RIEMANNIAN INTEGRAL OF SET-VALUED FUNCTION

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Lebesgue integral of a set-valued function is a conception of growing importance for the theories of the optimal control and of the differential games (see for example [1]). For solving concrete problems it will be often useful to reduce a Lebegue set-integral to a riemannien one. In this article we shall find sufficient and necessary conditions of existence of riemannien integral of a set-valued function and prove the equality between riemannien and Lebesgue set-integrals in the case when riemannien set-integral exists.

I. Notations. Definitions. Auxiliary Propositions.

Let R^n be an euclidean n-space, (x,y) - the scalar product of $x,y\in R^n$, $I=[a,b]\subset R^1$, $V=\{x\in R^n \mid \|x\|\leq 1\}$, $W=\{\psi\in R^n \mid \|\psi\|=1\}$. Let A and B be compact subsets of R^n . The Hausdorff distance is definded by formula: $h(A,B)=\min\{\varepsilon\geq 0 \mid A\subset B+\varepsilon V; B\subset A+\varepsilon V\}$. We shall denote by Ω^n the space of all compact subsets of R^n with Hausdorff metric in it. If $\{A_i\}\in\Omega^n$, then we shall define $\sum_{i=1}^{m} A_i=A_1+\ldots+A_m=\{a_1+\ldots+a_m \mid a_i\in A_i\}$; if $A\in\Omega^n$ - then $\text{extr}\, A$ is closure of extreme points of A and $|A|=h(A,0)$.

We shall say that $F:I\to\Omega^n$ is set-valued function. The conceptions of limit in the metric space Ω^n (denote $\lim$) and of continuity of function $F:I\to\Omega^n$ are defined as usual.

We shall say that map $F:I\to\Omega^n$ is convex-valued function if all the sets $F(t)\subset R^n$ are convex for every fixed $t\in I$. Support function of the set $A\subset R^n$ is the function $s(\psi,A)=\sup\{(\psi,x) \mid x\in A\}$. We shall consider the support function only for $\psi\in W$.

We shall say that map $F:I\to\Omega^n$ is Lebesgue measurable function on I if for any closed set $D\subset R^n$ the set $\{t\in I \mid F(t)\cap D\neq\emptyset\}$ is Lebesgue measurable set.

Definition 1 Lebesgue integral of the set-valued function $F:I\to\Omega^n$ on I (set-integral) is the set $L\int_I F(t)dt=\{w\in R^n \mid \exists r(t)\in F(t);$ $r(t)$ is Lebesgue measurable function on I and such that $w=L\int_I r(t)dt\}$.

Subdivision of an interval $I=[a,b]$ is finite set of numbers $\omega=\{t_1,t_2,\ldots,t_N\}$ such that $t_1=a,\ t_i<t_{i+1},\ t_N=b$. Let $\Delta t_i=t_{i+1}-t_i$, $\lambda=\max_{1\leq i\leq N-1}\{\Delta t_i\}$ be diameter of the subdivision ω . Let $\xi_i\in[t_i,t_{i+1}]$

be an arbitrary point. We shall consider in this article only bounded maps $F: I \to \Omega^n$, that means: $\exists \alpha > 0 : |F(t)| < \alpha \ \forall t \in I$.

Definition 2 Riemannien integral of the bounded set-valued function $F: I \to \Omega^n$ on I is the limit of sets: $R\int_I F(t)dt = \underset{\lambda \to 0}{LIM} \sum F(\xi_i)\Delta t_i$, provided this limit exists.

We shall use following well-known propositions:

Proposition 1. Convex-valued function $F: I \to \Omega^n$ is continuous iff it is bounded and support function $S(\Psi, F(t))$ is continuous by t for any $\Psi \in W$. ([2]).

Proposition 2. (C-property). Map $F: I \to \Omega^n$ is measurable on I iff for any $\varepsilon > 0$ there exists closed set $I_1 \subset I$ such that Lebesgue measure $m(I \setminus I_1) < \varepsilon$ and $F(t)$ is continuous on I_1. ([3]).

Proposition 3. Let $F: I \to \Omega^n$ be measurable function on I and let there exist the Lebesgue summable function $\mu(t) > 0$ on I such that $|F(t)| \le \mu(t)$. Then the Lebesgue integral of $F(t)$ is non empty convex compact subset of R^n and $L\int_I F(t)dt = L\int_I coF(t)dt$. (Here "co" means "convex").([4]).

2. The main lemmas.

Lemma 1. The bounded convex-valued function $F: I \to \Omega^n$ is continuous a.e. (almost everywhere) on I iff the function $f(t) = S(\Psi, F(t))$ is continuous a.e. on I for any $\Psi \in W$.

The proof follows from proposition 1.

