AN EXTREMAL CONTROL IN DIFFERENTIAL GAMES

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Let us consider a system

$$\frac{dx}{dt} = f(t, x, u, v); \; x \in R^n, \; u \in P \subset R^p, \; v \in Q \subset R^q. \quad (1)$$

Here P and Q are compact sets, $f(\cdot)$ is a continuous function, continuously differentiable in x. It is assumed that the following condition of uniform extendability of the solutions holds: for any bounded set $K \subset R^n$ there exists $\beta = \beta(K) > 0$ such that any solution $x(t) = x(t, t_*, x_*)$ of the equation

$$\frac{dx}{dt} \in \overline{co}\,\{f : f = f(t, x, u, v),\; u \in P,\; v \in Q\}$$

with the initial condition

$$t_* \in [t_0, \vartheta], \quad x_* \in K \quad \text{is}$$

uniformly bounded in the segment $[t_*, \vartheta]$: $\|x(t)\| \leq \beta$. Here the symbol $\|\cdot\|$ denotes Euclidean norm.

There is given a function $\omega(x, m)$ defined on the space $R^n \times M$ where M is a compact subset of R^m, which is continuous and continuously differentiable in x on domain $\omega \in (\omega_0, \omega^0)$, $\omega_0 < \omega^0$.

The first player by choosing a control $u \in P$, tries to minimize the value of the functional $\rho_M(x[\vartheta]) = \min_M \omega(x[\vartheta], m)$. The second player by choosing a control tries to maximize it.

Similarly to [1], any upper semicontinuous with respect to inclusion in (t, x) function $U(t, x) \subset P$ will be called an admissible strategy U of the first player. The admissible strategy V of the second player can be defined similarly; any upper semicontinuous with respect to inclusion in (t, x, v) function $U_v(t, x, v) \subset P$ will be called a counterstrategy U_v of the first player. We shall say that a function $x[t]$ is the motion generated by a strategy U

(counterstrategy $\mathfrak{U}_v$) if it is a solution of the equation

$$\frac{dx}{dt} \in \overline{co}\{f : f = f(t, x, u, v);\ u \in \mathfrak{U}(t, x),\ v \in Q\}$$

$$x[t_0] = x_0;$$

$$\frac{dx}{dt} \in \overline{co}\{f : f = f(t, x, u, v);\ u \in \mathfrak{U}_v(t, x, v),\ v \in Q\}$$

$$x[t_0] = x_0 .$$

Any solution of the equation

$$\frac{dx}{dt} \in \overline{co}\{f : f = f(t, x, u, v);\ u \in \mathfrak{U}(t, x), v \in V(t, x)\}$$

$$x[t_0] = x_0$$

will be called the motion generated by a pair of strategies $(\mathfrak{U}, V)$.

<u>Problem I.</u> There is required to build an optimal counter-strategy $\mathfrak{U}_v^\circ$ such that:

$$\max_{\{x[\cdot] | \mathfrak{U}_v^\circ\}} \min_M \omega(x[\vartheta], m) = \min_{\{\mathfrak{U}_v\}} \max_{\{x[\cdot] | \mathfrak{U}_v\}} \min_M \omega(x[\vartheta], m)$$

And if the condition

$$\min_P \max_Q s' f(t, x, u, v) = \max_Q \min_P s' f(t, x, u, v) \quad (2)$$

holds for every $s \in R^n$, $t \in [t_0, \vartheta]$, $x \in R^n$ it is required to find a solution in the class of admissible strategies.

<u>Problem 2.</u> It is required to find a pair $(\mathfrak{U}^\circ, V^\circ)$ of optimal strategies which satisfies

$$\min_{\{\mathfrak{U}\}} \max_{\{x[\cdot] | \mathfrak{U}\}} \min_M \omega(x[\vartheta], m) =$$

