DUAL MINIMAX PROBLEMS

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In 1972, L. McLinden [3,4] proposed a perturbation method in order to build the dual of a minimax problem. This method is similar to Rockafellar's for dual minimization problems but is much more complicated for several technical reasons. In particular, McLinden permanently works with classes of equivalent convex-concave functionals.

Our aim is to present an equivalent theory of duality for minimax problems, using only the classical duality theory for minimization problems and a notion of partial minimization.

I. NOTATIONS

The notations are essentially the same as in [1,2]. We denote by X and X' two locally convex topological linear spaces in duality ; $< x,x' >$ being the value of the bilinear form at $x \in X$ and $x' \in X'$. In the same way Y, Y' are in duality ; Y_1, Y_1' ; X_2, X_2' ; and so on ... . We consider functionals f defined on X (or X', Y, Y',...) with values in $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$. The set of functionals f defined on X which are the supremum of a family of continuous affine functionals will be denoted by $\Gamma(X)$. The conjugate of f will be denoted by f^* ; it is an element of $\Gamma(X')$. If a functional f is defined on a product XY, f^* means the conjugate of f with respect to the two variables ; it is an element of $\Gamma(X'Y')$. We will need a notion of partial conjugency ; for example, if f is defined on XY, $p_Y f$ will denote a functional defined on XY' which is obtained by partial conjugency with respect to $y \in Y$:

$$p_Y f(x,y') = \operatorname*{Sup}_{y \in Y} (< y,y' > - f(x,y)) \ .$$

The conjugency (or partial conjugency) operation will always be applied to convex functionals. Rather to define a notion of conjugency for concave functionals g, we will use the "change of sign operator" θ (such that $\theta g = -g$) and take the conjugate of θg . For example, if $f \in \Gamma(XY)$, $p_Y f$ is concave with respect to $x \in X$ and we can define $p_X\ \theta\ p_Y f$. It is easy to see that :

$$p_X\ \theta\ p_Y f(x',y') = f^*(x',y') = p_Y\ \theta\ p_X f(x',y') \ .$$

II. DUAL MINIMIZATION PROBLEMS AND DUAL MINIMAX PROBLEMS.

Following Rockafellar [5] , but using notations as in [1,2] , a pair of dual minimization problems is defined by two convex functionals $\varphi \in \Gamma(XY')$ and $\Psi \in \Gamma(X'Y)$ which are mutually conjugate (i.e. such that $\Psi = \varphi^*$ and $\varphi = \Psi^*$).
If we define $f(x) = \varphi(x,y' = 0)$ (*) and $g(y) = \Psi(x' = 0, y)$, the two dual problems are :

$$(P) \qquad \alpha = \operatorname*{Inf}_{x \in X} f(x)$$

$$(Q) \qquad \beta = \operatorname*{Inf}_{y \in Y} g(y)$$

We always have $-\beta \leq \alpha$. The variables y' and x' act as perturbatinn variables. If we consider the two families of minimization problems :

$$(P_{y'}) \qquad h(y') = \operatorname*{Inf}_{x \in X} \varphi(x,y')$$

$$(Q_{x'}) \qquad k(x') = \operatorname*{Inf}_{y \in Y} \Psi(x',y)$$

the problems (P) and (Q) correspond to the value zero of the perturbation variables y' and x'.

The duality between (P) and (Q) could also be defined by the class L of equivalent (see [5]) convex-concave functionals, the extreme elements of which are :

$$\bar{\ell}(x,y) = \theta p_{Y'}\varphi(x,y) = - \operatorname*{Sup}_{y' \in Y'} (< y,y' > - \varphi(x,y'))$$

$$\underline{\ell}(x,y) = p_{X'}\Psi(x,y) = \operatorname*{Sup}_{x' \in X'} (< x,x' > - \Psi(x',y))$$

$$L = \{\ell \mid \underline{\ell} \leq \ell \leq \bar{\ell}\}$$

The minimax problem (S) associated to the functionals φ and Ψ consists in finding $[\bar{x},\bar{y}] \in XY$ such that :

$$(S) \qquad \ell(\bar{x},y) \leq \ell(\bar{x},\bar{y}) \leq \ell(x,\bar{y}) \quad , \quad \text{for all } x \in X \text{ and } y \in Y$$

(the solutions of (S) are the same for all $\ell \in L$).

