CHARACTERISTICS OF SATURATION OF THE CLASS OF CONVEX FUNCTIONS

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I. Formulation of the problem

In many cases real-valued functions are essential only in respect of perfect pre-orders they induce on their domains. Thus we define the following equivalence relation in the space $F(G)$ of all real-valued functions given on one and the same set G :

$$(f \sim g) \Longleftrightarrow (f(x) \leqslant f(y) \Longleftrightarrow g(x) \leqslant g(y)).$$

Equivalence classes F_f corresponding to functions $f \in F(G)$ have the following simple functional description:

$$F_f = \{ u \circ f : \ u \in U(T_f) \},$$

where $U(T_f)$ is the set of all increasing functions defined on the numerical set $T_f = f(G) = \{ f(x) : \ x \in G \}$ and $u \circ f$ is a superposition on u and f, that is function $g(x) = u(f(x)), \ x \in G$.

As usual the saturation of the class $\Phi \subset F(G)$ related to the above-described equivalence is defined as the set

$$U\Phi = \bigcup_{\varphi \in \Phi} F_\varphi = \{ u \circ \varphi : \ \varphi \in \Phi, \ u \in U(T_\varphi) \}.$$

In other words a function $f \in F(G)$ belongs to saturation $U\Phi$ of the class $\Phi \subset F(G)$ if there exists such a function $u \in U(T_f)$ that $u \circ f \in \Phi$.

If the class $\Phi \subset F(G)$ coincides with its saturation $U\Phi$ then it's called saturated.

Hereafter, it will be considered that G is arbitrary (containing more than one point) relatively open convex set in a real vector space, that is such a convex set that for any $x, y \in G$ there exists $\lambda > 0$ for which $y + \lambda(y - x) \in G$.

By $V(G)$ we shall designate the class of convex functions, that is such functions $f \in F(G)$ that

$$f(\lambda x + (1-\lambda) y) \leqslant \lambda f(x) + (1-\lambda) f(y) \tag{I}$$

holds for arbitrary $x, y \in G$ and $\lambda \in (0, 1)$. By $W(G)$ we shall designate the class of quasi-convex functions, that is such functions $f \in F(G)$ that

$$f(\lambda x + (1-\lambda) y) \leqslant \max\{f(x), f(y)\} \tag{2}$$

holds for any $x, y \in G$, $\lambda \in (0, I)$ and the inequality is strict when $f(x) \neq f(y)$.

It is clear that the class of quasi-convex functions is saturated. Hence, it contains saturation $UV(G)$ of more narrow class of convex functions. But there always exist functions $f \in W(G)$ which do not belong to $UV(G)$. In connection with this the American mathematician W.Fenchel stated a well-known problem of characterizing quasi-convex functions that belong to saturation $UV(G)$ of the class of convex functions (see [I], p.115-137). The present report is devoted to solving that problem.

2. Auxiliary Functions

Let $F_c(G)$ be a subspace that consists of functions $f \in F(G)$ such that their traces on the cross-cuts of G with any straight line is continuous.

Let us note now that it's sufficient to solve the question on the quasi-convex functions, we are interested in, for more narrow class $W_c(G) = W(G) \cap F_c(G)$.

Indeed, for every function $f \in W(G) \setminus W_c(G)$ it is possible to construct an equivalent function $f_0 = v \circ f$, where $v \in U(T_f)$ coincides with Lebesque measure of set $T_f \cap [t_0, t]$ when $t \in T_f$ is greater than some fixed $t_0 \in T_f$ and if $t \leqslant t_0$ then it equals Lebesque measure of set $T_f \cap [t, t_0]$ multiplied by - I. If in this case $f_0 \bar{\in} F_c(G)$ then the initial function $f \in W(G)$ does not undoubtedly belong to the set $UV(G)$, we are interested in. Otherwise, the question is to examine function $f_0 \in W_c(G)$.

It's known (see [2]) that if traces of function $f \in F(G)$ on cross-cuts of the set G with any straight line are measurable then inequalities (I) and (2) are implications of related inequalities with some fixed $\lambda \in (0,1)$, for instance, with $\lambda = \frac{1}{2}$. Specifically, it's true for all functions from $UF_c(G)$. And that means we may study question concerning functions from $f \in W_c(G)$ in terms of auxiliary functions

$$\tau_f(t,t') = \sup\left\{ f\left(\frac{x+y}{2}\right) : x \in f^{-1}(t),\ y \in f^{-1}(t') \right\}$$

defined on $T_f \times T_f$.

Evidently, function $f \in W_c(G)$ belongs to set $UV(G)$ if and only if such a function $u: T_f \to R$ exists that inequalities

$$u(t') - u(t) > 0, \quad u(t) + u(t') - 2u(\tau_f(t,t')) \geq 0 \qquad (3)$$

hold for any $t < t'$ from T_f. Under this condition the function $u \in U(T_f)$ is automatically continuous.

All functions $f \in F_c(G)$ evidently have connected ranges. At that if function $f \in W_c(G)$ is not constant it cannot attain its maximum on relatively open convex set G. So, its range T_f either coincides with its open core $\mathring{T}_f$ or contains, besides, one additional point $\theta = \min_{x \in G} f(x)$.

We shall call the point $t^* \in T_f$ regular, the function $f \in W_c(G)$ beeing fixed, if for some $\varepsilon > 0$ there is such a function $u: T \to R$, where $T = T_f \cap (t^* - \varepsilon, t^* + \varepsilon)$, that for any $t < t'$ from T inequalities (3) hold.