$$= \max_{\{x[\cdot] | \mathfrak{U}^\circ\}} \min_M \omega(x[\vartheta], m) =$$

$$= \min_{\{x[\cdot] | V^\circ\}} \min_M \omega(x[\vartheta], m) =$$

$$= \max_{\{V\}} \min_{\{x[\cdot] | V\}} \min_M \omega(x[\vartheta], m) .$$

Let us call a set of all regular Borel measures $(r.B.m.)\, \eta(\cdot)$ on $[t_*, \vartheta] \times P \times Q$, having Lebesgue projection [2, 3] on $[t_*, \vartheta]$, a class $\{H(m(\cdot)), [t_*, \vartheta]\}$ of admissible open-loop controls of the first player on the segment $[t_*, \vartheta]$. A set of all $r.B.m.$ $\nu(\cdot)$ on $[t_*, \vartheta] \times Q$ having Lebesgue projection on $[t_*, \vartheta]$ will be called a class $\{E(m(\cdot)), [t_*, \vartheta]\}$ of admissible open-loop controls of the second player on the segment $[t_*, \vartheta]$. The set of all controls $\eta(\cdot) \in \{H(m(\cdot)), [t_*, \vartheta]\}$ coordinated [2, 3] with open-loop control $\nu(\cdot)$ of the second player we shall call a program $\{\Pi(\nu(\cdot)), [t_*, \vartheta]\}$.

To every open-loop control can be put in correspondence a program motion [2, 3] $\varphi(\cdot, t_*, x_*, \eta(\cdot))$ with initial condition (t_*, x_*). Let $G(\vartheta, t_*, x_*, \nu(\cdot))$ be a set of attainability [1, 3] for a program $\{\Pi(\nu(\cdot)), [t_*, \vartheta]\}$ at the moment ϑ

$$\varepsilon^\circ(t_*, x_*) = \max_{\{E(m(\cdot)), [t_*, \vartheta]\}} \min_{G(\vartheta, t_*, x_*, \nu(\cdot))} \min_{M} \omega(x, m). \quad (3)$$

Open-loop controls giving minimum and maximum (3) will be called optimal ones.

<u>Theorem I.</u> Let the following condition hold for a position (t_*, x_*) and open-loop control $\nu^*(\cdot) \in \{E(m(\cdot)), [t_*, \vartheta]\}$:

$$\min_{G(\vartheta, t_*, x_*, \nu^*(\cdot))} \min_{M} \omega(x, m) \in (\omega_\circ, \omega^\circ).$$

Then the optimal open-loop control $\eta^\circ(\cdot)$ in the program $\{\Pi(\nu^*(\cdot)), [t_*, \vartheta]\}$ satisfies the minimum principle

$$\int_\Delta \int_P \int_Q s_\circ'(t) f(t, \varphi^\circ(t), u, v)\, \eta^\circ(dt \times du \times dv) =$$

$$= \int_\Delta \int_Q \min_P s_\circ'(t) f(t, \varphi^\circ(t), u, v)\, \nu^*(dt \times dv)$$

where Δ is an arbitrary Borel subset of $[t_*, \vartheta]$,

$$\varphi^\circ(t) = \varphi(t, t_*, x_*, \eta^\circ(\cdot)),$$

$$s_\circ'(t) = \left[\frac{\partial}{\partial x} \omega(\varphi^\circ(\vartheta), m^\circ)\right]' S(\vartheta, t, \varphi^\circ(\cdot), \eta^\circ(\cdot)),$$

m° gives a minimum to $\min_M \omega(\varphi^\circ(\vartheta), m)$, $S(\vartheta, t, \varphi^\circ(\cdot), \eta^\circ(\cdot))$ is fundamental solution matrix for a corresponding system in variations.

<u>Theorem 2.</u> Let $\varepsilon^\circ(t_*, x_*) \in (\omega_\circ, \omega^\circ)$ and an open-loop control $\nu^\circ(\cdot) \in \{E(m(\cdot)), [t_*, \vartheta]\}_{opt}$ be such that an optimal control $\eta^\circ(\cdot)$ in the program $\{\Pi(\nu^\circ(\cdot)), [t_*, \vartheta]\}$ be unique, moreover a point $m^\circ \in M$, giving $\min_M \omega(\varphi^\circ(\vartheta), m)$, be also unique. Then for arbitrary Borel subset Δ

$$\int\limits_{\Delta}\int\limits_{P}\int\limits_{Q} s_0'(t) f(t,\varphi^\circ(t),u,v)\,\eta^\circ(dt\times du\times dv) =$$

$$= \int\limits_{\Delta} \max_{Q} \min_{P} s_0'(t) f(t,\varphi^\circ(t),u,v)\, m(dt).$$

Here notations $\varphi^\circ(t)$, $s_0(t)$ have the same meaning as in theorem 1.