Lemma 2. Let the suppositions of proposition 3 be satisfied. Then $S(\Psi, L\int_I F(t)dt) = L\int_I S(\Psi, F(t))dt$.

Proof. By proposition 3 the integral $L\int_I F(t)dt$ is convex compact, therefore for any $\Psi \in W$ there exists vector $x \in L\int_I F(t)dt$ such that $S(\Psi, L\int_I F(t)dt) = (\Psi, x)$. Then by definition 1 there exists measurable function $r(t) \in F(t)$ such that $x = L\int_I r(t)dt$. Hence

$$s(\Psi, L\int_I F(t)dt) = (\Psi, L\int_I r(t)dt) = L\int_I (\Psi, r(t))dt \le L\int_I S(\Psi, F(t))dt.$$

On the other hand let us consider map $R(t,\Psi) = \{x \in F(t) \mid S(\Psi, F(t)) = (\Psi, x)\}$. This map is measurable because $R(t,\Psi) = F(t) \cap Q(t,\Psi)$ where $Q(t,\Psi) = \{x \in R^n \mid s(\Psi, F(t)) = (\Psi, x)\}$. But the map $Q(t,\Psi)$ is measurable because $h(Q(t,\Psi), Q(t^*,\Psi)) = |S(\Psi, F(t)) - S(\Psi, F(t^*))|$. But if $R(t,\Psi)$ is measurable then there exists the measurable function $r(t) \in R(t,\Psi)$. Hence we have $S(\Psi, F(t)) = (\Psi, r(t))$. Finally

$$L\int_I S(\Psi, F(t))dt = L\int_I (\Psi, r(t))dt = (\Psi, L\int_I r(t)dt) \le S(\Psi, L\int_I F(t)dt).$$

Lemma 3. If convex-valued function $F: I \to \Omega^n$ is continuous a.e. on I then function $\mathbf{tend}\, F: I \to \Omega^n$ is continuous a.e. on I.

Proof. Let $W(\psi_1, \dots, \psi_K) = \{\psi \in W \mid (\psi, \psi_i) = 0 \,;\, 1 \le i \le K\}$. Let us consider a set of mutually orthogonal points $\psi_1, \dots, \psi_n$ from W. We shall define the maps $R(t, \psi_1, \dots, \psi_K)$ in such a way:

$$R(t, \psi_1) = \{x \in F(t) \mid s(\psi_1, F(t)) = (\psi_1, x)\},$$

$$R(t, \psi_1, \psi_2) = \{x \in R(t, \psi_1) \mid s(\psi_2, R(t, \psi_1)) = (\psi_2, x)\},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$R(t, \psi_1, \dots, \psi_n) = \{x \in R(t, \psi_1, \dots, \psi_{n-1}) \mid s(\psi_n, R(t, \psi_1, \dots, \psi_{n-1})) = (\psi_n, x)\}.$$

It is easy to show (as in lemma 2) that function $t \to R(t, \psi_1, \dots, \psi_n)$) is continuous a.e. on I and that $\mathbf{tend}\, F(t) =$

$$= cl \bigcup_{\psi_1 \in W} \bigcup_{\psi_2 \in W(\psi_1)} \dots \bigcup_{\psi_n \in W(\psi_1, \dots, \psi_{n-1})} R(t, \psi_1, \dots, \psi_n).$$

Lemma 4. Let function $F: I \to \Omega^n$ be summable on I by Riemann (that is there exists riemannien integral of $F(t)$). Then for any $\psi \in W$ the support function $f(t) = s(\psi, F(t))$ is summable by Riemann and $s(\psi, R\int_I F(t)dt) = R\int_I s(\psi, F(t))dt$.

Proof. It is easy to show that if $\underset{n\to\infty}{LIM}\, P_n = P$, then for any $\psi \in W$ $s(\psi, P) = \lim_{n\to\infty} s(\psi, P_n)$. From here follows

$$s(\psi, R\int_I F(t)dt) = s(\psi, \underset{\lambda\to 0}{LIM} \sum F(\xi_i)\Delta t_i) = \lim_{\lambda \to 0} s(\psi, \sum F(\xi_i)\Delta t_i) =$$

$$= \lim_{\lambda\to 0} \sum s(\psi, F(\xi_i))\Delta t_i = R\int_I s(\psi, F(t))dt.$$

Lemma 5. Let $\{F_\lambda\} \in \Omega^n$, $\lambda > 0$, $F \in \Omega^n$, $|F_\lambda| \le \alpha$. Let sets F_λ and F be convex and for any $\psi \in W$ $\lim_{\lambda\to 0} s(\psi, F_\lambda) = s(\psi, F)$.

Then there exists the limit of sets F_λ in space Ω^n and $\underset{\lambda\to 0}{LIM}\, F_\lambda = F$.