It's clear that if the function $f \in W_c(G)$ belongs to $UV(G)$ then all points of the set T_f are regular. The following inverse statement is also correct (its complete proof is given in [3]) :

<u>Theorem I</u>. If all points of set $\mathring{T}_f$ are regular for the function $f \in W_c(G)$ then the function is equivalent to some function $\varphi \in V(G)$.

So, if function $f \in W_c(G)$ does not belong to $U_c V(G) = F_c(G) \cap UV(G)$, then at least one interior point of set T_f is not regular. To complete the characterization of the set $U_c V(G)$ we are to find out necessary and sufficient regularity conditions of interior points of sets T_f corresponding to functions $f \in W_c(G)$.

3. Characteristics of regular points

Let us introduce auxiliary normed linear space of additive functions with finite supports belonging to some bounded connected set $T \subset R$. For the purpose let us assign to every $t \in T$ an additive function $\mu_t : 2^T \to R$ which equals 1 on sets $e \in 2^T$ containing t and equals 0 otherwise. Further let $\Phi_0(2^T)$ be the set of all finite linear combinations of introduced elementary functions with the coefficient sum equal to 0. The norm $\|\cdot\|_v$ in linear space $\Phi_0(2^T)$ is defined as the full variation of corresponding additive functions.

Let us assign now to every function $f \in W_c(G)$ and interval $T = [\alpha, \beta]$ from T_f a cone $K(f, T) \subset \Phi_0(2^T)$ which is the conical hull of the union of two sets:

$$A(f,T) = \{a_{tt'} = \mu_{t'} - \mu_t \ : \ t < t' \text{ from } T\},$$

$$B(f,T) = \{b_{tt'} = \mu_t + \mu_{t'} - 2\mu_{\tau_f(t,t')} : \ t < t' \text{ from } T\}.$$

According to one of non-classical separation theorems (see [4], theorem 9) and some properties derived from proof of theorem I of the preceding paragraph it is possible to demonstrate the validity of the following statements:

<u>Theorem 2</u>. Whatever a function $f \in W_c(G)$ and an interval $T = [\alpha, \beta]$ from T_f are, ($*$)-closure of the cone $K(f,T)$ coincides with closure of this cone in the topology of normed space $\Phi_0(2^T)$ and, hence, also with its closure in the strongest local-convex topology.

<u>Theorem 3</u>. The interior point t^* of the range T_f of the function $f \in W_c(G)$ is regular if and only if for some $\alpha < t^* < \beta$ from T_f the point $-a_{\alpha t^*}$ does not belong to the closure of the cone $K(f, [\alpha, \beta])$ in one of the three topologies of Theorem 2 (and , therefore, in all the topologies).

<u>Corollary I.</u> The necessary and sufficient regularity condition of inner point t^* of the range T_f of function $f \in W_c(G)$ is the existence of some $\alpha < t^* < \beta$ such that set $T = [\alpha, \beta]$ satisfies the condition

$$\rho_v(-a_{\alpha t^*}, K(f,T)) = \inf_{a \in K(f,T)} \|a + a_{\alpha t^*}\|_v > 0.$$

Corollary 2. If $f \in W_c(G)$ and all points of an interval $[\alpha_0, \beta_0]$ are regular, then for any $t^* \in (\alpha_0, \beta_0)$ inductively defined points

$$\alpha_i = \tau_f(\alpha_{i-1}, t^*), \quad \beta_i = \tau_f(t^*, \beta_{i-1}), \quad i = 1, 2, \dots$$

are such that for some natural number m inequalities

$$\tau_f(\alpha_i, \beta_{i+m}) < t^*$$

hold for all $i = 0, 1, 2, \dots$

With the help of the last corollary we can easily verify, for example, that for $G = (-3, 1)$ function

$$f(x) = \begin{cases} -x-2 & \text{if } -3 < x \leq -1 \\ x^3 & \text{if } -1 < x < 1 \end{cases}$$

from $W_c(G)$ is not equivalent to any convex function, because the point $0 \in \overset{\circ}{T_f}$ is not regular.

In conclusion let us mention one circumstance which is often useful for the practical solution of poblems of transformation of quasi-convex functions into the convex ones.

First of all it's clear that if functions $f \in W_c(G_1)$ and $g \in W_c(G_2)$ have the same range and $\tau_f(t, t') \leq \tau_g(t, t')$ is true for any $t < t'$ from $T_f = T_g$, then the function $u \in V(T_g)$ for which $u \circ g \in V(G_2)$, satisfies also the condition $u \circ f \in V(G_1)$. Specifically, it covers the case when $G_2 \subset R$ and function $g \in W_c(G_2)$ is increasing. Then the function $u = g^{-1} \in U(T_g) = U(T_f)$ can be taken as the transforming one for f.

Let us note here that if $T < R$ is a connected set without the largest element and a function $\tau : T \times T \to R$ is such that for any $t < t'$ from T

$$\tau(t, t) = t, \quad t < \tau(t, t') < t',$$

then for the existence of an increasing function g satisfying condition $\tau_g = \tau$, it's necessary and sufficient that for any t_1, t_2, t_3 and t_4 from T the following equality takes place

$$\tau(\tau(t_1, t_2), \tau(t_3, t_4)) = \tau(\tau(t_1, t_3), \tau(t_2, t_4)).$$

Besides, the function g we are interested in can be effectively constructed with the help of the function τ.

References.

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