(*) The notation $\varphi(x,y' = 0)$ instead of $\varphi(x,0)$ helps to recall that the variable which is taken equal to zero is $y' \in Y'$. This method will be very useful later for more complicated cases.

It is well-known that the following three propositions are equivalent :

(i) $[\bar{x},\bar{y}]$ is a solution of (S).

(ii) $\bar{x}$ is a solution of (P), $\bar{y}$ is a solution of (Q) and $\alpha = -\beta$.

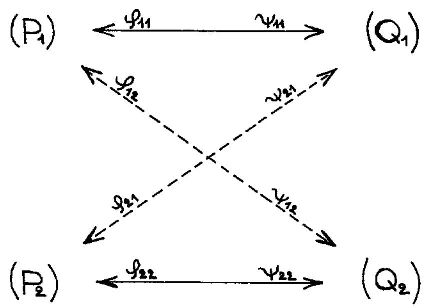
(iii) $f(\bar{x}) + g(\bar{y}) = 0$.

Thus, hereafter, a minimax problem will rather be given by a pair of mutually conjugate functionals φ and Ψ which are defined on a product of two spaces.

DEFINITION : Consider two minimax problems S_1 (represented by the two functionals φ_{11} and Ψ_{11}, and the associated minimization problems (P_1) and (Q_1)) and S_2 (represented in the same way by φ_{22} , Ψ_{22} , and the associated problems (P_2) and (Q_2)). We will say that S_1 and S_2 are dual if

- P_1 and Q_2 are dual (with respect to φ_{12} and Ψ_{12})
- P_2 and Q_1 are dual (with respect to φ_{21} and Ψ_{21})

This definition can be summarized by the following diagram :



In the next section we will show how this diagram can be easily obtained in the classical framework of duality for two minimization problems, using partial minimization.

III. DUAL MINIMAX PROBLEMS AND PARTIAL MINIMIZATION.

Let (P) and (Q) be two dual minimization problems corresponding to $\varphi \in \Gamma(XY')$ and $\Psi \in \Gamma(X'Y)$ (with $\Psi = \varphi^*$), but suppose that the variable $x \in X$ splits into two variables $x_1 \in X_1$ and $x_2 \in X_2$, i.e. X is the product X_1X_2. In the same way, suppose Y to be the product Y_1Y_2. The spaces $X = X_1X_2$ and $X' = X_1'X_2'$ are in duality and similarly the spaces $Y = Y_1Y_2$ and $Y' = Y_1'Y_2'$ are in duality.

The problem (P) can be written :

$$(P) \quad \alpha = \operatorname*{Inf}_{x_1 \in X_1} f(x_1,x_2)$$

with $f(x_1,x_2) = \varphi(x_1,x_2,y_1' = 0,\ y_2' = 0)$, while the problem (Q) is :

$$(Q) \quad \beta = \operatorname*{Inf}_{\substack{y_1 \in Y_1 \\ y_2 \in Y_2}} g(y_1,y_2)$$

with $g(y_1,y_2) = \Psi(x_1' = 0,\ x_2' = 0,\ y_1,y_2)$.

Now we construct a problem (P_1) which consists in minimizing over X_1 the functional f_1 which is the result of a partial minimization with respect to the variable x_2. Conversely we call (P_2) the problem of minimizing over X_2 the functional f_2 which is the result of a partial minimization with respect to x_1.

