A PROGAMMED CONSTRUCTION FOR THE POSITIONAL CONTROL

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Let the motion of a competitively controlled system be described by the differential equation

$$dx/dt = f(t,x,u,v) , \quad x(t_0) = x_0 , \tag{1}$$

where $x \in R^n$ is the phase vector of the system; u and v are the vectors controlling the actions of the players with restrictions $u[t] \in P \subset R^p$, $v[t] \in Q \subset R^q$; P and Q are compacts; the function $f(t,x,u,v)$ is continuous in the totality of the arguments and continuously differentiable in x . In addition, we will assume that the formulated in [1] condition of uniform extendability of the solutions for the equation (1) is fulfilled.

Given are some instant $\vartheta_0 > t_0$, a closed set $T \subset [t_0, \vartheta_0]$, a compact $\mathcal{M} = \{(\vartheta, m) : \vartheta \in T, \ m \in \mathcal{M}_\vartheta\}$ in R^{k+1} , where $\mathcal{M}_\vartheta = \{m : (\vartheta, m) \in \mathcal{M}\}$ and a function $\omega(\vartheta, x, m)$ defined on the set $\{(\vartheta, x, m) : (\vartheta, m) \in \mathcal{M}, x \in R^n\}$ is continuous in the totality of the arguments and continuously differentiable in x in the domain $\omega_0 < \omega < \omega^0$.

We will call a mixed strategy $\tilde{U} \div \mu_{\{t,x\}}(du)$ of the first player a function $\mu_{\{t,x\}}(du)$ which puts the Borel regular normed measures $\mu(du)$ on P in correspondence to any position $\{t,x\}$. Let us define a motion $x[t; t_0, x_0, \tilde{U}]$ generated by the strategy $\tilde{U}$ as any uniform limit of Euler splines $\tilde{x}_{\Delta^{(k)}}[t]$, for almost all $t \in [\tau_i^{(k)}, \tau_{i+1}^{(k)})$ satisfying the equation

$$d\tilde{x}_{\Delta^{(k)}}[t]/dt = \int_P \int_Q f(t, \tilde{x}_{\Delta^{(k)}}[t], u, v) \mu_{\{\tau_i^{(k)}, \tilde{x}_{\Delta^{(k)}}[\tau_i^{(k)}]\}}(du) \nu_t(dv) ,$$

$\tau_{i+1}^{(k)} - \tau_i^{(k)} \le \Delta^{(k)}$, $\Delta^{(k)} \to 0$ for $k \to \infty$, $\nu_t(dv)$ is a Borel regular weak measurable in t on $[t_0, \vartheta_0]$ function normed on Q, that is, a function

$$\alpha(t) = \int_Q \varphi(v)\, \nu_t(dv)$$

is a Lebesgue measurable function on $[t_0, \vartheta_0)$ for any arbitrary continuous function $\varphi(v) \in C(Q)$. Analogously a mixed strategy $\widetilde{V} \div \nu_{\{t,x\}}(dv)$ of the second player and a motion generated by this strategy and also a motion generated by the couple $\{\widetilde{U}, \widetilde{V}\}$ are defined.

<u>Problem I.</u> For a fixed position $\{t_0, x_0\}$ and a number c it is required to find a mixed strategy $\widetilde{U}$ which guarantees the inequality

$$\min_{\vartheta \in T} \min_{m_\vartheta \in \mathcal{M}_\vartheta} \omega(\vartheta, x[\vartheta], m) \leq c$$

for any motion $x[t; t_0, x_0, \widetilde{U}]$.

<u>Problem II.</u> For a fixed position $\{t_0, x_0\}$ and a number c it is required to find a mixed strategy $\widetilde{V}$ which guarantees the inequality

$$\min_{\vartheta \in T} \min_{m_\vartheta \in \mathcal{M}_\vartheta} \omega(\vartheta, x[\vartheta], m) \geq c$$

for any motion $x[t; t_0, x_0, \widetilde{V}]$.

Let us introduce an auxiliary programmed construction for solving these problems. Namely on the space of generalized programmed controls - Borel regular measures $\eta_t = \eta_t(du, dv)$ defined and normed on $P \times Q$ for all $t \in [t_0, \vartheta_0)$ and weak measurable in t on $[t_0, \vartheta_0)$ we will assign a totality of sets called programs.