For every position (t_*, x_*), $\varepsilon^\circ(t_*,x_*)\in(\omega_0,\omega^\circ)$, denote by $S_0(t_*,x_*)$ a set of all vectors s_0 of the form

$$s_0' = \left[\frac{\partial}{\partial x}\,\omega(\varphi_0(\vartheta), m_0)\right]' S(\vartheta, t_*, \varphi_0(\cdot), \eta_0(\cdot)), \tag{4}$$

where $\varphi_0(\cdot) = \varphi(\cdot, t_*, x_*, \eta_0(\cdot))$, $\eta_0(\cdot)\in\{\Pi(\nu_0(\cdot)), [t_*,\vartheta]\}$ $(\eta^\circ(\cdot), \nu_0(\cdot))$ are optimal for the position (t_*, x_*), m_0 gives $\min_M \omega(\varphi_0(\vartheta), m)$. It will be said that a game is perfectly regular if the set $S_0(t_*, x_*)$ consists of only one vector $s_0 = s_0(t_*,x_*)$ for every position (t_*,x_*), $\varepsilon^\circ(t_*,x_*)\in(\omega_0,\omega^\circ)$.

<u>Lemma I.</u> Let the game be perfectly regular. Then ε° is a continuously differentiable function in domain $\{\varepsilon^\circ\in(\omega_0,\omega^\circ), t\in(t_0,\vartheta)\}$ and

$$\left.\frac{\partial\varepsilon^\circ}{\partial x}\right|_{(t,x)} = s_0(t,x),\quad \left.\frac{\partial\varepsilon^\circ}{\partial t}\right|_{(t,x)} = -\max_{Q}\min_{P} s_0'(t,x)\, f. \tag{5}$$

Extremal strategies under the condition of perfect regularity on domain of differentiability are defined by the relations :

$$U^\circ(t_*,x_*) = \{u^\circ : u^\circ\in P,\ \max_{Q} s_0'(t_*,x_*) f(t_*,$$
$$x_*, u^\circ, v) = \min_{P}\max_{Q} s_0'(t_*,x_*) f(t_*,x_*,u,v)\},$$

$$V^\circ(t_*,x_*) = \{v^\circ : v^\circ\in Q,\ \min_{P} s_0'(t_*,x_*) f(t_*,$$
$$x_*, u, v^\circ) = \max_{Q}\min_{P} s_0'(t_*,x_*) f(t_*,x_*,u,v)\},$$

$$U^\circ_v(t_*,x_*,v) = \{u^\circ :\ u^\circ\in P,\ s_0'(t_*,x_*) f(t_*,$$
$$x_*, u^\circ, v) = \min_{P} s_0'(t_*,x_*) f(t_*,x_*,u,v)\}$$

Theorem 3. Let the game be perfectly regular and $\varepsilon^\circ(t_0, x_0) \in [\omega_0, \omega^\circ)$. Then a counterstrategy U°_v (and under condition (2) strategy U°) solves the problem 1. If (2) holds and $\varepsilon^\circ(t_0, x_0) \in (\omega_0, \omega^\circ)$, a pair ($U^\circ, V^\circ$) solves the problem 2 and $\varepsilon^\circ(t_0, x_0)$ is the cost of the positional game in pure strategies.

For every optimal for the position (t_*, x_*) open-loop control $\nu_0(\cdot)$ of the second player let us find a set $S_0(t_*, x_*, \nu_0(\cdot))$ of all vectors s_0 determined by (4) with fixed control $\nu_0(\cdot)$. Assume for every position (t_*, x_*) $t_* \in (t_0, \vartheta)$ and $\varepsilon^\circ \in (\omega_0, \omega^\circ)$ for every optimal open-loop control $\nu_0(\cdot)$ of the second player the set $S_0(t_*, x_*, \nu_0(\cdot))$ consists of the unique vector $s_0 = s_0(t_*, x_*, \nu_0(\cdot))$. This condition is fulfilled, for example, when

$$f(t, x, u, v) = A(t)x + f(t, u, v),$$

the set $M \subset R^n$ is convex and closed, and $\omega(x, m) = \|x - m\|$.
Theorem 4. Let $\varepsilon^\circ(t, x)$ be differentialable in the domain $\Gamma = \{t \in (t_0, \vartheta), \varepsilon^\circ \in (\omega_0, \omega^\circ)\}$. Then for every position $(t_*, x_*) \in \Gamma$ the set $S_0(t_*, x_*)$ consists of the unique vector $s_0 = s_0(t_*, x_*)$ and the partial derivatives of $\varepsilon^\circ(t, x)$ satisfy (5).

References

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