Proof of this lemma is similar to the proof of proposition 1.

Lemma 6. Let $F: I \to \Omega^n$ be convex-valued function on I and for any $\psi \in W$ the support function $f(t) = s(\psi, F(t))$ be summable by Riemann on I. Then $F(t)$ is summable by Riemann and $R\int_I F(t)dt = L\int_I F(t)dt$.

Proof. By lemma 2 we have $s(\psi, L\int_I F(t)dt) = L\int_I s(\psi, F(t))dt =$

$$= R\int_I s(\psi, F(t))dt = \lim_{\lambda\to 0} \sum s(\psi, F(\xi_i))\Delta t_i = \lim_{\lambda \to 0} s(\psi, \sum F(\xi_i)\Delta t_i).$$

From here and lemma 5 follows the proof.

Lemma 7. Let $P\in\Omega^n$. Then $R\int_I P\,dt=(b-a)\,co\,P$.

Proof. Let $\omega=\{t_1,\dots,t_N\}$ be subdivision of $I=[a,b]$. Consider sets $P_\lambda=\sum_{i=1}^{N-1}(P\cdot\Delta t_i)$. It is easy to show that $P_\lambda\subset(b-a)\,co\,P$. On the other hand let $x\in(b-a)\,co\,P$. By Carateodory's theorem there exist some points $x_i\in co\,P$ and numbers $\mu_i>0$, $i=1,\dots,K$; $K\le n+1$, $\sum_{i=1}^{K}\mu_i=1$, such that $x=(b-a)(\mu_1x_1+\dots+\mu_Kx_K)$. Let $|P|\le\alpha$. Denote

$$\mu(x)=(b-a)\min\{\mu_i\,|\,1\le i\le K\}\ \text{and}\ \lambda(x)=\min\{\mu(x),\tfrac{\varepsilon}{2n\alpha}\}.$$

It is obvious that $\lambda(x)>0$ and $x\in P_\lambda+\frac{\varepsilon}{2}V$ for any subdivision with diameter $\lambda\le\lambda(x)$. If $B_\varepsilon(x)=\{y\in R^n\,|\,\|x-y\|<\frac{\varepsilon}{2}\}$ then $B_\varepsilon(x)\subset P_\lambda+\varepsilon\cdot V$. If we take a finite covering of compact set $(b-a)\,co\,P$ by neighborhoods $B_\varepsilon(x)$ then we shall find $\lambda_0>0$ such that for any subdivision with $\lambda\le\lambda_0$: $(b-a)\,co\,P\subset P_\lambda+\varepsilon V$. Finally $h(P_\lambda,(b-a)\,co\,P)<\varepsilon$.

3. The main theorems.

Theorem 1. Let $F:I\to\Omega^n$ be a convex-valued function on I. Map $F:I\to\Omega^n$ is summable by Riemann on I iff $F:I\to\Omega^n$ is continuous a.e. on I. With such conditions riemannien integral of F is equal to Lebesgue integral of F.

Proof. It is well-known that a usual function is summable by Riemann on I iff it is continuous a.e. on I. Therefore the proof is consequence of lemmas 4, 1 and 6.

Theorem 2. If $F:I\to\Omega^n$ is continuous function on I then it is summable by Riemann and $R\int_I F(t)dt=R\int_I co\,F(t)dt$.

Proof. By theorem 1 and by continuity of $F(t)$ for any $\varepsilon>0$ there exists $\delta>0$ such that for $t,t_*\in I$, $|t-t_*|<\delta$, and for any subdivision $\omega_1=\{\tau_1,\dots,\tau_N\}$ with diameter δ we have:

$$F(t)\subset F(t_*)+\frac{\varepsilon}{3(b-a)}V\ ;\quad h\Big(\sum_{i=1}^{N-1}co\,F(\gamma_i)\Delta\tau_i\,,\ R\int_I co\,F(t)dt\Big)<\frac{\varepsilon}{3},$$

where $\gamma_i\in[\tau_i,\tau_{i+1}]$, $\Delta\tau_i=\tau_{i+1}-\tau_i$.

Let us consider subdivision $\omega_2=\{t_{11},t_{12},\dots,t_{1K_1}=t_{21},t_{22},\dots,t_{2K_2}=t_{31},\dots,t_{N1}\}$

with $t_{11}=\tau_1=a$; $t_{21}=\tau_2, \dots ,$ $t_{N1}=\tau_N=b$, and let diameter ω_2 be λ . The integral sum for ω_2 is of a form:

$\sum_{i=1}^{N-1}\sum_{j=1}^{K_i-1} F(\xi_{ij})\Delta t_{ij}$ where $\xi_{ij}\in[t_{ij},t_{ij+1}]$, $\Delta t_{ij}=t_{ij+1}-t_{ij}$.