Then let us define an elementary program $\{\mu_t \times \nu_t^*, [t_*, \vartheta)\}$ on $[t_*, \vartheta)$ as a set of all controls η_t on $[t_*, \vartheta)$ which are represented in the form of the direct product $\eta_t = \mu_t \times \nu_t^*$, where $\mu_t \in \{\mu_t\}$ and ν_t^* are weak measurable on $[t_*, \vartheta)$ Borel regular measures for any $t \in [t_*, \vartheta)$ defined and normed on P and Q respectively.

We will put in correspondence to each position $\{t_*, x_*\}$ $(t_* \in [t_0, \vartheta_0])$ the quantity

$$\tilde{\varepsilon}^0(t_*, x_*) = \min_{\vartheta \in T} \max_{\nu_t} \min_{\mu_t} \min_{m \in \mathcal{M}} \omega(\vartheta, x(\vartheta; t_*, x_*, \mu_{(\cdot)} \times \nu_{(\cdot)}), m) \qquad (2)$$

where $x(t) = x(t; t_*, x_*, \mu_{(\cdot)} \times \nu_{(\cdot)})$ is a programmed motion satisfying for almost all $t \in [t_*, \vartheta)$ the equation

$$dx/dt = \int\limits_P \int\limits_Q f(t,x,u,v)\mu_t(du)\nu_t(dv) \quad . \quad (3)$$

Given by(2) optimal programmed controls μ_t°, ν_t° and $m^\circ \in \mathcal{M}$, $\vartheta^\circ \in T$ exist on account of weak compactness in themselves of the elementary programs and the set $\{\nu_t\}_{[t_*,\vartheta)}$ of the controls ν_t on $[t_*, \vartheta)$. Incidentally under the weak convergence of the sequences we understand the convergence in $*$ - weak topology of the sequences of continuous linear functionals defined by the Borel regular measures $\mu^{(k)} = \mu_t^{(k)} \cdot m(dt)$, $\nu^{(k)} = \nu_t^{(k)} \cdot m(dt)$, $m(\cdot)$ - the Lebesgue measure on R^1.

We will say that the elementary program $\{\mu_t \times \nu_t^\circ, [t_*, \vartheta)\}$, where ν_t° is an optimal control for $\{t_*, x_*\}$, is regular in position $\{t_*, x_*\}$, if the problem (2) has an essentially unique solution μ_t° for the fixed control ν_t°, in addition, the minimal point $m^\circ \in \mathcal{M}$ is also unique.

There is valid the following assertion which is an analogy of the maximum principle [2] in the case under study.

Theorem I. Let the regularity condition of the program $\{\mu_t \times \nu_t^\circ, [t_*, \vartheta)\}$ be fulfilled and $\tilde{\varepsilon}^\circ(t_*, x_*) \in (\omega_\circ, \omega^\circ)$. Then for the optimal programmed motion $x^\circ(t) = x(t; t_*, x_*, \mu^\circ_{(\cdot)} \times \nu^\circ_{(\cdot)})$ there is the following minimax condition

$$\int\limits_P \int\limits_Q s^{\circ\prime}(\vartheta, t) f(t, x^\circ(t), u, v)\mu_t^\circ(du)\nu_t^\circ(dv) =$$

$$= \min_\mu \max_\nu \int\limits_P \int\limits_Q s^{\circ\prime}(\vartheta, t) f(t, x^\circ(t), u, v)\mu(du)\nu(dv)$$

for almost all $t \in [t_*, \vartheta]$. Here $s^{\circ\prime}(\vartheta, t) = -[\partial\omega(\vartheta, x^\circ(\vartheta), m^\circ)/\partial x]' \cdot S(\vartheta, t_*, x^\circ(\cdot), \mu^\circ_{(\cdot)} \times \nu^\circ_{(\cdot)})$, $S(t, t_*, x(\cdot), \mu_{(\cdot)} \times \nu_{(\cdot)})$ is the fundamental solution matrix of the first variational approximation equation for equation (3) computed on the motion $x(t) = x(t; t_*, x_*, \mu_{(\cdot)} \times \nu_{(\cdot)})$, the prime denotes transposition.

Let $\nu_t^{(k)} \xrightarrow{w} \nu_t$ and $\{\eta_t\}^*$ be a set of all controls η_t for each of which exists such a sequence $\{\mu_t^{(k)}\}$

that $\mu_t^{(k)} \times \nu_t^{(k)} \xrightarrow{w} \eta_t$. Then the weak closure of the set $\{\eta_t\}^*$ in $*$ - weak topology $C^*([t_*,\vartheta] \times P \times Q)$ we will call the program $\Pi(\nu_t^*)$. We will say that the program $\Pi(\nu_t^\circ)$ is optimal, if a sequence which forms it is maximizing for $\{t_*, x_*\}$. We will call ν_t° a regular control, if for any optimal $\Pi(\nu_t^\circ)$ the minimizing control η_t° in it is unique and also unique is $m^\circ \in \mathcal{M}$. Let us denote by $T^\circ(t_*, x_*)$ the set of the problem's (2) solutions ϑ° and by $\tilde{S}^\circ(t_*, x_*; \vartheta^\circ)$ the set of all vectors $s^\circ(\vartheta^\circ, t)$, $\vartheta^\circ \in T^\circ(t_*, x_*)$ corresponding to all kinds of η_t° for $\{t_*, x_*\}$.