In the same way we construct the problems (Q_1) and (Q_2). Let us write the four problems at the vertices of a square :

(P_1) : $\alpha = \operatorname*{Inf}_{x_1 \in X_1} f_1(x_1)$ with $f_1(x_1) = \operatorname*{Inf}_{x_2 \in X_2} f(x_1,x_2)$	(Q_1) : $\beta = \operatorname*{Inf}_{y_1 \in Y_1} g_1(y_1)$ with $g_1(y_1) = \operatorname*{Inf}_{y_2 \in Y_2} g(y_1,y_2)$
(P_2) : $\alpha = \operatorname*{Inf}_{x_2 \in X_2} f_2(x_2)$ with $f_2(x_2) = \operatorname*{Inf}_{x_1 \in X_1} f(x_1,x_2)$	(Q_2) : $\beta = \operatorname*{Inf}_{y_2 \in Y_2} g_2(y_2)$ with $g_2(y_2) = \operatorname*{Inf}_{y_1 \in Y_1} g(y_1,y_2)$

We will show that the duality relations described in the diagram of section II can be easily obtained :

a) Duality between (P_1) and (Q_1) :

The objective functionals of (P_1) and (Q_1) can be written :

$$f_1(x_1) = \operatorname*{Inf}_{x_2 \in X_2} \varphi(x_1,x_2,y_1' = 0,\ y_2' = 0)$$

$$g_1(y_1) = \operatorname*{Inf}_{y_2\in Y_2} \Psi(x_1' = 0,\ x_2' = 0,\ y_1, y_2)$$

The dual variable of y_1 being y_1', we consider the function of x_1 and y_1' defined by :

$$\varphi_{11}(x_1,y_1') = \operatorname*{Inf}_{x_2\in X_2} \varphi(x_1,x_2,y_1',y_2' = 0) = \theta p_{X_2}\, \varphi(x_1,x_2' = 0,\ y_1',y_2' = 0)$$

and similarly :

$$\Psi_{11}(x_1',y_1) = \operatorname*{Inf}_{y_2\in Y_2} \Psi(x_1',x_2' = 0,y_1,y_2) = \theta p_{Y_2}\, \Psi(x_1',x_2' = 0,\ y_1,y_2' = 0)$$

The functionals φ_{11} and Ψ_{11} are convex ; in general they do not belong to $\Gamma(X_1Y_1')$ and $\Gamma(X_1'Y_1)$ respectively, and they are not mutually conjugate. But under weak assumptions these properties can be satisfied. Let us consider for example the following two assumptions :

(H_1) for all $x_2' \in X_2'$, the functional

$$[x_1,y_1',y_2'] \to \theta p_{X_2}\, \varphi(x_1,x_2',y_1',y_2')$$

belongs to $\Gamma(X_1Y_1'Y_2')$.

(K_1) for all $y_2' \in Y_2'$, the functional

$$[x_1',x_2',y_1] \to \theta p_{Y_2} \Psi(x_1',x_2',y_1,y_2')$$

belongs to $\Psi(X_1'X_2'Y_1)$.

The assumption (H_1) means that the projection of the epigraph of φ (which is a closed convex set of $X_1X_2Y_1'Y_2'\mathbb{R}$) onto the space $X_1Y_1'Y_2'\mathbb{R}$ is closed. There are many sufficient conditions for obtaining this property.

PROPOSITION :

The assumption (H_1) implies that $\varphi_{11} = \Psi_{11}^*$ and the assumption (K_1) that $\Psi_{11} = \varphi_{11}^*$.

PROOF : We have :

$$\varphi_{11}^*(x_1',y_1) = (p_{Y_1'}\theta p_{X_1})\theta p_{X_2} \varphi(x_1',x_2' = 0,\ y_1,y_2' = 0)$$

$$\Psi_{11}(x_1',y_1) = \theta p_{Y_2} \Psi(x_1',x_2' = 0,\ y_1,y_2' = 0)$$

By assumption (K_1), if we take the conjugate of $\theta p_{Y_2}\Psi$ with respect to the variables $x_1'x_2'y_1$, and then the conjugate with respect to $x_1x_2y_1'$, we obtain $\theta p_{Y_2}\Psi$ itself :

$$(p_{Y_1'}\theta p_{X_1}\theta p_{X_2})(p_{Y_1}\theta p_{X_1'}\theta p_{X_2'})\theta p_{Y_2}\Psi = \theta p_{Y_2}\Psi \ .$$