On every segment $[\tau_i , \tau_{i+1}]$ while $\lambda\to 0$ the sum $\sum_{j=1}^{K_i-1} F(\gamma_i)\Delta t_{ij}$ tends to limit which is equal to $R\int_{\tau_i}^{\tau_{i+1}} F(\gamma_i)dt = co F(\gamma_i)\Delta\tau_i$.

Let us choose $\lambda_0>0$ such that for any ω_2 with diameter $\lambda\le\lambda_0$ for every $i=1,\dots,N-1$ we get: $h\left(\sum_{j=1}^{K_i-1}F(\gamma_i)\Delta t_{ij}, co F(\gamma_i)\Delta\tau_i\right)<\frac{\varepsilon}{3(N-1)}$.

From here $h\left(\sum_{i=1}^{N-1}\sum_{j=1}^{K_i-1}F(\gamma_i)\Delta t_{ij}, \sum_{i=1}^{N-1} co F(\gamma_i)\Delta\tau_i\right)<\frac{\varepsilon}{3}$. From $|\gamma_i-\xi_{ij}|<\delta$:

$$h\left(\sum_{i=1}^{N-1}\sum_{j=1}^{K_i-1}F(\xi_{ij})\Delta t_{ij}, \sum_{i=1}^{N-1}\sum_{j=1}^{K_i-1}F(\gamma_i)\Delta t_{ij}\right)\le\sum_{i=1}^{N-1}\sum_{j=1}^{K_i-1}h(F(\xi_{ij}),F(\gamma_i))\Delta t_{ij}<\frac{\varepsilon}{3}.$$

Hence for any $\lambda\le\lambda_0$: $h\left(\sum_{i=1}^{N-1}\sum_{j=1}^{K_i-1}F(\xi_{ij})\Delta t_{ij}, R\int_I co F(t)dt\right)<\varepsilon$.

Theorem 3. If map $F: I\to\Omega^n$ is continuous a.e. on I then it is summable by Riemann and $R\int_I F(t)dt = R\int_I co F(t)dt$.

Proof. Let I_1 be set of points of cease of $F(t)$ on I , then $mI_1=0$. Let $\alpha>0$ be such that $|F(t)|\le\alpha$ for $t\in I$. Let $\varepsilon>0$ and $\delta<\frac{\varepsilon}{6\alpha}$. There exists an open set I_2 such that $mI_2=\delta$ and $I_1\subset I_2\subset I$. For any subdivision ω the integral sum may be decomposed into two parts: $\sum F(\xi_i)\Delta t_i = \sum_I F(\xi_i)\Delta t_i + \sum_{II} F(\xi_i)\Delta t_i$, where $\sum_{II}$ consists of those indexes i for which $[t_i , t_{i+1}]\cap I_2\ne\emptyset$. By theorem 2 we can choose $\lambda_0>0$ such that for any subdivision with diameter $\lambda\le\lambda_0$ we'll have $\sum_{II}\Delta t_i<2\delta$ and $h(\sum_I F(\xi_i)\Delta t_i, R\int_{I\setminus I_2} co F(t)dt)<\frac{\varepsilon}{2}$.

Therefore
$$\begin{aligned} &h\left(R\int_I co F(t)dt, \sum F(\xi_i)\Delta t_i\right)\le h\left(R\int_{I\setminus I_2} co F(t)dt, \sum_I F(\xi_i)\Delta t_i\right)+\\ &+h\left(R\int_{I_2} co F(t)dt, \sum_{II}F(\xi_i)\Delta t_i\right)<\frac{\varepsilon}{2}+3\alpha\delta<\varepsilon. \end{aligned}$$

Theorem 4. The function $F: I\to\Omega^n$ is summable by Riemann iff the function $co F: I\to\Omega^n$ is continuous a.e. on I . Under this condition the following equalities hold:

$$R\int_I co F(t)dt = R\int_I F(t)dt = R\int_I \operatorname{tend} co F(t)dt; \quad R\int_I F(t)dt = L\int_I F(t)dt .$$

Proof. Let $co F(t)$ be continuous a.e. on I then $\operatorname{tend} co F(t)$ is continuous a.e. on I (lemma 3). By theorem 3 and by inclusion: $co F(t)\supset F(t)\supset \operatorname{tend} co F(t)$ we get the existence of riemannien integral of $F(t)$ and equality of three integrals. On the other hand if $F(t)$ is summable by Riemann then it is easy to show that $co\, F(t)$ is continuous a.e. on I (by lemmas 4 and 1). By equality of three integrals, by theorem 1 and proposition 3 we get:

$$R\int_I F(t)dt = L\int_I F(t)dt.$$

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