We shall suppose that for any $\{t_*, x_*\}$ $(t_\circ \le t_* \le \le \vartheta_\circ)$, $t_* \notin T^\circ(t_*, x_*)$, where $\tilde{\varepsilon}^\circ(t_*, x_*) \in (\omega_\circ, \omega^\circ)$ and any Borel regular normed measure $\nu^*(dv)$ on Q there exists an instant $\vartheta^\circ \in T^\circ(t_*, x_*)$ for which two following conditions are fulfilled.

A. Any control ν_t° is regular for $\{t_*, x_*\}$.

B. There exists a Borel regular normed measure $\mu^*(du)$ on P such that for any $s^\circ(\vartheta^\circ, t_*) \in \tilde{S}^\circ(t_*, x_*; \vartheta^\circ)$

$$\int_P \int_Q s^{\circ\prime}(\vartheta^\circ, t_*) f(t_*, x_*, u, v) \mu^*(du) \nu^*(dv) \le$$

$$\le \min_{\mu} \max_{\nu} \int_P \int_Q s^{\circ\prime}(\vartheta^\circ, t_*) f(t_*, x_*, u, v) \mu(du) \nu(dv) .$$

<u>Theorem 2.</u> Let $\tilde{\varepsilon}^\circ(t_\circ, x_\circ) \le c \in (\omega_\circ, \omega^\circ)$ and conditions A, B be fulfilled. Then the mixed strategy $\tilde{U}^{(e)}$ which is extremal [3] to the set $\tilde{W}_c = \{\{t, x\} : \tilde{\varepsilon}^\circ(t, x) \le c\}$ solves Problem I.

Let us denote by $\Sigma(t_*, x_*; \vartheta^\circ)$ the set of all ν_t° for $\{t_*, x_*\}$ and some $\vartheta^\circ \in T^\circ(t_*, x_*)$, $\tilde{S}^\circ(t_*, x_*) = = \bigcup_{T^\circ(t_*, x_*)} \tilde{S}^\circ(t_*, x_*; \vartheta^\circ)$. We shall now suppose that two following conditions are fulfilled for any $\{t_*, x_*\} : \tilde{\varepsilon}^\circ(t_*, x_*) \in (\omega_\circ, \omega^\circ)$ instead of A, B.

C. Sets $\Sigma(t_*, x_*; \vartheta^\circ)$ are upper weak semicontinuous in each point $\vartheta^\circ \in T^\circ(t_*, x_*)$.

D. There exists $\nu_*(dv)$ on Q for any $\mu_*(du)$ on P such that for any $s^\circ(\vartheta^\circ, t_*) \in \tilde{S}^\circ(t_*, x_*)$

$$\int\limits_P \int\limits_Q s^{\circ\prime}(\vartheta^\circ, t_*) f(t_*, x_*, u, v)\, \mu_*(du)\, \nu_*(dv) \geqslant$$

$$\geqslant \max_{\nu} \min_{\mu} \int\limits_P \int\limits_Q s^{\circ\prime}(\vartheta^\circ, t_*) f(t_*, x_*, u, v)\, \mu(du)\, \nu(dv).$$

Theorem 3. Let $\tilde{\varepsilon}^\circ(t_\circ, x_\circ) \geqslant c \in (\omega_\circ, \omega^\circ)$ and conditions C, D be fulfilled. Then the mixed strategy $\tilde{V}^{(c)}$ which is extremal to the set $\tilde{W}^{(c)} = \{\{t, x\} : \tilde{\varepsilon}^\circ(t, x) \geqslant c\}$ solves Problem II.

With conditions A – D fulfilled simultaneously, the situation of equilibrium takes place.

If the saddle point condition

$$\min_{u \in P} \max_{v \in Q} s' \cdot f(t, x, u, v) = \max_{v \in Q} \min_{u \in P} s' \cdot f(t, x, u, v)$$

for the minor game is fulfilled for any s, t, x, then the problems I, II are solvable in pure strategies.

R e f e r e n c e s

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