But $p_{Y_1}\theta p_{X_1'}\theta p_{X_2'}\theta p_{Y_2}\Psi = \Psi^* = \varphi$. It follows that $p_{Y_1'}\theta p_{X_1}\theta p_{X_2}\varphi = \theta p_{Y_2}\Psi$, hence $\varphi^*_{11} = \Psi_{11}$.

By similar arguments, we prove that assumption (H_1) implies $\Psi^*_{11} = \varphi_{11}$.

Thus, under (H_1) and (K_1), the functionals φ_{11} and Ψ_{11} are mutually conjugate and they define a duality between (P_1) and (Q_1). We will call (S_1) the corresponding minimax problem.

b) Duality between (P_2) and (Q_2) :
Proceeding in the same way, we define

$$\varphi_{22}(x_2,y_2') = \operatorname*{Inf}_{x_1\in X_1} \varphi(x_1,x_2,y_1' = 0,y_2') = \theta p_{X_1}\varphi(x_1' = 0,x_2,y_1' = 0,y_2')$$

$$\Psi_{22}(x_2',y_2) = \operatorname*{Inf}_{y_1\in Y_1} \Psi(x_1' = 0,x_2',y_1,y_2) = \theta p_{Y_1}\Psi(x_1' = 0,x_2',y_1' = 0,y_2)$$

and we consider the two assumptions :

(H_2) for all $x_1' \in X_1'$, the functional

$$[x_2,y_1',y_2'] \to \theta p_{X_1}\ \varphi(x_1',x_2,y_1',y_2')$$

belongs to $\Gamma(X_2Y_1'Y_2')$

(K_2) for all $y_1' \in Y_1'$, the functional

$$[x_1',x_2',y_2] \to \theta p_{Y_1}\ \Psi(x_1',x_2',y_1',y_2)$$

belongs to $\Gamma(X_1'X_2'Y_2)$.

The assumption (H_2) implies that $\varphi_{22} = \Psi^*_{22}$ and (K_2) that $\Psi_{22} = \varphi^*_{22}$. With (H_2) and (K_2) the two functionals φ_{22} and Ψ_{22} are mutually conjugate and define a duality between (P_2) and (Q_2). We will call (S_2) the corresponding minimax problem.

c) Duality between (S_1) and (S_2) :
In order to define a duality between (S_1) and (S_2) we have to define
1°/ a duality between the two minimization problems (P_1) and (Q_2) and
2°/ a duality between (P_2) and (Q_1).

Consider the two functionals :

$$\varphi_{12}(x_1,y_2') = \theta p_{X_2}\varphi(x_1,x_2' = 0,y_1' = 0,y_2')$$

$$\Psi_{12}(x_1',y_2) = \theta p_{Y_1}\Psi(x_1',x_2' = 0,y_1' = 0,y_2)$$

One can prove that the assumption (H_1) implies that $\varphi_{12} = \Psi_{12}^*$ and the assumption (K_2) that $\Psi_{12} = \varphi_{12}^*$. Thus the two assumptions (H_1) and (K_2) together imply that φ_{12} and Ψ_{12} are mutually conjugate and define a duality between (P_1) and (Q_2).

