \min\limits_{X^1 \subset X \subset X^2} \varphi(X)$;

$\tilde{X}$ is the arbitrary subset I, in particular it may be $\tilde{X} = X^0$;

$$\varphi_1(X) \equiv \varphi(X^1) - \sum_{i \in X^2 \setminus X^1} [\varphi(X^1) - \varphi(X^1 \cup i)];$$

$$\varphi_2(X^1) = \varphi(X^2) - \sum_{i \in X^2 \setminus X^1} [\varphi(X^2) - \varphi(X^2 \setminus i)].$$

There exists a large class of concrete problems of mathematical programming in which the corresponding functions $\varphi(X)$ satisfy the conditions (1) (for example, different types of problems of distribution [2,4]). For many of these problems effective algorithms have been elaborated for the determination of subset $\mathcal{D}(X^0, R) \subset \mathcal{D}$, that implement the generalized rejection rules [2,5].

2. $X \subset J = \{1, \ldots, n\}$, whereupon $|X| = m,\ m \le n.\ \mathcal{D} = \{X\}$, consequently $|\mathcal{D}| = C_n^m$.

For each $X \in \mathcal{D}$ let us correlate a quadratic system of linear equations:

$$\sum_{K \in X} y_K B_K = B \quad \text{, where}$$

$$B_K = (a_{K1}, \ldots, a_{Km}),\ B = (b_1, \ldots, b_m)^T.$$

Let us denote by $\mathcal{D}_S$ the set of all $X \in \mathcal{D}$, for which the corresponding system has a non-negative solution $\{y_K(X)\},\ K \in X$.

Let us examine the function

$$f(X) = \begin{cases} \sum\limits_{K \in X} c_K\, y_K(X), & \text{if } X \in \mathcal{D}_S \\ \infty, & \text{if } X \in \mathcal{D} \setminus \mathcal{D}_S \end{cases}$$

It is evident that the values X^0 and $f(X^0)$ in this case are determined by methods of linear programming. An algorithm has been elaborated [4] for the determination of the regions $\mathcal{D}(X^0, R)$ that is a modification of the algorithm proposed in [6].

3. $X=(x_1, \dots, x_n)$

$x_j \geqslant 0$ are integer numbers, satisfying the condition

$$\sum_{i=1}^{n} a_i(x_i) \leq b, \ \mathcal{D}=\{X\}.$$

$$f(X) = -\sum_{i=1}^{n} g_i(x_i)$$

The values X^0 and $f(X^0)$ may be determined by methods of dynamic programming [7]. The determination of region $\mathcal{D}(X^0, R)$ in [4] relies on a modification of the algorithm of Bellman for the case when $g_i(x_i) \geqslant 0$ and $a_i(x_i)$ are single-valued functions, $a_i(x_i)$ take on integer values in the presence of whole x_i, and $b \geqslant 0$ are integers.

Thus, for a large class of functions $\varphi(X)$ it turns out to be possible to determine the regions of R-stability of their optimal values. Primarily, this is of great practical significance for the solution of concrete problems of optimal planning - which in practice makes it possible to choose such a solution $\tilde{X} \in \mathcal{D}(X^0, R)$, that satisfies some additional conditions that had not been taken into account in the initial construction of the problem, or else are generally not formalizable. At the same time it is well known that $\varphi(\tilde{X}) > R$, if $\tilde{X} \notin \mathcal{D}(X^0, R)$.

Secondly, these functions $\varphi(X)$ may be used for the determination of optimal values of more "complex" functions $g(X)$. The approximational-combinatorial method for the solution of problems of discrete programming [4, 8], is based on this approach consisting of the following basic elements.

Let the determination of $Y \in \mathcal{D}$ be required, such that

$$g(Y) = \min_{X \in \mathcal{D}} g(X).$$

Let us assume that such a function $f(X)$ is known, for which there exist effective algorithms for the determination of the region $\mathcal{D}(X^0, R)$ and it has been established that $g(Y) \geqslant f(Y)$. Then a certain value $C \geqslant f(X^0)$ is chosen and the region $\mathcal{D}(X^0, R)$ is determined for $R = C - f(X^0)$. Then the element $\tilde{Y} \in \mathcal{D}(X^0, R)$ is found such that

$$g(\tilde{Y}) = \min_{X \in \mathcal{D}(X^0, R)} g(X).$$

If $g(\tilde{Y}) \leq C$, then $\tilde{Y} = Y$, $g(\tilde{Y}) = g(Y)$, that is, the problem is solved.

But if $g(\tilde{Y}) > C$, then $\tilde{Y}$ and $g(\tilde{Y})$ are taken as an approximate solution, whereupon $C < g(Y) \leq g(\tilde{Y})$.

Using this method problems of distribution that take into account communication, and territorial-production complexes, the distribution problem with Boolean variables and a series of others were solved [2,4,5,8].

References

1. V. P. Cherenin, "The Solution of Certain Combinatorial Problems of Optimal Planning by the Method of Sequential Calculations", Scientific-methods materials of the economico-mathematical seminar, LEMI AN SSSR, Issue 2, Moscow, Rotaprint, Gipromed, 1962, (Russian).

2. V. R. Khachaturov, Certain Questions and a Supplement to Distribution Problems of the method of Sequential Calculation, Dissertation for the Candidate of physico-mathematical science, Moscow, TsEMI AN SSSR, 1968. (Russian).

3. V. R. Khachaturov, "A Generalization of the Rejection Rules for the Solution of a Certain Class of Combinatorial Problems", Materials of the scientific conference on mathematics and mechanics, AS KazSSR, Alma-Ata, Institute of Economics and AN Kaz SSR, 1968. (Russian).

4. V. R. Khachaturov, "An Approximational-combinatorial Method and Certain of its Applications", Journal of Computer Mathematics and Mathematical Physics, Moscow, 1974, Vol. 14, No 6. (Russian).

5. N. D. Astakhov, V.E. Veselovskii, I. Kh. Sigal, V. R. Khachaturov, "Concerning the Experience in the Solution of Distribution Problems Modified by the Algorithm of Sequential Calculations", The thesis of the papers of the VI All-Union Conference on extremal problems, Tallin, AN EsSSR, 1973, (Russian).

6. K. G. Murty, "Solving the Fixed Charge Problem by Ranking the Extreme Points", Operations Research, 1968, vol. 16, No. 2, p. 268-279.

7. R. Bellman, S. Dreifus, Applied Problems of Dynamic Programming, Moscow, Nauka, 1965. (Russian).

8. V. R. Khachaturov, "Concerning the Approximal-combinatorial Method of Solving Problems of Mathematical Programming", Winter School on mathematical programming in the city of Drogobyche, Issue 3, Moscow, TsEMI AN SSSR, 1970, p. 657-674. (Russian).