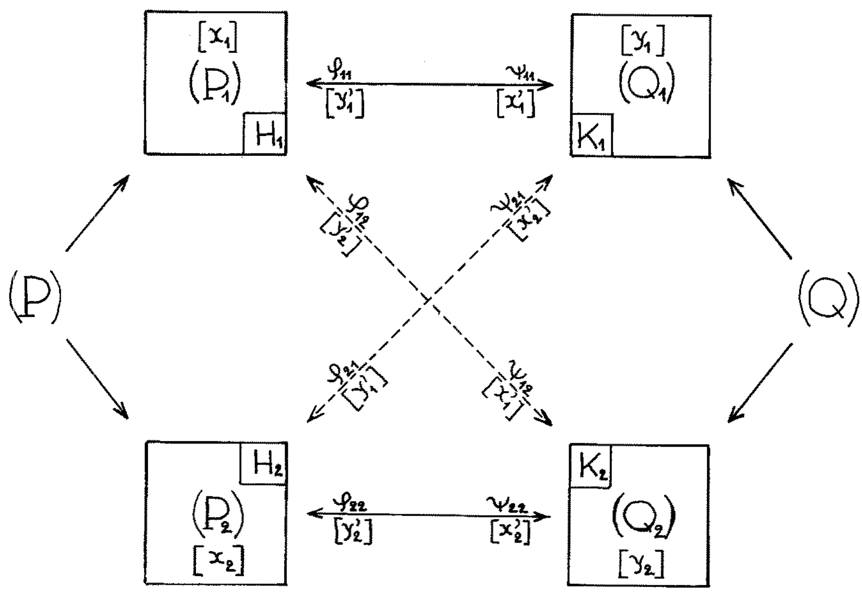
Similarly, we define :

$$\varphi_{21}(x_2,y_1') = \theta p_{X_1}\varphi(x_1' = 0,x_2,y_1',y_2' = 0) ,$$

$$\Psi_{21}(x_2',y_1) = \theta p_{Y_2}\Psi(x_1' = 0,x_2',y_1,y_2' = 0) .$$

With the two assumptions (H_2) and (K_1) these two functionals are mutually conjugate and they define a duality between (P_2) and (Q_1).

Finally, with the four assumptions (H_1), (H_2), (K_1) and (K_2), we have obtained two dual minimax problems (S_1), which is the pair (P_1), (Q_1), and (S_2), which is the pair (P_2), (Q_2). The problems (P_1), (P_2), and the problems (Q_1), (Q_2) are obtained by partial minimization of two problems, respectively (P) and (Q), which are dual of each other in the classical sense. This situation is summarized in the following diagram :



IV. STABILITY AND CHARACTERIZATION OF THE SOLUTIONS .

Using the characterization theorem of section II and the fact that (P_1) and (Q_2) are dual as well as (P_2) and (Q_1), we have the following result :

The pair $[\bar{x}_1,\bar{y}_1]$ is a solution of (S_1) and the pair $[\bar{x}_2,\bar{y}_2]$ is a solution of (S_2) if and only if the following two conditions hold :

$$f_1(\bar{x}_1) + g_2(\bar{y}_2) = 0 \qquad \text{and} \qquad f_2(\bar{x}_2) + g_1(\bar{y}_1) = 0$$

Using the terminology introduced in [1,2] , the problem (P_1) is stable with respect to the perturbation y'_2 if the functional :

$$h_{12}(y'_2) = \operatorname*{Inf}_{x_1 \in X_1} \varphi_{12}(x_1,y'_2)$$

is finite and continuous at $y'_2 = 0$.

In the same way, (Q_1) is stable with respect to x'_2 if the functional

$$k_{21}(x'_2) = \operatorname*{Inf}_{y_1 \in Y_1} \Psi_{21}(x'_2,y_1)$$

is finite and continuous at $x'_2 = 0$.

Speaking of the duality between (S_1) and (S_2), it is logical to say that (S_1) is stable if h_{12} and k_{21} are finite and continuous at $y'_2 = 0$ and $x'_2 = 0$ respectively.

We have then the following characterization theorem :

If (S_1) is stable, then $[\bar{x}_1,\bar{y}_1]$ is a solution of (S_1), if and only if there exists $[\bar{x}_2,\bar{y}_2] \in X_2\ Y_2$ such that :

$$f_1(\bar{x}_1) + g_2(\bar{y}_2) = 0 \qquad \text{and} \qquad f_2(\bar{x}_2) + g_1(\bar{y}_1) = 0 .$$

(obviously such a pair $[\bar{x}_2,\bar{y}_2]$ is a solution of (S_2)).

V. EXAMPLES.

In both examples we suppose that the assumptions (H_1),(H_2),(K_1) and (K_2) are satisfied.

Example 1 :

Suppose we have $X_1 = Y'_2$, $X'_1 = Y_2$, $X_2 = Y'_1$, $X'_2 = Y_1$ and let $\omega_1 \in \Gamma(X_1X_2)$ and $\omega_2 \in \Gamma(X_1X_2)$. Define :

$$\varphi(x_1,x_2,y'_1,y'_2) = \omega_1(x_1,x_2) + \omega_2(x_1-y'_1,\ x_2-y'_2)$$

Then the conjugate of φ is :

$$\Psi(x'_1,x'_2,y_1,y_2) = \omega_1^*(y_1+x'_1,\ y_2+x'_2) + \omega_2^*(-y_1,-y_2)$$

It follows that the minimax problem (S_1) is defined by :

$$\varphi_{11}(x_1,y_1') = \inf_{x_2 \in X_2} (\omega_1(x_1,x_2) + \omega_2(x_1-y_1',x_2))$$

$$\Psi_{11}(x_1',y_1) = \inf_{y_2 \in Y_2} (\omega_1^*(y_1+x_1',y_2) + \omega_2^*(-y_1,-y_2))$$

and (S_2) by :

$$\varphi_{22}(x_2,y_2') = \inf_{x_1 \in X_1} (\omega_1(x_1,x_2) + \omega_2(x_1,x_2-y_2'))$$

$$\Psi_{22}(x_2',y_2) = \inf_{y_1 \in Y_1} (\omega_1^*(y_1,y_2+x_2') + \omega_2^*(-y_1,-y_2))$$

(The functionals φ_{12} Ψ_{12} φ_{21} Ψ_{21} are defined in the same way).

The four problems which form (S_1) and (S_2) are :

(P_1) $\alpha = \inf_{x_1 \in X_1} f_1(x_1)$ $f_1(x_1) = \inf_{x_2 \in X_2} (\omega_1(x_1,x_2)+\omega_2(x_1,x_2))$	(Q_1) $\beta = \inf_{y_1 \in Y_1} g_1(y_1)$ $g_1(y_1) = \inf_{y_2 \in Y_2} (\omega_1^*(y_1,y_2)+\omega_2^*(-y_1,-y_2))$
(P_2) $\alpha = \inf_{x_2 \in X_2} f_2(x_2)$ $f_2(x_2) = \inf_{x_1 \in X_1} (\omega_1(x_1,x_2)+\omega_2(x_1,x_2))$	(Q_2) $\beta = \inf_{y_2 \in Y_2} g_2(y_2)$ $g_2(y_2) = \inf_{y_1 \in Y_1} (\omega_1^*(y_1,y_2)+\omega_2^*(-y_1,-y_2))$

<u>Example 2</u> :

Let $Y_1 = Y_1' = \mathbb{R}^m$ and $Y_2 = Y_2' = \mathbb{R}^n$. Suppose $\omega \in \Gamma(X_1X_2)$, $f_i \in \Gamma(X_1)$, $i=1,\dots,m$ and $g_j \in \Gamma(X_2)$, $j=1,\dots,n$.

Define :

$$\varphi(x_1,x_2,y_1',y_2') = \begin{cases} \omega(x_1,x_2) \text{ if } f_i(x_1)+y_{1i}' \leq 0 \;, & i=1,\dots,m \\ \qquad\qquad \text{and } g_j(x_2)+y_{2j}' \leq 0 \;, & j=1,\dots,n \\ +\infty \qquad \text{elsewhere.} & \end{cases}$$

The conjugate of φ has the following expression :

$$\Psi(x_1',x_2',y_1,y_2) = \begin{cases} \sup_{x_1x_2} (< x_1,x_1' > + < x_2,x_2' > - \omega(x_1,x_2) \\ \qquad\qquad - \sum_i y_{1i}\, f_i(x_1) \quad - \sum_j y_{2j}\, g_j(x_2)) \\ \qquad\qquad\qquad \text{if } y_{1i} \geq 0,\ i=1,\dots,m \text{ and } y_{2j} \geq 0,\ j=1,\dots,n \;, \\ +\infty \qquad \text{elsewhere.} \end{cases}$$

If we define :

$$C_1 = \{x_1 \in X_1 | f_i(x_1) \leq 0 \ , \ i=1,\dots,m\}$$

$$C_2 = \{x_2 \in X_2 | g_j(x_2) \leq 0 \ , \ j=1,\dots,n\}$$

the minimax problem (S_1) is given by :

$$\varphi_{11}(x_1,y_1') = \begin{cases} \inf\limits_{x_2 \in C_2} \omega(x_1,x_2) \text{ if } f_i(x_1) + y_{1i}' \leq 0 \ , \ i=1,\dots,m \\ +\infty \quad \text{elsewhere} \end{cases}$$

$$\Psi_{11}(x_1',y_1) = \begin{cases} \inf\limits_{y_2 \geq 0} \ \sup\limits_{\substack{x_1 \in X_1 \\ x_2 \in X_2}} \left(< x_1,x_1' > - \omega(x_1,x_2) - \sum\limits_{i=1}^{m} y_{1i} f_i(x_1) - \sum\limits_{j=1}^{n} y_{2j} g_j(x_2)\right) , \\ \qquad \text{if } y_{1i} \geq 0 \ , \ i=1,\dots,m \ , \\ +\infty \quad \text{elsewhere} \end{cases}$$

and the minimax problem (S_2) is given by :

$$\varphi_{22}(x_2,y_2') = \begin{cases} \inf\limits_{x_1 \in C_1} \omega(x_1,x_2) \text{ if } g_j(x_2) + y_{2j}' \leq 0 \ , \ j=1,\dots,n \ , \\ +\infty \quad \text{elsewhere} \end{cases}$$

$$\Psi_{22}(x_2',y_2) = \begin{cases} \inf\limits_{y_1 \geq 0} \ \sup\limits_{\substack{x_1 \in X_1 \\ x_2 \in X_2}} \left(< x_2,x_2' > - \omega(x_1,x_2) - \sum\limits_{i=1}^{m} y_{1i} f_i(x_1) - \sum\limits_{j=1}^{n} y_{2j} g_j(x_2)\right) , \\ \qquad \text{if } y_{2j} \geq 0 \ , \ j=1,\dots,n \ , \\ +\infty \quad \text{elsewhere.} \end{cases}$$

Finally, the four problems which compose (S_1) and (S_2) are :

(P_1) $\alpha = \inf\limits_{x_1 \in C_1} \omega_1(x_1)$ $\omega_1(x_1) = \inf\limits_{x_2 \in C_2} \omega(x_1,x_2)$	(Q_1) $\beta = \inf\limits_{y_1 \geq 0} \eta_1(y_1)$ $\eta_1(y_1) = -\inf\limits_{\substack{x_1 \in X_1 \\ x_2 \in C_2}} (\omega(x_1,x_2) + \sum\limits_i y_{1i} f_i(x_1))$
(P_2) $\alpha = \inf\limits_{x_2 \in C_2} \omega_2(x_2)$ $\omega_2(x_2) = \inf\limits_{x_1 \in C_1} \omega(x_1,x_2)$	(Q_2) $\beta = \inf\limits_{y_2 \geq 0} \eta_2(y_2)$ $\eta_2(y_2) = -\inf\limits_{\substack{x_1 \in C_1 \\ x_2 \in X_2}} (\omega(x_1,x_2) + \sum\limits_j y_{2j} g_j(x_2))$

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